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Pseudo-almost periodic solutions of neutral integral and differential equations with applications

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ABSTRACT

The existence and uniqueness of pseudo-almost periodic solutions to general neutral integral equations with deviations are obtained. For this, pseudo-almost periodic functions in two variables are considered. The results extend the corresponding ones to the convolution type integral equations. They are used to study pseudo-almost periodic solutions of general neutral differential equations and to the so-called scalar neutral logistic equation version.

1. Introduction

The existence of almost periodic, asymptotically almost periodic, pseudo-almost periodic solutions is among the most attractive topics in qualitative theory of differential equations due to their applications, especially in biology, economics and physics [1–5]. The concept of pseudo-almost periodicity, which is the central subject in this paper, was introduced by Zhang [6,7,5] in the early nineties. Since then, such a notion became of great interest to the classical almost periodicity in the sense of Bohr and Bochner. Thus such a concept is welcome for implementing another existing generalization of almost periodicity, the so-called asymptotically almost periodicity due to Frechet; see e.g. [1–4,8,5]. For more on the concepts of almost periodicity and pseudo-almost periodicity and related issues, we refer the reader to [1–4,6,7,5,9] (for both the almost periodicity and asymptotic almost periodicity) and to [10–19] (for the pseudo-almost periodicity).

In [20], Burton, studying the existence and uniqueness of periodic solutions to the logistic differential equation

$$u'(t) = au(t) + \beta u'(t-p) - Q(t, u(t), u(t-p)), \quad a \neq 0,\ |\beta| < 1,\ p > 0, \tag{1.1}$$

introduces the so-called neutral delay integral equations of advanced type

$$u(t) = f(u(t-p)) + \int_t^{\infty} C(t-s)Q(s, u(s), u(s-p))\mathrm{d}s + g(t). \tag{1.2}$$

This paper is concerned with the existence and uniqueness of pseudo-almost periodic and almost periodic solutions to an abstract integral equation of the form [21,20,22–25]

$$u(t) = f(t, u(t), u(h_0(t))) + \int_{\mathbb{R}} C(t, s, u(s), u(h(s)))\mathrm{d}s, \quad t \in \mathbb{R}$$

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or, more specifically, its advanced and delayed decomposition:

$$u(t) = f(t, u(t), u(h_0(t))) + \int_{-\infty}^{t} C_1(t, s, u(s), u(h_1(s)))\mathrm{d}s + \int_{t}^{\infty} C_2(t, s, u(s), u(h_2(s)))\mathrm{d}s, \quad t \in \mathbb{R}, \tag{1.3}$$

where $h_i : \mathbb{R} \to \mathbb{R}$ are continuous functions with $h_i(\mathbb{R}) = \mathbb{R}$ for $i = 0, 1, 2$ and $f : \mathbb{R} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$ and $C_i : \mathbb{R} \times \mathbb{R} \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, $i = 1, 2$ are jointly continuous. The cases

$$C_i(t, s, u, v) = \Lambda_i(t, s)\hat{C}_i(s, u, v), \quad i = 1, 2, \tag{1.4}$$

where $\Lambda_i(t, s)$ are $n \times n$ matrices and $\hat{C}_i : \mathbb{R}^2 \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$, are of special interest; see [20,22,24,25]. In particular, $\Lambda_i(t, s) = \Lambda_i(t - s)$ represents the convolution situation $C_i(t, s, u, v) = \Lambda_i(t - s)\hat{C}_i(s, u, v)$. Both cases appear naturally in the study of general neutral differential equations

$$y' = A(t)y + \frac{\mathrm{d}}{\mathrm{d}t}[f(t, y(t), y(h_0(t)))] + C(t, y(t), y(h_1(t))). \tag{1.5}$$

An interesting particular case in Eq. (1.3) is given by $h_i(t) = t + p_i$, p_i constant, $i = 0, 1, 2$ and the neutral integral equation of delayed and advanced type

$$u(t) = f(t, u(t), u(t + p_0)) + \int_{-\infty}^{t} C_1(t, s, u(s), u(s + p_1))\mathrm{d}s + \int_{t}^{\infty} C_2(t, s, u(s), u(s + p_2))\mathrm{d}s. \tag{1.6}$$

Some contributions related to pseudo-almost periodic solutions to abstract ordinary and partial differential equations have recently been made [10–13,2,14–17,4,26,19,6]. The existence of pseudo-almost periodic solutions to integral equations, especially those of the form Eq. (1.3) is, it seems, an untreated topic and this is the main motivation of present paper.

Due to the character of pseudo-almost periodic functions in the two variables t and s of kernels C_i we introduce some definitions of functions which could be understood as "weighted pseudo-almost periodic functions". These definitions represent very well the separated variables situation (1.4). So, under some suitable assumptions, the existence and uniqueness of a pseudo-almost periodic and almost periodic solution to Eq. (1.3) are obtained (Theorem 1). Next we make use of the previous results to prove the existence and uniqueness of a pseudo-almost periodic and almost periodic solutions to general neutral differential equation (1.5) and logistic type equations (Theorem 2).

2. Almost and pseudo-almost periodic functions

Let $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be a Banach space and let $(\mathrm{BC}(\mathbb{R}, \mathbb{Y}), \|\cdot\|_\infty)$ be the Banach space of bounded continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ endowed with the supremum norm $\|\phi\|_\infty = \sup_{t\in\mathbb{R}} \|\phi(t)\|_{\mathbb{Y}}$. For $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ another Banach space and a function $\lambda : \mathbb{R}^2 \to (0, \infty)$, $BC_\lambda(\mathbb{R}^2 \times \mathbb{X}, \mathbb{Y})$ will denote the vectorial space of continuous functions $f : \mathbb{R}^2 \times \mathbb{X} \to \mathbb{Y}$ such that f/λ is bounded. If $\Omega \subset \mathbb{X}$ is an open subset, then $\mathrm{BC}(\mathbb{R}^2 \times \Omega, \mathbb{Y})$ denotes the vectorial space of bounded continuous functions $F : \mathbb{R}^2 \times \Omega \to \mathbb{Y}$.

A function $f \in \mathrm{BC}(\mathbb{R}, \mathbb{Y})$ is called almost periodic [2–5] if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property:

$$\|f(t + \tau) - f(t)\|_{\mathbb{Y}} \leq \varepsilon, \quad \text{for every } t \in \mathbb{R}.$$

The number τ above is then called an ε-translation number of f, and the collection of such functions will be denoted $\mathrm{AP}(\mathbb{R}, \mathbb{Y})$. Similarly, a function $F \in \mathrm{BC}(\mathbb{R} \times \Omega, \mathbb{Y})$ is called almost periodic in $t \in \mathbb{R}$ uniformly in any $K \subset \Omega$ a bounded subset if for each $\varepsilon > 0$, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains a number τ with the following property: $\|F(t+\tau, x)-F(t, x)\|_{\mathbb{Y}} \leq \varepsilon$, for every $t \in \mathbb{R}$, $x \in K$. Here again, the number τ above is then called an ε-translation number of F, and the class of such functions will be denoted $\mathrm{AP}(\mathbb{R} \times \Omega, \mathbb{Y})$. $\mathrm{AP}(\mathbb{R}, \mathbb{Y})$ is a closed subspace of $\mathrm{BC}(\mathbb{R}, \mathbb{Y})$. For more on $\mathrm{AP}(\mathbb{R}, \mathbb{Y})$ (respectively, $\mathrm{AP}(\mathbb{R} \times \Omega, \mathbb{Y})$) and related issues, we refer to [2–5] and the references therein.

Definition 1. Let $\lambda : \mathbb{R}^2 \to (0, \infty)$ be a function. A function $F \in BC_\lambda(\mathbb{R}^2 \times \Omega, \mathbb{Y})$ will be called λ-almost periodic in $t, s \in \mathbb{R}$ uniformly in any bounded subset $K \subset \Omega$ if for each $\varepsilon > 0$, there exists $A_\varepsilon > 0$ such that for every rectangle $R_1 \times R_2 \subset \mathbb{R}^2$ of area A_ε there is a number $\tau \in R_1 \cap R_2$ with the following property:

$$\|F(t + \tau, s + \tau, x) - F(t, s, x)\|_{\mathbb{Y}} \leq \varepsilon c\lambda(t, s), \quad t, s \in \mathbb{R}, \ x \in K,$$

for $c > 0$ constant.

Again, the number τ above will be called an ε-translation number with respect to λ of F and the class of such functions F will be denoted $\mathrm{AP}_\lambda(\mathbb{R}^2 \times \Omega, \mathbb{Y})$. Particularly, we will need functions F in $\mathrm{AP}_\lambda(\mathbb{R}^2, \mathbb{Y})$, i.e. $F = F(t, s)$, independent on x. Note that for $\lambda = 1$: $\mathrm{AP}(\mathbb{R}^2 \times \Omega, \mathbb{Y}) = \mathrm{AP}_1(\mathbb{R}^2 \times \Omega, \mathbb{Y})$. Moreover, this definition harmonizes very well with the convolution case (1.4).

Now, we consider the ergodic terms:

$$\mathrm{PAP}^0(\mathbb{R},\mathbb{Y}) = \left\{ f \in \mathrm{BC}(\mathbb{R},\mathbb{Y}) : \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \|f(s)\|_{\mathbb{Y}}\,\mathrm{d}s = 0 \right\}.$$

Similarly, $\mathrm{PAP}^0(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ denotes the collection of functions $F \in \mathrm{BC}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ such that $\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\|F(t,u)\|_{\mathbb{Y}}\mathrm{d}t = 0$ uniformly in $u \in \mathbb{X}$.

Definition 2. For a function $\vartheta : \mathbb{R}^2 \to [0,\infty)$ and $F = F(t,s,x)$, we will say that $F \in \mathrm{PAP}^0_{\vartheta}(\mathbb{R}^2\times\mathbb{X},\mathbb{Y})$, if

$$\|F(t,s,x)\| \le \vartheta(t,s)\hat{F}(s,x), \quad t,s\in\mathbb{R},\ x\in\mathbb{X} \tag{2.1}$$

with $0 \le \hat{F}(s,x) \in \mathrm{PAP}^0(\mathbb{R}\times\mathbb{X},\mathbb{R})$.

Definition 3. Let $f \in \mathrm{BC}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$. f is called pseudo-almost periodic if $f = g + \phi$, where $g \in \mathrm{AP}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$ and $\phi \in \mathrm{PAP}^0(\mathbb{R}\times\mathbb{X},\mathbb{Y})$. g and ϕ are called the almost periodic component and the ergodic perturbation of f, respectively. The collection of such functions f will be denoted by $\mathrm{PAP}(\mathbb{R}\times\mathbb{X},\mathbb{Y})$.

We now equip the collection of pseudo-almost periodic functions from $\mathbb{R}$ into $\mathbb{Y}$, $\mathrm{PAP}(\mathbb{R},\mathbb{Y})$, with the supremum norm. It is well known that $(\mathrm{PAP}(\mathbb{R},\mathbb{Y}), \|\cdot\|_\infty)$ is a Banach space; see details in [13,2–5].

Definition 4. Let $\lambda,\vartheta : \mathbb{R}^2 \to [0,\infty)$ be two functions. A function $f : \mathbb{R}^2\times\mathbb{X}\to\mathbb{Y}$ is called (λ,ϑ) pseudo-almost periodic in $\mathbb{R}^2$ uniformly in $x\in\mathbb{X}$ if it can be expressed as $f = g+\phi$, where $g \in \mathrm{AP}_{\lambda}(\mathbb{R}^2\times\mathbb{X},\mathbb{Y})$ and, the ergodic component, $\phi \in \mathrm{PAP}^0_{\vartheta}(\mathbb{R}^2\times\mathbb{X},\mathbb{Y})$. The collection of such functions will be denoted by $\mathrm{PAP}_{(\lambda,\vartheta)}(\mathbb{R}^2\times\mathbb{X},\mathbb{Y})$.

A typical and very interesting example of $F \in \mathrm{PAP}_{(\lambda,\vartheta)}(\mathbb{R}^2\times\Omega,\mathbb{Y})$ is given by

$$F(t,s,x) = \Lambda(t,s)G(s,x),$$

which includes the convolution situation; see (1.4) and hypothesis (S) below. The matrix $\Lambda(t,s)$ could be a Green matrix associated to a differential operator; see Eqs. (1.5), (3.6) and (3.9), and [11–17].

Throughout the rest of the paper, the most of the times, we suppose that $\mathbb{X}=\mathbb{Y}=\mathbb{C}^n$, equipped with a suitable norm. However, when we deal with the pseudo-almost periodicity in $\mathbb{R}^2$ of the kernels C_i, $i=1,2$, in Eqs. (1.2), (1.3), (1.5) and (1.6), we choose $\mathbb{X}=\mathbb{C}^n\times\mathbb{C}^n$.

We require the following assumptions:

(C) For $i=0,1,2$, the functions $h_i:\mathbb{R}\to\mathbb{R}$ are continuous, $h_i(\mathbb{R})=\mathbb{R}$, and $u\in\mathrm{PAP}(\mathbb{R})$ implies $u(h_i)\in\mathrm{PAP}(\mathbb{R})$.

($\mathrm{L_f}$) The function $f:\mathbb{R}\times\mathbb{C}^{2n}\to\mathbb{C}^n$ is pseudo-almost periodic satisfying for some constant $L\in(0,1)$,

$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \le L\left(|x_1-x_2|+|y_1-y_2|\right), \quad t\in\mathbb{R},\ x_i,y_i\in\mathbb{C}^n.$$

($\mathrm{L_C}$) For $i=1,2$, there exist $\mu_i=\mu_i(t,s)$ such that for $t,s\in\mathbb{R}$, $x_i,y_i\in\mathbb{C}^n$, C_i satisfies the Lipschitz condition:

$$|C_i(t,s,x_1,y_1)-C_i(t,s,x_2,y_2)| \le \mu_i(t,s)\left(|x_1-x_2|+|y_1-y_2|\right),$$

where $\int_{-\infty}^{t}\mu_1(t,s)\mathrm{d}s+\int_{t}^{\infty}\mu_2(t,s)\mathrm{d}s\le\mu$, for $t\in\mathbb{R}$.

(PAP) For $i=1,2$, the functions C_i are (λ_i,θ_i) pseudo-almost periodic in $t,s\in\mathbb{R}$ uniformly if $(x,y)\in\mathbb{C}^{2n}$, that is we have decomposition:

$$C_i = Q_1^i + Q_2^i \quad \text{with } Q_1^i \in \mathrm{AP}_{\lambda_i}(\mathbb{R}^2\times\mathbb{C}^{2n},\mathbb{C}^n),\ Q_2^i\in\mathrm{PAP}^0_{\vartheta_i}(\mathbb{R}^2\times\mathbb{C}^{2n},\mathbb{C}^n),$$

$$\text{i.e. } |Q_2^i(t,s,x,y)| \le \theta_i(t,s)\hat{Q}_2^i(s,x,y).$$

(I) For some constants $\alpha_i,\theta_i>0$ $i=1,2$, the functions λ_i,ϑ_i: $\mathbb{R}^2\to[0,\infty)$ satisfy

$$\int_{-\infty}^{t}\lambda_1(t,s)\mathrm{d}s\le\alpha_1, \qquad \int_{t}^{\infty}\lambda_2(t,s)\mathrm{d}t\le\alpha_2, \quad t\in\mathbb{R}, \tag{2.2}$$

$$\int_{s}^{r}\vartheta_1(t,s)\mathrm{d}t\le\theta_1, \qquad \int_{-r}^{s}\vartheta_2(t,s)\mathrm{d}t\le\theta_2, \quad \text{for } |s|\le r. \tag{2.3}$$

(AP) For $i=1,2$, the functions C_i are λ_i-almost periodic in $t,s\in\mathbb{R}$ uniformly in $(x,y)\in\mathbb{C}^{2n}$, where λ_i satisfy (2.2).

Specially interesting are the cases (1.4): $C_i(t,s,u,v)=\Lambda_i(t,s)\hat{C}_i(s,u,v)$, $i=1,2$, for which conditions ($\mathrm{L_C}$), (PAP) and (I) follow from the following condition (S).

(S) For $i=1,2$:

(a) $\hat{C}_i(s,u,v)$ are pseudo-almost periodic in s uniformly in u,v and there exist constants $L_i=L_i(\hat{C}_i)$ such that for all $s\in\mathbb{R}$, $u_k,v_k\in\mathbb{C}^n$:

$$\left|\hat{C}_i(s,u_1,v_1)-\hat{C}_i(s,u_2,v_2)\right| \le L_i\left(|u_1-u_2|+|v_1-v_2|\right).$$

(b) $\Lambda_i(t,s) \in \mathrm{AP}_{\lambda_i}(\mathbb{R}^2)$, where $\lambda_i : \mathbb{R}^2 \to (0,\infty)$ satisfy (2.2), and

$$\sup_{t\in\mathbb{R}} \int_{-\infty}^{t} |\Lambda_1(t,s)|\,\mathrm{d}s = \mu_1, \qquad \sup_{t\in\mathbb{R}} \int_{t}^{\infty} |\Lambda_2(t,s)|\,\mathrm{d}s = \mu_2 \quad \text{and}$$
$$\int_{s}^{r} |\Lambda_1(t,s)|\,\mathrm{d}t \le \theta_1, \qquad \int_{-r}^{s} |\Lambda_2(t,s)|\,\mathrm{d}t \le \theta_2, \quad \text{for } |s| \le r.$$

In particular, for the convolution case $\Lambda_i(t,s) = \Lambda_i(t-s)$, the conditions (b) become $\Lambda_1 \in L^1(0,\infty)$, $\Lambda_2 \in L^1(-\infty,0)$.

3. Existence of almost periodic and pseudo-almost periodic solutions

So, for the neutral integral equation (1.3), we obtain:

Theorem 1. *Under assumptions* (C), ($\mathrm{L_f}$), ($\mathrm{L_C}$), (PAP) *and* (I) *with* $2(L+\mu) < 1$, *the neutral integral equation* (1.3) *has a unique pseudo-almost periodic solution. In particular, if* (C) *and* ($\mathrm{L_f}$) *are true for almost periodic functions and* (PAP) *is replaced by* (AP), *then the neutral integral equation* (1.3) *has a unique almost periodic solution.*

Before proving Theorem 1, we establish the following technical lemma:

Lemma. *Let*

$$\begin{aligned}\mathbb{F}_1(u)(t) &:= \int_{-\infty}^{t} C_1(t,s,u(s),u(h_1(s)))\mathrm{d}s, \quad t\in\mathbb{R},\\ \mathbb{F}_2(u)(t) &:= \int_{t}^{\infty} C_2(t,s,u(s),u(h_2(s)))\mathrm{d}s, \quad t\in\mathbb{R}.\end{aligned} \tag{3.1}$$

Under assumptions (C), ($\mathrm{L_C}$), (PAP) *and* (I), *the functions* $\mathbb{F}_i$ *map* $\mathrm{PAP}(\mathbb{R})$ *into itself. In particular,* $\mathbb{F}_i$ *map* $\mathrm{AP}(\mathbb{R})$ *into itself, if* (PAP) *is replaced by* (AP) *and* (C) *is true for almost periodic functions.*

Proof. We prove only the case $i = 2$. For $i = 1$, the proof is similar. Let $i = 2$, $C_2 = C$ and $\mathbb{F}_2 = \mathbb{F}$. Let $u \in \mathrm{PAP}(\mathbb{R})$. By hypothesis (C), $t \to u(h_2(t))$ is pseudo-almost periodic. Using ($\mathrm{L_C}$) and (PAP), from the composition theorems, it follows that the function $(t,s) \to C(t,s,u(s),u(h_2(s)))$ is pseudo-almost periodic in t,s; see, e.g., [13,2,14,18]. From (PAP) and (I), we have the decomposition

$$C = Q_1 + Q_2, \quad Q_1 \in \mathrm{AP}_{\lambda}(\mathbb{R}^2 \times \mathbb{C}^{2n}, \mathbb{C}^n) \text{ and } Q_2 \in \mathrm{PAP}^0_{\vartheta}(\mathbb{R}^2 \times \mathbb{C}^{2n}, \mathbb{C}^n),$$

where $\lambda, \vartheta : \mathbb{R}^2 \to (0,\infty)$ satisfy (2.2) and (2.3). Then

$$M_1(u)(t) := \int_{t}^{\infty} Q_1(t,s,u(s),u(h_2(s)))\mathrm{d}s \tag{3.2}$$

is the almost periodic component of $\mathbb{F}u(t)$ and its ergodic component is

$$M_2(u)(t) := \int_{t}^{\infty} Q_2(t,s,u(s),u(h_2(s)))\mathrm{d}s. \tag{3.3}$$

In fact, both integrals (3.2), (3.3) exist by ($\mathrm{L_C}$). We will prove that $M_1u \in \mathrm{AP}(\mathbb{R})$ and $M_2u \in \mathrm{PAP}^0(\mathbb{R})$. By (PAP) and (I), $Q_1(t,s,u(s),u(h_2(s))) \in \mathrm{AP}_{\lambda}(\mathbb{R}^2)$, where $\lambda : \mathbb{R}^2 \to (0,\infty)$ satisfies (2.2). Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that every rectangle $R = R_1 \times R_2 \subset \mathbb{R} \times [t,\infty)$ with area $A(R) < \delta$, there is $\tau \in R_1 \cap R_2$, for which

$$|Q_1(t+\tau,s+\tau,u(s+\tau),u(h_2(s+\tau))) - Q_1(t,s,u(s),u(h_2(s)))| \le \varepsilon c\lambda(t,s) \tag{3.4}$$

for $t,s \in \mathbb{R}$ and some constant c. Since

$$M_1(u)(t+\tau) = \int_{t}^{\infty} Q_1(t+\tau,s+\tau,u(s+\tau),u(h_2(s+\tau)))\mathrm{d}s,$$

(3.2) and (3.4) imply that $|M_1(u)(t+\tau) - M_1(u)(t)| \le \varepsilon c\alpha$, for every $t \in \mathbb{R}$. Then $M_1(u) \in \mathrm{AP}(\mathbb{R})$.

Now, we show that $M_2(u) \in \mathrm{PAP}^0(\mathbb{R})$. By (3.3), it is clear that $t \to M_2(u)(t)$ is a bounded continuous function. By (PAP), the ergodic component Q_2 satisfies $|Q_2(t,s,x,y)| \le \vartheta(t,s)\hat{Q}_2(s,x,y)$, where $\int_{-r}^{s} \vartheta(t,s)\mathrm{d}t \le \theta$ for all $|s| \le r$. Finally, (PAP) and (I) imply

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} |M_2(u)(t)|\,\mathrm{d}t = 0. \tag{3.5}$$

In fact, $\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} |M_2(u)(t)|\mathrm{d}t \le l_1 + l_2$, where

$$l_1 := \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \mathrm{d}t \left(\left|\int_t^r Q_2(t,s,u(s),u(h_2(s)))\mathrm{d}s\right|\right), \quad \text{and}$$

$$l_2 := \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \mathrm{d}t \left(\left|\int_r^\infty Q_2(t,s,u(s),u(h_2(s)))\mathrm{d}s\right|\right).$$

Moreover, by changing the order of integration, (1.6) and $\hat{Q}_2(\cdot,u(\cdot),u(h_2(\cdot))) \in \mathrm{PAP}^0(\mathbb{R})$ imply:

$$l_1 \le \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \hat{Q}_2(s,u(s),u(h_2(s)))\mathrm{d}s \left(\int_{-r}^{s} \vartheta(t,s)\mathrm{d}t\right)$$
$$\le \theta \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \hat{Q}_2(s,u(s),u(h_2(s)))\mathrm{d}s = 0,$$

and similarly,

$$l_2 \le \lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r} \hat{Q}_2(s,u(s),u(h_2(s)))\mathrm{d}s \left(\int_{-r}^{r} \vartheta(t,s)\mathrm{d}t\right) = 0.$$

So, (3.5) follows. Respect to the second assertion, the situation where the functions are all almost periodic is clearly included in the previous proof. Thus the demonstration is complete. □

Proof of Theorem 1. For $u \in \mathrm{PAP}(\mathbb{R})$, define the nonlinear operator

$$\mathbb{A}(u)(t) := f(t,u(t),u(h_0(t))) + \int_{-\infty}^{t} C_1(t,s,u(s),u(h_1(s)))\mathrm{d}s + \int_t^\infty C_2(t,s,u(s),u(h_2(s)))\mathrm{d}s, \quad t \in \mathbb{R}.$$

From the composition theorems of pseudo-almost periodic functions in [13,2,14,18], we have $f(\cdot,u(\cdot),u(h_0(\cdot))) \in \mathrm{PAP}(\mathbb{R})$. Thus, by the previous lemma, $\mathbb{A}$ maps $\mathrm{PAP}(\mathbb{R})$ into itself and $M_1^i u$ and $M_2^i u$, $i = 1, 2$ are respectively the almost periodic and ergodic perturbation components of functions $\mathbb{F}_i u$, $i = 1, 2$ in $\mathbb{A}(u)$.

Finally, $\mathbb{A} : \mathrm{PAP}(\mathbb{R}) \to \mathrm{PAP}(\mathbb{R})$ has a unique fixed point. For $u, v \in \mathrm{PAP}(\mathbb{R})$, $(\mathrm{L_f})$ and $(\mathrm{L_C})$ imply $|\mathbb{A}(u)(t) - \mathbb{A}(v)(t)| \le 2(L+\mu)\|u-v\|_\infty$, since

$$\begin{aligned}|\mathbb{A}(u)(t) - \mathbb{A}(v)(t)| &\le 2L\|u-v\|_\infty + \int_{-\infty}^{t} |C_1(t,s,u(s),u(h_1(s))) - C_1(t,s,v(s),v(h_1(s)))|\,\mathrm{d}s \\ &\quad + \int_t^\infty |C_2(t,s,u(s),u(h_2(s))) - C_2(t,s,v(s),v(h_2(s)))|\,\mathrm{d}s \\ &\le 2L\|u-v\|_\infty + \int_{-\infty}^{t} \mu_1(t,s)\,(|u(s)-v(s)| + |u(h_1(s)) - v(h_1(s))|)\,\mathrm{d}s \\ &\quad + \int_t^\infty \mu_2(t,s)\,(|u(s)-v(s)| + |u(h_2(s)) - v(h_2(s))|)\,\mathrm{d}s.\end{aligned}$$

As $2(L+\mu) < 1$, the operator $\mathbb{A}$ is a contraction and has a unique fixed point, which obviously is the only pseudo-almost periodic solution to the integral equation (1.3). The assertion corresponding to almost periodic situation is obviously included in the above development. Then the proof is complete. □

In the separated variables situation (4), when condition (S) holds, the next corollary is a straightforward consequence of Theorem 1.

Corollary 1. *Under assumptions* (C), $(\mathrm{L_f})$, (S), *and* $2L+L_1\mu_1+L_2\mu_2 < 1$, *the integral equation* (1.3) *has a unique pseudo-almost periodic solution. In particular, if* (C), $(\mathrm{L_f})$ *and* (S), *part* (a) *hold for almost periodic functions, then neutral integral equation* (1.3) *has a unique almost periodic solution.*

Moreover, we can study a general neutral differential equation (1.5)

$$y'(t) = A(t)y + [f(t,y(t),y(h_0(t)))]' + Q(t,y(t),y(h_1(t))), \tag{3.6}$$

where

$$x' = A(t)x \tag{3.7}$$

has an exponential dichotomy and the function $t \to f(t,y(t),y(h_0(t)))$ is supposed differentiable. Indeed, if G is the Green matrix of the linear system (3.7), any solution of the integral equation

$$y(t) = f(t,y(t),y(h_0(t))) + \int_{-\infty}^{\infty} G(t,s)\,(A(s)f(s,y(s),y(h_0(s))) + Q(s,y(s),y(h_2(s))))\,\mathrm{d}s \tag{3.8}$$

is solution of the neutral differential equation (3.6).

By simplicity, we consider $A(t) = A$ constant and the assumptions:

(E) The eigenvalues λ of the constant matrix A satisfy $Re\lambda \neq 0$ and the Green operator has the norm $\sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} |G(t, s)| \mathrm{d}s = \mu < \infty$.

(L) $Q = Q(t, u, v)$ and $f_A(t, u, v) = A(t)f(t, u, v)$ are pseudo-almost periodic and for all $t \in \mathbb{R}, u_i, v_i \in \mathbb{C}^n,\ i = 1, 2$ satisfy

$$|Q(t, u_1, v_1) - Q(t, u_2, v_2)| \le L_Q(|u_1 - u_2| + |v_1 - v_2|), \quad L_Q \text{ constant},$$
$$|A(t)(f(t, u_1, v_1) - f(t, u_2, v_2))| \le L_A(|u_1 - u_2| + |v_1 - v_2|), \quad L_A \text{ constant}.$$

Theorem 2. *If* (C), (E) *and* (L) *are fulfilled and* $2L + (L_A + L_Q)\mu < 1$, *then the neutral differential equation* (3.6) *has a unique pseudo-almost periodic solution. In particular, if* (C) *holds for almost periodic functions and* f_A *and* Q *are only almost periodic functions, then the neutral differential equation* (3.6) *has a unique almost periodic solution.*

Proof. If A satisfies condition (E), then $\Phi(t) = \exp(tA)$ is the fundamental matrix and $\Phi(t)\Phi^{-1}(s) = \Phi(t - s)$. Let P be a projection matrix such that $G_1(t, s) = G_1(t - s) = \Phi(t - s)P \to 0$ as $t \to \infty$. So, the Green matrix $G = G(t, s)$ is given by: $G(t, s) = G_1(t - s)$ for $t \ge s$ and $G(t, s) = G_2(t - s) = -\Phi(t - s)(I - P)$ for $t < s$. Hypothesis (I) is provided with $\mu_i(t, s) = \lambda_i(t, s) = \vartheta_i(t, s) = |G_i(t - s)|$. Now, Theorem 2 follows at once from Theorem 1. □

Remark 1. Using technical lemmas about linear differential equations with almost periodic coefficients (see for example, Fink [3]), Theorem 2 is easily extended to an exponentially dichotomic system (3.7) with almost periodic matrix $A(t)$. In this case, we must take $\lambda_1(t, s) = e^{\frac{-(t-s)}{2\mu}}$, $\lambda_2(t, s) = e^{\frac{(t-s)}{2\mu}}$.

An interesting particular case of (3.6) is given by

$$u' = Au + Bu'(h_0(t)) + Q(t, u(t), u(h_2(t))), \tag{3.9}$$

when B is a constant matrix and $h_0' = 1$, implying the following:

Corollary 2. *If* (C) *and* (E) *hold,* Q *satisfies* (L) *and* $[(1 + |A|)|B| + 2L_Q]\mu < 1$, *then the conclusions of Theorem* 2 *follow.*

Finally, for the scalar neutral logistic equation (1.1), we deduce

Corollary 3. *If* (C) *holds and* (Q) *satisfies* (L) *with* $|\beta|(1 + |a|) + 2L_Q < |a|$, *the logistic equation* (1.1) *has a unique pseudo-almost periodic solution. In particular, if* Q *is an only almost periodic function, then the logistic equation* (1.1) *has a unique almost periodic solution.*

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