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h-asymptotic stability by fixed point in neutral nonlinear differential equations with delay

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ABSTRACT

We use fixed-point theory to obtain h-asymptotic stability results about the zero solution of nonlinear differential equations with functional delay. Neither boundedness of the deviations nor a fixed sign on the coefficients functions are asked. An explicit estimation of the solutions is obtained and a necessary and sufficient condition is determined. Classical results are improved and generalized.

1. Introduction

This research is motivated by a number of difficulties encountered when we study the stability by means of Lyapunov's direct method. Much of these difficulties disappear by applying fixed point theory [1–10]. While Lyapunov's direct method usually requires pointwise conditions, our results ask averaging conditions. Lyapunov functions have been the main tool used to obtain boundedness, stability and the existence of periodic solutions of differential equations, differential equations with functional delays and functional differential equations (see [11–13]). As an example, in the study of differential equations with functional delays by using Lyapunov functionals, many difficulties arise if the delay is unbounded (see [14,8]). Even more difficult it is to obtain necessary and sufficient conditions. Many authors have examined particular problems which have offered great difficulties for that theory and have presented solutions by means of various fixed point theorems for the last ten years. Burton [1–3] and Burton and Furumochi [4–6] have shown that many of these problems can be solved using fixed point theory. For a complete framework of stability by fixed point theory, see the new book [15] and the reference therein.

In this paper we look the scalar neutral differential equation

$$\begin{aligned} x'(t) &= a(t)x(t) + b(t)x'(\gamma_1(t)) + g(t, x(t), x(\gamma_2(t))), \quad t \geq \tau \\ x(t) &= \phi(t), \quad t \leq \tau, \end{aligned} \tag{1}$$

where $a(t)$ may change sign, $a(t)$, $b(t)$, $\gamma_i(t)$, $i = 1, 2$ and $g(t, x, y)$ are continuous in their respective arguments. We assume the local existence and uniqueness of the solutions $x(t) = x(t, \tau, \phi)$ of (1). We obtain necessary and sufficient conditions under which the asymptotic stability is ensured. The stability is not necessarily uniform and the result gives an explicit

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estimation for the solutions. Precisely, the h-stability [16] of the zero solution of (1) is proved, namely for a positive constant c, we have

$$|x(t,\tau,\phi)| \leq c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1}, \quad \text{for } t \geq \tau \geq 0, \|\phi\| \leq \delta, \tag{2}$$

where

$$h_\sigma(t) = \exp\left(\sigma \int_0^t a(s)\mathrm{d}s\right), \quad 0 < \sigma < 1. \tag{3}$$

This function is bounded if

$$\limsup_{t\to\infty} \int_0^t a(s)\mathrm{d}s < \infty \tag{4}$$

and converges to zero if

$$\lim_{t\to\infty} \int_0^t a(s)\mathrm{d}s = -\infty. \tag{5}$$

Furthermore, the condition

$$\int_{t_1}^t a(s)\mathrm{d}s \leq K, \ \text{for } t \geq t_1 \geq 0, \tag{6}$$

is associated with the uniform stability of Eq. (1). Note that (2) implies the boundedness of the solutions $x(t,\tau,\phi)$ and the stability of the null-solution.

The integral equation associated to (1) has a linear part and other nonlinear parts, for that it is natural to use Krasnoselskii's fixed point theorem. Once the correct mapping is constructed, then the analysis in this paper is based on an appropriate choice of the invariant set where (2) must be true. To our knowledge this general type of result is new (see [1,5,6,17,9]). Our Eq. (1) generalizes the work of [1,4,8,10] where the Banach's fixed point is used and, in addition, we obtain a precise bound for the behavior, see [8] and the new book [15].

The rest of the paper is organized as follows. In the next section, Krasnoselskii's fixed point theorem, some definitions and preliminary results are presented. Section 3 is devoted to establish the main results for the stability of the zero solution of (1). Finally, in Section 4 we show an illustrative example.

2. Fixed-point theorems and preliminaries

Now, we state the Krasnosleskii's fixed point theorem, which will be useful in Section 3. A statement of this theorem can be found in [18].

Theorem A. *Let S be a closed, bounded convex, non-empty set of a Banach space E. Suppose that Γ_1 and Γ_2 map S into E and that*

(1) $\Gamma_1 x + \Gamma_2 y \in S$ *for all* $x, y \in S$.
(2) Γ_1 *is completely continuous on S, and*
(3) Γ_2 *is a contraction on S.*

Then, there exists $z \in S$ such that $\Gamma_1 z + \Gamma_2 z = z$.

Let $\mathbb{R}$ and $\mathbb{R}^+$ denote, respectively the set of real numbers and nonnegative real numbers, $|x|$ the absolute value for $x \in \mathbb{R}$. Let the delay functions d_i, $d_i(t) = t - \gamma_i(t) \geq 0$, $i = 1, 2$, which can be both bounded or unbounded delays. We define

$$\rho_i(\tau) := \inf_{t\geq\tau} \gamma_i(t), \quad i = 1, 2; \quad \rho_-(\tau) = \min\{\rho_1(\tau), \rho_2(\tau)\}.$$

Considering the initial closed interval $[\rho_-(\tau), \tau]$ ($(-\infty, \tau]$ if $\rho_-(\tau) = -\infty$), Eq. (1) has a natural vectorial space for the initial conditions:

$$BC(\tau) = \{\phi : [\rho_-(\tau), \tau] \to \mathbb{R} |\ \phi \text{ is a bounded continuous function}\}.$$

Denote $\mathbb{B}$ to $BC([\rho_-(\tau), \infty), \mathbb{R})$ the Banach space of bounded and continuous real functions, with the supreme norm $\|\cdot\|$. For the compactness in the space $\mathbb{B}$ an equi-convergence criterion is useful, see, [5,19].

Lemma 1. *Let*

$$S_\phi := \{y \in \mathbb{B} : y(t) = \phi(t), t \in [\rho_-(\tau), \tau], \ |y(t)| \leq q(t), \ t \geq \tau\},$$

for fixed ϕ, and $|\phi(\tau)| \leq q(\tau)$. If $\lim_{t\to\infty} q(t) = 0$ and S_ϕ is an equi-continuous set on every interval $[\tau, n]$, $n \in \mathbb{N}$, then S_ϕ is a compact convex nonempty subset of $\mathbb{B}$.

We need find an admissible map for (1), to apply the fixed point theory. To obtain the desired map, we consider the linear differential equation

$$\begin{aligned} &x'(t) = a(t)x(t) + b(t)x'(\gamma_1(t)) + f(t), \quad t \geq \tau \\ &x(t) = \phi(t), \quad t \in [\rho_-(\tau), \tau], \end{aligned} \tag{7}$$

where $a(t), f(t)$ are continuous, $b(t)$ is continuously differentiable and γ_1 is twice continuously differentiable such that $\gamma_1'(t) \neq 0,\ \forall t \in \mathbb{R}$.

Lemma 2. *$x(t) = x(t, \tau, \phi)$ is a solution of Eq. (7) if and only if*

$$x(t) = \Psi_\phi(\tau)h_1(t)h_1(\tau)^{-1} + B(t)x(\gamma_1(t)) + \int_\tau^t h_1(t)h_1(u)^{-1}w(u)x(\gamma_1(u))\mathrm{d}u + \int_\tau^t h_1(t)h_1(u)^{-1}f(u)\mathrm{d}u, \tag{8}$$

where

$$\begin{aligned} &\Psi_\phi(\tau) = \Big(\phi(\tau) - B(\tau)\phi(\gamma_1(\tau))\Big), \qquad B(t) = \frac{b(t)}{\gamma_1'(t)}, \\ &\omega(u) = B(u)a(u) - B'(u), \qquad h_1(t) = e^{\int_0^t a(s)\mathrm{d}s}. \end{aligned} \tag{9}$$

Proof. By Eq. (7) we have

$$\frac{d}{dt}[x(t) - x(\gamma_1(t))B(t)] = a(t)[x(t) - x(\gamma_1(t))B(t)] + w(t)x(\gamma_1(t)) + f(t)$$

and (8) follows from the variation of parameters formula. □

3. Stability via fixed-point theory

In this section we prove our main result, so we recall the Eq. (1)

$$x'(t) = a(t)x(t) + b(t)x'(\gamma_1(t)) + g(t, x(t), x(\gamma_2(t))),$$

where

$$g(t, 0, 0) = 0, \qquad \gamma_i(t) \leq t, \quad i = 1, 2;\ t \in \mathbb{R}.$$

Let $\sigma \in (0, 1)$ and $D_i, M_i : \mathbb{R} \to \mathbb{R}^+$, $i = 1, 2$, defined by

$$D_i(t) := \exp\left\{-\int_{\gamma_i(t)}^t a(s)\mathrm{d}s\right\}, \qquad M_i(t) := \max\{1, D_i^\sigma(t)\}. \tag{10}$$

Next, we state our assumptions:

(H1)
- The function $a : \mathbb{R} \to \mathbb{R}$, satisfies (4) and (6).
- The functions $\gamma_i : \mathbb{R}^+ \to \mathbb{R},\ i = 1, 2$ are continuous and
$$\lim_{t\to\infty} \gamma_1(t) = \infty, \tag{11}$$
- γ_1 and $B = b/\gamma_1'$ are continuously differentiable.

(H2)
- g is a continuous function, say: $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$. For any $\varepsilon > 0$, there exist $\delta > 0$ and $\lambda : \mathbb{R} \to \mathbb{R}^+$ function such that $|x_1 - x_2|, |y_1 - y_2| < \delta$ implies
$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq \varepsilon\lambda(t), \quad t \in \mathbb{R}^+. \tag{12}$$
Moreover, there exists $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ a nondecreasing continuous function for which $\nu(0^+) = 0$ and
$$|g(t, x, y)| \leq \lambda(t)\nu(|x| + |y|)[|x| + |y|], \quad t, s \in \mathbb{R}. \tag{13}$$

(H3)
- For a positive constant $\sigma < 1$ the functions λ and M_2 satisfy:
$$\sup_{t\geq 0} \int_0^t h_{1-\sigma}(t)h_{1-\sigma}(u)^{-1}\lambda(u)M_2(u)\mathrm{d}u < \infty. \tag{14}$$
- The functions $B = b/\gamma_1'$, ω and M_1 verify
$$|B(t)|M_1(t) + \int_0^t h_{1-\sigma}(t)h_{1-\sigma}(u)^{-1}|\omega(u)|M_1(u)\mathrm{d}u \leq \alpha < 1, \quad t \geq 0. \tag{15}$$

Theorem 1. *If* (H1), (H2) *and* (H3) *hold. Then:*

(i) *Assuming the condition (5), the zero solution of (1) is h_σ-asymptotically stable, namely, there exists $\delta > 0$ such that $\|\phi\| \le \delta$ implies*

$$|x(t,\tau,\phi)| \le c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1}, \quad t \ge \tau \ge 0, \tag{16}$$

where $c = (1 + \|B\|)e^{(1-\sigma)K}$, K given by (6).

(ii) *The zero solution of (1) is asymptotically stable if and only if (5) is true.*

Proof. *Sufficiency*. Assuming that (5) is valid. For any $\delta > 0$, the set

$$S = \{x \in \mathbb{B} : \ x(t) = \phi(t), \ t \in [\rho_-(\tau), \tau], |x(t)| \text{ satisfies (16) and } \|\phi\| \le \delta\},$$

is a closed, convex and nonempty subset of $\mathbb{B}$. From Lemma 2, we define the map P by

$$(Px)(t) = \phi(t), \quad \rho_-(\tau) \le t \le \tau,$$

and

$$(Px)(t) = \Psi(\phi) h_1(t) h_1(\tau)^{-1} + B(t) x(\gamma_1(t)) + \int_\tau^t h_1(t) h_1(u)^{-1} \omega(u) x(\gamma_1(u)) \mathrm{d}u + \int_\tau^t h_1(t) h_1(u)^{-1} g(u, x(u), x(\gamma_2(u))) \mathrm{d}u, \quad t \ge \tau.$$

From (H1) and (14), we can see that $PS \subset \mathbb{B}$. We consider the operators Γ_1, Γ_2 by

$$\Gamma_1 x(t) = \Psi_\phi(\tau) h_1(t) h_1(\tau)^{-1} + B(t) x(\gamma_1(t)) + \int_\tau^t h_1(t) h_1(u)^{-1} \omega(u) x(\gamma_1(u)) \mathrm{d}u,$$

$$\Gamma_2 x(t) = \int_\tau^t h_1(t) h_1(u)^{-1} g(u, x(u), x(\gamma_2(u))) \mathrm{d}u.$$

We note that $\Gamma_1 x + \Gamma_2 x = Px$ so, we shall use Theorem A to ensure the existence of a fixed point of P. We claim that $\Gamma_1 : S \to \mathbb{B}$ is a contractive mapping. Let $x, y \in \mathbb{B}$, from (13), (15) and the fact $M_1(t) \ge 1$, we have

$$\begin{aligned} |\Gamma_1 x(t) - \Gamma_1 y(t)| &\le |B(t)| \, |x(\gamma_1(t)) - y(\gamma_1(t))| + \int_\tau^t |h_1(t) h_1(u)^{-1}| |\omega(u)| |x(\gamma_1(u)) - y(\gamma_1(u))| \mathrm{d}u \\ &\le \alpha \|x - y\|, \quad \alpha < 1. \end{aligned}$$

Next, we will show that there exists a $\delta > 0$ such that for any $x, y \in S = S(\delta)$ with $\|\phi\| \le \delta$ we have $\Gamma_1 x + \Gamma_2 y \in S$, i.e., there exists S a convex closed non-empty subset of $\mathbb{B}$, such that $PS \subset S$. We denote $R = R(\phi) := 2c\|\phi\| \, \|h_\sigma\|$. Let $x, y \in S$ the

$$\begin{aligned} |\Gamma_1 x(t) + \Gamma_2 y(t)| \le\ & |\Psi_\phi(\tau)| h_1(t) h_1(\tau)^{-1} + |B(t)| |x(\gamma_1(t))| \\ &+ \int_\tau^t h_1(t) h_1(u)^{-1} |\omega(u)| |x(\gamma_1(u))| \mathrm{d}u + \int_\tau^t h_1(t) h_1(u)^{-1} |g(u, y(u), y(\gamma_2(u)))| \mathrm{d}u. \end{aligned}$$

We note that $h_1(t) = h_\sigma(t) h_{1-\sigma}(t)$ and $h_\sigma(\gamma_i(t)) h_\sigma(t)^{-1} = D_i^\sigma(t)$, $i = 1, 2$. So, from $x, y \in S$, we have

$$\begin{aligned} |\Gamma_1 x(t) + \Gamma_2 y(t)| \le\ & |\Psi_\phi(\tau)| h_1(t) h_1(\tau)^{-1} \\ &+ c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1} \left[|B(t)| D_1^\sigma(t) + \int_\tau^t h_{1-\sigma}(t) h_{1-\sigma}(u)^{-1} |\omega(u)| D_1^\sigma(u) \mathrm{d}u \right] \\ &+ c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1} \int_\tau^t h_{1-\sigma}(t) h_{1-\sigma}(u)^{-1} \lambda(u) \nu(R) (1 + D_2^\sigma(u)) \mathrm{d}u. \end{aligned}$$

From (10) and (11), we obtain

$$|\Gamma_1 x(t) + \Gamma_2 y(t)| \le |\Psi_\phi(\tau)| h_1(t) h_1(\tau)^{-1} + c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1} \left(\alpha + 2\nu(R) \int_\tau^t h_{1-\sigma}(t) h_{1-\sigma}(u)^{-1} \lambda(u) M_2(u) \mathrm{d}u \right).$$

From assumptions on ν in (H2), there exists $\delta > 0$ such that for $\|\phi\| \le \delta$ we obtain

$$2\nu(R) \int_\tau^t h_{1-\sigma}(t) h_{1-\sigma}(u)^{-1} \lambda(u) M_2(u) \mathrm{d}u \le 1 - \alpha.$$

So, for $x, y \in S$ with $\|\phi\| \le \delta$ we have

$$|\Gamma_1 x(t) + \Gamma_2 y(t)| \le |\Psi_\phi(\tau)| h_1(t) h_1(\tau)^{-1} + c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1}, \quad t \ge \tau \ge 0.$$

So, by choice of δ small enough, we get

$$|\Gamma_1 x(t) + \Gamma_2 y(t)| \le c\|\phi\| h_\sigma(t) h_\sigma(\tau)^{-1}, \quad t \ge \tau \ge 0.$$

So for $\phi \in BC(\tau)$ such that $\|\phi\| \le \delta$, we have that $PS \subset S$, so S is a convex non-empty invariant set for the operator $\Gamma_1 + \Gamma_2$. Actually, from assumptions, for any τ fixed, $h_\sigma(t)h_\sigma(\tau)^{-1} \to 0$ as $t \to \infty$.

Now, we prove that operator Γ_2 is completely continuous. For the continuity of Γ_2, let $\varepsilon > 0$. From (12) there exists $\xi > 0$ such that: $x, y \in S$ and $\|x - y\| < \xi$, imply

$$\begin{aligned}|\Gamma_2 x(t) - \Gamma_2 y(t)| &\le \int_\tau^t h_1(t)h_1(u)^{-1} |g(u, x(u), x(\gamma_2(u))) - g(u, y(u), y(\gamma_2(u)))| \mathrm{d}u \\ &\le \varepsilon I,\end{aligned}$$

where

$$I := \sup_{t \ge 0} \int_0^t h_1(t)h_1(s)^{-1}\lambda(s)\mathrm{d}s, \tag{17}$$

which exists by (14). Moreover, $\Gamma_2 S$ is equi-continuous on each compact interval. Indeed, let $x \in S$, then

$$\begin{aligned}|\Gamma_2 x(t) - \Gamma_2 x(v)| \le &\int_\tau^v |g(u, x(u), x(\gamma_2(u)))||h_1(t)h_1(u)^{-1} - h_1(v)h_1(u)^{-1}|\mathrm{d}u \\ &+ \int_v^t |g(u, x(u), x(\gamma_2(u)))| h_1(t)h_1(u)^{-1}\mathrm{d}u,\end{aligned}$$

from (13) we obtain

$$\begin{aligned}|\Gamma_2 x(t) - \Gamma_2 x(v)| \le\ &2\nu(R) h_\sigma(v)h_\sigma(\tau)^{-1} \int_\tau^v h_{1-\sigma}(v)h_{1-\sigma}(u)^{-1}\lambda(u)M_2(u)\left|1 - h_1(t)h_1(v)^{-1}\right| \mathrm{d}u \\ &+ 2\nu(R) h_\sigma(t)h_\sigma(\tau)^{-1} \int_v^t h_{1-\sigma}(t)h_{1-\sigma}(u)^{-1}\lambda(u)M_2(u)\mathrm{d}u,\end{aligned}$$

for every $x \in S$. From condition (15) and the fact that function $h_1(u)$ is continuous and positive in every compact, we have for every $\epsilon > 0$, there exists $\xi > 0$ such that $|t - v| < \xi$ implies

$$|\Gamma_2 x(t) - \Gamma_2 x(v)| \le \epsilon, \quad \forall x \in S.$$

Therefore, $\Gamma_2 S$ is equi-continuous on every interval $[\tau, \tau + n]$, and it is clear $\Gamma_2 S \subset S$. So we can apply the Lemma 1, which guarantees Γ_2 is a compact operator. Therefore, from Theorem A, we obtain for $\|\phi\| < \delta$ there exists a solution $x(t, \tau, \phi)$ of (1) and $x(t, \tau, \phi) \in S$, namely,

$$|x(t, \tau, \phi)| \le c\|\phi\| h_\sigma(t)h_\sigma(\tau)^{-1}, \quad t \ge \tau \ge 0.$$

Therefore, the zero solution of (1) is stable and asymptotically stable.

Necessity. Conversely, suppose (5) fails. From (H2) and (H3) there exists $r > 0$ such that $\alpha + 2\nu(2r)I < 1$, where I is given by (17). From (4), we have that $\|h_1\| := \|h_1\|_\infty$ is well-defined and there is a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\int_0^{t_n} a(s)\mathrm{d}s$ is convergent as $n \to \infty$, i.e. there exists $\ell > 0$ such that $|\int_0^{t_n} a(s)\mathrm{d}s| \le \ell$ for $n \ge 1$. We consider

$$\eta(u) := |B(u)a(u) - B'(u)| + 2\lambda(u)\nu(2r).$$

From (14) and (15), we have that $\int_0^{t_n} e^{\int_u^{t_n} a(s)\mathrm{d}s}\eta(u)\mathrm{d}u$ is bounded, so there exists a constant $\kappa > 0$ such that

$$\int_0^{t_n} e^{\int_u^{t_n} a(s)\mathrm{d}s}|\eta(u)|\mathrm{d}u = \int_0^{t_n} h_1(t_n)h_1(u)^{-1}\eta(u)\mathrm{d}u \le \kappa.$$

This yields

$$\int_0^{t_n} h_1(u)^{-1}\eta(u)\mathrm{d}u \le \kappa h_1(t_n)^{-1} = \kappa e^{-\int_0^{t_n} a(s)\mathrm{d}s} \le \kappa e^{\ell}.$$

Therefore, the sequence $\{\int_0^{t_n} h_1(u)^{-1}\eta(u)\mathrm{d}u\}_{n\in\mathbb{N}}$ is bounded, so there exists a subsequence of $\{t_n\}$, which we call again $\{t_n\}$, such that

$$\lim_{n\to\infty} \int_0^{t_n} h_1(u)^{-1}\eta(u)\mathrm{d}u = \gamma, \quad \gamma \in \mathbb{R}^+.$$

We can choose a positive integer m_0 large enough that

$$\int_{t_{m_0}}^{t_n} h_1(u)^{-1}\eta(u)\mathrm{d}u < \frac{1-\alpha-2\nu(2r)I}{4\|h_1\|^2}, \quad n \ge m_0.$$

We consider the solution $x(t) = x(t, t_{m_0}, \phi)$ with $\|\phi\| < \delta$. Then we can choose $\delta \le r$ such that:

$$\frac{|\Psi_\phi(t_{m_0})|\|h_1\|}{1-\alpha-2\nu(2r)I} \le r.$$

Lemma 2 implies

$$|x(t)| \le |\Psi_\phi(t_{m_0})|e^{\int_{t_{m_0}}^t a(s)\mathrm{d}s} + \alpha\|x\| + 2\|x\|\nu(2r)I.$$

Then we have

$$|x(t)| \le \frac{|\Psi_\phi(t_{m_0})|\|h_1\|}{1-\alpha-2\nu(2r)I} =: \rho \le r$$

for all $t \ge t_{m_0}$. Moreover from (8), we have for $t_n \ge t_{m_0}$ and $\phi \ne 0$:

$$\begin{aligned}
|x(t_n) - B(t_n)x(\gamma_1(t_n))| &\ge |\Psi_\phi(t_{m_0})|h_1(t_n)h(t_{m_0})^{-1} \\
&\quad - \left|\int_{t_{m_0}}^{t_n} h_1(t_n)h(u)^{-1}\left[\omega(u)x(\gamma_1(u))\mathrm{d}u + g(u, x(u), x(\gamma_2(u)))\right]\right| \\
&\ge |\Psi_\phi(t_{m_0})|h_1(t_n)h(t_{m_0})^{-1} - 2\rho\int_{t_{m_0}}^{t_n} h_1(t_n)h(u)^{-1}\eta(u)\mathrm{d}u \\
&= h_1(t_n)h(t_{m_0})^{-1}\left(|\Psi_\phi(t_{m_0})| - 2\rho\int_{t_{m_0}}^{t_n} h_1(t_{m_0})h_1(u)^{-1}\eta(u)\mathrm{d}u\right) \\
&\ge h_1(t_n)h(t_{m_0})^{-1}\left(|\Psi_\phi(t_{m_0})| - 2\rho\|h_1\|\int_{t_{m_0}}^{t_n} h_1(u)^{-1}\eta(u)\mathrm{d}u\right) \\
&\ge h_1(t_n)h(t_{m_0})^{-1}\left(|\Psi_\phi(t_{m_0})| - \frac{1}{2}|\Psi_\phi(t_{m_0})|\right) \\
&= \frac{1}{2}|\Psi_\phi(t_{m_0})|e^{\int_0^{t_n} a(s)\mathrm{d}s}e^{-\int_0^{t_{m_0}} a(s)\mathrm{d}s} \ge \frac{1}{2}|\Psi_\phi(t_{m_0})|e^{-2\ell} > 0.
\end{aligned} \tag{18}$$

On the other hand, if the zero solution of (1) is asymptotically stable, then $x(t) = x(t, t_{m_0}, \phi) \to 0$ as $t \to \infty$. From $\gamma_1(t) \to \infty$ as $t \to \infty$ and (15), we have

$$x(t_n) - B(t_n)x(\gamma_1(t_n)) \to 0 \quad \text{as } n \to \infty,$$

which contradicts (18). Hence condition (5) is necessary for the asymptotic stability of the zero solution of (1). The proof is complete. □

Remark 1. For $\tau = 0$, (16) is reduced to

$$|x(t, 0, \phi)| \le c\|\phi\|h(t), \quad t \ge 0, \ \|\phi\| \le \delta$$

and the result follows with the only condition

$$\int_0^t a(s)\mathrm{d}s \le 0, \quad \text{for } t \ge 0,$$

which does not imply (6). See for instance [11]. Note that function a can take negative and positive values. Thus, in our results, much Eq. (1) which are not uniformly stable are included. This is the case of the Eq. (1) with

$$a(t) = \sin(\log t) + \cos(\log t) - \alpha, \quad 1 < \alpha < \sqrt{2}.$$

Several papers are generalized [11,2,6].

Remark 2. The functions $D_i(t)$ $(i = 1, 2)$ in (10) can be unbounded (see [1]). If $a(\cdot) \le 0$ then they are bounded and disappear ($M_i \equiv 1$). In our work, functions $D_i(t)$ were important to achieve a proper invariant space, therefore to obtain an estimate for the behavior of the solutions from (1). In [2–4,7,8] they get results on the stability of the zero solution, but not an estimate of the asymptotic behavior. The use of these functions can improve the works mentioned. Note that (14) shows the close relation between the delay, the stability of the linear equation, and the possibility of getting an estimate.

Remark 3. Obviously, similar results can be obtained when g is a locally Lipschitz function, i.e.

$$|g(t,x_1,y_2)-g(t,x_1,x_2)|\le\lambda(t)\left(|x_1-x_2|+|y_1-y_2|\right),$$

$t\ge 0$, $|x_i|,\ |y_i|\le r$, $i=1,2$ (see [11,15,7,8,10]).

4. Example

To illustrate our result, consider the neutral nonlinear differential equation

$$x'(t)=\left(\sin(t)-\frac{1}{1+t}\right)x(t)+\frac{\beta}{1+2t}x'(t/2)+\ln\left(1+\frac{x(t/3)^2}{1+t}\right), \tag{19}$$

for $t\ge\tau\ge 0$, and with initial condition $x(t)=\phi(t),\ t\le\tau$. The above neutral delay differential equation has the form of (1), where

$$a(t)=\sin(t)-\frac{1}{1+t},\qquad b(t)=\frac{\beta}{1+2t},\qquad g(t,x,y)=\ln\left(1+\frac{y^2}{1+t}\right).$$

So we give conditions on the constant β, to apply our result to (19). It is clear that (H1) holds. We note that

$$g(t,x,y)=\ln\left(1+\frac{x(t/3)^2}{1+t}\right)\le\frac{x(t/3)^2}{1+t},$$

so $\lambda(t)=\frac{1}{1+t},\ \nu(x)=x$. Therefore (H2) holds. By a straightforward computation we have

$$\int_u^t a(s)\mathrm{d}s=\ln\left(\frac{1+u}{1+t}\right)+\cos(u)-\cos(t). \tag{20}$$

So, we obtain

$$h_{\frac12}(t)h_{\frac12}(u)^{-1}=\sqrt{\frac{1+u}{1+t}}\exp\left\{\frac{\cos(u)-\cos(t)}{2}\right\},\quad t\ge u\ge 0,$$

it follows that

$$\int_0^t h_{\frac12}(t)h_{\frac12}(u)^{-1}\lambda(u)\mathrm{d}u\le e\int_0^t\sqrt{\frac{1+u}{1+t}}\frac{1}{(1+u)}\mathrm{d}u=2e,\quad t\ge 0.$$

We have verified (14). From (20), the functions $D_i(t),\ i=1,2$ are

$$D_i(t)=\exp\left\{-\int_{\gamma_i(t)}^t a(s)\mathrm{d}s\right\}=\left(\frac{1+t}{1+\gamma_i(t)}\right)\exp\{\cos(t)-\cos(\gamma_i(t))\}.$$

Therefore, we have that $D_i^{\frac12}(t)=\sqrt{\frac{(1+t)}{1+\gamma_i(t)}}\exp\left\{\frac{\cos(t)-\cos(\gamma_i(t))}{2}\right\}$, $i=1,2$ for $t\ge 0$. We note that $B(t)=\frac{2\beta}{1+2t},\ B'(t)=\frac{-4\beta}{(1+2t)^2}$, so

$$|B(t)|\le 2|\beta|,\qquad M_1(t)\le\sqrt{2}e,\qquad M_2(t)\le\sqrt{3}e,\quad t\ge 0.$$

Next, we must estimate

$$\int_0^t h_{\frac12}(t)h_{\frac12}(u)^{-1}|w(u)|\mathrm{d}u,$$

where $w(u)=B(u)a(u)-B'(u)$, since above we have:

$$\int_0^t h_{\frac12}(t)h_{\frac12}(u)^{-1}|B'(u)|\mathrm{d}u\le 2|\beta|e\int_0^t\frac{2}{(1+2u)^2}\mathrm{d}u\le 2|\beta|e,\quad t\ge 0,$$

and

$$\int_0^t h_{\frac12}(t)h_{\frac12}(u)^{-1}|B(u)||a(u)|\mathrm{d}u\le 2e\int_0^t\sqrt{\frac{1+u}{1+t}}\frac{2|\beta|}{1+2u}\mathrm{d}u\le 4e|\beta|,\quad t\ge 0.$$

Then we obtain

$$|B(t)|M_1(t) + \int_0^\infty h_{\frac{1}{2}}(t)h_{\frac{1}{2}}(u)^{-1}|w(u)|M_1(u)\mathrm{d}u \le 2^{3/2}e|\beta|(1+3e), \quad t \ge 0.$$

Therefore the assumption (15) becomes

$$2^{3/2}e|\beta|(1+3e) < 1.$$

So we have shown

Corollary 1. *If* $|\beta| < [2^{3/2}e(1+3e)]^{-1}$ *in* (19)*, then for* $|\phi|$ *small enough the solution* $x(t,\tau,\phi)$ *of* (19) *satisfies for* $t \ge \tau \ge 0$

$$|x(t,\tau,\phi)| \le 2(1+2|\beta|)\|\phi\|\sqrt{\frac{1+\tau}{1+t}}\exp\left\{\frac{1}{2}(\cos(t)-\cos(\tau))\right\}.$$

In the same way, the examples in [7] can be done more precisely, showing an explicit estimate.

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