# A hydrodynamical model for relativistic spin quantum plasmas

Felipe A. Asenjo,[1,a)] Víctor Muñoz,[1,2] J. Alejandro Valdivia,[1,2] and Swadesh M. Mahajan[3]
[1]Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, 7800024 Santiago, Chile
[2]Centro para el Desarrollo de la Nanociencia y la Nanotecnología (CEDENNA), 9170124 Santiago, Chile
[3]Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA

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Based on the one-body particle-antiparticle Dirac theory of electrons, a set of relativistic quantum fluid equations for a spin half plasma is derived. The particle-antiparticle nature of the relativistic particles is explicit in this fluid theory, which also includes quantum effects such as spin. The nonrelativistic limit is shown to be in agreement with previous attempts to develop a spin plasma theory derived from the Pauli Hamiltonian. Harnessing the formalism to the study of electromagnetic mode propagation, conceptually new phenomena are revealed; the particle-antiparticle effects increase the fluid opacity to these waves, while the spin effects tend to make the fluid more transparent. © *2011 American Institute of Physics*. [doi:10.1063/1.3533448]

## I. INTRODUCTION

After an extended hiatus following an initial spurt of activity,[1–5] theoretical studies of quantum plasmas have staged an impressive comeback in the past few years. When the de Broglie wavelength of the charged particles is comparable to the spatial scales of the system, the quantum nature of the plasma constituents cannot be ignored and quantum effects could affect the collective behavior in a profound way. In this context, a proper quantum treatment may be crucial. Examples can be found in nanoscale physics,[6] astrophysical,[7] and high-energy laser[8] systems.

Quantum plasmas have been described in many ways.[9] First, insights into the dynamics of the fluid models of a quantum plasma were, however, obtained when the electrons were assumed to obey the Schrödinger equation, i.e., the electron spin was neglected. A Madelung decomposition of the wave function then leads to a one-body fluid theory. This model has been studied extensively[10–13] and quantum corrections to a host of collective motions have been derived, which include propagating mode solutions,[14–18] dusty quantum plasma modes,[19,20] instabilities,[21,22] nonlinear wave interactions and modulation,[23,24] quantum ion-acoustic waves,[25,26] and shock waves,[27,28] among others.

The very important next step of including spin in a quantum fluid theory for plasmas was taken recently;[29,30] a fluid theory for spin plasmas is developed from the Pauli Hamiltonian and generalizing the Madelung decomposition for the two component spinor wave function. This nonrelativistic spin theory produces new effects in, for example, propagation modes,[31] magnetosonic solitons,[32] instabilities,[33] and metamaterials.[34] All these quantum effects are expected to be important in low temperature and very dense plasmas. However, it was recently shown that even for high temperature plasmas, under specific conditions, quantum features such as spin cannot be neglected, and they can be significant.[35]

In the wake of these interesting developments, it is natural to ask whether a relativistic quantum electron fluid will display some unusual properties. One has good reasons to expect so because a unification of quantum mechanics and relativity fundamentally couples the electron (particle) and positron (antiparticle) states.

Relativistic quantum plasmas have been investigated, invoking a variety of models: (1) covariant treatments that use Fermi–Dirac statistics and Wigner function techniques to find dispersion and fluctuations[36,37] by constructing kinetic models from the Dirac equation; and (2) second quantized quantum electrodynamics models describing a relativistic degenerate electron Fermi gas[38] or a particle-antiparticle plasma.[39] Similar techniques are harnessed to deal with the decay of photons and plasmons into neutrino pairs.[40] Besides, extensive results for magnetized and unmagnetized relativistic quantum plasmas have been obtained principally by kinetic covariant methods;[41] the results include calculating the dispersion of transverse and longitudinal modes, and resonance conditions for fermionic and bosonic plasmas.[41,42]

All the previously mentioned works have studied the relativistic quantum plasmas with known quantum electrodynamics tools. In this paper, we present a novel way to study this kind of plasmas. We model a relativistic quantum plasma casting the Dirac equation into a fluid description. This allows us to compare the new phenomena introduced by relativistic and quantum effects with the classical fluid plasma descriptions. Thus, for instance, in a relativistic quantum description, there cannot be a pure electron fluid as there must always be an appropriate mixture of positrons. Fortunately, this and other features of relativistic quantum mechanics are fully contained in the Dirac equation, which will be the starting point to develop our theory for a relativistic quantum fluid. In order to do so, we will first associate the relativistic fluidlike variables corresponding to the particle-antiparticle state with the bilinear covariants of the Dirac theory. This is done using the fluid description of the Dirac theory developed by Takabayasi.[43] Then, we construct an *N*-body theory with a complete set of covariant fluid equations, following the procedure introduced recently by Marklund and Brodin[29]

[a)]Electronic mail: fasenjo@levlan.ciencias.uchile.cl.

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for the Pauli equation (nonrelativistic). This yields the basic set of fluid dynamical equations of a relativistic quantum charged plasma made up of particles with spin 1/2. Although the formalism will apply to any spin half fluid, electron-positron fluids will be our primary interest.

In Sec. II, we derive, starting from the Dirac equation of a charged fluid in an electromagnetic field, the basic covariant equations obeyed by the appropriately chosen one-body fluid variables. In Sec. III, we construct the $N$-body theory, leading to the final fluid equations. Then we discuss several properties of the system and also put the theory in the more familiar vectorial form (Sec. IV). In Sec. V, we show that in the nonrelativistic limit, the systems described in Refs. 29 and 30 are recovered. In Sec. VI, the formalism is applied to study the propagation of an electromagnetic wave through this electron-positron plasma. Finally, in Sec. VII, we present our conclusions.

## II. FLUID FORMALISM FOR ONE-BODY PARTICLE-ANTIPARTICLE STATE

The "fluidization" of Dirac equation, via constructing observables from bilinear covariants, was first done by Takabayasi.[43–47] We begin by reviewing his work to develop a better understanding of the meaning of the fluid variables before we construct the complete $N$-body plasma theory.

Let us start from the Dirac equations obeyed by the bispinor fields $\psi$ and $\bar{\psi}=\psi^\dagger\gamma^0$ immersed in an electromagnetic field,

$$i\gamma^\mu\partial_\mu\psi - e\gamma^\mu A_\mu\psi - m\psi = 0, \tag{1}$$

$$i\partial_\mu\bar{\psi}\gamma^\mu + e\bar{\psi}\gamma^\mu A_\mu + m\bar{\psi} = 0, \tag{2}$$

where $\gamma^\mu$ are the four by four Dirac matrices, $A_\mu$ is the electromagnetic four-potential, and $\mu=0,1,2,3$. Throughout this paper the Greek (Latin) indices run from 0 to 3 (from 1 to 3), the Minkowski metric is $g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, and definitions $e\equiv e/(\hbar)$ and $m\equiv m_e c/\hbar$ (where $m_e$ is the electron mass and the speed of light has been taken as $c=1$) will be used. Notice that the normalized mass is the inverse of the Compton wavelength. The Dirac equation represents a spin half particle-antiparticle system. In effect, the wave function has four components (each of the particle and antiparticle contributing a two component spinor), thus accommodating the inextricable particle-antiparticle coupling characteristic of relativistic quantum systems.

To construct the fluid description for the Dirac equation, it is necessary to identify appropriate Lorentz tensors that behave like familiar fluid observables. The most straightforward path is to take advantage of the existence of the 16 bilinear covariants of this theory. Let $\bar{\psi}\gamma^A\psi$ represent the set of 16 bilinear covariants, where $\gamma^A$ can be any of $\hat{1},\gamma^5,\gamma^\mu,\gamma^\mu\gamma^5$ and $\sigma^{\mu\nu}=(i/2)[\gamma^\mu,\gamma^\nu]$. They transform, respectively, as a scalar, a pseudoscalar, a four-vector, a pseudo-four-vector, and a second rank tensor. The pseudoscalar (vector) transforms exactly like a scalar (vector) under the proper Lorentz group but has opposite parity. Our basic fluid observables will be constructed as Lorentz covariants. We will soon find that fluid velocity and the fluid momentum constructed in the theory are not simply related; their complicated relationship indicates certain deeper aspects of the theory.

Multiplying Eq. (1) on the left by $\bar{\psi}\gamma^A$ and multiplying Eq. (2) on the right by $\gamma^A\psi$, and adding and subtracting the results yields the starting equations for the current formalism,

$$i\bar{\psi}\gamma^A\gamma^\mu\partial_\mu\psi + i\partial_\mu\bar{\psi}\gamma^\mu\gamma^A\psi - e\bar{\psi}\gamma^A\gamma^\mu A_\mu\psi + e\bar{\psi}\gamma^\mu A_\mu\gamma^A\psi = 0, \tag{3}$$

$$i\bar{\psi}\gamma^A\gamma^\mu\partial_\mu\psi - i\partial_\mu\bar{\psi}\gamma^\mu\gamma^A\psi - e\bar{\psi}\gamma^A\gamma^\mu A_\mu\psi - e\bar{\psi}\gamma^\mu A_\mu\gamma^A\psi - 2m\bar{\psi}\gamma^A\psi = 0. \tag{4}$$

The next step is to cast Eqs. (3) and (4) in terms of fluid variables that, for a fully covariant theory, must be Lorentz tensors. We begin by finding a suitable expression for "density;" the total fluid density $\rho$ must naturally contain both the scalar $\Omega=\bar{\psi}\psi$ and the pseudoscalar $\bar{\Omega}=i\bar{\psi}\gamma_5\psi$ densities. The positive definite symmetric scalar,

$$\rho = \sqrt{\Omega^2+\bar{\Omega}^2} \tag{5}$$

seems an obvious choice. Besides, one defines the pseudoscalar parameter,

$$\theta = \tan^{-1}\left(\frac{\bar{\Omega}}{\Omega}\right), \tag{6}$$

which represents in some sense the "measure" of the particle-antiparticle mixing inherent in the Dirac theory.

It is the nonzero $\theta$ that destroys, in this model, the conventional direct relationship between fluid velocity and momentum. We now introduce the following notation. For the bilinear covariants, we set the vector $S_\mu=\bar{\psi}\gamma_\mu\psi$, the pseudovector $\bar{S}_\mu=\bar{\psi}\gamma_5\gamma_\mu\psi$, the tensor $M^{\mu\alpha}=\bar{\psi}\sigma^{\mu\alpha}\psi$, and the pseudotensor $\bar{M}^{\mu\alpha}=i\bar{\psi}\gamma_5\sigma^{\mu\alpha}\psi$. New operator derivatives are defined as $\delta_\mu(\bar{\psi}\gamma^A\psi)=i(\bar{\psi}\gamma^A\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma^A\psi)$ and $\delta^*_\mu(\bar{\psi}\gamma^A\psi)=\delta_\mu(\bar{\psi}\gamma^A\psi)-2eA_\mu\bar{\psi}\gamma^A\psi$. Finally, the derived tensorial quantities are denoted as the vector $J_\mu=(1/2m)\delta_\mu{}^*\Omega$, the pseudovector $\bar{J}_\mu=(1/2m)\delta_\mu{}^*\bar{\Omega}$, the tensor $T_\nu{}^\mu=(1/2m)\delta_\nu{}^*S^\mu$, the pseudotensor $\bar{T}_\nu{}^\mu=(1/2m)\delta_\nu{}^*\bar{S}^\mu$, the third rank tensor $N_{\alpha,}{}^{\mu\alpha}=(1/2m)\delta_\alpha{}^*M^{\mu\alpha}$, and the corresponding pseudotensor $\bar{N}_{\alpha,}{}^{\mu\alpha}=(1/2m)\delta_\alpha{}^*\bar{M}^{\mu\alpha}$.

With these definitions, we can rewrite Eqs. (3) and (4) for each bilinear covariant in a succinct and revealing form. Tedious but straightforward algebra yields

$$\partial_\mu S^\mu = 0, \tag{7}$$

$$\partial_\mu\bar{S}^\mu = -2m\bar{\Omega}, \tag{8}$$

$$T^\mu_\mu = \Omega, \tag{9}$$

$$\bar{T}^\mu_\mu = 0, \tag{10}$$

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$$\frac{1}{2m}\partial_\alpha M^{\mu\alpha} + J^\mu - S^\mu = 0, \quad (11)$$

$$\frac{1}{2m}\partial_\alpha \bar{M}^{\alpha\mu} - \bar{J}^\mu = 0, \quad (12)$$

$$N_{\alpha,}{}^{\mu\alpha} - \frac{1}{2m}\partial^\mu \Omega = 0, \quad (13)$$

$$\bar{N}_{\alpha,}{}^{\mu\alpha} - \frac{1}{2m}\partial^\mu \Omega + \bar{S}^\mu = 0, \quad (14)$$

$$T^{\mu\nu} - T^{\nu\mu} = \frac{1}{2m}\varepsilon^{\mu\nu\alpha\beta}\partial_\alpha \bar{S}_\beta, \quad (15)$$

$$\varepsilon^{\mu\nu\alpha\beta}\bar{T}_{\alpha\beta} = M^{\mu\nu} - \frac{1}{2m}(\partial^\mu S^\nu - \partial^\nu S^\mu). \quad (16)$$

However, the fluid features of this set of equations are still far from clear. To display them in a more usual form, we define the fluid dynamical four-velocity density,

$$v_\mu = \frac{1}{\rho}S_\mu, \quad (17)$$

and in an analogous way, the four-spin density,

$$w_\mu = \frac{1}{\rho}\bar{S}_\mu, \quad (18)$$

such that they satisfy the constraints,

$$v_\mu v^\mu = 1, \quad w_\mu w^\mu = -1, \quad v_\mu w^\mu = 0. \quad (19)$$

We define another dynamical variable,

$$k^\mu = \frac{1}{\rho^2}(\Omega J^\mu + \bar{\Omega}\bar{J}^\mu), \quad (20)$$

which satisfies

$$\partial_\mu k_\nu - \partial_\nu k_\mu = \frac{-i}{2m}\varepsilon^{\alpha\beta\gamma\delta}v_\alpha w_\beta(\partial_\mu v_\gamma \partial_\nu v_\delta - \partial_\mu w_\gamma \partial_\nu w_\delta) - \frac{e}{m}F_{\mu\nu}, \quad (21)$$

with $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$. The four-vector $k^\mu$ is interpreted as the four-momentum of the fluid. In terms of these variables, the set of Eqs. (7)–(16) can be rewritten to obtain what may be viewed as the system representing the relativistic quantum fluid,[43–46]

$$\partial_\mu(\rho v^\mu) = 0, \quad (22)$$

$$\partial_\mu(\rho w^\mu) = -2m\rho \sin\theta, \quad (23)$$

$$\partial_\mu \theta w^\mu - 2mk_\mu v^\mu - i\varepsilon_{\mu\alpha\beta\lambda}v_\alpha w_\beta \partial^\mu v_\lambda = -2m\cos\theta, \quad (24)$$

$$\partial_\mu \theta v^\mu - 2mk_\mu w^\mu - i\varepsilon_{\mu\alpha\beta\lambda}v_\alpha w_\beta \partial^\mu w_\lambda = 0, \quad (25)$$

$$v_\nu \partial^\nu(\rho v_\mu) - w_\nu \partial^\nu(\rho w_\mu) = \partial_\mu \rho - \rho w^\alpha \partial_\nu v_\alpha(v^\nu w_\mu - v_\mu w^\nu) + 2im\rho\varepsilon_{\mu\nu\alpha\beta}v^\nu w^\alpha k^\beta, \quad (26)$$

$$v^\nu \partial_\nu w^\mu = w_\nu(\partial^\nu v^\mu - \partial^\mu v^\nu) + ig^{\mu\beta}\varepsilon_{\beta\nu\kappa\lambda}v^\nu w^\kappa \partial^\lambda \theta. \quad (27)$$

An explanatory remark is in order. Equations (22)–(27) are really the various $\gamma^A$ "moments" of the Dirac equation, manipulated and expressed in terms of what appears to be reasonable Lorentz covariant fluid variables. At this stage, these equations, however complicated, represent only a single Dirac particle. To conform to the standard notion of fluid equations, we will eventually have to carry out an appropriate ensemble average. With that promise soon to be fulfilled, we will continue calling the preceding set as fluid equations.

As this fluid is charged, we must add the Maxwell equations to close the system,

$$\partial_\nu F^{\nu\mu} = -e\rho v^\mu. \quad (28)$$

The collection of 20 equations, formed by Eqs. (19) and (21)–(28), will be called set II. The total number of independent equations is 20 because Eqs. (26) and (35) each involves only two linearly independent equations. Set II must be solved for the 20 variables that describe the relativistic quantum fluid for one particle-antiparticle state $(\rho,\theta,v^\mu,w^\mu,k^\mu,F^{\mu\nu})$.

These fluid equations are covariant and fully quantum relativistic. Note the strong nonlinearity of these equations and the huge difference from a traditional fluid theory. Of course, this is simply due to the deep physical difference between Dirac theory and "classical" theories. One such difference is the first-order nature of the Dirac equation, which leads to the velocity to be defined in a form that does not have a direct correspondence with the nonrelativistic velocity.

Set II is just an intermediate step in our quest; it does not quite show, explicitly and in a revealing form, the evolution of either four-velocity or four-spin densities. Since the aim is to construct fluid equations for a relativistic quantum plasma, we need to put set II in a more appropriate form. It is not difficult to show that Eqs. (24)–(26) can be solved for

$$k^\mu = v^\mu \cos\theta - \frac{1}{2m}\partial_\alpha \theta(v^\alpha w^\mu - v^\mu w^\alpha) - \frac{i}{2m\rho}\varepsilon^{\mu\alpha\kappa\lambda}\partial_\alpha(\rho v_\kappa w_\lambda). \quad (29)$$

All of the previous sets of equations have been obtained by rewriting the Dirac equation in terms of fluidlike Lorentz covariant variables. It is also possible to construct another equivalent set of equations in an alternative and somewhat more familiar approach by writing the energy-momentum tensor of the fluid and then using conservation equations to find the evolution equations for the four-velocity and four-spin density. The symmetric Dirac Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}\bar{\psi}(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu - m)\psi - \tfrac{1}{2}(i\partial_\mu\bar{\psi}\gamma^\mu + e\bar{\psi}\gamma^\mu A_\mu + \bar{\psi}m)\psi, \quad (30)$$

yields the Dirac equations (1) and (2). Note that $\mathcal{L}=0$ on the solution. The energy-momentum tensor is

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$$\Theta_\nu{}^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} D_\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} D_\nu{}^* \bar{\psi} - \delta_\nu{}^\mu \mathcal{L},$$

where $D_\mu = \partial_\mu + ieA_\mu$. Using the Lagrangian, it may be shown that $\Theta_{\mu\nu} = mT_{\mu\nu}$.

Writing $T_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ in terms of the fluid variables,

$$T_{\mu\nu} = \rho k_\mu v_\nu - \frac{\rho}{2m}(\partial_\mu \theta w_\nu - i\varepsilon_{\nu\alpha\beta\lambda} v^\alpha w^\beta \partial_\mu v^\lambda), \quad (31)$$

$$\bar{T}_{\mu\nu} = \rho k_\mu w_\nu - \frac{\rho}{2m}(\partial_\mu \theta v_\nu - i\varepsilon_{\nu\alpha\beta\lambda} v^\alpha w^\beta \partial_\mu w^\lambda), \quad (32)$$

our final set of equations is

$$\partial_\mu(\rho v^\mu) = 0, \quad (33)$$

$$\partial_\mu(\rho w^\mu) = -2m\rho \sin\theta, \quad (34)$$

$$v^\nu \partial_\nu w^\mu = w_\nu(\partial^\nu v^\mu - \partial^\mu v^\nu) + ig^{\mu\beta}\varepsilon_{\beta\nu\kappa\lambda} v^\nu w^\kappa \partial^\lambda \theta, \quad (35)$$

$$k^\mu = v^\mu \cos\theta - \frac{1}{2m}\partial_\alpha \theta (v^\alpha w^\mu - v^\mu w^\alpha) - \frac{i}{2m\rho}\varepsilon^{\mu\alpha\kappa\lambda}\partial_\alpha(\rho v_\kappa w_\lambda), \quad (36)$$

$$\rho v^\nu \partial_\nu k_\mu = -\frac{e\rho}{m}F_{\mu\nu}v^\nu + \frac{1}{2m}\partial^\nu(\rho \partial_\mu \theta w_\nu - i\rho \varepsilon_{\nu\alpha\beta\lambda} v^\alpha w^\beta \partial_\mu v^\lambda), \quad (37)$$

$$\partial_\mu(\rho k^\mu w^\nu) = -\frac{e}{m}\rho w_\mu F^{\mu\nu} + \frac{1}{2m}\partial_\mu(\rho \partial^\mu \theta v^\nu) - \frac{i}{2m}g^{\nu\lambda}\varepsilon_{\lambda\alpha\beta\gamma}\partial_\mu(\rho v^\alpha w^\beta \partial^\mu w^\gamma), \quad (38)$$

$$\varepsilon_{\kappa\lambda\mu\nu} v^\kappa w^\lambda \partial^\mu k^\nu = \frac{-i}{4m}\varepsilon_{\kappa\lambda\mu\nu}\varepsilon^{\alpha\beta\gamma\delta} v^\kappa w^\lambda v_\alpha w_\beta (\partial^\mu v_\gamma \partial^\nu v_\delta - \partial^\mu w_\gamma \partial^\nu w_\delta) - \frac{e}{2m}\varepsilon_{\kappa\lambda\mu\nu} v^\kappa w^\lambda F^{\mu\nu}, \quad (39)$$

plus Eq. (19) and Maxwell equations (28). Most of these are conservation laws. For instance, Eq. (37) is the conservation of the velocity density,

$$\partial^\nu \Theta_{\mu\nu} = -e\rho v^\nu F_{\mu\nu},$$

while Eq. (38) is the conservation equation for the four-spin density,[43]

$$m\partial^\mu \bar{T}_{\mu\nu} = -e\rho w^\mu F_{\mu\nu}.$$

Incidentally, Eq. (21) is equivalent to Eqs. (37)–(39).[43] Noticing that Eqs. (35) and (36) correspond to 6 independent equations, whereas Eqs. (37) and (38) are equivalent to 5 independent equations, it turns out that the set of Eqs. (19), (28), and (33)–(39) yields 20 independent equations for the 20 unknown variables.

## III. FLUID FORMALISM FOR *N*-BODY PARTICLE-ANTIPARTICLE STATE

Hitherto, we have derived the relativistic quantum fluid equations for a one-body particle-antiparticle state using the previous formalism developed in Ref. 43. Now, we focus on constructing the dynamics of a collection of $N$ such one-body states. To the best of our knowledge, this is the first derivation of a many-body relativistic quantum fluid plasma, whose constituents are Dirac bispinors representing a fully coupled fermion-antifermion state.

We set out to derive the fluid equations for a relativistic quantum plasma described by an $N$-body bispinor wave function $\Psi(\mathbf{r}_1, \mathbf{r}_1, \ldots, \mathbf{r}_N)$. Since spin half particles obey Fermi–Dirac statistics, this $N$-particle spinor must be written as a $4^N \times 4^N$ Slater determinant of $N$ one-particle Dirac states,

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}}\begin{vmatrix} \psi_{a_1}(\mathbf{r}_1) & \psi_{a_1}(\mathbf{r}_2) & \cdots & \psi_{a_1}(\mathbf{r}_N) \\ \psi_{a_2}(\mathbf{r}_1) & \psi_{a_2}(\mathbf{r}_2) & \cdots & \psi_{a_2}(\mathbf{r}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{a_N}(\mathbf{r}_1) & \psi_{a_N}(\mathbf{r}_2) & \cdots & \psi_{a_N}(\mathbf{r}_N) \end{vmatrix},$$

where each one-particle wave function $\psi_j(\mathbf{r})$ satisfies the Dirac equation, and the subscripts $j = a_1, a_2, \ldots, a_N$ label the quantum state in which the Dirac particle resides. This $N$-body bispinor satisfies the generalized Dirac equation $\mathbf{H}\Psi = i\partial_t \Psi$, where $\mathbf{H}$ is an $N$-particle generalization of the Dirac Hamiltonian and where the Dirac matrices have been generalized to $4^N \times 4^N$ matrices.[48]

When there are no particle-particle interactions (no condensation, superconductivity, entanglement, etc.), the $N$-particle Dirac equation can be decomposed into $N$ independent Dirac equations, where each Dirac particle is described by the four-component spinor $\psi_\alpha$ in the quantum state $\alpha$.

Fluid equations are obtained by averaging over these $N$-particle states. Following the procedure introduced recently in Ref. 29 for the Pauli equation (nonrelativistic), we let $p_\alpha$ be the probability for a Dirac particle of being in the state $\alpha$. The probability will depend on the nature of the ensemble used, but no explicit expressions for $p_\alpha$ are necessary for our calculations. All that we need is the implicit existence of a representative ensemble. We define the fluid density $n$ in the fluid rest frame as

$$n = \sum_\alpha p_\alpha \rho_\alpha.$$

Likewise, the ensemble average of any quantity $x_\alpha^\mu$ may be defined as

$$X^\mu = \langle x_\alpha^\mu \rangle = \frac{1}{n}\sum_\alpha p_\alpha \rho_\alpha x_\alpha^\mu.$$

Thus, the fluid four-velocity is $U^\mu = \langle v_\alpha^\mu \rangle$ and the total spin density is $W^\mu = \langle w_\alpha^\mu \rangle$. We also define the rest frame microscopic four-velocity $z_\alpha^\mu = v_\alpha^\mu - U^\mu$ and the rest frame microscopic spin density $s_\alpha^\mu = w_\alpha^\mu - W^\mu$, with $\langle z_\alpha^\mu \rangle = 0 = \langle s_\alpha^\mu \rangle$.

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From now on, the subscript $\alpha$ representing the state of the Dirac particle will be dropped for notation simplicity. An ensemble average automatically implies that it is over the one-particle states. Taking the ensemble average of the set of Eqs. (19), (28), and (33)–(39), and using Eq. (36) in every equation, we obtain

$$\partial_\mu(nU^\mu)=0, \tag{40}$$

$$\partial_\mu(nW^\mu)=-2mn\langle\sin\theta\rangle, \tag{41}$$

$$\begin{aligned}U^\nu\partial_\nu W^\mu &= W_\nu(\partial^\nu U^\mu-\partial^\mu U^\nu)+ig^{\mu\beta}\varepsilon_{\beta\nu\kappa\lambda}U^\nu W^\kappa\langle\partial^\lambda\theta\rangle\\ &\quad-\langle v^\nu\partial_\nu s^\mu\rangle+\langle w_\nu(\partial^\nu z^\mu-\partial^\mu z^\nu)\rangle\\ &\quad+ig^{\mu\beta}\varepsilon_{\beta\nu\kappa\lambda}(\langle v^\nu s^\kappa\partial^\lambda\theta\rangle+\langle z^\nu\partial^\lambda\theta\rangle W^\kappa),\end{aligned} \tag{42}$$

$$\begin{aligned}\langle\cos\theta\rangle U^\mu\partial_\mu W^\nu &= \frac{e}{m}W_\mu F^{\nu\mu}-\left\langle\frac{i}{2m\rho}g^{\nu\lambda}\varepsilon_{\lambda\alpha\beta\gamma}\partial_\mu(\rho v^\alpha w^\beta\partial^\mu w^\gamma)\right\rangle+\left\langle\frac{1}{2m\rho}\partial_\mu(\rho\partial^\mu\theta v^\nu+\rho w^\nu\partial_\alpha\theta(v^\alpha w^\mu-v^\mu w^\alpha)\right.\\ &\quad\left.+iw^\nu\varepsilon^{\mu\alpha\kappa\lambda}\partial_\alpha(\rho v_\kappa w_\lambda))\right\rangle-\langle\cos\theta z^\mu\partial_\mu w^\nu\rangle-U^\mu\langle\cos\theta\partial_\mu s^\nu\rangle+\langle\sin\theta w^\nu v^\mu\partial_\mu\theta\rangle,\end{aligned} \tag{43}$$

$$\begin{aligned}\langle\cos\theta\rangle U^\nu\partial_\nu U_\mu &= -\frac{e}{m}F_{\mu\nu}U^\nu+\frac{1}{2m}\langle v^\nu\partial_\nu(\partial_\alpha\theta(v^\alpha w_\mu-v_\mu w^\alpha))\rangle+\frac{i}{2m}\left\langle v^\nu\partial_\nu\left(\frac{1}{\rho}g_{\mu\beta}\varepsilon^{\beta\alpha\kappa\lambda}\partial_\alpha(\rho v_\kappa w_\lambda)\right)\right\rangle\\ &\quad+\left\langle\frac{1}{m\rho}\partial^\nu(\rho\partial_\mu\theta w_\nu-i\rho\varepsilon_{\nu\alpha\beta\lambda}v^\alpha w^\beta\partial_\mu v^\lambda)\right\rangle-\langle\cos\theta z^\nu\partial_\nu v_\mu\rangle-U^\nu\langle\cos\theta\partial_\nu z_\mu\rangle+\langle\sin\theta v_\mu v^\nu\partial_\nu\theta\rangle,\end{aligned} \tag{44}$$

$$\begin{aligned}&\varepsilon_{\kappa\lambda\mu\nu}\langle v^\kappa w^\lambda\partial^\mu k^\nu\rangle\\ &\quad=\frac{-i}{4m}\varepsilon_{\kappa\lambda\mu\nu}\varepsilon^{\alpha\beta\gamma\delta}\langle v^\kappa w^\lambda v_\alpha w_\beta(\partial^\mu v_\gamma\partial^\nu v_\delta-\partial^\mu w_\gamma\partial^\nu w_\delta)\rangle\\ &\qquad-\frac{e}{2m}\varepsilon_{\kappa\lambda\mu\nu}\langle v^\kappa w^\lambda\rangle F^{\mu\nu},\end{aligned} \tag{45}$$

plus the macroscopic Maxwell equations,

$$\partial_\mu F^{\mu\nu}=-enU^\nu. \tag{46}$$

The group of Eqs. (40)–(46), a complete, many-particle description of a covariant relativistic quantum plasma, is the primary new result of this paper. We can see that this plasma is highly nonlinear and the terms have an implicit sum over all the quantum states. The velocity and the four-spin density appear mixed in the equations, revealing their strong coupling.

We now discuss the current density of Eq. (46) in the perspective of the nonrelativistic Pauli approach to spin plasmas. In nonrelativistic quantum theories for plasmas, the total current density is composed by the usual current and the spin magnetization current.[34,49]

For a Dirac fluid, the current $enU^\nu$ contains these two current densities in an intrinsic manner. Executing a Gordon decomposition of the four-velocity,[48] it is possible to show that $enU^\nu=J^\mu_{\text{out}}+J^\mu_{\text{in}}$, where $J^\mu_{\text{out}}$ is an outer current density that is equivalent to what is found in the Klein–Gordon theory, while $J^\mu_{\text{in}}$ is an internal current density arising from internal degrees of freedom of the electron, which are related to the spin and to the Zitterbewegung effect. This internal current density is given by

$$J^\mu_{\text{in}}=\frac{ie\hbar}{2m}\langle\partial_\nu(\bar\psi\gamma^\mu\gamma^\nu\psi)\rangle. \tag{47}$$

From here, it is straightforward to obtain an expression for one of the two terms constituting the spatial components of the current density given in Eq. (47), namely,

$$\frac{e\hbar}{2m}\gamma^0\,\nabla\times(nW^i), \tag{48}$$

which, with the definition $\mathbf{S}^i=(\hbar/2)W^i$ for the spin vector, is completely equivalent to the expression for the spin magnetization current density found, for instance, in Ref. 49. The other term is a polarization term due to the electric moment of the internal degrees of freedom.[48]

The preceding discussion clearly demonstrates that the definition of the four-velocity and four-current density (46) automatically contains the relativistic generalization of what would be the magnetization current density due to a Pauli spin fluid.

## IV. VECTORIAL DESCRIPTION

The set of Eqs. (40)–(45) is manifestly covariant. It is useful, though, to put it in the standard vectorial notation. We follow the usual procedure by rewriting the macroscopic four-velocity as

$$U^\mu=[\gamma_f,\gamma_f\mathbf{V}], \tag{49}$$

where $\mathbf{V}$ is the vector-three velocity of the fluid and $\gamma_f=U^0=\langle v^0_\alpha\rangle$ is the average relativistic factor. Using the constraint (19) and the definition for $z^\mu$, one readily obtains $U_\mu U^\mu=1-\langle z_\mu z^\mu\rangle$, which, when combined with Eq. (49), yields

$$\gamma_f=\gamma(1-\langle z_\mu z^\mu\rangle)^{1/2}, \tag{50}$$

where $\gamma=(1-\mathbf{V}^2)^{-1/2}$.

The factor $\gamma_f$ is, by definition, the average relativistic factor associated with the $N$ particle fluid. There are two

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contributions to this: one is given by the particle velocities in the electromagnetic field and the other by random thermal motion, as given by the pressurelike term $\langle z_\mu z^\mu\rangle$. In a typical hot relativistic fluid theory, the relativistic factor is corrected by the enthalpy density of the fluid.[50] Thus, the thermal correction in Eq. (50) is related to the enthalpy of the system.

We apply the same procedure to define a macroscopic vector spin density $\mathbf{S}$,

$$\mathbf{S}^k = \frac{\hbar}{2}\langle w^k\rangle, \tag{51}$$

and then the macroscopic four-spin density is $W^\mu = (W^0,(2/\hbar)\mathbf{S})$, where $W^0=\langle w^0\rangle$. It can be shown that the four-spin density satisfies

$$W_\mu W^\mu = -1 - \langle s_\mu s^\mu\rangle.$$

In addition, the single particle constraint $v_\mu w^\mu=0$ leads to the relation $U_\mu W^\mu = -\langle z_\mu s^\mu\rangle$ between the macroscopic four-velocity and the macroscopic four-spin density. Thus, $\mathbf{V}$ and $\mathbf{S}$ represent the macroscopic quantities for the velocity vector and the spin vector of our fluid, respectively. They are the spatial part of their corresponding relativistic four-vectors. When the thermal effects vanish, the normalization conditions and the constraints on the macroquantities are the same as the ones obeyed by the microquantities [Eq. (19)].

With macroscopic quantities properly defined, we are able to write our equations in a vectorial form. We shall display here only the dynamical evolution equations for the velocity and the spin, although all macroscopic fluid equations can be written in a similar way. For instance, the continuity equation (40) becomes

$$\partial_0(n\gamma_f) + \nabla\cdot(n\gamma_f\mathbf{V}) = 0. \tag{52}$$

The spatial part of the equation for the evolution of velocity (44) reads as

$$mn\frac{d}{dt}(\gamma_f\mathbf{V}) = -\frac{en}{\langle\cos\,\theta\rangle}\gamma_f(\mathbf{E}+\mathbf{V}\times\mathbf{B}) - \frac{1}{\langle\cos\,\theta\rangle}\nabla\cdot\Pi + \frac{1}{\langle\cos\,\theta\rangle}\mathbf{F}_Q, \tag{53}$$

with the operator $d/dt=\gamma_f\partial_0+\gamma_f\mathbf{V}\cdot\nabla$. The quantity $\Pi$ is the relativistic quantum pressure tensor,

$$\Pi^{ij} = mn\langle\cos\,\theta z^i z^j\rangle. \tag{54}$$

In the nonrelativistic quantum limit ($\theta$=0), this tensor reduces to the one in Ref. 29; it also reduces to the well-known classical pressure tensor in the appropriate limit. The term $\mathbf{F}_Q/\langle\cos\,\theta\rangle$ is the relativistic quantum force. This highly nonlinear force includes all the relativistic quantum corrections to the fluid dynamics and is given by

$$\begin{aligned}\mathbf{F}_Q^k =& -m\partial_0(n\langle\cos\,\theta z^0 z^k\rangle) + mn\left\langle\frac{1}{\rho}\partial_\nu(\rho\cos\,\theta z^\nu)z^k\right\rangle - mn\langle\cos\,\theta z^\nu\rangle\partial_\nu(\gamma_f\mathbf{V}^k) - mnU^\nu\langle\cos\,\theta\partial_\nu z^k\rangle\\ &+\frac{n}{2}\langle v^\nu\partial_\nu(\partial_\alpha\theta(v^\alpha w^k - v^k w^\alpha))\rangle + \frac{in}{2}\left\langle v^\nu\partial_\nu\left(\frac{1}{\rho}\varepsilon^{k\alpha\beta\lambda}\partial_\alpha(\rho v_\beta w_\lambda)\right)\right\rangle + n\left\langle\frac{1}{2\rho}\partial^\nu(\rho\partial^k\theta w_\nu - i\rho\varepsilon_{\nu\alpha\beta\lambda}v^\alpha w^\beta\partial^k v^\lambda)\right\rangle\\ &+ mn\langle\sin\,\theta v^k v^\nu\partial_\nu\theta\rangle.\end{aligned} \tag{55}$$

It is expected that macroscopically the largest relativistic quantum effects will be in this quantum force.

The spatial part of Eq. (43) may be written as

$$n\frac{d}{dt}\mathbf{S} = \frac{en}{m\langle\cos\,\theta\rangle}\left(\frac{\hbar}{2}W^0\mathbf{E} + \mathbf{S}\times\mathbf{B}\right) - \frac{\hbar}{2m\langle\cos\,\theta\rangle}\nabla\cdot K + \frac{\hbar}{2\langle\cos\,\theta\rangle}\Xi_Q, \tag{56}$$

where $K$ is the relativistic quantum thermal-spin coupling tensor,

$$K^{ij} = mn\langle\cos\,\theta z^i s^j\rangle, \tag{57}$$

which reduces, in the nonrelativistic limit ($\theta$=0), to its equivalent in Ref. 29. The $\hbar\Xi_Q/(2\langle\cos\,\theta\rangle)$ term is the relativistic quantum nonlinear correction to the spin evolution,

$$\begin{aligned}\Xi_Q^k =& -\partial_0(n\langle\cos\,\theta z^0 s^k\rangle) + n\left\langle\frac{1}{\rho}\partial_\nu(\rho\cos\,\theta z^\nu)s^k\right\rangle - \frac{2n}{\hbar}\langle\cos\,\theta z^\nu\rangle\partial_\nu\mathbf{S}^k - nU^\nu\langle\cos\,\theta\partial_\nu s^k\rangle\\ &+ n\langle\sin\,\theta w^k v^\nu\partial_\nu\theta\rangle + \frac{n}{2m}\left\langle\frac{1}{\rho}\partial_\nu(\rho\partial^\nu\theta v^k)\right\rangle + \frac{n}{2m}\left\langle\frac{1}{\rho}\partial_\nu(\rho w^k\partial_\alpha\theta(v^\alpha w^\nu - v^\nu w^\alpha))\right\rangle + \frac{in}{2m}\left\langle\frac{1}{\rho}\partial_\nu(w^k\varepsilon^{\nu\alpha\gamma\lambda}\partial_\alpha(\rho v_\gamma w_\lambda))\right\rangle\\ &- \frac{in}{2m}\left\langle\frac{1}{\rho}g^{k\lambda}\varepsilon_{\lambda\alpha\beta\gamma}\partial_\nu(\rho v^\alpha w^\beta\partial^\nu v^\gamma)\right\rangle.\end{aligned} \tag{58}$$

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From Eq. (56), we see that in this relativistic quantum fluid theory the spin can be coupled with the electric field through the zero component of the four-vector spin density. However, this coupling vanishes in the nonrelativistic theory, where $W^0=0$.

The full set of macroscopic fluid equations is given by Eqs. (40)–(42), (45), (46), (53), and (56). Our model is deeply different from any other model for quantum fluids. As in other models, our formalism has a quantum force $\vec{F}_Q$ in the momentum equation and also a quantum spin correction $\vec{\Xi}_Q$ in the spin density equation. However, the forces $\vec{F}_Q$ and $\vec{\Xi}_Q$ derived here are more complex and general; they explicitly display effects that arise from the quantum relativistic nature of the particles. Besides, the Dirac equation is an equation for particle-antiparticle states. This is manifested in a renormalization of the mass and pressure. This is a pure relativistic quantum effect, and it does not have any classical counterpart. Related to this is the fact that, whereas in a nonquantum formalism an electron-positron plasma can be formulated as a two-fluid theory, in the relativistic quantum case the same plasma is described by a one-fluid formalism.

## V. NONRELATIVISTIC LIMIT OF THE THEORY

It is expected that our relativistic quantum theory produces the correct quantum nonrelativistic limit epitomized in the fluid description based on the Pauli theory. The nonrelativistic spin effects in quantum plasmas have been recently studied; a spin evolution equation has been derived and it has been shown that the spin affects the evolution of velocity.[29,30]

A fundamental difference between the relativistic and the nonrelativistic quantum theory is that in the former theory the particle velocity defined via Eq. (17) does not reduce to the nonrelativistic velocity. This phenomenon is related to the so-called "Zitterbewegung" (see, for example, Ref. 51) originating in the mixing of the particle and antiparticle states. The profound consequence is that in relativistic quantum mechanics, the momentum is not "equivalent" to the velocity.

What is reduced to the nonrelativistic velocity in the appropriate limit, however, is the four-momentum $k^\mu$ given by Eq. (20).[29,30,52] Thus, care must be taken in the choice of variables when one wishes to find the nonrelativistic (NR) limit of the Dirac fluid equations.

Obtaining the NR limit of our formalism is quite tedious. However, we will carry out a few illustrative calculations in order to outline the procedure. It is obvious that the NR will be characterized by $\theta=0$ (no antiparticles) and $v_k \ll 1$. Using the definition for $k^\mu$ [Eq. (36)] and given that $m \equiv mc/\hbar$, we obtain an expression for the spatial NR velocity,

$$v_i = k_i - \frac{1}{mc\rho}\varepsilon_{i\alpha\lambda}\partial_\alpha\left(\rho\frac{\hbar S^\lambda}{2}\right), \tag{59}$$

where $\hbar S^k/2 = -i\hbar w^k/2$ is the NR definition for the spin density given in Refs. 29, 30, 52, and 53, and $\rho$ is the NR density. Thus, the spatial component of $k^i$ reduces to the nonrelativistic velocity $v^i$ defined in Refs. 29, 30, 52, and 53.

We now turn our attention to the entire equation for the velocity. In the NR limit, all terms on the right hand side of Eq. (37), except the relativistic Lorentz force $-e\rho F_{i\nu}v^\nu$, vanish. Using Eq. (59) for the NR velocity, we can split the relativistic Lorentz force into $-e\rho F_{i\nu}k^\nu$ (the nonrelativistic Lorentz force), a divergence, and $-(e\rho/2mc^2)S^k\partial_i B_k$; the latter is the Pauli spin contribution to the equation of motion. Carrying out similar manipulations, the NR limit of the energy-momentum tensor (31) takes the form

$$\Theta_{\mu\nu} = m\rho k_\mu v_\nu - \tfrac{1}{2}\varepsilon_{\nu\lambda\beta}\partial_\mu(\rho S^\beta)v^\lambda. \tag{60}$$

The last term, after using Eq. (59), can be written as

$$\frac{\hbar}{4mc\rho}\partial_\mu\rho\partial_\nu\rho - \frac{\hbar\rho}{4mc}\partial_\mu S^\beta\partial_\nu S^\beta, \tag{61}$$

plus a divergence. The first term in Eq. (61) represents the first of two contributions, which make up the Bohm potential. The other contribution to the Bohm potential appears because, in the NR limit, the Pauli Lagrangian does not vanish.[52] The second term in Eq. (61) represents the spin-spin interaction. Then, and following Ref. 52, the one-body equation of motion (37) can be shown to reproduce the NR one-body velocity found in Refs. 29, 30, and 52.

We end this section by showing that our theory produces the correct NR evolution equation for the spin. Since $\theta=0$, $w^0 \ll 1$, $v^0=1$, $v^i \ll 1$, Eq. (38) leads to

$$\frac{dS^k}{dt} = \frac{e}{mc}S_iF^{ki} + \frac{\hbar}{2m\rho}\varepsilon_{k\beta\gamma}\partial_j(\rho S^\beta\partial_j S^\gamma), \tag{62}$$

which is the same result as in Refs. 29, 30, and 52.

It is important to notice that the nonrelativistic limit obtained here corresponds to the one-body evolution equations. The complete plasma theory will emerge upon taking ensemble averages of these equations.

## VI. ELECTROMAGNETIC WAVES IN A DIRAC FLUID

In order to illustrate how quantum relativistic effects may affect collective plasma motions, we will now apply the formalism developed in this paper to solve a simple problem. The idea is to demonstrate qualitative new features that quantum relativity is capable of bringing to collective plasma motions. In particular, for the purpose of illustration, we will study a particular electromagnetic wave propagating in a relativistic quantum, cold electron-positron plasma, a problem which has been extensively studied by various authors.[54]

We start by considering a plasma with constant velocity $\mathbf{V}_0$ and constant density $n_0$. In such a plasma, all electromagnetic fields are zero. Let the velocity and the density of the fluid be perturbed to the first order $\mathbf{V}=\mathbf{V}_0+\mathbf{V}_1$ and $n=n_0+n_1$, and linearize the plasma equations with respect to these perturbations. We assume that a transverse electromagnetic wave, $\mathbf{k}\cdot\mathbf{E}=0$ and $\mathbf{k}\cdot\mathbf{B}=0$, can propagate in the plasma and satisfies $\mathbf{V}_0\cdot\mathbf{V}_1=0$ so that the relativistic Lorentz factor $\gamma$ is constant. The waves also satisfy $\mathbf{k}\cdot\mathbf{V}_0=0$ and $\mathbf{V}_0\times\mathbf{B}=0$.

For the sake of simplicity, we choose $\mathbf{E}=E\hat{z}$, $\mathbf{B}=B\hat{x}$, $\mathbf{V}_0=V_0\hat{x}$, $\mathbf{V}_1=V_y\hat{y}+V_z\hat{z}$, and $\mathbf{k}=k\hat{y}$, although other choices are possible. From Eq. (53), the linearized equation of motion is

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$$\gamma\frac{\partial \mathbf{V}_1}{\partial t} = -\frac{e}{m_e\langle\cos\,\theta\rangle}\mathbf{E} + \frac{\hbar}{m_e n_0 \gamma\langle\cos\,\theta\rangle}\mathbf{F}_Q. \quad (63)$$

In Eq. (63), the relativistic quantum effects directly appear through the factor $\langle\cos\,\theta\rangle$, which measures the mixing of the antiparticle states. There are many additional quantum relativity effects contained in the rather complicated quantum force $\mathbf{F}_Q$ [see Eq. (55)]; however, dealing with all the richness implicit in $\mathbf{F}_Q$ is beyond the scope of this study. Instead, we will attempt to make some very rough but reasonable assumptions in order to extract those parts of the quantum force $\mathbf{F}_Q$ relevant for our simple example. The first approximation is to regard $\theta$ as a constant, being fully aware that it is a dynamical variable. More precisely, we neglect its variation on the electromagnetic wave time and space scales. Besides, neglecting all thermal effects, the quantum force can be written as

$$\mathbf{F}_Q^k = \frac{in}{2}\left\langle v^\nu\partial_\nu\left(\frac{1}{\rho}\varepsilon^{k\alpha\beta\lambda}\partial_\alpha(\rho v_\beta w_\lambda)\right)\right\rangle - \frac{in}{2}\left\langle\frac{1}{\rho}\partial^\nu(\rho\varepsilon_{\nu\alpha\beta\lambda}v^\alpha w^\beta\partial^k v^\lambda)\right\rangle. \quad (64)$$

Moreover, for the transverse electromagnetic wave considered, the quantum force of Eq. (63) further simplifies to

$$\frac{\hbar\mathbf{F}_Q}{m_e n_0\gamma\langle\cos\,\theta\rangle} = \frac{i}{m_e\langle\cos\,\theta\rangle}U^\nu\partial_\nu\left[-\partial_0(\mathbf{V}_1\times\mathbf{S}) + (\nabla\times\mathbf{S}) - \frac{\hbar}{2}(\nabla\times W^0\mathbf{V}_1)\right] + \frac{i\gamma}{m_e\langle\cos\,\theta\rangle}\{\partial_0[\mathbf{V}_1\cdot(\mathbf{S}\times\partial_j\mathbf{V}_1)] - \nabla\cdot(\mathbf{S}\times\partial_j\mathbf{V}_1) + W^0\,\nabla\,\cdot(\mathbf{V}_1\times\partial_j\mathbf{V}_1)\}, \quad (65)$$

where $\mathbf{S}$ is the macroscopic spin-density vector defined in Eq. (51).

We need a similar simplification of Eq. (56) for the spin vector. We can focus on scale lengths similar to the Larmor radius[30] and neglect all the terms quadratic in $w^\mu$. Further assuming that the spin inertia can be neglected for frequencies below cyclotron frequency, the cold plasma solution of Eq. (56) is

$$\mathbf{S} = -\frac{\hbar}{2}\left(\frac{\mathbf{B}}{|\mathbf{B}|} - W^0\frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{B}|^2}\right), \quad (66)$$

where we have assumed that the spin will be antiparallel to the magnetic field in order to minimize the magnetic moment energy. For the particular case of the transverse electromagnetic wave considered, the macroscopic spin is $\mathbf{S}=-(\hbar/2\gamma)\times(\hat{x}-W^0\hat{y})$.

Substituting this solution in Eq. (65) and linearizing Eq. (63) on the perturbed variables, the velocity evolution equation becomes

$$-i\omega\gamma\mathbf{V}_1 = -\frac{e}{m_e\langle\cos\,\theta\rangle}\mathbf{E} + \frac{i\hbar\gamma}{2m_e\langle\cos\,\theta\rangle}\times[\omega^2\hat{x}\times\mathbf{V}_1 - \omega(\omega+k)W^0(\hat{y}\times\mathbf{V}_1)] + \frac{i\hbar\gamma}{2m_e\langle\cos\,\theta\rangle}k^2[(\mathbf{V}_1\times\hat{x})\cdot\hat{y}]\hat{y}, \quad (67)$$

which can be solved for $V_z$, yielding

$$V_z = \frac{-ie}{m_e\omega\gamma\langle\cos\,\theta\rangle}E\left[1+(\omega^2-k^2)\left(\frac{\hbar}{2m_e\langle\cos\,\theta\rangle}\right)^2\right]^{-1}. \quad (68)$$

Substituting $V_z$ in the Maxwell equation $\nabla\times\nabla\times\mathbf{E} = 4\pi en\gamma\partial_t V_z\hat{z} - \partial_t^2\mathbf{E}$, one finds the dispersion relation,

$$\omega^2 - k^2 = \omega_p^2\left[\langle\cos\,\theta\rangle^2 + \frac{\hbar^2}{4m_e^2}(\omega^2-k^2)\right]^{-1}, \quad (69)$$

with the plasma frequency defined as

$$\omega_p^2 = \frac{4\pi e^2 n_0\langle\Omega_\alpha/\rho_\alpha\rangle}{m_e}. \quad (70)$$

This definition is justified because

$$n_0\left\langle\frac{\Omega_\alpha}{\rho_\alpha}\right\rangle = \sum_\alpha p_\alpha\Omega_\alpha \quad (71)$$

is the average of the scalar density, which in the nonrelativistic limit becomes the usual total fluid density.

The quantum relativistic dispersion relation (69) differs from the standard electromagnetic dispersion relation $\omega^2 - k^2 = \omega_p^2$ in two striking ways:

(1) Quantum relativistic effects, through the term $1/\langle\cos\,\theta\rangle^2$, cause an effective increase in the plasma frequency, thus increasing the plasma opacity. We do not see $\hbar$ explicitly in this term because $\theta$ was regarded as a constant, and it is actually a dynamical variable, which should be found by simultaneously solving the entire system of fluid equations. To understand the increased opacity of the plasma, it is useful to remind that a nonzero $\theta$ [see Eq. (6)], which is a complicated function of quantities such as the energy of the particles, the fine structure constant, etc., implies a finite density antiparticle fluid interacting with the particle fluid. For a nonrelativistic plasma ($\theta=0$), there is only the pure electron fluid (in a neutralizing immobile ion background) but as the fluid energy increases, the electron fluid becomes contaminated with a higher and higher fraction of electron-positron pairs. In the extreme relativistic limit ($\theta=\pi/4$), the amplitudes of the scalar and the pseudoscalar densities are approximately the same, $1/\langle\cos\,\theta\rangle^2 = 2$ and the dispersion relation becomes

$$\omega^2 - k^2 = 2\omega_p^2,$$

the classical result for an electron-positron pair plasma. It is expected but still amazing that the Dirac fluid simulates both the behavior of a classical electron fluid and that of the classical electron-positron fluid in the two appropriate limits. The marriage of quantum mechanics

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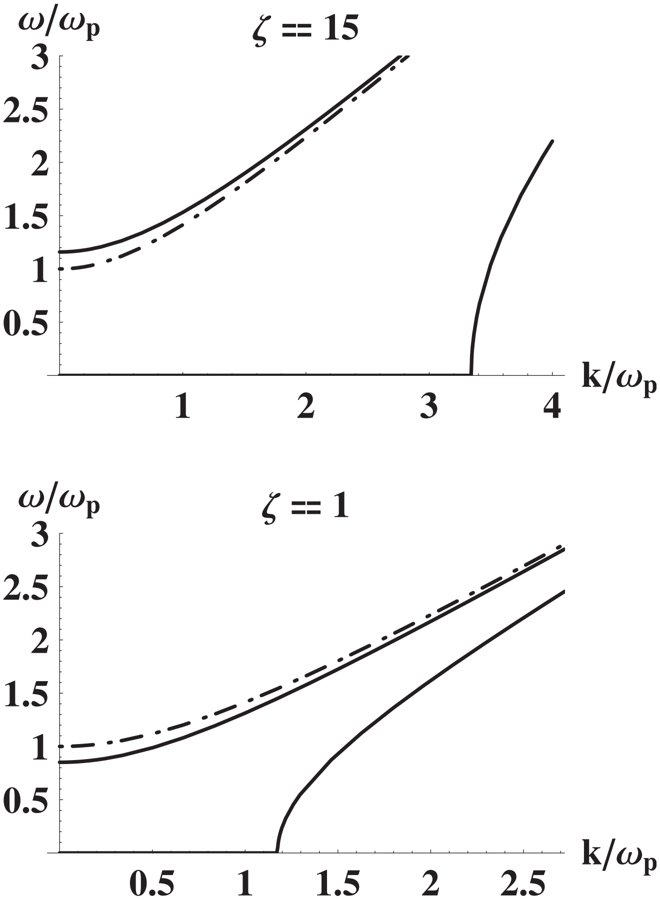


FIG. 1. Spin effect in electromagnetic branches of the dispersion relation (72) for $\theta=\pi/5$. The dotted line is the classic electromagnetic dispersion relation, while the full lines are $x_{(+)}$ and $x_{(-)}$ branches of Eq. (72). The $x_{(+)}$ branch propagates from $y=0$. The $x_{(-)}$ branch propagates from $2y^2=\zeta\langle\cos\,\theta\rangle^2+\sqrt{\zeta^2\langle\cos\,\theta\rangle^4+4\zeta}$.

and special relativity does, indeed, produce unique and qualitatively deep new results with no analog in either quantum (but not relativistic) or relativistic (but not quantum) fluids.

(2) The second effect in Eq. (69), proportional to $\hbar^2$, is a more conventional spin dependent quantum effect (which, to the best of our knowledge, is derived here for the first time). Since it tends to increase the denominator on the right hand side of Eq. (69), the spin dependent correction tends to induce plasma transparency, that is, the opposite of the mixing angle effect discussed above. For a mildly relativistic system ($\theta=0$), this term goes as $(\hbar\omega/m_e)^2=(\lambda_c/\lambda)^2$, where $\lambda=2\pi/\omega$ and $\lambda_c=2\pi\hbar/m_e$ are the wavelength and the Compton length, respectively. The quantum mechanical induced transparency effect can be very large for high frequency waves, especially when the wave has more energy than the electron rest mass.

The dispersion relation (69) can be readily solved for the normalized frequency $x=\omega/\omega_p$,

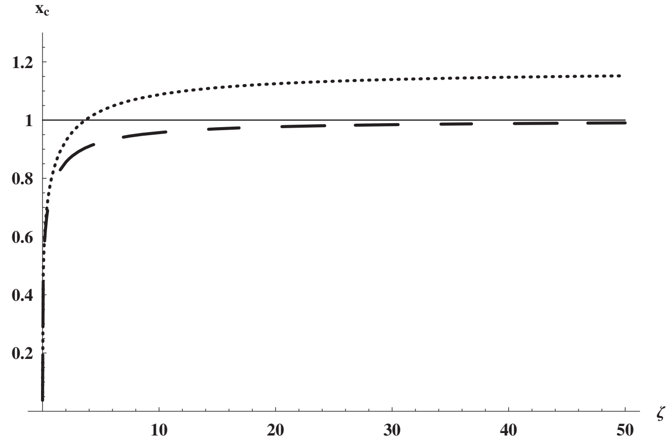


FIG. 2. A plot of the effective critical frequency $x_c$, solution of Eq. (73), as a function of $\zeta$ for $\theta=0$ (dashed line) and $\theta=7\pi/40$ (dotted line). The solid line represents the critical frequency for the classical dispersion relation. When $\zeta$ is large, the dominant effect is opacity. Instead, when $\zeta$ is small, plasma transparency is enhanced.

$$x_{(\pm)}^2=y^2+\frac{\zeta\langle\cos\,\theta\rangle^2}{2}\left(\pm\sqrt{1+\frac{4}{\zeta\langle\cos\,\theta\rangle^4}}-1\right) \tag{72}$$

in terms of the parameters $y=k/\omega_p$ and $\zeta=4m_e^2/\hbar^2\omega_p^2$. The frequency $x_{(+)}$ is for the usual electromagnetic branch now with relativistic quantum corrections. The frequency $x_{(-)}$ represents a new electromagnetic branch that appears only due to the relativistic quantum nature of this plasma. This mode can only propagate when $2y^2>\zeta\langle\cos\,\theta\rangle^2+\sqrt{\zeta^2\langle\cos\,\theta\rangle^4+4\zeta}$. In the $\zeta\to 0$ limit, the branch associated with $x_-$ disappears.

In Fig. 1, we see the spin effect in electromagnetic branches $x_{(+)}$ and $x_{(-)}$ of the dispersion relation (72) for different values of $\zeta$ and $\theta=\pi/5$. The dotted line is the classical electromagnetic dispersion relation $x^2=y^2+1$; the full line is for $x_{(+)}$ and $x_{(-)}$ of Eq. (72). When $\zeta$ decreases, the spin effect increases. In the $x_{(+)}$ branch, the spin effect produces a reduction of the effective critical frequency, which produces an increase of the plasma transparency. When $\zeta\to 0$, the wave becomes a light wave. On the other hand, we see that the spin effect in the $x_{(-)}$ branch is related to the group of wave numbers in which the wave does not propagate.

It is easy to see that spin effects are most important when $\zeta$ is small, i.e., when the quantum energy of plasma waves related to $\omega_p$ is much larger than the electron rest mass, which, in turn, may occur, for instance, in very high density plasmas. The frequency cutoff for the $x_{(+)}$ branch occurs at an effective critical frequency $x_c$, below which there is no wave propagation. It can be calculated by setting $y=0$, which yields

$$x_c^2=\frac{1}{\langle\cos\,\theta\rangle^2+(x_c^2/\zeta)}, \tag{73}$$

clearly showing that the first term (related to antiparticle mixing with classical value equal to Eq. (1) always increases plasma opacity, while the second one (related to spin) increases plasma transparency.

In Fig. 2, we plot the effective critical frequency $x_c$ as a function of the parameter $\zeta$, as given by Eq. (73). The solid

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line represents the classical electromagnetic dispersion relation $x^2=y^2+1$ with critical frequency identically equal to unity. The dashed line corresponds to $\theta=0$, and the dotted line to $\theta=7\pi/40$, an arbitrary choice just to illustrate the effects discussed in this section. For all nonzero $\theta$, $x_c>1$ in the limit $\zeta\gg 1$, where the spin effects are negligible. In this regime, the dominant effect is purely quantum relativistic, namely, the opacity produced by the electron-positron pair fraction (the $\langle\cos\,\theta\rangle$ effect). For the special case $\theta=0$, only transparency inducing quantum effects pertain. The plasma tends to be more and more transparent as $\zeta\to 0$ and the critical frequency $x_c\ll 1$. In this regime, the spin effect can overwhelm the $\langle\cos\,\theta\rangle$ effect, and the net effect is a greater plasma transparency. Here, the electromagnetic wave asymptotically becomes a light wave (electromagnetic wave in vacuum).

One must, however, warn the reader that for a classical plasma, the relativistically induced transparency effect is somewhat subtle; the kinematic and the thermal motions exert qualitatively different influences; the relativistic thermal motion actually decreases the effective plasma frequency (as do the spin effect mentioned above), while the directed plasma motion cannot because the two sides of the dispersion relation must be covariant. For a hot quantum relativistic plasma, the dispersion relation for the electromagnetic wave is modified to

$$\omega^2-k^2=\frac{\omega_p^2}{\Gamma_T\left[\langle\cos\,\theta\rangle^2+\dfrac{\hbar^2}{4m_e^2}(\omega^2-k^2)\right]},\tag{74}$$

where $\Gamma_T$ is the thermal relativistic factor,[50] which can be related to $\gamma_f/\gamma$ of Eq. (50).

## VII. CONCLUSIONS

Starting from the Dirac equation, we have developed a new fluid formalism for relativistic quantum spin half plasmas. The theory is highly nonlinear and much more complicated (as expected) than theories that are either classical relativistic or quantum but nonrelativistic. New features, inherent to the Dirac equation, become manifest in the macroscopic fluid equations: the explicit appearance of spin, the intrinsic particle-antiparticle coupling, and the nontrivial relationship between momentum and velocity are prime examples. These effects emerge in the theory through new parameters peculiar to the relativistic quantum mechanics. The new theory encompasses known theories and can be reduced to them in appropriate limits.

The new formalism opens a wide new field for investigation. Not only can it be used to calculate relativistic corrections to what may be already known in quantum plasmas, but the theory must be examined for qualitative new results that have no analog in the simpler versions. One of the obvious results is that a quantum relativistic electron plasma can never be just an electron plasma; it must always be an interacting electron-positron system.

Although the primary objective of this paper was the development and derivation of the system of equations satisfied by a quantum relativistic fluid, we have also solved a conceptually important but algebraically simple problem; we have derived the dispersion relation for an electromagnetic wave sustained by such a fluid. We find that the electron-positron coupling, a necessary feature of this system based on the Dirac equation, produces correction to the usual electromagnetic modes and introduces a new branch, which does not have a classical counterpart. Besides, the electron-positron nature of this fluid tends to increase the plasma opacity, while the spin effects make the fluid more transparent. To the best of our knowledge, both these effects are new. The increase in opacity comes into existence through the combination of quantum mechanics and special relativity and is unique to the current system. The spin induced transparency can become particularly strong when the radiation and the plasma quanta have energies in excess of the rest mass energy of the electron.

We expect that this formalism and the results enumerated above will find wide-ranging applications in cosmic plasmas as well as in plasmas produced by intense laser beams.

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