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Idempotents in plenary train algebras

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ABSTRACT

In this paper we study plenary train algebras of arbitrary rank. We show that for most parameter choices of the train identity, the additional identity $(x^2 - \omega(x)x)^2 = 0$ is satisfied. We also find sufficient conditions for A to have idempotents.

1. Introduction

Nonassociative algebras arise in population genetic models in a quite natural way. For more information see Worz-Busekros [6], Lyubich [4] and Reed [5]. In particular Gutierrez [3] shows that every genetic algebra is a plenary train algebra.

Plenary powers are defined inductively by $x^{(1)} = x$ and $x^{(n+1)} = (x^{(n)})^2$. The pair (A, ω) is called a baric algebra if $\omega : A \to K$ is a nontrivial homomorphism. If a baric algebra (A, ω) satisfies an identity of the form

$$x^{(n)} = \alpha_1 \omega(x)^{2^{n-1}-1} x + \alpha_2 \omega(x)^{2^{n-1}-2} x^2 + \cdots + \alpha_{n-1} \omega(x)^{2^{n-2}} x^{(n-1)}, \tag{1}$$

where $\sum_{i=1}^{n-1} \alpha_i = 1$, then we call it a plenary train algebra. We will further assume that A is commutative.

An important question in nonassociative algebras in general and in train algebras in particular is the existence of idempotents. Given an idempotent we may better understand the algebra studying

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its Peirce decomposition. For train algebras this was done by Gutierrez [3]. In addition to this mathematical importance, idempotents are also significant in the biological application since they represent a genetic equilibrium.

2. Main section

Lemma 1. *Let A be any baric algebra with weight function ω. If A satisfies the identity*

$$\left(x^2-\omega(x)x\right)^2=0, \tag{2}$$

then, for any integers $i, j>0$ and for any element x of weight 1:

$$\left(x^{(i)}-x^{(j)}\right)^2=0. \tag{3}$$

Proof. We proceed by induction on $n=|i-j|$. The case $n=0$ is obvious. The case $n=1$ is a direct consequence of (2). We start by expanding and linearizing (2):

$$4(xx)(xy)-2\omega(y)x(xx)-2\omega(x)y(xx)-4\omega(x)x(xy)+2\omega(x)\omega(y)(xx)+2\omega(x)\omega(x)(xy)=0.$$

When $\omega(x)=\omega(y)=1$ this shortens to

$$\left(4x^2-4x\right)(xy)+2xy-2x^2y-2xx^2+2x^2=0. \tag{4}$$

Our inductive hypothesis is that (3) holds for all x of weight 1 and for all i, j such that $|i-j|<n$:

$$2x^{(i)}x^{(j)}=x^{(i+1)}+x^{(j+1)}. \tag{5}$$

Replacing $y=x^{(n)}$ in (4) we get

$$\left(4x^2-4x\right)\left(xx^{(n)}\right)+2xx^{(n)}-2x^2x^{(n)}-2xx^2+2x^2=0.$$

Using (5) on the first occurrence of $xx^{(n)}$

$$\left(2x^2-2x\right)\left(x^2+x^{(n+1)}\right)+2xx^{(n)}-2x^2x^{(n)}-2xx^2+2x^2=0.$$

Again using (5) where appropriate

$$\begin{aligned}&2x^{(3)}+\left(x^{(3)}+x^{(n+2)}\right)-\left(x^2+x^{(3)}\right)-2xx^{(n+1)}+\left(x^2+x^{(n+1)}\right)\\&\quad-\left(x^{(3)}+x^{(n+1)}\right)-\left(x^2+x^{(3)}\right)+2x^2=0.\end{aligned}$$

Collecting similar terms

$$x^{(n+2)}-2xx^{(n+1)}+x^2=\left(x^{(n+1)}-x\right)^2=0.$$

This proves (3) for $|i-j|=n$. $\square$

Theorem 2. *Let A be a plenary train algebra of rank n with defining identity*:

$$x^{(n)} = \alpha_1\omega(x)^{2^{n-1}-1}x + \alpha_2\omega(x)^{2^{n-1}-2}x^2 + \cdots + \alpha_{n-1}\omega(x)^{2^{n-1}}x^{(n-2)}, \tag{6}$$

where $\sum_{i=1}^{n-1}\alpha_i = 1$. Let

$$\lambda = \sum_{i=1}^{n-1}(n-i)\alpha_i.$$

Assume A satisfies $(x^2 - \omega(x)x)^2 = 0$. If $\lambda \neq 0$ then A has idempotents.

Proof. Let x be any weight one element of A and let

$$b_k = \sum_{i=1}^{k}\alpha_i, \qquad b = \sum_{k=1}^{n-1} b_k x^{(k)}.$$

Notice that $\sum_{k=1}^{n-1} b_k = \lambda$ and that $b_{n-1} = 1$. Next we calculate b^2:

$$\begin{aligned} b^2 &= \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} b_k b_j x^{(k)} x^{(j)} \\ &= \frac{1}{2}\sum_{k=1}^{n-1}\sum_{j=1}^{n-1} b_k b_j \big(x^{(k+1)} + x^{(j+1)} - \big(x^{(k)} - x^{(j)}\big)^2\big). \end{aligned}$$

Using Lemma 1, $(x^{(k)} - x^{(j)})^2 = 0$,

$$b^2 = \frac{1}{2}\left(\sum_{k=1}^{n-1}\sum_{j=1}^{n-1} b_k b_j x^{(k+1)} + \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} b_k b_j x^{(j+1)}\right).$$

Switching the indices of the first sum and using that $\sum b_k = \lambda$,

$$b^2 = \sum_{k=1}^{n-1}\sum_{j=1}^{n-1} b_k b_j x^{(j+1)} = \lambda\sum_{j=1}^{n-1} b_j x^{(j+1)}.$$

From the plenary identity and noticing that $b_{n-1} = 1$,

$$b^2 = \lambda\left(\sum_{j=1}^{n-2} b_j x^{(j+1)} + \sum_{k=1}^{n-1}\alpha_k x^{(k)}\right).$$

Collecting terms and using the definition of the b_k,

$$b^2 = \lambda\left(\alpha_1 x + \sum_{k=2}^{n-1}(b_{k-1} + \alpha_k)x^{(k)}\right) = \lambda\left(\sum_{k=1}^{n-1} b_k x^{(k)}\right) = \lambda b.$$

We conclude that $e = \frac{b}{\lambda}$ is an idempotent in A. □

We may notice that in the previous proof the hypothesis $(x^2 - \omega(x)x)^2 = 0$ is not fully used. A sufficient condition would be $\sum_{k<j<n} b_k b_j (x^{(k)} - x^{(j)})^2 = 0$, where the b_k are defined as in the proof of the theorem.

Lemma 3. *Let A be a baric algebra. If all weight one elements $x \in A$ satisfy the equation*:

$$x^{(k)} = \sum_{i=1}^{n-1} \beta_i x^{(i)}, \tag{7}$$

for some fixed $k \geqslant n$ and $\sum \beta_i = 1$, then they also satisfy

$$\sum_{1 \leqslant i<j<n} \beta_i \beta_j \left(x^{(i)} - x^{(j)}\right)^2 = 0. \tag{8}$$

Proof. Let

$$S = 2 \sum_{1 \leqslant i<j<n} \beta_i \beta_j \left(x^{(i)} - x^{(j)}\right)^2.$$

We can turn (8) into a full double sum by adding some trivially zero terms where $i = j$:

$$S = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j \left(x^{(i)} - x^{(j)}\right)^2.$$

Expanding the squared terms

$$S = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j \left(x^{(i)}\right)^2 + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j \left(x^{(j)}\right)^2 - 2 \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_i \beta_j x^{(i)} x^{(j)}.$$

Changing the summation order and factoring the sums

$$S = \sum_{j=1}^{n-1} \beta_j \sum_{i=1}^{n-1} \beta_i x^{(i+1)} + \sum_{i=1}^{n-1} \beta_i \sum_{j=1}^{n-1} \beta_j x^{(j+1)} - 2 \sum_{i=1}^{n-1} \beta_i x^{(i)} \sum_{j=1}^{n-1} \beta_j x^{(j)}.$$

Using (7) and that $\sum \beta_i = 1$

$$S = 2 \sum_{j=1}^{n-1} \beta_j x^{(j+1)} - 2\left(x^{(k)}\right)^2.$$

Using (7) again for x^2 in place of x

$$S = 2x^{(k+1)} - 2x^{(k+1)} = 0. \qquad \square$$

Lemma 4. *Let A be a plenary train algebra of rank n with defining identity*:

$$x^{(n)} = \sum_{i=1}^{n-1} \alpha_i \omega(x)^{2^{n-1}-2^{i-1}} x^{(i)},$$

where $\sum_{i=1}^{n-1} \alpha_i = 1$. *Consider an element* $x \in A$ *of weight one and let its plenary powers up to* $x^{(n-1)}$ *be the spanning set of a vector space where* $x^{(i)} = (0 \dots 1 \dots 0)$ *has a one in the ith-position. Then we can express* $x^{(k+1)}$ *in terms of this spanning set by* $(1, 0, 0, 0, \dots, 0)A^k$ *where*

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \dots & & \\ & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{n-2} & \alpha_{n-1} \end{pmatrix}.$$

Proof. The proof goes by induction on k. For $k = 0$ there is nothing to prove. So we assume

$$x^{(k)} = \sum_{i=1}^{n-1} \beta_i x^{(i)} = (\beta_1, \beta_2, \beta_3, \dots, \beta_{n-2}, \beta_{n-1}) = (1, 0, 0, 0, \dots, 0)A^{k-1}.$$

Replacing x by x^2 we have

$$\begin{aligned} x^{(k+1)} &= \sum_{i=1}^{n-1} \beta_i x^{(i+1)} = \sum_{i=2}^{n-1} \beta_{i-1} x^{(i)} + \beta_{n-1} \sum_{i=1}^{n-1} \alpha_i x^{(i)} \\ &= (0, \beta_1, \beta_2, \dots, \beta_{n-3}, \beta_{n-2}) + \beta_{n-1}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n-2}, \alpha_{n-1}) \\ &= (\beta_1, \beta_2, \beta_3, \dots, \beta_{n-2}, \beta_{n-1})A \\ &= (1, 0, 0, 0, \dots, 0)A^k. \qquad \square \end{aligned}$$

Theorem 5. *Let A be a plenary train algebra of rank n with defining identity*:

$$x^{(n)} = \sum_{i=1}^{n-1} \alpha_i \omega(x)^{2^{n-1}-2^{i-1}} x^{(i)}.$$

Let $\lambda_1, \dots, \lambda_{n-1}$ *be the eigenvalues of the matrix A defined in Lemma* 4 (*the* λ_k *are the nonzero roots of the associative polynomial* $x^n - \sum \alpha_i x^i$). *If all the products* $\lambda_i \lambda_j$ *are distinct then A satisfies* $(x^2 - \omega(x)x)^2 = 0$ *and A has idempotents.*

Proof. Using Lemma 3 and Lemma 4 we get identities

$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \beta_{ki} \beta_{kj} \big(x^{(i)} - x^{(j)}\big)^2 = 0,$$

where $(\beta_{k1}, \beta_{k2}, \beta_{k3}, \dots, \beta_{kn-2}, \beta_{kn-1}) = e_1 A^{k-1}$ and k is any positive integer. So we have a homogeneous system of identities satisfied by the squares $(x^{(i)} - x^{(j)})^2$. In matrix form this can be written as

$$\langle (e_1 A^{k-1})^T e_1 A^{k-1}, U \rangle = 0,$$

where U is the symmetric matrix such that $U_{ij} = (x^{(i)} - x^{(j)})^2$, and where the angled bracket of two matrices X, Y stands for $\langle X, Y\rangle = \sum_{i,j} X_{ij} Y_{ij}$. Now consider $v_1, \dots, v_{n-1}$ eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \dots, \lambda_{n-1}$ of A, and write $e_1 = \sum c_i v_i$ as a linear combination of them. Since $e_k = e_1 A^{k-1} = \sum \lambda_i^{k-1} c_i v_i$ we notice that the $c_i v_i$ also form a basis of eigenvectors for A, so we may assume that $c_i = 1$ for every i. Then

$$\begin{aligned} 0 = \langle (e_1 A^k)^T e_1 A^k, U \rangle &= \left\langle \left(\sum_{i=1}^{n-1} \lambda_i^k v_i \right)^T \sum_{i=1}^{n-1} \lambda_i^k v_i, U \right\rangle \\ &= \left\langle \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (\lambda_i \lambda_j)^k v_i^T v_j, U \right\rangle \\ &= \sum_{1 \leqslant i \leqslant j < n} (\lambda_i \lambda_j)^k \langle (v_i^T v_j + v_j^T v_i), U \rangle. \end{aligned}$$

Since this holds for all k, the Vandermonde determinant says that for each $1 \leqslant i \leqslant j < n$ we have

$$\langle (v_i^T v_j + v_j^T v_i), U \rangle = 0.$$

Using the symmetry of U,

$$2 \langle (v_i^T v_j), U \rangle = 0.$$

Since the v_i form a basis for the $(n-1)$-dimensional row space, the matrices $v_i^T v_j$ form a basis for the space of all $(n-1) \times (n-1)$ matrices. To verify this, it suffices to show that they are linearly independent. In fact, if $\sum r_{ij} v_i^T v_j = 0$ then multiplying by any v_k on the left we get $\sum_j (\sum_i r_{ij} v_k v_i^T) v_j$. Since the v_j are linearly independent, $\sum_i r_{ij} v_k v_i^T = 0$ for every k, j. Now since the v_k form a basis $\sum_i r_{ij} v_i^T = 0$, and finally since the v_i^T are linearly independent, $r_{ij} = 0$ for every i, j.

Finally, this shows that U is orthogonal to a basis for the space of all matrices, so $U = 0$ and in particular $(x^{(i)} - x^{(j)})^2 = 0$ for every i, j. Finally, to use Theorem 2 we need to check that $\lambda = \sum (n-i)\alpha_i \neq 0$. We will show that this just means that 1 is not a repeated eigenvalue of A and so it is part of the hypothesis. We factor the associative polynomial:

$$\begin{aligned} x^n - \sum_{i=1}^{n-1} \alpha_i x^i = \sum_{i=1}^{n-1} \alpha_i (x^n - x^i) &= \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i (x^{k+1} - x^k) \\ &= (x-1) \sum_{i=1}^{n-1} \sum_{k=i}^{n-1} \alpha_i x^k. \end{aligned}$$

Evaluating the right factor at $x = 1$ we get

$$\sum_{i=1}^{n-1}\sum_{k=i}^{n-1}\alpha_i = \sum_{i=1}^{n-1}(n-i)\alpha_i.$$

So A has idempotents by Theorem 2. □

As an illustration we consider some small cases:

Example 6 *($n = 3$).* Let A be a plenary train algebra satisfying

$$x^{(3)} = \alpha x + (1-\alpha)x^2.$$

The nonzero roots of the polynomial $x^3 - (1-\alpha)x^2 - \alpha x$ are 1 and $-\alpha$ so by Theorem 5 we can guarantee that A has an idempotent as long as $1, -\alpha, \alpha^2$ are all different, that is $\alpha \notin \{0, 1, -1\}$. Furthermore, for every x of weight 1, we know an idempotent to be

$$\frac{1}{\alpha+1}\left(\alpha x + x^2\right).$$

Notice that when $\alpha = 0$, the formula still works and x^2 is an idempotent. When $\alpha = 1$ we do not find an idempotent in this way but Etherington [2] showed that there are idempotents in this case. Etherington also showed that when $\alpha = -1$ there may not be any idempotents.

Example 7 *($n = 4$).* Let A be a plenary train algebra satisfying

$$x^{(4)} = \alpha x + \beta x^2 + \gamma x^{(3)},$$

where $\alpha + \beta + \gamma = 1$. Lets assume that $1, \lambda, \mu$ are the nonzero roots of $x^4 - \gamma x^3 - \beta x^2 - \alpha x = 0$ so that $\alpha = \lambda\mu$, $\beta = -(\lambda\mu + \lambda + \mu)$, $\gamma = \lambda + \mu + 1$. Theorem 5 says that A has an idempotent as long as $1, \lambda, \mu, \lambda\mu, \lambda^2, \mu^2$ are all distinct, that is $\lambda\mu(\lambda^2-1)(\mu^2-1)(\lambda^2-\mu^2)(\lambda-\mu^2)(\lambda^2-\mu)(\lambda\mu-1) \neq 0$.
Furthermore, in this case, we know an idempotent to be

$$\frac{1}{3\alpha+2\beta+\gamma}\left(\alpha x + (\alpha+\beta)x^2 + (\alpha+\beta+\gamma)x^{(3)}\right).$$

One may notice again that the given condition is not necessary and to answer the question it suffices to show that $\alpha(\alpha+\beta)x + \alpha(\alpha+\beta+\gamma)x^2 + (\alpha+\beta)(\alpha+\beta+\gamma)x^{(3)} = 0$ (see the proof of Theorem 2). For this we have to solve a linear algebra problem. We need to know whether the vector

$$\begin{pmatrix} \alpha(\alpha+\beta) & \alpha(\alpha+\beta+\gamma) & (\alpha+\beta)(\alpha+\beta+\gamma) \end{pmatrix}$$

is in the row space of the following matrix:

$$\begin{pmatrix} \alpha\beta & \alpha\gamma & \beta\gamma \\ \alpha\gamma(\alpha+\beta\gamma) & \alpha\gamma(\beta+\gamma^2) & (\alpha+\beta\gamma)(\beta+\gamma^2) \\ \alpha(\beta+\gamma^2)(\alpha\gamma+\beta(\beta+\gamma^2)) & \alpha(\beta+\gamma^2)(\alpha+\beta\gamma+\gamma(\beta+\gamma^2)) & (\alpha\gamma+\beta(\beta+\gamma^2))(\alpha+\beta\gamma+\gamma(\beta+\gamma^2)) \end{pmatrix}.$$

The coefficients of this matrix are obtained applying Lemma 3 to the train identity and to the higher order identities from Lemma 4. It turns out that this is the case as long as $(\beta-1)(\alpha-1) \neq 0$. We also need that $3\alpha + 2\beta + \gamma \neq 0$. Finally, in terms of the eigenvalues, the condition is

$$\left(\lambda^2 - 1\right)\left(\mu^2 - 1\right)(\lambda\mu - 1) \neq 0.$$

This result was obtained recently by Labra and Suazo [1].

Example 8 *($n = 5$).* Let A be a plenary train algebra satisfying

$$x^{(5)} = \alpha x + \beta x^2 + \gamma x^{(3)} + (1 - \alpha - \beta - \gamma)x^{(4)}.$$

Skipping the details, using Maxima to solve the linear system, we know an idempotent to be

$$\frac{1}{3\alpha + 2\beta + \gamma + 1}\left(\alpha x + (\alpha + \beta)x^2 + (\alpha + \beta + \gamma)x^{(3)} + x^{(4)}\right),$$

as long as $(\alpha + \gamma - 1)(\alpha\gamma + \alpha\beta - \alpha - \beta + 1)(3\alpha + 2\beta + \gamma + 1) \neq 0$. In terms of the eigenvalues, the condition is

$$\left(\lambda^2 - 1\right)\left(\mu^2 - 1\right)\left(\nu^2 - 1\right)(\lambda\mu - 1)(\lambda\nu - 1)(\mu\nu - 1) \neq 0.$$

3. Open problem

One question that remains open is to find precise necessary and sufficient conditions for a plenary train algebra to have idempotents.

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