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Nilpotency of commutative finitely generated algebras satisfying $L_x^3 + \gamma L_{x^3} = 0$, $\gamma = 1, 0$

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ABSTRACT

This paper deals with two varieties of commutative non-associative algebras. One variety satisfies $L_x^3 + L_{x^3} = 0$. The other variety satisfies $L_x^3 = 0$. We prove that in either variety, any finitely generated algebra is nilpotent. Our results require characteristic $\neq 2, 3$.

1. Introduction

For power-associative algebras, an element a is nilpotent if $a^n = 0$ for some n. An algebra is called a *nilalgebra* if every element is nilpotent. Albert's conjecture [1] was that every commutative, finite dimensional, power-associative nilalgebra is nilpotent. In [20] Suttles gave an example of a commutative power-associative nilalgebra of dimension 5 which was not nilpotent. This counterexample is solvable. Since Albert's conjecture was now known to be false, it was modified to "every commutative, finite dimensional, power associative nilalgebra is solvable". That is, if $A^{(1)} = A$ and $A^{(n+1)} = A^{(n)}A^{(n)}$, for $n > 1$, must there exist a k such that $A^{(k)} = 0$? This modified conjecture is still open. It has been solved for dimensions 1 through 8 (see [4–7,10,11,14] and [16]). It has also been solved when the nilindex is close to the dimension (see [3,6] and [15]). There are some partial results for dimensions 9 and 10 (see [12]).

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If an algebra is not power-associative, then the concept of nilpotency needs further clarification. When Gerstenhaber [13] states that $a^n = 0$, he means all products of n factors of a, no matter how associated, have to be zero. Others [17,18] pick a particular association, usually left tapped. In their papers $a^4 = 0$ means $a(a(aa)) = 0$. For commutative algebras with characteristic 0 or sufficiently large, Gerstenhaber proved that $a^n = 0$ implies that $L_a^{2n-3} = 0$. L_a is *left multiplication by* a, *i.e.* $L_a(x) = ax$. This established a connection between the nilpotency of an element a and the nilpotency of L_a.

Subsequent authors began studying commutative algebras where L_a is assumed nilpotent for all a in the algebra.

Gutiérrez Fernández [17] showed that finite dimensional commutative algebras satisfying the identity $x(x(xy)) = 0$ were nilpotent.

These various definitions of *nil* opened a new approach to the “Albert conjecture”.

Instead of approaching the Albert conjecture by putting assumptions on the dimension, one can assume additional identities. In a commutative algebra A, the identity $a^2 = 0$ means that $A^{(2)} = 0$ for characteristic $\neq 2$. The identity $a^3 = 0$ means the algebra is Jordan. The possible identities of degree 4 are given in Osborn [19] and Carini, Hentzel, Piacentini-Cattaneo [2]. In [18], Hentzel and Labra consider commutative algebras which simultaneously satisfy both $x(x(xx)) = 0$ and $\beta\{x(y(xx)) - x(x(xy))\} + \gamma\{y(x(xx)) - x(x(xy))\} = 0$. With some restrictions on β, γ and the characteristic, they show that there is an ideal I of the algebra A satisfying $A(AI) = 0$ and A/I is power associative. So with the possible exception of five special cases, these algebras are very close to being power associative. The exceptional cases were studied in the paper using computational techniques. For the cases $(\beta, \gamma) = (1, -1)$ and $(\beta, \gamma) = (1, +1)$ the major lemma, $A(A((xx)(xx))) = 0$, was not true. This identified these cases as interesting and warranting additional attention. The case $(\beta, \gamma) = (1, -1)$ corresponds to $L_x^3 + L_{x^3} = 0$. The case $(\beta, \gamma) = (1, +1)$ corresponds to the case $L_x^3 = 0$.

Let A be the free commutative (but not associative) algebra with k generators. Let $\mathrm{Dim}[n, k]$ be the dimension of the subspace of A which is spanned by terms of degree less than n. Thus $\mathrm{Dim}[n, k]$ is the number of distinct monomials of A with degree less than n.

Let the x_i be monomials in A and $L_{x_1} L_{x_2} \cdots L_{x_n}$ be a string of left multiplications by monomials x_i. The *length of the string* is n. *The total degree of the string* is $\sum_{i=1}^n \deg(x_i)$. The *max degree* is the maximum of $\{\deg(x_1), \deg(x_2), \ldots, \deg(x_n)\}$.

This paper studies two varieties of non-associative algebras concurrently. The first variety satisfies characteristic $\neq 2, 3$ and the identity

$$L_x L_x L_x + L_{(xx)x} = 0, \tag{1}$$

whose linearizations are

$$L_x L_x L_y + L_x L_y L_x + L_y L_x L_x + L_{(xx)y} + 2L_{(yx)x} = 0 \tag{1$'$}$$

and

$$L_x L_x L_y + 2L_y L_x L_x + L_x L_{xy} + L_y L_{xx} + L_{(xy)x} = 0. \tag{1$''$}$$

The second variety satisfies characteristic $\neq 2$ and the identity

$$L_x L_x L_x = 0. \tag{2}$$

So, A satisfies the identity $x(x(xy)) = 0$ whose linearization is

$$L_z L_x L_y + L_x L_z L_y + L_z L_{xy} + L_{(xy)z} + L_{(yz)x} + L_x L_{yz} = 0. \tag{2$'$}$$

A string is called *reducible* if it is expressible as a linear combination of strings of the same total degree but of shorter lengths. This is done in the first case using only the identities (1), (1′) and (1″), and in the second case using only the identities (2) and (2′). If X and Y are strings of the same length and same total degree, we use $X \equiv Y$ to mean that $X–Y$ is reducible. That is, $X–Y$ is expressible as a linear combination of strings each of which has the same total degree as X and Y but which have length less than the length of X and Y. When $X \equiv Y$ we say X is equivalent to Y.

2. Reducing strings

Lemma 1. *$L_x L_{x_1} \cdots L_{x_n} L_x$ is equivalent to a linear combination of strings where the L_x's are adjacent. This requires characteristic $\neq 2, 3$ in the first case and characteristic $\neq 2$ in the second case.*

Proof. Case 1. We use induction: In the string $L_x L_y L_x$, the distance that the L_x's are apart is 1. Using $(1')$ we have $L_x L_y L_x \equiv -L_x L_x L_y - L_y L_x L_x$.

Assume that the result is true if the distance that the L_x's are apart is less than k. The "$\circ$" notation means $L_u \circ L_v = L_u L_v + L_v L_u$.

$$L_x L_{x_1} L_{x_2} L_{x_3} \cdots L_{x_k} L_x = (L_x \circ L_{x_1}) L_{x_2} L_{x_3} \cdots L_{x_k} L_x - L_{x_1} L_x L_{x_2} L_{x_3} \cdots L_{x_k} L_x$$
$$\equiv -2 L_{x_2} (L_x \circ L_{x_1}) L_{x_3} \cdots L_{x_k} L_x - L_{x_1} L_x L_{x_2} L_{x_3} \cdots L_{x_k} L_x,$$

using $(1'')$.

Altogether, there are three strings represented in the above expression. In each of these three strings, the L_x's are less than k apart.

By induction each is equivalent to a linear combination of strings where the L_x's are adjacent.

Case 2. The identity $(2')$ reduces a string where the distance that the L_x's are apart is one. Assume that the result is true if the distance that the L_x's are apart is less than k.

$$L_x L_{x_1} L_{x_2} \cdots L_{x_k} L_x = (L_x L_{x_1} L_{x_2}) \cdots L_{x_k} L_x$$
$$\equiv -L_{x_1} L_x L_{x_2} \cdots L_{x_k} L_x \quad \text{using } (2').$$

By induction this is equivalent to a linear combination of strings where the L_x's are adjacent. $\square$

Lemma 2. *The string $L_x L_x L_y L_y$ is reducible for characteristic $\neq 2, 3$ in the first case and characteristic $\neq 2$ in the second case.*

Proof. Case 1. Subtracting $(1'')$ from $(1')$ we obtain the identity

$$L_y L_x L_x \equiv L_x L_y L_x. \tag{1'''}$$

Therefore:

$$
\begin{aligned}
&(L_x L_x L_y) L_y \\
&\quad \equiv -2(L_y L_x L_x) L_y \quad \text{using the identity } (1'') \\
&\quad \equiv -2(L_x L_y L_x) L_y \quad \text{using the identity } (1''') \\
&\quad \equiv -2 L_x (L_y L_x L_y) \\
&\quad \equiv -2 L_x (L_x L_y L_y) \quad \text{using the identity } (1''') \text{ with } x \text{ and } y \text{ interchanged} \\
&\quad \equiv -2 L_x L_x L_y L_y.
\end{aligned}
$$

So, $3 L_x L_x L_y L_y$ is reducible. Since the characteristic is $\neq 2, 3$, $L_x L_x L_y L_y$ is reducible.

Case 2. $L_x L_x L_y L_y \equiv (L_x L_x L_y) L_y \equiv 0$ using the identity $(2')$ with $x = z$ and characteristic $\neq 2$. $\square$

Lemma 3. *$L_x L_x L_x$ is reducible.*

Proof. $L_x L_x L_x = -L_{x^3}$ using identity (1) and $L_x L_x L_x = 0$ using the identity (2). □

Lemma 4. *$L_x L_x L_{x_1} L_{x_2} \cdots L_{x_k} L_y L_y$ is reducible for characteristic $\neq 2, 3$ in the first case and $\neq 2$ in the second case.*

Proof. We use induction on k. If $k = 0$, $L_x L_x L_y L_y$ is reducible using Lemma 2. Assume that the result is true for strings of length less than k. Then

Case 1.

$$\begin{aligned} L_x L_x L_{x_1} \cdots L_{x_{k-1}} L_{x_k} L_y L_y &\equiv -2 L_{x_1} L_x L_x L_{x_2} \cdots L_{x_{k-1}} L_{x_k} L_y L_y \quad &&\text{by identity } (1'') \\ &\equiv 0 &&\text{by induction.} \end{aligned}$$

Case 2.

$$L_x L_x L_{x_1} \cdots L_{x_{k-1}} L_{x_k} L_y L_y \equiv 0$$

because $L_x L_x L_{x_1} \equiv 0$ by identity $(2')$ with $x = z$ and characteristic $\neq 2$. □

Lemma 5. *Let A be the free commutative (but not associative) algebra with k generators. Then any string of total degree $\geqslant$ to $n \operatorname{Dim}[n,k]$ is reducible to a linear combination of strings whose max degree is $\geqslant n$ or which have an adjacent pair of identical L_{x_i}'s. We assume characteristic $\neq 2, 3$ in the first case and characteristic $\neq 2$ in the second case.*

Proof. For purposes of this proof, a string is completely reduced if:

(a) its max degree $\geqslant n$
or
(b) it has a pair of adjacent identical L_{x_i}'s.

A string will be completely reducible if it can be reduced to a linear combination of strings which are completely reduced.

Suppose there are strings of total degree $\geqslant n \operatorname{Dim}[n,k]$ which are not completely reducible. Let N_0 be the minimal length of all such strings. Let S_0 be one of the strings of length N_0, with total degree $\geqslant n \operatorname{Dim}[n,k]$, which cannot be completely reduced.

Any string of total degree $\geqslant n \operatorname{Dim}[n,k]$ and length $\leqslant \operatorname{Dim}[n,k]$ must have max degree $\geqslant n$. Therefore $N_0 > \operatorname{Dim}[n,k]$.

Any string of length $> \operatorname{Dim}[n,k]$ and max degree $< n$, is longer than the number of distinct monomials of degree $< n$ in the free commutative ring with k generators. Therefore S_0 must have two L_{x_i}'s which are identical. Using Lemma 1, S_0 is equivalent to a linear combination of strings which have the same total degree, and either have a pair of adjacent identical L_{x_i}'s or have shorter length. By the minimality of N_0, these shorter strings are completely reducible.

Therefore, S_0 must be completely reducible. This contradiction proves that every string of total degree $\geqslant n \operatorname{Dim}[n,k]$ is completely reducible and this proves Lemma 5. □

Theorem 1. *Let A be the free commutative (but not associative) algebra with k generators. Then any string of total degree $\geqslant 2n \operatorname{Dim}[n,k] + (n-2)$ is reducible to a linear combination of strings of max degree greater than or equal to n. We assume characteristic $\neq 2, 3$ in the first case and characteristic $\neq 2$ in the second case.*

Proof. Our goal is to reduce any string of sufficiently high total degree to a linear combination of strings of the same total degree, but with max degree $\geqslant n$. If a string has high total degree but low max degree, then it must have a long length. Since the number of possible terms of degree $< n$ is $\mathrm{Dim}[n,k]$, when the length is $> \mathrm{Dim}[n,k]$, at least one of the L_{x_i}'s has to be repeated.

Using Lemmas 1 through 5, the length of the string can be shortened while keeping the same total degree. This process is more carefully explained in the following paragraph. The result is that any irreducible string of high enough total degree must have max degree $\geqslant n$.

Let S be a string of total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ which has max degree $< n$. Divide S into two strings $S = S'S''$ so that $n\,\mathrm{Dim}[n,k] \leqslant$ total degree of $S' \leqslant n\,\mathrm{Deg}[n,k] + (n-2)$. This can be done because the max degree of S is $< n$.

One continues adjoining successive terms to S' until the total degree of S' is $\geqslant n\,\mathrm{Dim}[n,k]$. Since the last added term has degree $< n$, the resulting S' will have total degree $\leqslant n\,\mathrm{Deg}[n,k] + (n-2)$. The total degree of $S' \geqslant n\,\mathrm{Deg}[n,k]$ by construction. The total degree of $S'' \geqslant 2n\,\mathrm{Dim}[n,k] + (n-2) - (n\,\mathrm{Deg}[n,k] + (n-2)) = n\,\mathrm{Dim}[n,k]$. By Lemma 5, S' and S'' can be reduced to strings whose max degree is $\geqslant n$ or that have an adjacent pair of identical L_{x_i}'s.

S is equivalent to a linear combination of products of the reduced strings coming from S' and the reduced strings coming from S''.

When these reduced strings are multiplied together, each product will have max degree $\geqslant n$, or else will have two pair of adjacent multiplications of identical L_{x_i}'s. By Lemma 4, such strings are reducible.

If a string has total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ and it has max degree $< n$, then it is reducible. This means that any string of total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ is reducible to a linear combination of strings of max degree $\geqslant n$. $\square$

3. Nilpotency

In this section A will be a commutative algebra satisfying the identities (1) or (2). We will prove that if A is finitely generated and satisfies one of these identities, then A is nilpotent. Our results require characteristic $\neq 2, 3$ for the first case and characteristic $\neq 2, 3$ in the second case. Notice that now both cases have the same assumption on characteristic.

Case 1. A satisfies (1) and A commutative implies that A satisfies the identity

$$\big((yx)x\big)x + y\big((xx)x\big) = 0. \tag{3}$$

We define the function $J(x,y,z)$ by $J(x,y,z) = (xy)z + (yz)x + (zx)y$.

In [9, Theorem 5] we prove the following result:

Theorem 2. *Let A be a commutative algebra over a field of characteristic $\neq 2, 3$, that satisfies identity (3). Let W be the linear subspace of A generated by the elements of the form $J(x,y,z)$ with $x, y, z \in A$. Then W is an ideal of A and $W^2 = 0$.*

It is known (see [21, p. 114]) that a finitely generated commutative algebra satisfying $x^3 = 0$ is nilpotent. Let k be the number of generators and let n be the degree of nilpotence of a commutative algebra on k generators which satisfies $x^3 = 0$.

Theorem 3. *Any finitely generated commutative algebra of characteristic $\neq 2, 3$ satisfying identity (1) will be nilpotent of index at most $2^{4n\,\mathrm{Dim}[n,k]+2(n-2)}$.*

Proof. Any product of total degree $\geqslant 2^{4n\,\mathrm{Dim}[n,k]+2(n-2)}$ is expressible as a string of length $\geqslant 4n\,\mathrm{Dim}[n,k] + 2(n-2)$.

By Theorem 1, any string of total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ in a finitely generated commutative algebra is reducible to a linear combination of strings in which one of the factors is of degree greater than or equal to n. Passing to the homomorphic image satisfying identity (1), this factor of degree greater than n must lie in the ideal generated by all cubes and therefore must lie in W (see Theorem 2 for definition of W).

If we let the length of the string be twice as long, then there will be two factors from W. On multiplying these strings out, the result will be zero because $W^2 = 0$. This finishes the proof of Theorem 3. It is immediate that a string of length $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ will have total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ because the length of a string is $\leqslant$ its total degree. $\square$

Case 2. A satisfies identity (2) and A commutative implies that A satisfies the identity

$$\big((yx)x\big)x = 0. \tag{4}$$

Replacing y by x we get that A satisfies the identity $((xx)x)x = 0$. Linearizing this identity we get $2((yx)x)x + ((xx)y)x + ((xx)x)y = 0$. This linearization requires characteristic $\neq 2, 3$. For characteristic $\neq 2$ identity (4) is equivalent to identity

$$\big(y(xx)\big)x + y\big((xx)x\big) = 0. \tag{5}$$

Identity (4) was studied by Correa and Hentzel in [8], and by Gutiérrez Fernández in [17]. In the first it was shown that commutative, finitely generated algebras satisfying (4) are solvable. In the second the author proves that commutative finite dimensional algebras satisfying (4) are nilpotent. We will use the following two polynomial identities that appear in [8, Lemma 1 and its proof]:

$$\big(x^2y\big)y = \big(y^2x\big)x, \tag{6}$$

$$J(xy, z, w) = J(x, y, zw). \tag{7}$$

Theorem 4. *Let A be a commutative algebra over a field of characteristic $\neq 2, 3$, that satisfies identity* (4). *Let $I = \{x \in A \mid J(x, b, c) = 0,$ for all $b, c \in A\}$. Then*:

(i) *I is an ideal of A.*
(ii) *$(xy)i = -(xi)y - x(yi)$ for all i in I.*
(iii) *$x^3 I = 0$ for all x in A.*
(iv) *$\mathrm{Ann}(I) = \{x \in A \mid xI = 0\}$ is an ideal of A.*
(v) *The ideal generated by all cubes annihilates I.*
(vi) *Let $H = \mathrm{Ann}(I) \cap I$. Then H is an ideal and $H^2 = 0$.*

Proof. (i) I is clearly a linear subspace. From identity (7), if i is in I, then for every a, b, x in A, $J(ix, a, b) = J(i, x, ab) = 0$. This shows that I absorbs multiplication.

(ii) For i in I and x, y in A, $0 = J(i, x, y) = (ix)y + (xy)i + (yi)x$, which gives $(xy)i = -(xi)y - x(yi)$.

(iii) For every x in A and i in I we have using (i) and (ii)

$$\begin{aligned}
\big((xx)x\big)i &= -\big((xx)i\big)x - (xx)(xi) \\
&= \big((xi)x\big)x + \big(x(xi)\big)x + \big(x(xi)\big)x + x\big(x(xi)\big) \\
&= 4\big((ix)x\big)x \\
&= 0.
\end{aligned}$$

(iv) $\mathrm{Ann}(I)$ is clearly a linear subspace. For a in $\mathrm{Ann}(I)$, x in A and i in I;

$$(ax)i = -(ai)x - a(xi) = 0$$

because xi is in I and $aI = 0$. This shows $\mathrm{Ann}(I)$ absorbs multiplication.

(v) All the cubes lie in $\mathrm{Ann}(I)$ by Part (iii). Therefore the ideal generated by all the cubes also lies in $\mathrm{Ann}(I)$.

(vi) The intersection of ideals is an ideal and $H^2 \subset \mathrm{Ann}(I)I = 0$. $\square$

Throughout this section, we shall use I and H for these particular ideals.

Lemma 6. *Let A be a commutative algebra over a field of characteristic $\neq 2, 3$ that satisfies identity* (4). *Then $aJ(bc, d, e) - J(a(bc), d, e)$ is an element of H, for all a, b, c, d, e in A.*

Proof. By commutativity and the symmetry of the arguments of J, we need only show that $u = aJ(b^2, c, c) - J(ab^2, c, c)$ is in H for all a, b, c in A. The proof is done in three parts:

(i) u is in I.
(ii) $uI = 0$.
(iii) u is in H.

(i): We have to prove that $J(u, x, x) = 0$. That is,

$$J\big(J\big(ab^2, c, c\big), x, x\big) = J\big(aJ\big(b^2, c, c\big), x, x\big).$$

Linearizing (6) we get:

$$2\big((xz)y\big)y = \big(y^2 z\big)x + \big(y^2 x\big)z. \tag{8}$$

Using the definition of J, the fact that J is symmetric and linear on its three arguments, (7) and (8), we get:

$$\begin{aligned}
J\big(J\big(ab^2, c, c\big), x, x\big) &\overset{(7)}{=} J\big(J\big(a, b^2, c^2\big), x, x\big) \\
&= J\big(\big(ab^2\big)c^2 + \big(b^2c^2\big)a + \big(c^2a\big)b^2, x, x\big) \\
&= J\big(\big(ab^2\big)c^2, x, x\big) + J\big(\big(b^2c^2\big)a, x, x\big) + J\big(\big(ac^2\big)b^2, x, x\big) \\
&\overset{(7)}{=} J\big(ab^2, c^2, x^2\big) + J\big(b^2c^2, a, x^2\big) + J\big(ac^2, b^2, x^2\big) \\
&\overset{(7)}{=} J\big(a, b^2, c^2x^2\big) + J\big(a, b^2c^2, x^2\big) + J\big(b^2, x^2, ac^2\big) \\
&= J\big(b^2, a, c^2x^2\big) + J\big(a, b^2c^2, x^2\big) + J\big(b^2, x^2, ac^2\big) \\
&\overset{(7)}{=} J\big(b, b, a\big(c^2x^2\big)\big) + J\big(a, b^2c^2, x^2\big) + J\big(b, b, x^2\big(ac^2\big)\big) \\
&= J\big(b, b, a\big(c^2x^2\big) + x^2\big(ac^2\big)\big) + J\big(a, b^2c^2, x^2\big) \\
&= J\big(b, b, \big(c^2x^2\big)a + \big(c^2a\big)x^2\big) + J\big(a, b^2c^2, x^2\big) \\
&\overset{(8)}{=} J\big(b, b, 2\big(\big(ax^2\big)c\big)c\big) + J\big(a, b^2c^2, x^2\big) \\
&\overset{(7)}{=} 2J\big(b^2, \big(ax^2\big)c, c\big) + J\big(a, b^2c^2, x^2\big) \\
&= 2J\big(b^2, c, \big(ax^2\big)c\big) + J\big(a, b^2c^2, x^2\big) \\
&\overset{(7)}{=} 2J\big(b^2c, ax^2, c\big) + J\big(a, b^2c^2, x^2\big) \\
&= 2J\big(ax^2, b^2c, c\big) + J\big(a, b^2c^2, x^2\big)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(7)}{=} 2J\big(a, x^2, \big(b^2c\big)c\big) + J\big(a, b^2c^2, x^2\big) \\
&= 2J\big(a, \big(b^2c\big)c, x^2\big) + J\big(a, b^2c^2, x^2\big) \\
&= J\big(a, 2\big(b^2c\big)c + b^2c^2, x^2\big) \\
&= J\big(a, J\big(b^2, c, c\big), x^2\big) \\
&\overset{(7)}{=} J\big(aJ\big(b^2, c, c\big), x, x\big).
\end{aligned}
$$

This proves (i).

(ii) Since $J(A, A, A)$ is contained in the ideal generated by all cubes, we have that u is in the ideal generated by all cubes. By Theorem 4 Part (v), $uI = 0$.

(iii) It follows from (i) and (ii) and the definition of H. $\square$

Theorem 5. *Let A be a commutative algebra over a field of characteristic $\neq 2, 3$ that satisfies identity* (4). *Then $J(A^5, A, A) \subset H$.*

Proof. We need to prove the three following statements:

(i) $J((((AA)A)A)A, A, A) \subset H$.
(ii) $J(((AA)(AA))A, A, A) \subset H$.
(iii) $J((AA)((AA)A), A, A) \subset H$.

(i): Using Lemma 6, the symmetry of the arguments of J, and (7) we get the following congruences modulo H:

$$
\begin{aligned}
J\big(a\big(b(cd)\big), x, x\big) &\equiv aJ\big(b(cd), x, x\big) \\
&\equiv aJ\big(b, cd, x^2\big) \\
&\equiv J\big(b, cd, ax^2\big) \\
&\equiv J\big(b(cd), a, x^2\big) \\
&\equiv bJ\big(cd, a, x^2\big) \\
&\equiv bJ\big((cd)a, x, x\big) \\
&\equiv bJ\big(a(cd), x, x\big) \\
&\equiv J\big(b\big(a(cd)\big), x, x\big).
\end{aligned}
$$

It follows that $J(a(b(c(de))), x, x)$ is symmetric on a, b, c modulo H, and so, is zero, from identity (4). Therefore we get (i).

(ii): From identity (4) we have: $b^2(x(xc)) + x(b^2(xc)) + x(x(b^2c)) = 0$. From Part (i) we know that $J(x(x(b^2c)), a, a)$ is in H (5 taps are zero modulo H). Therefore by (7) and Lemma 6 we get the following congruences modulo H:

$$
\begin{aligned}
&J\big(a, a, b^2\big(x(xc)\big) + x\big(b^2(xc)\big)\big) \equiv 0, \\
&J\big(a, a, b^2\big(x(xc)\big)\big) + J\big(a, a, x\big(b^2(xc)\big)\big) \equiv 0, \\
&J\big(a^2, b^2, x(xc)\big) + J\big(a^2, x, b^2(xc)\big) \equiv 0, \\
&J\big(a^2, b^2, x(xc)\big) + J\big(a^2x, b^2, xc\big) \equiv 0,
\end{aligned}
$$

$$J\big(a^2, b^2, x(xc)\big) + xJ\big(a^2, b^2, xc\big) \equiv 0,$$
$$J\big(a^2, b^2, x(xc)\big) + J\big(a^2, b^2, x(xc)\big) \equiv 0.$$

So, $2J(a^2, b^2, x(xc)) \in H$. Characteristic $\neq 2$ gives $J(a^2, b^2, x(xc)) \in H$, and by linearization we get $J(a^2, b^2, p(qc)) \equiv -J(a^2, b^2, q(pc))$. This is used three times in the following sequence of calculations.

$$\begin{aligned} J\big(a^2, b^2, x(yc)\big) &\equiv J\big(a^2, b^2, -y(xc)\big) \\ &\equiv J\big(a^2, b^2, -y(cx)\big) \\ &\equiv J\big(a^2, b^2, +c(yx)\big) \\ &\equiv J\big(a^2, b^2, +c(xy)\big) \\ &\equiv J\big(a^2, b^2, -x(cy)\big) \\ &\equiv J\big(a^2, b^2, -x(yc)\big). \end{aligned}$$

Thus $2J(a^2, b^2, x(yc)) \in H$. Characteristic $\neq 2$ gives $J(a^2, b^2, x(yc)) \in H$. By commutativity we have $J(AA, AA, A(AA)) \subset H$. Then from (7) we have $J(((AA)(AA))A, A, A) = J((AA)(AA), A, AA) = J(AA, AA, A(AA)) \subset H$.

(iii) We have using (7), commutativity, and the symmetry of the arguments of J:

$$\begin{aligned} J\big((AA)\big((AA)A\big), A, A\big) &= J\big(AA, (AA)A, AA\big) \\ &\equiv AJ(AA, AA, AA) \\ &\equiv AJ\big((AA)(AA), A, A\big) \\ &\equiv J\big(\big((AA)(AA)\big)A, A, A\big) \\ &\equiv 0 \quad \text{by Part (ii).} \end{aligned}$$

This proves the theorem. □

Lemma 7. *Let A be a commutative algebra over a field of characteristic $\neq 2, 3$ that satisfies identity* (2). *Then any product involving three cubes is zero.*

Proof. This proof resembles that of Theorem 4 applied to the quotient ring A/H. The ideals are defined in the obvious way. We emphasize this connection by calling the ideals $\bar{I}$, $\mathrm{Ann}(\bar{I})$ and $\overline{H}$. These ideals are ideals of A, not of A/H. Because there are significant differences in the details, we find it necessary to give the proof independently. There are nine steps in the proof. The congruences are modulo H.

(i) $J(x^3A, A, A) \subset H$.
(ii) $J(x^3, A, A)A \subset H$.
(iii) $wy^3 = -J(w, y, y)y$, for any $w, y \in A$.
(iv) $x^3y^3 \in H$ for all $x, y \in A$.
(v) $\bar{I} = \{x \in A \mid J(x, A, A) \subset H\}$ is an ideal of A.
(vi) $x^3A \subset \bar{I}$.
(vii) $\mathrm{Ann}(\bar{I}) = \{x \in A \mid x\bar{I} \subset H\}$ is an ideal of A.
(viii) $x^3 \in \mathrm{Ann}(\bar{I})$.
(ix) Let C be the ideal generated by all cubes. Then $C^2 \subset H$.

We start proving each of these nine parts:

(i)

$$\begin{aligned} J\big(yx^3, c, c\big) &= -J\big(\big(y(xx)\big)x, c, c\big) \quad \text{by (5)} \\ &\equiv -xJ\big(y(xx), c, c\big) \quad \text{by Lemma 6} \\ &= -xJ\big(y, xx, c^2\big) \quad \text{by (7)} \\ &= -xJ\big(x^2, y, c^2\big) \\ &\equiv -J\big(x^3, y, c^2\big) \quad \text{by Lemma 6} \\ &= -J\big(x^3y, c, c\big) \quad \text{by (7).} \end{aligned}$$

It follows that $2J(yx^3, c, c) \in H$. By characteristic $\neq 2$, we have (i).

(ii) This follows directly from Lemma 6 and Part (i).

(iii)

$$\begin{aligned} wy^3 &= -\big(wy^2\big)y \quad \text{from (5)} \\ &= -J(w, y, y)y + 2\big((wy)y\big)y \\ &= -J(w, y, y)y \quad \text{by (4).} \end{aligned}$$

(iv) $x^3y^3 = -J(x^3, y, y)y \in H$ by Part (iii) and Part (ii).

(v) Let $\bar{I} = \{x \in A \mid J(x, A, A) \subset H\}$. From (7), $\bar{I}$ is an ideal of A.

(vi) $x^3A \subset \bar{I}$ from Part (i).

(vii) It is clear that $\mathrm{Ann}(\bar{I})$ is a linear subspace. We will now prove that it absorbs multiplication. Suppose $\bar{a} \in \mathrm{Ann}(\bar{I})$, $\dot{\bar{i}} \in \bar{I}$ and $x \in A$. Then:

$$\begin{aligned} 0 &\equiv J(\dot{\bar{i}}, \bar{a}, x) \\ &\equiv (\dot{\bar{i}}\bar{a})x + (\bar{a}x)\dot{\bar{i}} + (x\dot{\bar{i}})\bar{a} \\ &\equiv (\bar{a}x)\dot{\bar{i}} \quad \text{because } x\dot{\bar{i}} \in \bar{I} \text{ and } \bar{a}\bar{I} \subset H. \end{aligned}$$

Since $(\bar{a}x)\dot{\bar{i}} \equiv 0$, $\mathrm{Ann}(\bar{I})$ absorbs multiplication and is an ideal of A.

(viii)

$$\begin{aligned} \bar{I}x^3 &\subset J(\bar{I}, x, x)x \quad \text{by Part (iii)} \\ &\subset H \quad \text{by definition of } \bar{I} \text{ and the fact that } H \text{ is an ideal.} \end{aligned}$$

We now prove (ix). $\overline{H} = \mathrm{Ann}(\bar{I}) \cap \bar{I}$ is an ideal. By Parts (vi) and (viii) it contains x^3A. Therefore $\langle x^3 \rangle \subset x^3 + \overline{H}$, where $\langle x^3 \rangle$ denotes the ideal of A generated by x^3. Furthermore $\overline{H}^2 \subset H$. It follows that

$$\begin{aligned} \langle x^3 \rangle \langle y^3 \rangle &\subset \big(x^3 + \overline{H}\big)\big(y^3 + \overline{H}\big) \\ &\subset x^3y^3 + x^3\overline{H} + y^3\overline{H} + \overline{H}\,\overline{H} \\ &\subset x^3y^3 + x^3\bar{I} + y^3\bar{I} + \bar{I}\,\mathrm{Ann}(\bar{I}) \\ &\subset H \quad \text{by Parts (iv) and (viii).} \end{aligned}$$

It then follows that $C^2 \subset H$.

To show that any product involving three cubes, is zero, we need to show $\langle CC\rangle C = 0$. But:

$$\begin{aligned} \langle CC\rangle C &\subset HC \quad \text{by Part (ix)} \\ &\subset IC \quad \text{since } H \subset I \\ &= 0 \quad \text{by Theorem 4 Part (v).} \qquad \square \end{aligned}$$

Theorem 6. *Any finitely generated commutative algebra of characteristic $\neq 2, 3$ satisfying identity* (2) *will be nilpotent of index at most $2^{6n\,\mathrm{Dim}[n,k]+3(n-2)}$.*

Proof. This proof is very similar to the proof of Theorem 3.

Any product of total degree $\geqslant 2^{6n\,\mathrm{Dim}[n,k]+3(n-2)}$ is expressible as a string of length $\geqslant 6n\,\mathrm{Dim}[n,k] + 3(n-2)$.

By Theorem 1, any string of total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ in a finitely generated commutative algebra is reducible to a linear combination of strings in which one of the factors is of degree $\geqslant n$. Passing to the homomorphic image satisfying identity (2), this factor of degree $\geqslant n$ must lie in the ideal generated by all cubes.

If we let the string be three times as long, then there will be three factors from the ideal generated by all cubes. On multiplying these strings out, the result will be zero by Lemma 7.

It is immediate that a string of length $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ will have total degree $\geqslant 2n\,\mathrm{Dim}[n,k] + (n-2)$ because the length of a string is $\leqslant$ its total degree. $\square$

Remark. The condition of finitely generated is necessary. In fact, there exists an example due to Zhevlakov [21, Example 1, p. 82], of a commutative not finitely generated algebra A over a field of characteristic $\neq 2, 3$, that satisfies $x^3 = 0$ and $A^2 A^2 = 0$ but is not nilpotent. It is easy to prove that this algebra also satisfies identity (2).

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