Physica A 393 (2014) 391–403

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Option pricing, stochastic volatility, singular dynamics and constrained path integrals

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HIGHLIGHTS

- We study stochastic volatility models for extremely correlated cases.
- For each model we determine the underlying singular classical dynamics.
- We apply Dirac's method to singular Hamiltonian systems.
- We compute propagators in Euclidean time, for several different stochastic volatility models.

ARTICLE INFO

Article history:
Received 12 January 2013
Received in revised form 6 August 2013
Available online 3 September 2013

Keywords:
Option pricing
Stochastic volatility
Quantum mechanics
Singular Lagrangian systems
Dirac's method
Constrained Hamiltonian path integrals

ABSTRACT

Stochastic volatility models have been widely studied and used in the financial world. The Heston model (Heston, 1993) [7] is one of the best known models to deal with this issue. These stochastic volatility models are characterized by the fact that they explicitly depend on a correlation parameter ρ which relates the two Brownian motions that drive the stochastic dynamics associated to the volatility and the underlying asset. Solutions to the Heston model in the context of option pricing, using a path integral approach, are found in Lemmens et al. (2008) [21] while in Baaquie (2007,1997) [12,13] propagators for different stochastic volatility models are constructed. In all previous cases, the propagator is not defined for extreme cases $\rho = \pm 1$. It is therefore necessary to obtain a solution for these extreme cases and also to understand the origin of the divergence of the propagator. In this paper we study in detail a general class of stochastic volatility models for extreme values $\rho = \pm 1$ and show that in these two cases, the associated classical dynamics corresponds to a system with second class constraints, which must be dealt with using Dirac's method for constrained systems (Dirac, 1958,1967) [22,23] in order to properly obtain the propagator in the form of a Euclidean Hamiltonian path integral (Henneaux and Teitelboim, 1992) [25]. After integrating over momenta, one gets an Euclidean Lagrangian path integral without constraints, which in the case of the Heston model corresponds to a path integral of a repulsive radial harmonic oscillator. In all the cases studied, the price of the underlying asset is completely determined by one of the second class constraints in terms of volatility and plays no active role in the path integral.

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http://dx.doi.org/10.1016/j.physa.2013.08.057

1. Introduction

The Black–Scholes model [1,2], one of the cornerstones of current financial theory, assumes by default that market volatility is constant. But the analysis of the actual economically-financial data implies, as a matter of fact, that volatility varies en time [2]. As Fisher Black himself remarked:

"Suppose we use the standard deviation of possible future returns on a stock as a measure of its volatility. Is it reasonable to take that volatility as a constant over time? I think not".

To address this problem in the context of the Black–Scholes standard model, the concept of smile has been developed [2–5]. In this approach, the volatility smile as a function of the underlying asset price is determined from the empirical data. Other more sophisticated models that try to capture the variation in volatility are stochastic volatility models [2,6]. Here it is assumed that the market volatility behaves as a random variable determined by a Brownian motion. This movement is different from a second Brownian motion that dictates the dynamics of the underlying asset in the Black–Scholes model. The correlation between both movements is parameterized by the correlation coefficient ρ. This fact, together with the hypothesis of no arbitrage and a self financing portfolio, imply that the option price satisfies a partial differential equation in two spatial dimensions (corresponding to the option price and the value of the volatility) and one time variable. The equation has a potential term similar to that of a quantum particle in the presence of an external electromagnetic field. Different stochastic models can be constructed depending on the shape of the potential term. The best known ones are the Heston model, the Hull and White model and the model of Ornstein–Uhlenbeck [2,7,8]. Other variations of these stochastic volatility models include incorporating a "jump diffusion" term, which gives rise to integro-differential equations [9] for the price of the option.

Furthermore, in recent years path integrals techniques have been increasingly applied to obtain solutions of the Black–Scholes equation [10–16] while numerical techniques are presented in Refs. [17–20]. In the context of stochastic volatility models, in Ref. [21] the propagator has been calculated for the Heston model via path integrals, obtaining closed form solutions for the price of the option. This result depends on the value of the correlation coefficient ρ and the proposed solution is indeterminate for extreme cases where $\rho = \pm 1$. The same behaviour appears in Refs. [12,13] where propagators for different stochastic volatility models are constructed. So, it is then interesting to study in detail what happens to the propagator when the correlation coefficient takes its extreme values $\rho = \pm 1$.

This article shows that, when looking at stochastic volatility models as Euclidean quantum mechanical systems, the classical mechanics underlying the bidimensional Schrödinger equation at $\rho = \pm 1$, is a system with constraints, reminiscent of an optimal control problem. Due to the presence of links, it is necessary to resort to the Dirac method [22–24] for the correct description of this constrained system, both in the classical and the quantum levels. Applying Dirac method shows that the links are second class and the propagator must be calculated in terms of a Hamiltonian path integral with second class constraints [25].

2. Stochastic volatility models

Stochastic volatility models are used to evaluate options prices and generalizes the Black–Scholes model to the non constant volatility case. Between the different plethora of models, the best know are the Heston model [7], the CEV model [26,27], the SBR volatility model [28], the GARCH model [29], the 3/2 model, the Hull and White model [8] and the Chen model [30]. Some stochastic volatility models are even capable to capture some important statistical properties of real markets, called stylized facts, such as the autocorrelation and leverage effect [31–34].

To account this stylized facts for the real financial data, empirical analysis implies that $|\rho|$ must be of the order of 0.5 [31,32]. Although, from a statistical point of view, the extreme case $\rho = \pm 1$ is not realized in the real world, these values can be satisfied for "outliers" events in the sense of Ref. [35], and from a structural perspective, is necessary understand the behaviour of the stochastic volatility models for the full range of its parameters values.

In order make contact with results for $\rho \neq \pm 1$ studied in literature [21,12,13], we consider first a wide class of stochastic models that are related to the Heston model, but our methods and analysis can be applied to an arbitrary stochastic volatility model. So, we consider a class of stochastic volatility models that are characterized by the following stochastic differential equations associated [3,2] to the underlying price $S(t)$ and the volatility $v(t)$ respectively

$$dS = \mu(S, t)dt + \sqrt{v}SdW_1 \tag{1}$$

$$dv = \alpha(S, v, t)dt + \sigma\beta(S, v, t)\sqrt{v}dW_2 \tag{2}$$

where it is assumed that the Brownian motions dW_1 and dW_2 have a correlation factor given by

$$\langle dW_1 \; dW_2 \rangle = \rho. \tag{3}$$

These equations, together with the assumption of no-arbitrage and using a self financing portfolio constructed from the underlying asset and options, imply [3,2] the following equation for the option price $U(S, v, t)$

$$\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v\beta S\frac{\partial^2 U}{\partial S\partial v} + \frac{1}{2}\sigma^2\beta^2 v\frac{\partial^2 U}{\partial v^2} + r\left(S\frac{\partial U}{\partial S} - U\right) + (\alpha - \phi\beta\sqrt{v})\frac{\partial U}{\partial v} = 0 \tag{4}$$

where the function $\phi = \phi(S, v, t)$ is called the "market price of risk". As mentioned in Ref. [3] one can use a risk neutral drift α in (1) and all dependence in ϕ disappears in (4). In fact, in all examples presented at the end of this paper it is assumed that $\phi = 0$, but the analysis is performed in general and can be accommodate a non trivial ϕ function.

Under the change of variables

$$x = \ln S. \tag{5}$$

Eq. (4) may be written in the (x, v, t) space as

$$\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial x^2} + \rho\sigma v\beta\frac{\partial^2 U}{\partial x\partial v} + \frac{1}{2}\sigma^2\beta^2 v\frac{\partial^2 U}{\partial v^2} + \left(r - \frac{1}{2}v\right)\frac{\partial U}{\partial x} - rU + (\alpha - \phi\beta\sqrt{v})\frac{\partial U}{\partial v} = 0. \tag{6}$$

The Heston model corresponds to the special case where $\alpha(x, v, t) = -\lambda_0(v - v_0),\ \phi = 0,\ \beta = 1$ where λ_0 is a constant. The propagator of Eq. (6) for the Heston model has been obtained in Ref. [21] and it is well defined for $-1 < \rho < 1$. For $\rho = \pm 1$ the propagator diverges, so it is interesting to explore what happens in these limiting cases. The same is true in the results presented in Refs. [12,13], where the propagator for a class of stochastic volatility is analysed. In fact, for the case where $\rho \to \pm 1$ the propagator integrand of both models tend to a Dirac delta function, which means that the propagator tends to 1 in this limit.

In this article we study the propagator of Eq. (6) for the case where the correlation coefficient ρ takes its extreme values $\rho = \pm 1$ for a wide stochastic volatility model class, that is, for arbitrary functions $\alpha(x, v, t)$, $\phi(x, v, t)$ and $\beta(x, v, t)$. To this end, we first define the wave function Ψ by

$$\Psi(x, v, t) = U(x, v, t)e^{r(T-t)} \tag{7}$$

in such a way that the Ψ dynamics is given by

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2}v\frac{\partial^2 \Psi}{\partial x^2} + \rho\sigma v\beta\frac{\partial^2 \Psi}{\partial x\partial v} + \frac{1}{2}\sigma^2\beta^2 v\frac{\partial^2 \Psi}{\partial v^2} + \left(r - \frac{1}{2}v\right)\frac{\partial \Psi}{\partial x} + (\alpha - \phi\beta\sqrt{v})\frac{\partial \Psi}{\partial v} = 0. \tag{8}$$

Note that Eq. (8) can be interpreted as a Schrödinger type equation in Euclidean time

$$\hat{H}\Psi(x, v, t) = \frac{\partial \Psi(x, v, t)}{\partial t} \tag{9}$$

where the Hamiltonian operator is given by

$$\hat{H} = -\frac{1}{2}v\frac{\partial^2}{\partial x^2} - \rho\sigma v\beta\frac{\partial^2}{\partial x\partial v} - \frac{1}{2}\sigma^2\beta^2 v\frac{\partial^2}{\partial v^2} - \left(r - \frac{1}{2}v\right)\frac{\partial}{\partial x} - (\alpha - \phi\beta\sqrt{v})\frac{\partial}{\partial v} \tag{10}$$

so one can write the propagator for Eq. (9) in terms of path integrals in a way which is analogous to the quantum mechanical description. We will work directly in Euclidean time, so we need to make some considerations about the Euclidean formalism.

3. Euclidean framework

In this section the analysis for the path integral and classical dynamics associated to the Eq. (9) is done in the Euclidean time.

3.1. Euclidean path integral

Consider the Euclidean Schrödinger type equation

$$\hat{H}|\Psi(t)\rangle = \frac{\partial |\Psi(t)\rangle}{\partial t} \tag{11}$$

where $\hat{H}$ is a time independent Hamiltonian. The solution of this equation, for a final condition $|\Psi(T)\rangle = |\Phi_0\rangle$ is (see Ref. [12] for details)

$$|\Psi(t)\rangle = e^{-\hat{H}(T-t)}|\Phi_0\rangle. \tag{12}$$

The wave function can be written as

$$\Psi(x, t) = \int K(x, x', T - t)\Phi_0(x')dx' \tag{13}$$

where the propagator

$$Z = K(x, x', \tau) = \langle x|e^{-\hat{H}\tau}|x'\rangle \tag{14}$$

and $\tau = T - t$ is the forward time. By dividing the time τ in N time steps of length $\epsilon = \tau/N$ the propagator can be written as

$$Z = \langle x|e^{-\hat{H}\epsilon}e^{-\hat{H}\epsilon}\dots e^{-\hat{H}\epsilon}|x'\rangle \tag{15}$$

by inserting the identity operators

$$\hat{I} = \int |x\rangle\langle x|dx \tag{16}$$

$$\hat{I} = \int |p_x\rangle\langle p_x|\frac{dp_x}{2\pi} \tag{17}$$

the propagator is

$$Z = \prod_{i=1}^{N-1}\int dx_i \prod_{j=1}^{N}\int \frac{dp_j}{2\pi}\ \langle x|e^{-\hat{H}\epsilon}|p_N\rangle\langle p_N|x_{N-1}\rangle\dots\langle x_2|e^{-\hat{H}\epsilon}|p_2\rangle\langle p_2|x_1\rangle\langle x_1|e^{-\hat{H}\epsilon}|p_1\rangle\langle p_1|x'\rangle \tag{18}$$

but using the fact that

$$\langle p|x\rangle = e^{-ipx} \tag{19}$$

$$\langle x|p\rangle = e^{ipx} \tag{20}$$

$$\langle x|e^{-\hat{H}\epsilon}|p\rangle = e^{-H(x,p)\epsilon}\langle x|p\rangle \tag{21}$$

where $H(x, p)$ is the classical Hamiltonian associated to the Hamiltonian operator $\hat{H}$ through the relation

$$\langle x|\hat{H}|p\rangle = H(x, p)\langle x|p\rangle \tag{22}$$

so the propagator is

$$Z = \prod_{i=1}^{N-1}\int dx_i \prod_{j=1}^{N}\int \frac{dp_j}{2\pi}\ e^{-H(x_N,p_N)\epsilon+ip_N(x_N-x_{N-1})}\dots e^{-H(x_2,p_2)\epsilon+ip_2(x_2-x_1)}\ e^{-H(x_1,p_1)\epsilon+ip_1(x_1-x_0)} \tag{23}$$

where $x_N = x$ and $x_0 = x'$. So finally

$$Z = \prod_{i=1}^{N-1}\int dx_i \prod_{j=1}^{N}\int \frac{dp_j}{2\pi}\ e^{\sum_{k=1}^{N}\left[ip_k\frac{(x_k-x_{k-1})}{\epsilon}-H(x_k,p_k)\right]\epsilon} \tag{24}$$

or in the limit $\epsilon \to 0$

$$Z = \int \mathcal{D}x \int \mathcal{D}p_x\ e^{S[x,p_x]} \tag{25}$$

where

$$S[x, p_x] = \int (ip_x\dot{x} - H(x, p_x))d\tau \tag{26}$$

is the Euclidean classical action and $\dot{x}$ means time derivative respect to τ. For a time dependent Hamiltonian, Eq. (25) is still valid, for details see for example [36].

3.2. Euclidean classical Hamiltonian equations

The classical Euclidean Hamiltonian equations can be deduced by imposing that the classical trajectories optimize the classical action:

$$\delta S = S[x + \delta x, p_x + \delta p_x] - S[x, p_x] = 0 \tag{27}$$

but

$$S[x + \delta x, p_x + \delta p_x] = \int [i(p_x + \delta p_x)(\dot{x} + \delta\dot{x}) - H(x + \delta x, p_x + \delta p_x)]d\tau. \tag{28}$$

By expanding the Hamiltonian and assuming that the extremes of trajectories are fixed, one gets

$$S[x + \delta x, p_x + \delta p_x] = S[x, p_x] + \int \left(i\dot{x} - \frac{\partial H(x, p_x)}{\partial p_x}\right)\delta p_x d\tau + \int \left(-i\dot{p}_x - \frac{\partial H(x, p_x)}{\partial x}\right)\delta x d\tau + \cdots \tag{29}$$

hence

$$\delta S = \int \left(i\dot{x} - \frac{\partial H(x, p_x)}{\partial p_x} \right) \delta p_x \mathrm{d}\tau + \int \left(-i\dot{p}_x - \frac{\partial H(x, p_x)}{\partial x} \right) \delta x \mathrm{d}\tau + \cdots \tag{30}$$

so the classical Hamiltonian equations in Euclidean time reads

$$i\dot{x} = \frac{\partial H(x, p_x)}{\partial p_x}, \tag{31}$$

$$i\dot{p}_x = -\frac{\partial H(x, p_x)}{\partial x}. \tag{32}$$

In n dimensions, the Poisson Brackets may be defined by

$$\{A(x, p), B(x, p)\} \equiv \sum_i \left(\frac{\partial A(x, p)}{\partial x^i} \frac{\partial B(x, p)}{\partial p_i} - \frac{\partial B(x, p)}{\partial x^i} \frac{\partial A(x, p)}{\partial p_i} \right), \tag{33}$$

to rewrite Hamilton's equations in the form

$$i\dot{x}^j = \{x^j, H\}, \tag{34}$$

$$i\dot{p}_k = \{p_k, H\}. \tag{35}$$

4. Euclidean classical Hamiltonian dynamics for stochastic volatility models

The Hamiltonian operator, acting on a wavefunction Ψ, for a generic stochastic volatility model is

$$\hat{H}\Psi = -\frac{1}{2} v \frac{\partial^2 \Psi}{\partial x^2} - \rho\sigma v \beta \frac{\partial^2 \Psi}{\partial x \partial v} - \frac{1}{2}\sigma^2 \beta^2 v \frac{\partial^2 \Psi}{\partial v^2} - \left(r - \frac{1}{2} v \right) \frac{\partial \Psi}{\partial x} - (\alpha - \phi\beta\sqrt{v}) \frac{\partial \Psi}{\partial v} \tag{36}$$

and in this case

$$\langle p_x\, p_v | x\, v \rangle = \mathrm{e}^{-ip_x x - ip_v v} \tag{37}$$

$$\langle x\, v | p_x\, p_v \rangle = \mathrm{e}^{ip_x x + ip_v v} \tag{38}$$

$$\langle x\, v | \hat{H} | p_x\, p_v \rangle = H(x, v, p_x, p_v)\mathrm{e}^{ip_x x + ip_v v} \tag{39}$$

where $H(x, v, p_x, p_v)$ is the classical Hamiltonian

$$H(x, v, p_x, p_v) = \frac{1}{2} v p_x^2 + \rho\sigma v \beta p_x p_v + \frac{1}{2}\sigma^2\beta^2 v p_v^2 - A_x(v) p_x - A_v(x, v, \tau) p_v \tag{40}$$

where the functions A_x and A_v are defined by

$$A_x(v) = i\bar{A}_x(v) = i\left(r - \frac{1}{2} v \right) \tag{41}$$

$$A_v(x, v, \tau) = i\bar{A}_v(x, v, \tau) = i(\alpha(x, v, \tau) - \phi(x, v, \tau)\beta(x, v, \tau)\sqrt{v}) \tag{42}$$

and are similar to a vector electromagnetic potentials in two spatial dimensions. Here and in the rest of the paper, $A_x(v)$ and $A_v(x, v, \tau)$ will denote imaginary potentials whereas $\bar{A}_x(v)$ and $\bar{A}_v(x, v, \tau)$ will denote real ones. The classical Euclidean Hamiltonian equations in this case are

$$i\dot{x} = \frac{\partial H}{\partial p_x} \quad i\dot{p}_x = -\frac{\partial H}{\partial x} \tag{43}$$

$$i\dot{v} = \frac{\partial H}{\partial p_v} \quad i\dot{p}_v = -\frac{\partial H}{\partial v}. \tag{44}$$

For the classical Hamiltonian (40), the Euclidean Hamiltonian equations read

$$i\dot{x} = v p_x + \rho\sigma v \beta p_v - A_x \tag{45}$$

$$i\dot{v} = \rho\sigma\beta v p_x + \sigma^2 v \beta^2 p_v - A_v \tag{46}$$

$$i\dot{p}_x = \frac{\partial A_v}{\partial x} p_v \tag{47}$$

$$i\dot{p}_v = -\frac{1}{2} p_x^2 - \rho\sigma p_x p_v - \frac{1}{2}\sigma^2 p_v^2 + \frac{\partial A_x}{\partial v} p_x + \frac{\partial A_v}{\partial v} p_v. \tag{48}$$

The first two Hamiltonian equations may be written as

$$\begin{pmatrix} i\dot{x}+A_x \\ i\dot{v}+A_v \end{pmatrix} = \mathbf{M}\begin{pmatrix} p_x \\ p_v \end{pmatrix} = \begin{pmatrix} v & \rho\sigma v\beta \\ \rho\sigma v\beta & \sigma^2 v\beta^2 \end{pmatrix}\begin{pmatrix} p_x \\ p_v \end{pmatrix}$$

note that determinant of the matrix **M** is

$$\det(\mathbf{M}) = \sigma^2 v^2 \beta^2 (1-\rho^2) \tag{49}$$

which vanishes for $\rho = \pm 1$, which means that momenta cannot be expressed in terms of velocities, which is similar to a well known fact for singular Lagrangian systems. Note that for $\rho = \pm 1$ the classical Hamiltonian reduces to

$$H^{\pm} = \frac{1}{2} v (p_x \pm \sigma\beta p_v)^2 - A_x p_x - A_v p_v. \tag{50}$$

5. Euclidean classical Lagrangian singular dynamics

For $\rho = \pm 1$ the velocities may be expressed in terms of momenta by

$$\begin{pmatrix} i\dot{x}+A_x \\ i\dot{v}+A_v \end{pmatrix} = \begin{pmatrix} v & \pm\sigma v\beta \\ \pm\sigma v\beta & \sigma^2 v\beta^2 \end{pmatrix}\begin{pmatrix} p_x \\ p_v \end{pmatrix}$$

so that velocities are constrained by

$$(i\dot{x}+A_x) \mp \frac{(i\dot{v}+A_v)}{\sigma\beta} = 0, \tag{51}$$

which is the same as

$$(\dot{x}+\bar{A}_x) \mp \frac{(\dot{v}+\bar{A}_v)}{\sigma\beta} = 0. \tag{52}$$

Define the Euclidean Lagrangian by

$$L^{\pm} = i\dot{x}p_x + i\dot{v}p_v - H^{\pm} \tag{53}$$

one get

$$L^{\pm} = \frac{(i\dot{x}+A_x)^2}{2v} = -\frac{(\dot{x}+\bar{A}_x)^2}{2v} \tag{54}$$

subject to the real constraint (52). In this way, the Lagrangian theory associated to the singular Hamiltonian $H^{\pm}$ is equivalent to an optimal control problem [24], where the state variables (x, v) are subject to a Lanchester equation given by constraint (52).

One can now apply Dirac's method [22,23] to construct the Hamiltonian formalism associated to the Lagrangian theory. One must consider the Lagrangian with constraints

$$\tilde{L}^{\pm} = \frac{(i\dot{x}+A_x)^2}{2v} + \lambda\left[(i\dot{x}+A_x) \mp \frac{(i\dot{v}+A_v)}{\sigma\beta}\right] \tag{55}$$

and the definition of the canonical momenta in Euclidean time are

$$ip_x = \frac{\partial\tilde{L}^{\pm}}{\partial\dot{x}} = i\frac{(i\dot{x}+A_x)}{v} + i\lambda \tag{56}$$

$$ip_v = \frac{\partial\tilde{L}^{\pm}}{\partial\dot{v}} = \mp i\frac{\lambda}{\sigma\beta} \tag{57}$$

$$ip_\lambda = \frac{\partial\tilde{L}^{\pm}}{\partial\dot{\lambda}} = 0. \tag{58}$$

The above equations imply the existence of two constraints for the Euclidean canonical momenta

$$\Phi_1^{\pm} = p_v \pm \frac{\lambda}{\sigma\beta} \simeq 0 \tag{59}$$

$$\Phi_2 = p_\lambda \simeq 0. \tag{60}$$

The Dirac's extended Hamiltonian is given by

$$H^{\pm}_{ext} = ip_x\dot{x} + ip_v\dot{v} + ip_\lambda\dot{\lambda} - \tilde{L}^{\pm} + \mu_1\Phi_1^{\pm} + \mu_2\Phi_2 \tag{61}$$

$$H^{\pm}_{ext} = ip_x\dot{x} + ip_v\dot{v} + ip_\lambda\dot{\lambda} - \frac{(i\dot{x}+A_x)^2}{2v} - \lambda\left[(i\dot{x}+A_x) \mp \frac{(i\dot{v}+A_v)}{\sigma\beta}\right] + \mu_1\left[p_v \pm \frac{\lambda}{\sigma\beta}\right] + \mu_2[p_\lambda] \tag{62}$$

which written in terms of the momenta turns out to be

$$H^{\pm}_{ext} = \frac{1}{2}v(p_x \pm \sigma\beta p_v)^2 - A_x p_x - A_v p_v + \mu_1\left[p_v \pm \frac{\lambda}{\sigma\beta}\right] + \mu_2[p_\lambda]. \tag{63}$$

The (time evolution) consistency of the constraints imply

$$i\dot{\Phi}_1^{\pm} = \{\Phi_1^{\pm}, H^{\pm}_{ext}\} = 0 \tag{64}$$

$$i\dot{\Phi}_2 = \{\Phi_2, H^{\pm}_{ext}\} = 0 \tag{65}$$

which yield

$$-\frac{\partial H^{\pm}_{ext}}{\partial v} \pm \left[\frac{\lambda}{\sigma}\frac{\partial}{\partial x}\left(\frac{1}{\beta}\right)(v(p_x \pm \sigma\beta p_v) - A_x) + \frac{\lambda}{\sigma}\frac{\partial}{\partial v}\left(\frac{1}{\beta}\right)(v(p_x \pm \sigma\beta p_v)(\pm\sigma\beta) - A_v + \mu_1) + \frac{1}{\sigma\beta}\mu_2\right] = 0 \tag{66}$$

$$\mp \frac{1}{\sigma\beta}\mu_1 = 0. \tag{67}$$

If $\sigma\beta \neq 0$ then the Lagrange multipliers μ_1, μ_2 may be solved for using the last two equations and the consistency equations generate no new constraints. There are only two primary constraints. The Poisson bracket for the two primary constraints Φ_1 and Φ_2 turns out to be

$$\{\Phi_1^{\pm}, \Phi_2\} = \pm\frac{1}{\sigma\beta} \tag{68}$$

so that the Dirac matrix is invertible when $\sigma\beta \neq 0$, a condition that is necessary to ensure the existence of propagation, as discussed later. Then the two constraints $\Phi_1^{\pm}$ and Φ_2 are second class and can be eliminated by using Dirac brackets. So, one can set $\Phi_1^{\pm}$ and Φ_2 strongly equal to zero, that is,

$$\Phi_1^{\pm} = p_v \pm \frac{\lambda}{\sigma\beta} = 0 \tag{69}$$

$$\Phi_2 = p_\lambda = 0 \tag{70}$$

so the extended Hamiltonian can be written as

$$H^{\pm}_{ext} = H^{\pm} \tag{71}$$

which coincides with the original expression (50). Thus, the Euclidean Lagrangian and Hamiltonian theories are equivalent.

6. Constrained Euclidean path integral

The Euclidean path integral for systems with second class constraints is given in phase space by Ref. [25]

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v\mathcal{D}\lambda\frac{\mathcal{D}p_x}{2\pi}\frac{\mathcal{D}p_v}{2\pi}\frac{\mathcal{D}p_\lambda}{2\pi}\sqrt{\det(\mathbf{C})}\delta(\Phi_1^{\pm})\delta(\Phi_2)\exp(S^{\pm}[x, v, \lambda, p_x, p_v, p_\lambda]) \tag{72}$$

where $\mathbf{C}$ is the Dirac's matrix constructed out of second class constraints according to

$$\mathbf{C} = (C_{ij}) = \{\Phi_i, \Phi_j\} \tag{73}$$

and

$$S^{\pm}[x, v, \lambda, p_x, p_v, p_\lambda] = \int [i\dot{x}p_x + i\dot{v}p_v + i\dot{\lambda}p_\lambda - H^{\pm}(x, v, \lambda, p_x, p_v, p_\lambda)]\mathrm{d}\tau \tag{74}$$

is the Hamiltonian action. In our case, Dirac's matrix is

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} \{\Phi_1^{\pm}, \Phi_1^{\pm}\} & \{\Phi_1^{\pm}, \Phi_2\} \\ \{\Phi_2, \Phi_1^{\pm}\} & \{\Phi_2, \Phi_2\} \end{pmatrix} = \begin{pmatrix} 0 & \pm\frac{1}{\sigma\beta} \\ \mp\frac{1}{\sigma\beta} & 0 \end{pmatrix}$$

therefore,

$$\det(\mathbf{C}) = \frac{1}{\sigma^2\beta^2} \tag{75}$$

so

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v\mathcal{D}\lambda \frac{\mathcal{D}p_x}{2\pi}\frac{\mathcal{D}p_v}{2\pi}\frac{\mathcal{D}p_\lambda}{2\pi}\frac{1}{\sigma\beta}\delta\left(p_v \pm \frac{\lambda}{\sigma\beta}\right)\delta(p_\lambda)\exp(S^{\pm}[x, v, \lambda, p_x, p_v, p_\lambda]) \tag{76}$$

where

$$S^{\pm}[x, v, \lambda, p_x, p_v, p_\lambda] = \int \left[ip_x\dot{x} + ip_v\dot{v} + ip_\lambda\dot{\lambda} - \frac{1}{2}v(p_x \pm \sigma\beta p_v)^2 + A_x p_x + A_v p_v\right]\mathrm{d}\tau. \tag{77}$$

Integration on p_λ yields

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v\mathcal{D}\lambda \frac{\mathcal{D}p_x}{2\pi}\frac{\mathcal{D}p_v}{2\pi}\frac{1}{2\pi}\frac{1}{\sigma\beta}\delta\left(p_v \pm \frac{\lambda}{\sigma\beta}\right) \times \exp\left(\int\left[ip_x\dot{x} + ip_v\dot{v} - \frac{1}{2}v(p_x \mp \sigma\beta p_v)^2 + A_x p_x + A_v p_v\right]\mathrm{d}\tau\right) \tag{78}$$

whereas integration on p_v gives rise to

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v\mathcal{D}\lambda \frac{\mathcal{D}p_x}{2\pi}\frac{1}{(2\pi)^2}\frac{1}{\sigma\beta}\exp\left(\int\left[ip_x\dot{x} \mp \frac{\lambda}{\sigma\beta}i\dot{v} - \frac{1}{2}v(p_x - \lambda)^2 + A_x p_x \mp \frac{\lambda A_v}{\sigma\beta}\right]\mathrm{d}\tau\right). \tag{79}$$

The integrand in the exponential of the action may be rewritten as

$$-\frac{v}{2}\left[p_x + \left(\lambda + \frac{(i\dot{x} + A_x)}{v}\right)\right]^2 + \frac{(i\dot{x} + A_x)^2}{2v} + \lambda\left[(i\dot{x} + A_x) \mp \frac{(i\dot{v} + A_v)}{\sigma\beta}\right] \tag{80}$$

so that integrating on p_x one gets

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v\mathcal{D}\lambda \frac{1}{(2\pi)^3}\frac{1}{\sigma\beta}\sqrt{\frac{2\pi}{v\mathrm{d}\tau}}\exp\left(\int\left[\frac{(i\dot{x} + A_x)^2}{2v}\right] + \lambda\left[(i\dot{x} + A_x) \mp \frac{(i\dot{v} + A_v)}{\sigma\beta}\right]\mathrm{d}\tau\right). \tag{81}$$

If instead of the imaginary potential $A = i\bar{A}$ one uses the real $\bar{A}$, the propagator reads

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v\mathcal{D}\lambda \frac{1}{(2\pi)^3}\frac{1}{\sigma\beta}\sqrt{\frac{2\pi}{v\mathrm{d}\tau}}\exp\left(\int\left[-\frac{(\dot{x} + \bar{A}_x)^2}{2v}\right] + i\lambda\left[(\dot{x} + \bar{A}_x) \mp \frac{(\dot{v} + \bar{A}_v)}{\sigma\beta}\right]\mathrm{d}\tau\right). \tag{82}$$

Integration on λ yields

$$Z^{\pm} = \int \mathcal{D}x\mathcal{D}v \frac{1}{(2\pi)^3}\frac{1}{\sigma\beta}\sqrt{\frac{2\pi}{v\mathrm{d}\tau}}\delta\left[(\dot{x} + \bar{A}_x) \mp \frac{(\dot{v} + \bar{A}_v)}{\sigma\beta}\right]\exp\left(\int -\frac{(\dot{x} + \bar{A}_x)^2}{2v}\mathrm{d}\tau\right). \tag{83}$$

Finally integrating on x yields

$$Z^{\pm} = \int \mathcal{D}v \frac{1}{(2\pi)^3}\frac{1}{\sigma\beta}\sqrt{\frac{2\pi}{v\mathrm{d}\tau}}\ \exp\left(-\int\frac{(\dot{v} + \bar{A}_v)^2}{2v\sigma^2\beta^2}\mathrm{d}\tau\right). \tag{84}$$

In this last expression, functions β and $\bar{A}_v$ must be evaluated in $x = x(\tau) = x_v(\tau)$, where $x_v(\tau)$ is the solution to the inhomogeneous equation

$$\dot{x} = -\bar{A}_x(v) \pm \frac{\dot{v} + \bar{A}_v(x, v, \tau)}{\sigma\beta(x, v, \tau)} \tag{85}$$

that is, in (84), $\bar{A_v}$ and β are given by

$$\bar{A}_v = \bar{A}_v(x_v(\tau), v, \tau) \tag{86}$$

$$\beta = \beta(x_v(\tau), v, \tau). \tag{87}$$

The change of variables

$$u(\tau) = 2\sqrt{v(\tau)} \tag{88}$$

in Eq. (84), leads us to the propagator written in terms of the variable u according to

$$Z^{\pm} = \frac{1}{(2\pi)^2} \int \mathcal{D}u \sqrt{\frac{1}{2\pi \mathrm{d}\tau} \left(\frac{1}{\sigma \beta(u,\tau)}\right)^2} \exp\left(-\int \frac{1}{2}\left(\frac{1}{\sigma\beta(u,\tau)}\right)^2 \left[\dot{u} + \frac{2\bar{A}_u(u,\tau)}{u}\right]^2 \mathrm{d}\tau\right) \tag{89}$$

where

$$\bar{A}_u(u,\tau) = \alpha\left(x_{\frac{u^2}{4}}(\tau), \frac{u^2}{4}, \tau\right) - \phi\left(x_{\frac{u^2}{4}}(\tau), \frac{u^2}{4}, \tau\right) \beta\left(x_{\frac{u^2}{4}}(\tau), \frac{u^2}{4}, \tau\right) \frac{u}{2} \tag{90}$$

and

$$\beta(u,\tau) = \beta\left(x_{\frac{u^2}{4}}(\tau), \frac{u^2}{4}, \tau\right). \tag{91}$$

When β does not depend explicitly on time ($\beta = \beta(u)$), one can change variables by

$$\theta = F(u) = \int \frac{1}{\sigma\beta(u)} \mathrm{d}u \tag{92}$$

so that

$$\dot{\theta} = \frac{\dot{u}}{\sigma\beta(u)} \tag{93}$$

and the propagator expressed in terms of the new variables θ reads

$$Z^{\pm} = \frac{1}{(2\pi)^2} \int \mathcal{D}\theta \sqrt{\frac{1}{2\pi \mathrm{d}t}} \exp\left(-\int \frac{1}{2}[\dot{\theta} + V(\theta)]^2 \mathrm{d}t\right) \tag{94}$$

where $V(\theta)$ may be obtained from

$$V(\theta) = \frac{2\bar{A}_u(u)}{\sigma\beta(u)u} \tag{95}$$

evaluated at $u = u(\theta) = F^{-1}(\theta)$ where $F^{-1}(\theta)$ is defined by expression (92). Specific examples of applications of the latter form of the propagator are presented in the next section.

7. Some special cases

In this section we discuss special cases such as

7.1. Free particle

Consider the stochastic equation for volatility given by

$$\mathrm{d}v = \sigma\sqrt{v}\mathrm{d}W_2 \tag{96}$$

in such a way that $\alpha(x, v, t) = 0$, $\beta(x, v, t) = 1$ and take $\phi(x, v, t) = 0$. The vector potential $A_v = 0$, which implies that $A_u = 0$ and from Eq. (95) one gets $V(\theta) = 0$. Here $\beta(x, u, t) = 1$ and therefore the propagator, in this case, is

$$Z = \frac{1}{(2\pi)^2} \int \mathcal{D}\theta \sqrt{\frac{1}{2\pi \mathrm{d}t}} \exp\left(-\int \frac{1}{2}\dot{\theta}^2 \mathrm{d}t\right) \tag{97}$$

which is the unit mass free particle propagator in quantum mechanics or the diffusion equation propagator

$$\frac{\partial\psi(u,t)}{\partial t} = D\frac{\partial^2\psi(u,t)}{\partial u^2} \tag{98}$$

with diffusion constant $D = \frac{1}{2}$.

7.2. Heston model

For the Heston model the stochastic equation for volatility is

$$dv = -\lambda_0(v - v_0)dt + \sigma\sqrt{v}dW_2 \tag{99}$$

take $\alpha(x, v, t) = -\lambda_0(v - v_0)$, $\beta(x, v, t) = 1$ and $\phi(x, v, t) = 0$, with constant λ_0. The vector potential A_v is a function linear in the volatility

$$A_v(x, v, t) = -\lambda_0(v - v_0) \tag{100}$$

or, written in terms of $A_u(u, t)$

$$A_u(u, t) = -\lambda_0\left(\frac{u^2}{4} - v_0\right) \tag{101}$$

with

$$\beta(x, u, t) = 1. \tag{102}$$

The change of variables (92) is, in this case,

$$\theta = \int \frac{du}{\sigma} = \frac{u}{\sigma} \tag{103}$$

and its inverse relation

$$u = \sigma\theta. \tag{104}$$

The potential $V(\theta)$ is, in this case,

$$V(\theta) = \left.\frac{-2\lambda_0(u^2/4 - v_0)}{\sigma u}\right|_{u=\sigma\theta} = -\frac{\lambda_0\theta}{2} + \frac{2\lambda_0 v_0}{\sigma^2\theta} \tag{105}$$

so that the propagator reads

$$Z = \frac{1}{(2\pi)^2}\int \mathcal{D}\theta\sqrt{\frac{1}{2\pi\,dt}}\,\exp\left(-\int \frac{1}{2}\left[\dot{\theta} - \frac{\lambda_0\theta}{2} + \frac{2\lambda_0 v_0}{\sigma^2\theta}\right]^2 dt\right) \tag{106}$$

which, up to total derivatives, is equivalent to

$$Z = \frac{1}{(2\pi)^2}\int \mathcal{D}\theta\sqrt{\frac{1}{2\pi\,dt}}\exp\left(-\int \left[\frac{1}{2}\dot{\theta}^2 + \frac{\lambda_0^2}{8}\theta^2 + \frac{2\lambda_0^2 v_0^2}{\sigma^4\theta^2}\right]dt\right) \tag{107}$$

which corresponds to the Euclidean path integral for a repulsive radial harmonic oscillator with angular momentum $l = \frac{2\lambda_0 v_0}{\sigma^2}$.

7.3. Hull & White model

For the Hull & White model, the stochastic equation for volatility is

$$dv = a(b - v)dt + \sigma v dW_2 \tag{108}$$

so take $\alpha(x, v, t) = a(b - v)$, $\beta(x, v, t) = \sqrt{v}$ and $\phi(x, v, t) = 0$. Therefore,

$$A_u(u, t) = a\left(b - \frac{u^2}{4}\right) \tag{109}$$

and

$$\beta(u, t) = \frac{u}{2}. \tag{110}$$

The change of variables (92) is

$$\theta = \int \frac{2du}{\sigma u} = \frac{1}{\sigma}\ln u^2 \tag{111}$$

or its inverse relation

$$u = e^{\frac{\sigma\theta}{2}}. \tag{112}$$

The potential $V(\theta)$ in this case, is

$$V(\theta) = \left(\frac{4ab}{\sigma u^2} - \frac{a}{\sigma} \right) \Big|_{u=e^{\frac{\sigma\theta}{2}}} = \frac{4ab}{\sigma} e^{-\sigma\theta} - \frac{a}{\sigma} \tag{113}$$

so the propagator reads

$$Z = \frac{1}{(2\pi)^2} \int \mathcal{D}\theta \sqrt{\frac{1}{2\pi \, dt}} \, \exp\left(- \int \frac{1}{2} \left[\dot{\theta} + \frac{4ab}{\sigma} e^{-\sigma\theta} - \frac{a}{\sigma} \right]^2 dt \right). \tag{114}$$

7.4. 3/2 model

For this model, the stochastic equation for volatility is

$$dv = (av - bv^2)dt + \sigma v^{3/2} dW_2 \tag{115}$$

take $\alpha(x, v, t) = (av - bv^2)$, $\beta(x, v, t) = v$ and $\phi = 0$, as always, then the vector potential is

$$A_u(u, t) = \left(a\frac{u^2}{4} - b\frac{u^4}{16} \right) \tag{116}$$

and

$$\beta(u, t) = \frac{u^2}{4}. \tag{117}$$

The change of variables (92) is

$$\theta = \int \frac{4du}{\sigma u^2} = \frac{-4}{\sigma u} \tag{118}$$

or its inverse relation

$$u = \frac{-4}{\sigma\theta}. \tag{119}$$

The potential $V(\theta)$ is

$$V(\theta) = \left(\frac{2a}{\sigma u} - \frac{2bu}{4\sigma} \right) \Big|_{u=\frac{-4}{\sigma\theta}} = -\frac{a\theta}{2} + \frac{2b}{\sigma^2\theta} \tag{120}$$

and the propagator turns out to be

$$Z = \frac{1}{(2\pi)^2} \int \mathcal{D}\theta \sqrt{\frac{1}{2\pi \, dt}} \, \exp\left(- \int \frac{1}{2} \left[\dot{\theta} - \frac{a\theta}{2} + \frac{2b}{\sigma^2\theta} \right]^2 dt \right) \tag{121}$$

which is identical to the Heston model propagator with $\lambda_0 = a$ and $\lambda_0 v_0 = b$.

7.5. Baaquie α model

In this model, the stochastic equation for volatility is [12]

$$dv = (\lambda_0 + \mu_0 v)dt + \sigma v^{\alpha} dW_2 \tag{122}$$

take $\alpha(x, v, t) = (\lambda_0 + \mu_0 v)$, $\beta(x, v, t) = v^{\alpha - 1/2}$ and $\phi = 0$. The vector potential is

$$A_u(u, t) = \left(\lambda_0 + \mu_0 \frac{u^2}{4} \right) \tag{123}$$

and

$$\beta(u, t) = \left(\frac{u}{2} \right)^{2\alpha - 1}. \tag{124}$$

The change of variables (92) is

$$\theta = \int \frac{2^{2\alpha-1} du}{\sigma u^{2\alpha-1}} = \frac{2^{2\alpha-1} u^{2-2\alpha}}{\sigma(2 - 2\alpha)} \tag{125}$$

or its inverse relation

$$u = 2[\sigma\theta(1-\alpha)]^{\frac{1}{2(1-\alpha)}}. \tag{126}$$

The potential $V(\theta)$ in this case is

$$V(\theta) = \lambda_0\, \sigma^{\frac{1}{\alpha-1}}\, [\theta(1-\alpha)]^{\frac{\alpha}{\alpha-1}} + \mu_0(1-\alpha)\theta \tag{127}$$

and the propagator reads

$$Z = \frac{1}{(2\pi)^2} \int \mathcal{D}\theta \sqrt{\frac{1}{2\pi\, \mathrm{d}t}}\ \exp\left(-\int \frac{1}{2}\, [\dot{\theta} + \lambda_0\, \sigma^{\frac{1}{\alpha-1}}\, [\theta(1-\alpha)]^{\frac{\alpha}{\alpha-1}} + \mu_0(1-\alpha)\theta]^2 \mathrm{d}t\right) \tag{128}$$

which for $\alpha = \frac{1}{2}$ is the same as the Heston model propagator with $\lambda_0 \to \lambda_0 v_0$ and $\mu_0 \to -\lambda_0$.

8. Conclusions and further research

Thus, stochastic volatility models with $\rho = \pm 1$ viewed from the standpoint of a physical system is a constrained system. The constraints happen to be second class according to Dirac's classification. For proper evaluation of the propagator of this singular case, one must resort to a constrained Hamiltonian path integral which resembles those used in gauge theories in particle physics. Once the momenta path integral are performed, one gets an unconstrained effective Lagrangian path integral. The effective Lagrangian, in the case of the Heston model, may be associated to a repulsive radial oscillator Lagrangian.

In all cases, the price of the underlying asset is completely determined by one of the second class constraints in terms of volatility and plays no active role in the path integral. Obviously, the propagator for these special values of the correlation parameter cannot be obtained as a limiting case from the propagator for $\rho \neq \pm 1$.

In forthcoming articles will show that the appearance of second class constraints for $\rho = \pm 1$ is a universal property of all stochastic volatility models given by the general form of the differential stochastic system (1), (2). We will explore whether the propagator for the models studied in this paper can be computed in a closed form or numerically, and will use the results to obtain the price of a call or put, for each case. It is also interesting to develop in the future a semi-classical approximation for these systems, to obtain approximate expressions for the propagators and for the option prices.

Acknowledgements

We would like to sincerely thank the referees for many useful comments and bibliographical suggestions which helped us to improve this article.

References

[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, J. Political Economy 8, 31 (1973) 637–654.
[2] Wilmott Paul, Paul Wilmott on Quantitative Finance, Wiley, 2000.
[3] Gatheral Jim, The Volatility Surface, Wiley, 2006.
[4] B. Dupire, Pricing with a Smile Risk Magazine, Incisive Media, 1994.
[5] B. Dupire, Pricing and hedging with smiles, in: M.A.H. Dempster, S.R. Pliska (Eds.), Mathematics of Derivative Securities, Cambridge University Press, 1997.
[6] Alireza Javaheri, Inside of Volatility Arbitrage, Wiley, 2005.
[7] S. Heston, A close form solution for option with stochastic volatility, with applications to bond and currency options, Rev. Financial Studies 6 (1993) 327–343.
[8] John Hull, Alan White, The pricing of options on assets with stochastic volatilities, J. Finance XLII NO. 2 (1987).
[9] Rama Cont, Peter Tankov, Financial Modelling with Jump Processes, Chapman and Hall, 2004.
[10] E. Bennati, M. Rosa-Clot, S. Taddei, Int. J. Theor. Appl. Finance 2 (1999) 381.
[11] Vadim Linetsky, The path integral approach to financial modelling and option pricing, Comput. Econ. v11 (1998) 129–163.
[12] Belal E. Baaquie, Quantum Finance: Path Integrals and Hamiltonians for Option and Interest Rates, Cambridge, 2007.
[13] Belal E. Baaquie, A path integral to option price with stochastic volatility: some exact results, J. Phys. I France 7 (12) (1997) 1733–1753.
[14] M. Contreras, R. Pellicer, M. Villena, A. Ruiz, A quantum model for option pricing: when Black–Scholes meets Schrodinger and its semi-classic limit, Physica A v329 (23) (2010) 5447–5459.
[15] Jan W. Dash, Quantitative Finance and Risk Management: A Physicist Approach, World Scientific, 2004.
[16] J.P.A. Devreese, D. Lemmens, J. Tempere, Path integral approach to Asian options in the Black–Scholes model, q-fin.PR (2011) arXiv:0906.4456v3.
[17] M. Rosa-Clot, S. Taddei, Int. J. Theor. Appl. Finance 5 (2002) 123.
[18] G. Montagna, O. Nicrosini, N. Moreni, Physica A 310 (2002) 450.
[19] B.E. Baaquie, C. Coriano, M. Srikant, Physica A 334 (2004) 531.
[20] G. Bormetti, et al., Quant. Finance 6 (2006) 55.
[21] D. Lemmens, M. Wouters, J. Tempere, A path integral approach to closed-form option pricing formulas with applications to stochastic volatility and interest rate models, q-fin.PR (2008) arXiv:0806.0932v1.
[22] Paul M. Dirac, Generalized Hamiltonian dynamics, Proc. Roy. Soc. (London) A246 (1958) 326.
[23] Paul M. Dirac, Lectures on Quantum Mechanics, Yeshiva University, 1967.
[24] Sergio A. Hojman, Optimal Control and Dirac's Theory of Singular Hamiltonian Systems, in preparation.
[25] Marc Henneaux, C. Teitelboim, Quantization of Gauge Systems, Princeton University Press, 1992.
[26] J. Cox, Notes on Option Pricing I: Constant Elasticity of Diffusions, Stanford University, 1975, Unpublished draft.

[27] D.C. Emanuel, J.D. MacBeth, Further results of the constant elasticity of variance call option pricing model, J. Financ. Quant. Anal. 4 (1984) 533–553.
[28] Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, Diana E. Woodward, Managing smile risk, Wimott magazine.
[29] Tim Bollerslev, Generalized autoregressive conditional heteroskedasticity, J. Econometrics 31 (1986) 307–327.
[30] Lin Chen, Stochastic mean and stochastic volatility: a three-factor model of the term structure of interest rates and its application to the pricing of interest rate derivatives, Financ. Mark. Instit. Instrum. 5 (1996) 1–88.
[31] D. Delpini, G. Bormetti, Phys. Rev. E 83 (2011).
[32] G. Bormetti, et al., J. Stat. Mech. (2008) P11013.
[33] J. Masoliver, J. Perello, Quant. Finance 6 (2006) 423.
[34] J. Perello, J. Masoliver, N. Anento, Physica A 344 (2004) 134.
[35] Didier Sornette, Why Stocks Markets Crash, Princeton University Press, 2003, Chapter 3.
[36] Hagen Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets, fourth ed., World Scientific, 2006.