

TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATIONS OF THE GROUP $GL(3, \mathbb{F}_q)$

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ABSTRACT. We describe the tensor products of two irreducible linear complex representations of the group $G = GL(3, \mathbb{F}_q)$ in terms of induced representations by linear characters of maximal torii and also in terms of classical and generalized Gelfand-Graev representations. Our results include MacDonal’s conjectures for G and at the same time they are extensions to G of finite counterparts to classical results on tensor products of holomorphic and antiholomorphic representations of the group $SL(2, \mathbb{R})$. Moreover they provide an easy way to decompose these tensor products, with the help of Frobenius reciprocity. We also state some conjectures for the general case of $GL(n, \mathbb{F}_q)$.

1. INTRODUCTION

This paper studies the tensor product of two irreducible linear complex representations of the group $G = GL(3, \mathbb{F}_q)$, generalizing previous work [1] of the first two named authors on $GL(2, \mathbb{F}_q)$ and $SL(2, \mathbb{F}_q)$.

Our main idea is to express tensor products of irreducible representations in terms of induced representations. Indeed, this paper adds to the experimental evidence which strongly suggests that, for classical groups, tensor products of generic irreducible representations are essentially induced representations by suitable linear characters from either of the involved torii. Here “essentially” means “up to lower dimensional correcting terms”.

General results of this type are of interest for several reasons.

First, for finite classical groups, tensor products of irreducible representations realized as induced representations of this kind, have appeared already in the context of the famous MacDonal’s conjectures, which state, in the case of $GL(n, \mathbb{F}_q)$, that the tensor product of the canonical Steinberg representation St with a generic irreducible representation $R_T(\theta)$, associated to a character θ , in general position, of a maximal torus T of $GL(n, \mathbb{F}_q)$, equals the representation induced by θ from T to $GL(n, \mathbb{F}_q)$. These conjectures were proved much later by Deligne and Lusztig as a corollary of their construction in [4].

Second, it was pointed out to us by A. Guichardet, that remarkably enough theoretical physicists, like Rideau [6], have been interested on their own in describing tensor products of irreducible unitary representations of a same series for classical real Lie groups like $SL(2, \mathbb{R})$, proving that the tensor product of holomorphic and antiholomorphic discrete series representations of $SL(2, \mathbb{R})$ is given by a suitable

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induced representation from the corresponding torus. Rideau's results were extended to principal series representations as well by Guichardet, and we realized that analogous results would hold for $GL(2, \mathbb{F}_q)$ if adequate correcting terms were introduced. So in fact, Rideau's results and MacDonal'd's conjectures dwell under the same roof. A complete description of tensor products of irreducible representations of $GL(2, \mathbb{F}_q)$ and $SL(2, \mathbb{F}_q)$ in terms of induced representations, was then given in [1].

Third, tensor products of irreducible representations may be decomposed quite easily, via Frobenius reciprocity, once you have described them in terms of induced representations (see section 4 below).

Fourth, the realization of the tensor product of two irreducible representations as an induced representation from a linear character also allows us to guess and glean interesting relations between special functions of various sorts. Indeed, such an induced representation may be looked upon as a "twisted" natural representation, for which spherical functions may be calculated, in the multiplicity-free case, as in [8]. For instance, recent work [5] on the relationship between classical Kloosterman sums and the so called Legendre sums and Soto-Andrade sums for $G = PGL(2, q)$, may be understood as a consequence of the fact that

$$St \otimes St = (Ind_{T \uparrow G} \mathbf{1}) \oplus St$$

where T denotes the anisotropic torus of G , if you recall that this induced representation is just the natural multiplicity-free representation of G associated to its homographic action on the (double cover of) finite Poincaré's upper half plane, whose spherical functions are given by the last two aforementioned sums [8]. Also Gel'fand-Graev representations appear in this way (see subsection 3.2 below).

Fifth, there is, on the other hand, a non obvious but close connections between tensor products and Gelfand models for the classical groups. Recall that a Gelfand model for a group G is any representation of G which decomposes with multiplicity one as the sum of all the irreducible representations of G .

Intriguing experimental evidence suggests that quite often Gelfand models or "quasi-models" may be obtained as tensor products of Steinberg representations. The first case is $G = PGL(2, q)$, where the tensor square of the Steinberg representation affords a quasi-model for G , where only the sign representation is missing. Analogous results seem to hold for $PGL(n, q)$. It is indeed very interesting to realize a Gelfand model or quasi-model as an induced representation from a linear character, specially from the unit character. In the latter case we say that we have a geometrical Gelfand model or quasi-model.

Sixth, although the problem of decomposing tensor products in the p-adic case was treated in [2, 3, 9], apparently tensor products of two different series of representations have not been studied, and the description of this tensor products as induced representation by linear characters from torii has not been worked out.

So, in this paper we describe tensor products of two irreducible representations of the group $G = GL(3, \mathbb{F}_q)$ as induced representations by linear characters from torii of G , up to adequate lower dimensional correcting terms. We notice that the situation is more complex than in dimension 2, and subtler in some cases (see for example case 10 of theorem 1 below). We concentrate here on "generic" irreducible representations, since the "non generic" or "degenerate" lower dimensional irreducible representations may be obtained from limiting cases of the generic ones, as

in lemma 2. We show then how to get formulas for the non generic representations from the generic ones.

Depending on the series of representations to which an irreducible representation belongs, a torus is associated in a natural way, and it is the one we use in the corresponding formula. Thus in the cases where we deal with the tensor product of two irreducible representations belonging to two different series, we give two descriptions, one in terms of each torus.

However, another approach is also possible in this case: to express our tensor product in terms of the intersection of the two different torii (the center $Z(G)$ of G) times the upper unipotent subgroup N . More specifically, let α be a character of $k^\times = Z(G) = Z$ and let ψ be a non trivial character of k^\times seen as character on N by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \psi(x + y).$$

The classical Gelfand-Graev representation of G is the representation $Ind_{ZN}^G(\alpha\psi)$. We call *generalized Gelfand-Graev representation* of G , any representation of the form $Ind_{ZN_1}^G(\alpha\psi)$, where N_1 is the subgroup of N defined by the condition $x = 0$. Proposition 1 below describes the tensor product of a discrete series representation and a principal series representation of G as the sum of these two Gelfand-Graev representations, classical and generalized.

A similar result should be obtained in the general case of dimension n . We state this as a conjecture, after proposition 1 below.

The paper is organized as follows: section 2 gives the preliminaries, mainly the character table of the group G ; section 3 is devoted to our main results.

2. PRELIMINARIES

2.1. Notations. Let \mathbb{F}_q be the finite field of $q = p^n$ elements, p prime. Then the quadratic and cubic extensions of \mathbb{F}_q are \mathbb{F}_{q^2} and \mathbb{F}_{q^3} respectively. We identify $z \in \mathbb{F}_{q^3}$ with the \mathbb{F}_q -automorphism of \mathbb{F}_{q^3} given by $x \mapsto zx$, and we identify $w \in \mathbb{F}_{q^2}$ with the \mathbb{F}_q -automorphism of \mathbb{F}_{q^2} given by $x \mapsto wx$. We also denote by z the matrix of the above automorphism with respect to the basis $\{1, \sigma, \sigma^2\}$, σ a generator of \mathbb{F}_{q^3} over \mathbb{F}_q . Similarly, for w in \mathbb{F}_{q^2} , w also denotes the matrix of the automorphism $x \mapsto wx$ with respect to the basis $\{1, \tau\}$, τ a generator of \mathbb{F}_{q^2} over \mathbb{F}_q .

The above defines a monomorphism of $\mathbb{F}_{q^3}^\times$ into G whose image is the anisotropic torus T_a .

We denote by T_i the isotropic torus of G ; thus T_i is the image of $\mathbb{F}_q^\times \times \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ in G . Similarly we denote by T_m the intermediate torus

$$T_m = \left\{ \begin{pmatrix} x & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{F}_q^\times, x \in \mathbb{F}_{q^2}^\times \right\}.$$

Finally, we set L equal to the intermediate Levi subgroup $GL(2, \mathbb{F}_q) \times \mathbb{F}_q^\times$.

We adopt the convention that η denotes a representation of G as well as its character.

2.2. The conjugacy classes of $G = GL(3, \mathbb{F}_q)$. The elements

$$T^a = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}; \quad T_1^a = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}; \quad T_{11}^a = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix}$$

$$T^{ab} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}; \quad T_1^{ab} = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}; \quad T^{abc} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

(where $a \neq b$, $a \neq c$, $b \neq c$) are representatives of different conjugacy classes in G . In addition, if we set

$$T^{\varkappa a} = \begin{pmatrix} \varkappa & 0 \\ 0 & a \end{pmatrix}; \quad T^z = z \quad (\text{where } a \in \mathbb{F}_q^\times, \varkappa \in \mathbb{F}_{q^2}^\times \text{ and } z \in \mathbb{F}_{q^3}^\times),$$

then the set of 3 by 3 matrices $\{T^a, T_1^a, T_{11}^a, T^{ab}, T_1^{ab}, T^{abc}, T^{\varkappa a}, T^z\}$ is a full set of representatives of the conjugacy classes of G .

Moreover the following holds.

Lemma 1. *We have*

- (1) For $T = T_1^a, T_{11}^a, T_1^{ab}$, we have $\{X \in G \mid XTX^{-1} \in T_i\} = \phi$ and also $\{X \in G \mid XT^{\varkappa a}X^{-1} \in T_i\} = \{X \in G \mid XT^zX^{-1} \in T_i\} = \phi$.
- (2) $\{X \in G \mid XT^{ab}X^{-1} \in T_i\} =$

$$= \left\{ \left(\begin{pmatrix} r & s & 0 \\ t & u & 0 \\ 0 & 0 & k \end{pmatrix}, \begin{pmatrix} k & 0 & 0 \\ 0 & r & s \\ 0 & t & u \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \right. \\ \left. \begin{pmatrix} r & 0 & s \\ 0 & k & 0 \\ t & 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mid k \in \mathbb{F}_q^\times, \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL(2, \mathbb{F}_q) \right\}$$

$$\text{We note that } \begin{pmatrix} 0 & 0 & k \\ r & s & 0 \\ t & u & 0 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ 0 & r & s \\ 0 & t & u \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} r & s & 0 \\ 0 & 0 & k \\ t & u & 0 \end{pmatrix} = \begin{pmatrix} r & 0 & s \\ 0 & k & 0 \\ t & 0 & u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- (3) $\{X \in G \mid XT^{abc}X^{-1} \in T_i\} = T_i \times S_3$
- (4) For $T = T_1^a, T_{11}^a, T_1^{ab}$ we get $\{X \in G \mid XTX^{-1} \in T_a\} = \phi$; moreover $\{X \in G \mid XT^{abc}X^{-1} \in T_a\} = \phi$ and $\{X \in G \mid XT^{\varkappa a}X^{-1} \in T_a\} = \phi$
- (5) $\{X \in G \mid XT^zX^{-1} \in T_a\} = T_a \times \Gamma_3$
where Γ_3 is the Galois group of the cubic extension generated by Frobenius automorphism, acting naturally on T_a .
- (6) For $T = T_1^a, T_{11}^a, T_1^{ab}$, we get $\{X \in G \mid XTX^{-1} \in T_m\} = \phi$, and also $\{X \in G \mid XT^zX^{-1} \in T_m\} = \phi$
- (7) $\{X \in G \mid XT^{ab}X^{-1} \in T_m\} = L$
- (8) $\{X \in G \mid XT^{\varkappa a}X^{-1} \in T_m\} = T_m \times \left\{ \begin{pmatrix} \Gamma_2 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

where Γ_2 is (isomorphic to) the Galois group of the quadratic extension of the field \mathbb{F}_q , acting through Frobenius automorphism of the quadratic extension.

2.3. The characters of $G = GL(3, \mathbb{F}_q)$. The following table gives the irreducible characters of the group G . This result leans on the work of Steinberg [7].

Character Table of $GL(3, \mathbb{F}_q)$

Elementary divisors of conjugacy classes	Representatives	χ_α^1	$\chi_\alpha^{q^2+q}$	$\chi_\alpha^{q^3}$
$x - a, x - a, x - a$	T^a	$\alpha^3(a)$	$(q^2 + q) \alpha^3(a)$	$q^3 \alpha^3(a)$
$1, x - a, (x - a)^2$	T_1^a	$\alpha^3(a)$	$q \alpha^3(a)$	0
$1, 1, (x - a)^3$	T_{11}^a	$\alpha^3(a)$	0	0
$1, x - a, (x - a)(x - b)$	T^{ab}	$\alpha(a^2b)$	$(q + 1) \alpha(a^2b)$	$q \alpha(a^2b)$
$1, 1, (x - a)^2(x - b)$	T_1^{ab}	$\alpha(a^2b)$	$\alpha(a^2b)$	0
$1, 1, (x - a)(x - b)(x - c)$	T^{abc}	$\alpha(abc)$	$2\alpha(abc)$	$\alpha(abc)$
$1, 1, (x - a)(x - \varkappa)(x - \varkappa^q)$	$T^{\varkappa a}$	$\alpha(a \varkappa \varkappa^q)$	0	$-\alpha(a \varkappa \varkappa^q)$
$1, 1, (x - z)(x - z^q)(x - z^{q^2})$	T^z	$\alpha(z z^q z^{q^2})$	$-\alpha(z z^q z^{q^2})$	$\alpha(z z^q z^{q^2})$
$a, b, c \in \mathbb{F}_q^\times, a \neq b \neq c \neq a$ $\varkappa \in \mathbb{F}_{q^2} - \mathbb{F}_q, z \in \mathbb{F}_{q^3} - \mathbb{F}_q$				
Number of characters		$q - 1$	$q - 1$	$q - 1$
Series		Principal series	Principal series	Principal series

Representatives	$\chi_{\alpha, \beta}^{q^2+q+1}$	$\chi_{\alpha, \beta}^{q(q^2+q+1)}$
T^a	$(q^2 + q + 1) (\alpha \beta^2)(a)$	$q (q^2 + q + 1) (\alpha \beta^2)(a)$
T_1^a	$(q + 1) (\alpha \beta^2)(a)$	$q (\alpha \beta^2)(a)$
T_{11}^a	$(\alpha \beta^2)(a)$	0
T^{ab}	$(q + 1) \alpha(a) \beta(ab) + \beta^2(a) \alpha(b)$	$(q + 1) \alpha(a) \beta(ab) + q \beta^2(a) \alpha(b)$
T_1^{ab}	$\alpha(a) \beta(ab) + \beta^2(a) \alpha(b)$	$\alpha(a) \beta(ab)$
T^{abc}	$\sum_{(a,b,c)} \alpha(a) \beta(bc)$	$\sum_{(a,b,c)} \alpha(a) \beta(bc)$
$T^{\varkappa a}$	$\alpha(a) \beta(\varkappa \varkappa^q)$	$-\alpha(a) \beta(\varkappa \varkappa^q)$
T^z	0	0
Number of characters	$(q - 1)(q - 2)$	$(q - 1)(q - 2)$
Series	Principal series	Principal Series

Representatives	$\chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)}$	$\chi_{\alpha, \lambda}^{q^3-1}$	$\chi_\varphi^{(q-1)(q^2-1)}$
T^a	$(q + 1) (q^2 + q + 1) (\alpha \beta \gamma)(a)$	$(q^3 - 1) \alpha(a) \lambda(a)$	$(q - 1) (q^2 - 1) \varphi(a)$
T_1^a	$(2q + 1) (\alpha \beta \gamma)(a)$	$-\alpha(a) \lambda(a)$	$-(q - 1) \varphi(a)$
T_{11}^a	$(\alpha \beta \gamma)(a)$	$-\alpha(a) \lambda(a)$	$\varphi(a)$
T^{ab}	$(q + 1) \sum_{(\alpha, \beta, \gamma)} (\alpha \beta)(a) \gamma(b)$	$(q - 1) \alpha(b) \lambda(a)$	0
T_1^{ab}	$\sum_{(\alpha, \beta, \gamma)} (\alpha \beta)(a) \gamma(b)$	$-\lambda(a) \alpha(b)$	0
T^{abc}	$\sum_{(\alpha, \beta, \gamma)} \alpha(a) \beta(b) \gamma(c)$	0	0
$T^{\varkappa a}$	0	$-\alpha(a) (\lambda + \lambda^q)(\varkappa)$	0
T^z	0	0	$(\varphi + \varphi^q + \varphi^{q^2})(z)$
		$\lambda \in (\mathbb{F}_{q^2}^\times)^\wedge$ $\lambda \neq \lambda^q$	$\varphi \in (\mathbb{F}_{q^3}^\times)^\wedge$ $\varphi \neq \varphi^q$
Number of characters	$\frac{1}{6} (q - 1)(q - 2)(q - 3)$	$\frac{1}{3} q (q - 1)^2$	$\frac{1}{3} q (q^2 - 1)$
	Principal series	Intermediate series	Discrete series

The following lemma describes the decomposition of the character obtained from an irreducible character in the "degenerate" case, i.e., when two or more of the (distinct) parameters are now taken to be equal.

Lemma 2. *We have*

- (1) $\chi_{\alpha \circ N_3}^{(q-1)(q^2-1)} = \chi_\alpha^1 - \chi_\alpha^{q^2+q} + \chi_\alpha^{q^3}$
- (2) $\chi_{\alpha, \beta \circ N_2}^{q^3-1} = \chi_{\alpha, \beta}^{q^3+q^2+q} - \chi_{\alpha, \beta}^{q^2+q+1}$
- (3) $\chi_{\alpha, \alpha \circ N_2}^{q^3-1} = \chi_\alpha^{q^3} - \chi_\alpha^1$
- (4) $\chi_{\alpha, \alpha}^{q(q^2+q+1)} = \chi_\alpha^{q^3} + \chi_\alpha^{q^2+q}$
- (5) $\chi_{\alpha, \alpha}^{q^2+q+1} = \chi_\alpha^1 + \chi_\alpha^{q^2+q}$
- (6) $\chi_{\alpha, \beta, \beta}^{(q+1)(q^2+q+1)} = \chi_{\alpha, \beta}^{q^2+q+1} + \chi_{\alpha, \beta}^{q(q^2+q+1)}$
- (7) $\chi_{\alpha, \alpha, \alpha}^{(q+1)(q^2+q+1)} = \chi_\alpha^1 + 2\chi_\alpha^{q^2+q} + \chi_\alpha^{q^3}$

Proof. Follows from the character table of G . □

3. TENSOR PRODUCTS OF IRREDUCIBLE CHARACTERS OF THE GROUP G .

3.1. Description in terms of induced representations from torii. In what follows, we denote by $\tilde{\alpha}$ an extension to \mathbb{F}_q^\times (or to $\mathbb{F}_{q^2}^\times$, whichever is the case) of the character $\alpha \in \widehat{\mathbb{F}_q^\times}$. On the other hand, for characters λ and μ of \mathbb{F}_q^\times (or of $\mathbb{F}_{q^2}^\times$), $\lambda|_q$ denotes the restriction to \mathbb{F}_q^\times of the character λ and μ^q stands for $\mu \circ F$, where F is the Frobenius \mathbb{F}_q^\times -automorphism of \mathbb{F}_q^\times (or $\mathbb{F}_{q^2}^\times$), given by $F(x) = x^q$.

Theorem 1. *With notations as above, we have:*

- (1) i. $\chi_{\alpha, \lambda}^{q^3-1} \otimes \chi_{\beta, \gamma, \delta}^{(q+1)(q^2+q+1)} = \text{Ind}_{T_m}^G (\gamma\delta\lambda, \alpha\beta) + \chi_{\alpha\beta, \tilde{\alpha}\tilde{\delta}\lambda}^{q^3-1} +$
 $+ \left(\chi_{\alpha\gamma, \tilde{\beta}\tilde{\delta}\lambda}^{q^3-1} + \chi_{\alpha\delta, \tilde{\beta}\tilde{\gamma}\lambda}^{q^3-1} \right) \otimes \chi_1^{q^2+q}$
- ii. $\chi_{\alpha\lambda, (\alpha \circ N_2)\lambda^2}^{q^3-1} \otimes \chi_{\beta, \gamma, \delta}^{(q+1)(q^2+q+1)} = \text{Ind}_{T_i}^G (\alpha\lambda\beta, \alpha\lambda\gamma, \alpha\lambda\delta) - \chi_{\alpha\lambda\beta, \alpha\lambda\gamma, \alpha\lambda\delta}^{(q+1)(q^2+q+1)}$
- (2) $\chi_{\alpha, \lambda}^{q^3-1} \otimes \chi_{\beta, \mu}^{q^3-1} = \text{Ind}_{T_m}^G (\lambda\mu, \alpha\beta) - \chi_{\alpha\beta, \lambda\mu^q}^{q^3-1}$
- (3) i. $\chi_{\alpha, \lambda}^{q^3-1} \otimes \chi_\varphi^{(q-1)(q^2-1)} = \text{Ind}_{T_a}^G (\tilde{\alpha}\tilde{\lambda}|_\varphi) - \chi_{\tilde{\alpha}\tilde{\lambda}|_\varphi}^{(q-1)(q^2-1)}$
- ii. $\chi_{\alpha, \lambda}^{q^3-1} \otimes \chi_\varphi^{(q-1)(q^2-1)} = \text{Ind}_{T_m}^G (\lambda, \alpha\varphi) + \chi_{\alpha\varphi, \lambda}^{q^3-1} \otimes \left(\chi_1^1 - \chi_1^{q^2+q} \right)$
- (4) $\chi_\delta^{q^3} \otimes \chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)} = \text{Ind}_{T_i}^G (\alpha\delta, \beta\delta, \gamma\delta)$
- (5) $\chi_\alpha^{q^3} \otimes \chi_\varphi^{(q-1)(q^2-1)} = \text{Ind}_{T_a}^G (\alpha \circ N_3) \varphi$
- (6) $\chi_\alpha^{q^3} \otimes \chi_{\beta, \lambda}^{q^3-1} = \text{Ind}_{T_m}^G ((\alpha \circ N_2) \lambda, \alpha\beta)$
- (7) i. $\chi_\alpha^{q^3} \otimes \chi_\beta^{q^3} = \text{Ind}_{T_m}^G (\alpha\beta \circ N_2, \alpha\beta) + \chi_{\alpha\beta}^{q^3}$
- ii. $\chi_\alpha^{q^3} \otimes \chi_\beta^{q^3} = \text{Ind}_{T_i}^G (\alpha\beta, \alpha\beta, \alpha\beta) - \chi_{\alpha\beta}^{q^3} \otimes \left(2\chi_1^{q^2+q} + \chi_1^1 \right)$
- iii. $\chi_\alpha^{q^3} \otimes \chi_\beta^{q^3} = \text{Ind}_{T_a}^G \alpha\beta \circ N_3 + \chi_{\alpha\beta}^{q^3} \otimes \left(\chi_1^{q^2+q} - \chi_1^1 \right)$
- (8) i. $\chi_\varphi^{(q-1)(q^2-1)} \otimes \chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)} = \text{Ind}_{T_a}^G \varphi \left(\widetilde{\alpha\beta\gamma} \right) + \chi_{\varphi(\widetilde{\alpha\beta\gamma})}^{(q-1)(q^2-1)} \otimes$
 $\left(2\chi_1^{q^2+q} + \chi_1^1 \right)$

$$\begin{aligned}
 & \text{ii. } \chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)} = \text{Ind}_{T_i}^G(\alpha, \beta, \varphi\gamma) + \chi_{\alpha, \beta, \varphi\gamma}^{(q+1)(q^2+q+1)} \\
 & \quad \otimes (\chi_1^1 - \chi_1^{q^2+q}) \\
 (9) \quad & \chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\psi}^{(q-1)(q^2-1)} = \\
 & \quad \text{Ind}_{T_a}^G \varphi\psi + \chi_{\varphi\psi^q}^{(q-1)(q^2-1)} + \chi_{\varphi\psi^{q^2}}^{(q-1)(q^2-1)} - \chi_{\varphi\psi}^{(q-1)(q^2-1)} \otimes (\chi_1^{q^2+q} + \chi_1^1) \\
 (10) \quad & \chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)} \otimes \chi_{\delta, \varepsilon, \eta}^{(q+1)(q^2+q+1)} = \\
 & \quad \text{Ind}_{T_i}^G(\alpha\delta, \beta\varepsilon, \gamma\eta) + \sum_{(\delta\varepsilon\eta) \in S_3 - \{I\}} \chi_{\alpha\delta, \beta\varepsilon, \gamma\eta}^{(q+1)(q^2+q+1)} + \\
 & \quad \left(\chi_{\alpha\varepsilon, \beta\eta, \gamma\delta}^{(q+1)(q^2+q+1)} + \chi_{\alpha\eta, \beta\delta, \gamma\varepsilon}^{(q+1)(q^2+q+1)} \right) \otimes (\chi_1^{q^2+q} - 2\chi_1^1)
 \end{aligned}$$

Proof. The proof consists in computing the relevant characters on the different conjugacy classes of G (See section 2). Specifically, the computations rely on lemma 1 and the character table of G .

As an example of the above, we present a proof of 8.i.

- a) In the case of the conjugacy classes given by $T^a, T_1^a, T_{11}^a, T^{ab}, T_1^{ab}, T^{abc}$ and $T^{\ast a}$, the result follows directly from both the character table and lemma 1.
- b) We have

$$\begin{aligned}
 & \left(\chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)} \right) (T^z) = \left[(\varphi + \varphi^q + \varphi^{q^2})(z) \right] 0 = 0 \\
 & \text{Also, using the character table and part 7 of Lemma 1, we have} \\
 & \left(\text{Ind}_{T_a}^G \varphi(\widetilde{\alpha\beta\gamma}) + \chi_{\varphi(\widetilde{\alpha\beta\gamma})}^{(q-1)(q^2-1)} \otimes (2\chi_1^{q^2+q} + \chi_1^1) \right) (T^z) = \\
 & \frac{1}{|T_a|} \sum_{\substack{X \in G \\ XT^zX^{-1} \in T_a}} \varphi(\widetilde{\alpha\beta\gamma})(XT^zX^{-1}) - \varphi(\widetilde{\alpha\beta\gamma})(z) - \left(\varphi(\widetilde{\alpha\beta\gamma}) \right)^q(z) - \\
 & \left(\varphi(\widetilde{\alpha\beta\gamma}) \right)^{q^2}(z)
 \end{aligned}$$

Now, by part 5 of lemma 1, the above expression becomes

$$\begin{aligned}
 & \frac{1}{|T_a|} \sum_{X \in T_a \times \Gamma_3} \varphi(\widetilde{\alpha\beta\gamma})(XT^zX^{-1}) - \varphi(\widetilde{\alpha\beta\gamma})(z) - \left(\varphi(\widetilde{\alpha\beta\gamma}) \right)^q(z) - \\
 & \left(\varphi(\widetilde{\alpha\beta\gamma}) \right)^{q^2}(z) = \frac{|T_a|}{|T_a|} \left(\varphi(\widetilde{\alpha\beta\gamma})(z) + \varphi(\widetilde{\alpha\beta\gamma})(z^q) + \varphi(\widetilde{\alpha\beta\gamma})(z^{q^2}) \right) \\
 & - \varphi(\widetilde{\alpha\beta\gamma})(z) - \left(\varphi(\widetilde{\alpha\beta\gamma}) \right)^q(z) - \left(\varphi(\widetilde{\alpha\beta\gamma}) \right)^{q^2}(z) = 0.
 \end{aligned}$$

□

Finally, using lemma 2, the next corollary address the case of irreducible representations that arise from limit cases of previous formulas where now some parameters coincide.

Corollary 1. *With notations as above, we have*

$$\begin{aligned}
 (1) \quad & \chi_{\alpha}^{q^3} \otimes \left[\chi_{\beta, \gamma}^{q^2+q+1} + \chi_{\beta, \gamma}^{q(q^2+q+1)} \right] = \text{Ind}_{T_i}^G(\alpha\beta, \alpha\gamma, \alpha\gamma) \\
 (2) \quad & \chi_{\alpha}^{q^3} \otimes \chi_{\beta}^{q^2+q} = 2\chi_{\alpha\beta}^{q^3} + \text{Ind}_{T_m}^G(\alpha\beta \circ \mathbf{N}_2, \alpha\beta) - \text{Ind}_{T_a}^G(\alpha \circ \mathbf{N}_3)(\beta \circ \mathbf{N}_3)
 \end{aligned}$$

Proof. The first formula follows from 4 of theorem 1 above, taking two parameters to be equal. To prove the second, it is enough to use parts 7 and 5 of theorem 1 and to reduce terms applying part 1 of Lemma 2. \square

3.2. Description in terms of Gelfand-Graev representations. The next proposition describes the tensor product of a discrete and a principal series representation of G as the sum of the classical and a generalized Gelfand-Graev representation of G .

We denote by N the standard unipotent subgroup of G , and by N_2 the subgroup of N whose $(1, 2)$ entry is 0, i.e.,

$$N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in k \right\}, \quad N_2 = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid y, z \in k \right\}.$$

For a non trivial character φ of k^+ , we also denote by φ the character on N defined by $\varphi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right) = \varphi(x + y)$.

Proposition 1. *With notations as above, we have*

$$\chi_\psi^{(q-1)(q^2-1)} \otimes \chi_{\beta,\gamma,\delta}^{(q+1)(q^2+q+1)} = \text{Ind}_{ZN}^G(\psi\beta\gamma\delta)\varphi + \text{Ind}_{ZN_2}^G(\psi\beta\gamma\delta)\varphi$$

Proof. We compute first the involved characters on the conjugacy class of T_1^a :

$$\left[\chi_\psi^{(q-1)(q^2-1)} \otimes \chi_{\beta,\gamma,\delta}^{(q+1)(q^2+q+1)} \right] (T_1^a) = -(q-1)(2q+1) (\psi\beta\gamma\delta) (a).$$

$$\text{Let } D_1 = \{X \mid XT_1^a X^{-1} \in ZN_2\} = \left\{ \begin{pmatrix} p & q & r \\ l & m & n \\ 0 & t & 0 \end{pmatrix} \in G \right\}, \text{ and}$$

$$\text{let } D_2 = \left\{ \begin{pmatrix} p & q & r \\ 0 & m & n \\ 0 & t & s \end{pmatrix} \mid s \neq 0, p \neq 0, ms - tn \neq 0 \right\}, \text{ we see that,}$$

$$D = \{X \mid XT_1^a X^{-1} \in ZN\} = D_1 \cup D_2 = \left\{ \begin{pmatrix} p & q & r \\ l & m & n \\ 0 & t & s \end{pmatrix} \in G \mid ls = 0 \right\},$$

$$\begin{aligned} & \text{so} \\ & \left[\text{Ind}_{ZN}^G(\psi\beta\gamma\delta)\varphi + \text{Ind}_{ZN_2}^G(\psi\beta\gamma\delta)\varphi \right] (T_1^a) = \\ & \frac{1}{|ZN|} \sum_{X \in D} (\psi\beta\gamma\delta)\varphi(XT_1^a X^{-1}) + \frac{1}{|ZN_2|} \sum_{X \in D_1} (\psi\beta\gamma\delta)\varphi(XT_1^a X^{-1}) = \\ & = \left[\frac{1}{|ZN|} + \frac{1}{|ZN_2|} \right] \sum_{\substack{t \in \mathbb{F}_q^\times, q, m \in \mathbb{F}_q \\ pn - lr \neq 0}} (\psi\beta\gamma\delta)(a) \varphi\left(\frac{a^{-1}l}{t}\right) + \\ & + \frac{1}{|ZN|} \sum_{\substack{s, p \in \mathbb{F}_q^\times; r, q \in \mathbb{F}_q \\ ms - tn \neq 0}} (\psi\beta\gamma\delta)(a) \varphi\left(\frac{a^{-1}ps}{ms - tn}\right) = \\ & = \left[\frac{q+1}{q^3(q-1)} (-q^3(q-1)^2) + \frac{1}{q^3(q-1)} (-q^4(q-1)^2) \right] (\psi\beta\gamma\delta)(a) \\ & = -(q-1)(2q+1) (\psi\beta\gamma\delta)(a). \end{aligned}$$

The computations on the conjugacy class of T_{11}^a is similar to the above one. In this case the sum corresponding to ZN_2 has no support.

The remaining cases are straightforward. \square

The above proposition suggests that for $G = GL(n, \mathbb{F}_q)$ the tensor product of a cuspidal (discrete) and a principal series representation may be expressed as the direct sum of $\frac{(n-2)(n-1)}{2} + 1$ packets of generalized Gelfand-Graev representations of the same dimension, the first one consisting simply of the classical Gelfand-Graev representation and the last one consisting only of the generalized Gelfand-Graev representation induced from Z times the upper unipotent subgroup with all non zero upper diagonal entries in the last column.

To describe our general conjecture more precisely we introduce some notations:

Let $G = GL(n, \mathbb{F}_q)$ and Z be the center of G . Let α be any character of Z and φ any non trivial character of \mathbb{F}_q^+ . We extend as usual φ to a character, still denoted by φ , of the standard upper unipotent subgroup N , or any of its subgroups, by $\varphi(u_{ij}) = \varphi(u_{12} + u_{23} + \dots + u_{(n-1)n})$ for $u = (u_{ij}) \in N$.

For any subgroup N' of N , we put

$$\Gamma_{N'}(\alpha\varphi) = \text{Ind}_{ZN'}^G(\alpha\varphi).$$

Let $\tilde{n} = 1 + 2 + \dots + (n-2) = \frac{(n-2)(n-1)}{2}$.

We define the family of numbers $c_j(n)$ for $0 \leq j \leq \tilde{n}$ by

$$(q+1)(q^2+q+1)\dots+(q^{n-2}+q^{n-1}+\dots+1) = \sum_{0 \leq j \leq \tilde{n}} c_j(n)q^j$$

We will call *Gelfand-Graev interpolating family*, any family $\{\mathcal{N}_i\}_{0 \leq i \leq \tilde{n}}$ of sets of subgroups of the standard upper unipotent subgroup N of G such that, for all i :

- a. $|\mathcal{N}_i| = c_i(n)$
- b. each subgroup $N' \in \mathcal{N}_i$ is defined by the vanishing of i upper unipotent entries, so that $[N : N'] = q^i$.

Notice that \mathcal{N}_0 consists only of N and $\mathcal{N}_{\tilde{n}}$ contains only one subgroup, of order $q^{(n-1)}$.

Conjecture 1. *Fix a non trivial character φ of \mathbb{F}_q^+ . There exists a Gelfand-Graev interpolating family $\{\mathcal{N}_i\}_{0 \leq i \leq \tilde{n}}$ such that*

$$\begin{aligned} & \chi_{\psi}^{(q-1)(q^2-1)\dots(q^{n-1}-1)} \otimes \chi_{\beta_1, \dots, \beta_n}^{(q+1)(q^2+q+1)\dots(q^{n-1}+q^{n-2}+\dots+q+1)} = \\ & = \bigoplus_{0 \leq j \leq \tilde{n}} \left[\bigoplus_{N' \in \mathcal{N}_j} \Gamma_{N'}((\psi\beta_1 \dots \beta_n)\varphi) \right] \end{aligned}$$

4. APPLICATION: CLEBSCH-GORDAN COEFFICIENTS FOR THE TENSOR PRODUCT OF TWO CUSPIDAL REPRESENTATIONS OF G .

Finally, we notice that the decomposition of tensor products of irreducible representations in irreducible constituents can be easily computed, once you have described these tensor products in terms of induced representations.

For example, consider the case of the tensor product $\chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\psi}^{(q-1)(q^2-1)}$ of two cuspidal (discrete series) representations of G .

If $\langle \Omega, \Theta \rangle_G$ stands for the usual inner product of two complex valued functions on G , i.e.,

$$\langle \Omega, \Theta \rangle_G = \frac{1}{|G|} \sum_{t \in G} \Omega(t) \overline{\Theta(t)},$$

where $\overline{\Theta(t)}$ is the conjugate of the complex number $\Theta(t)$, then we have that the multiplicities of the irreducible representations of G in this tensor product are given as follows:

$$\begin{aligned}
(1) & \left\langle \chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\psi}^{(q-1)(q^2-1)}, \chi_{\alpha}^1 \right\rangle = 1 \text{ if and only if } \varphi\psi = \alpha \circ \mathbf{N}_3 \\
(2) & \left\langle \chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\psi}^{(q-1)(q^2-1)}, \chi_{\alpha, \beta, \gamma}^{(q+1)(q^2+q+1)} \right\rangle_G = \langle \varphi\psi, \alpha\beta\gamma \rangle_{k^\times} \\
(3) & \left\langle \chi_{\varphi}^{(q-1)(q^2-1)} \otimes \chi_{\psi}^{(q-1)(q^2-1)}, \chi_{\Lambda}^{(q-1)(q^2-1)} \right\rangle = \\
& \langle \varphi\psi, \Lambda \rangle_{K^\times} + \langle \varphi\psi, \Lambda^q \rangle_{K^\times} + \left\langle \varphi\psi, \Lambda^{q^2} \right\rangle_{K^\times} + (q-3) \langle \varphi\psi, \Lambda \rangle_{k^\times} + \\
& \left\langle \chi_{\varphi\psi^q}^{(q-1)(q^2-1)}, \chi_{\Lambda}^{(q-1)(q^2-1)} \right\rangle + \left\langle \chi_{\varphi\psi^{q^2}}^{(q-1)(q^2-1)}, \chi_{\Lambda}^{(q-1)(q^2-1)} \right\rangle = \\
& \langle \varphi\psi, \Lambda \rangle_{K^\times} + \langle \varphi\psi, \Lambda^q \rangle_{K^\times} + \left\langle \varphi\psi, \Lambda^{q^2} \right\rangle_{K^\times} + \langle \varphi\psi^q, \Lambda \rangle_{K^\times} + \langle \varphi\psi^q, \Lambda^q \rangle_{K^\times} + \left\langle \varphi\psi^q, \Lambda^{q^2} \right\rangle_{K^\times} \\
& + \left\langle \varphi\psi^{q^2}, \Lambda \right\rangle_{K^\times} + \left\langle \varphi\psi^{q^2}, \Lambda^q \right\rangle_{K^\times} + \left\langle \varphi\psi^{q^2}, \Lambda^{q^2} \right\rangle_{K^\times} + (q-3) \langle \varphi\psi, \Lambda \rangle_{k^\times}
\end{aligned}$$

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