Math. Z. (2014) 276:397–420
DOI 10.1007/s00209-013-1206-1

Mathematische Zeitschrift

Polarizations on abelian subvarieties of principally polarized abelian varieties with dihedral group actions

Herbert Lange · Rubí E. Rodríguez · Anita M. Rojas

Received: 27 September 2012 / Accepted: 1 July 2013 / Published online: 3 August 2013

Abstract For any $n \geq 2$ we study the group algebra decomposition of an $([\frac{n}{2}]+1)$-dimensional family of principally polarized abelian varieties of dimension n with an action of the dihedral group of order $2n$. For any odd prime p, $n = p$ and $n = 2p$ we compute the induced polarization on the isotypical components of these varieties and some other distinguished subvarieties. In the case of $n = p$ the family contains a one-dimensional family of Jacobians. We use this to compute a period matrix for Klein's icosahedral curve of genus 5.

Keywords Principally polarized abelian variety · Group algebra decomposition · Induced polarization

Mathematics Subject Classification (2010) 14K02 · 14K12 · 32G20

1 Introduction

In the nineteenth century the decomposition of an abelian variety was expressed in terms of reducible abelian integrals and their theta functions. However most authors, starting perhaps with Abel, were mainly looking (in our terminology) for elliptic factors of Jacobian varieties (see the last chapter of Krazer's book [7]). It was only relatively recently that further decom-

The second author was supported by Fondecyt Grant 1100767, the third author by Fondecyt Grant 1100113.

H. Lange
Mathematisches Institut, Universität Erlangen-Nürnberg, Erlangen, Germany
e-mail: lange@mi.uni-erlangen.de

R. E. Rodríguez
Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile
e-mail: rubi@mat.puc.cl

A. M. Rojas (✉)
Departamento de Matemáticas Facultad de Ciencias, Universidad de Chile, Santiago, Chile
e-mail: anirojas@uchile.cl

positions were studied, mostly for polarized abelian varieties with an action by a finite group. However, apart from Prym varieties and Prym-Tyurin varieties, there are only few examples for which the polarization on the subvariety induced from the given polarization was determined. In the present paper we propose to do this for a family of principally polarized abelian varieties with an action of the dihedral group D_n of order $2n$ for n an odd prime as well as twice a prime.

Let (A, L) be a polarized abelian variety and $B \subset A$ an abelian subvariety. The polarization L on A restricts to a polarization on B which we call the induced polarization. After some preliminaries we outline in Sect. 2 a method to compute the induced polarization; that is, to find an adequate basis for the lattice defining the subvariety and to compute the matrix for the induced polarization with respect to this basis. Moreover, we show that, given any finite group G and any faithful integral representation of G of degree n, there exists, for any type $D = (d_1, \ldots, d_n)$, a one-dimensional family of polarized abelian varieties of dimension n and type D such that G acts faithfully on it with the given representation.

In Sect. 3 we define for any $n \geq 2$, an $([\frac{n}{2}] + 1)$-dimensional family $\mathcal{A}_n$ of principally polarized abelian varieties of dimension n which admit an action of the dihedral group D_n of order $2n$. The abelian varieties are explicitely given by period matrices. Then we consider the isotypical and group algebra decompositions (for the definitions see Sect. 2.1) and determine the induced polarizations of the components, for n an odd prime in Sect. 4, for n twice an odd prime in Sect. 6 and for $n = 4$ in Sect. 7. The methods of proof are slightly different in Sects. 4 and 6. For $n = p$ we use two abelian subvarieties which one can directly read off the period matrices, whereas for $n = 2p$ we use abelian subvarieties associated to subgroups of D_{2p} which admit a principal polarization.

In Sect. 5 we study the Jacobians contained in $\mathcal{A}_p$: we show that for an odd prime p there is exactly one irreducible family of Jacobians contained in our family (Corollary 5.3). We use this to compute explicitly a period matrix for the Jacobian of Klein's icosahedral curve of genus 5 (Theorem 5.8).

Notation Let V/Λ be a complex torus of dimension g and $\{\alpha_1, \ldots, \alpha_{2g}\}$ a basis of the lattice Λ. For any rational $2g \times 2g$-matrix $M = (m_{ij})$ we identify the j-th column with the element $\sum_{i=1}^{2g} m_{ij}\alpha_i$. Then we denote by $\langle M \rangle_{\mathbb{Z}}$ the lattice generated by the columns of M and define

$$\langle M \rangle_\Lambda := (\langle M \rangle_{\mathbb{Z}} \otimes \mathbb{Q}) \cap \Lambda$$

and

$$\langle M \rangle_{\mathbb{C}} := \text{the complex vector space generated by the columns of } M.$$

2 Preliminaries

We recall some notions and results from other papers which we need in the sequel and add some additional material.

2.1 The isotypical decomposition

(see [2, Section 13.6]). Let A be a complex abelian variety of dimension g with a faithful action by a finite group G. The action induces a homomorphism of $\mathbb{Q}$-algebras

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A).$$

We denote an element of the rational group algebra and its images in $\mathrm{End}_{\mathbb{Q}}(A)$ by the same letter.

Any element $\alpha \in \mathbb{Q}[G]$ defines an abelian subvariety

$$A^{\alpha} := \mathrm{Im}(m\alpha) \subset A$$

where m is some positive integer such that $m\alpha \in \mathrm{End}(A)$. This definition does not depend on the chosen integer m.

Let

$$\mathbb{Q}[G] = Q_1 \times \cdots \times Q_r$$

denote the decomposition of $\mathbb{Q}[G]$ as a product of simple $\mathbb{Q}$-algebras Q_i. The factors Q_i correspond canonically to the finite dimensional irreducible rational representations W_i of the group G, in the sense that G acts on Q_i via (a multiple of) W_i. The corresponding decomposition of $1 \in \mathbb{Q}[G]$,

$$1 = e_1 + \cdots + e_r$$

with central idempotents e_i induces an isogeny

$$A^{e_1} \times \cdots \times A^{e_r} \to A \tag{2.1}$$

which is given by addition. Note that the components A^{e_i} are G-stable complex subtori of A with $\mathrm{Hom}_G(A^{e_i}, A^{e_j}) = 0$ for $i \neq j$.

If W_i is the irreducible rational representation of G corresponding to e_i, we also denote

$$A_{W_i} := A^{e_i}.$$

The decomposition (2.1) is called the *isotypical decomposition* of the complex G-abelian variety A. The idempotents e_i are determined as follows: Let χ_i be the character of one of the irreducible $\mathbb{C}$-representations associated to W_i and K_i the field

$$K_i = \mathbb{Q}(\chi_i(g), g \in G).$$

Then

$$e_i = \frac{\deg \chi_i}{|G|} \sum_{g \in G} \mathrm{tr}_{K_i|\mathbb{Q}}(\chi_i(g^{-1}))g. \tag{2.2}$$

The isotypical components A_{W_i} can be decomposed further. According to Schur's Lemma

$$D_i := \mathrm{End}_G(W_i)$$

is a skew-field of finite dimension

$$n_i = \frac{\deg \chi_i}{m_i} \tag{2.3}$$

over $\mathbb{Q}$, where m_i denotes the Schur index of χ_i (see [4]). It is easy to see that there is a set of primitive idempotents $\{q_{i1}, \ldots, q_{in_i}\}$ in $Q_i \subset \mathbb{Q}[G]$ such that

$$e_i = q_{i1} + \cdots + q_{in_i}.$$

Moreover, the abelian subvarieties $A^{q_{ij}}$ are mutually isogenous for fixed i and $j = 1, \ldots, n_i$. If B_{W_i} denotes one of them, we get an isogeny

$$B_{W_i}^{n_i} \to A_{W_i}$$

for every $i = 1, \ldots, r$. Combining with (2.1) we get an isogeny

$$B_{W_1}^{n_1} \times \cdots \times B_{W_r}^{n_r} \to A, \tag{2.4}$$

which is called the *group algebra decomposition* of the G-abelian variety A. Note that, whereas (2.1) is uniquely determined, (2.4) is not. It depends on the choice of the q_{ij} as well as the choice of the B_{W_i}.

2.2 The abelian subvariety associated to a subgroup

Any subgroup H of G defines an idempotent of $\mathbb{Q}[G]$

$$e_H = \frac{1}{|H|} \sum_{h \in H} h$$

which in turn defines an abelian subvariety

$$A_H := A^{e_H}$$

that we call the abelian subvariety *associated to* H (see [5]). We also need the following remark for which we refer to [4, Theorem 4.4].

Remark 2.1 If W_i is a rational irreducible representation of a group G such that

$$\langle W_i, \rho_H \rangle = 1,$$

where ρ_H is the representation of G induced by the trivial representation of H, then the idempotent

$$e_{W_i}\, e_H = e_H\, e_{W_i}$$

is primitive in $\mathbb{Q}[G]$.

2.3 Polarized abelian varieties

Now suppose that a line bundle L on A defines a polarization on A, i.e. (A, L) is a polarized abelian variety. The polarization induces the Rosati involution $'$ on $\mathrm{End}_{\mathbb{Q}}(A)$ in the usual way. According to [2, Theorem 5.3.1] the symmetric idempotents (with respect to Rosati) are in 1–1 correspondence with the abelian subvarieties of A and according to [2, Proposition 13.6.5] the idempotents e_i are symmetric whenever the group G respects the polarization, i.e. g^*L is algebraically equivalent to L for any $g \in G$. Moreover we have

Proposition 2.2 *Suppose that the action of the group G on A respects the polarization. For any subgroup H of G the idempotent e_H is symmetric with respect to the Rosati involution.*

Proof Let $\phi_L : A \to \widehat{A}, a \mapsto t_a^*L \otimes L^{-1}$ denote the isogeny onto the dual abelian variety $\widehat{A}$ associated to the line bundle L. The assumption implies that

$$\phi_L = \phi_{g^*L} = \widehat{g}\phi_L g$$

for every $g \in G$. This gives

$$g' = \phi_L^{-1}\widehat{g}\phi_L = g^{-1}$$

and hence

$$e'_H = \frac{1}{|H|} \sum_{h \in H} h' = \frac{1}{|H|} \sum_{h \in H} h^{-1} = e_H.$$

□

Clearly, if e is a symmetric idempotent of $\mathrm{End}_{\mathbb{Q}}(A)$, then so is $f = 1 - e$. The fact that $e + f = 1$ implies for the corresponding abelian subvarieties that the addition map induces an isogeny

$$A^e \times A^f \to A.$$

The subvariety A^f is called the *complementary abelian subvariety of* A^e with respect to the polarization L. The fact that the idempotents e_i and e_H of above are symmetric with respect to any polarization on A immediately implies the following proposition.

Proposition 2.3 *The complementary abelian subvarieties of the abelian subvarieties A_{W_i} and A_H of A are independent of the polarization L.*

2.4 The induced polarization on an abelian subvariety

Let (A, L) be a polarized abelian variety of dimension g with associated isogeny $\phi_L : A \to \widehat{A}$. Recall that there are uniquely determined positive integers $d_1, \ldots, d_g$ with $d_i | d_{i+1}$ for $i = 1, \ldots, g-1$ such that

$$\mathrm{Ker}\,\phi_L \simeq (\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_g\mathbb{Z})^2.$$

The tuple $(d_1, \ldots, d_g)$ is called the *type* of the polarization L, and the exponent d_g of the group $\mathrm{Ker}\,\phi_L$ is called the *exponent* of the polarization L. The polarization induces a polarization $L|_B$ on every abelian subvariety B of A which we call the *induced polarization* (without further mentioning the given polarization L). In this section we outline an algorithm to compute the type of the induced polarization, developed in [11] for the case of Jacobians.

Suppose $A = V/\Lambda$ with V a complex vector space and Λ a lattice of maximal rank in V. The first Chern class of the line bundle L can be considered as an integer valued alternating form on Λ whose elementary divisors give the type of the polarization L. Let $(d_1, \ldots, d_g)$ be the type of the polarization L. A *symplectic basis* for this polarization is a basis of Λ with respect to which the alternating form is given by the matrix

$$J_D := \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

with $D = \mathrm{diag}(d_1, \ldots, d_g)$.

Let $\rho_a : G \to \mathrm{GL}(V)$ and $\rho_r : G \to \mathrm{GL}(\Lambda \otimes \mathbb{Q})$ denote the analytic and rational representations of the action of G on A as well as their extensions to $\mathbb{Q}[G]$. For any $\alpha \in \mathbb{Q}[G]$ the sublattice of Λ defining the abelian subvariety A^α is given by

$$\Lambda^\alpha := \langle \rho_r(\alpha) \rangle_\Lambda.$$

Given a symplectic basis of Λ, we denote the matrix of $\rho_r(g)$ with respect to this basis by the same symbol. Since the action of G respects the polarization, we have

$$\rho_r(g)^t \cdot J_D \cdot \rho_r(g) = J_D.$$

This just means that $\rho_r(g) \in \mathrm{Sp}_{2g}^D(\mathbb{Z})$.

Now choose any basis of the lattice Λ^α. If $h = \dim A^\alpha$, then expressing the elements of this basis in terms of the symplectic basis of Λ, we get a $(2g \times 2h)$- integer matrix P_α which defines the canonical embedding

$$i_\alpha : A^\alpha \hookrightarrow A.$$

With these notations we have,

Proposition 2.4 *Suppose we are given a symplectic basis of Λ, then for any $\alpha \in \mathbb{Q}[G]$ the type of the induced polarization on the abelian subvariety A^α is given by the elementary divisors of the alternating matrix*

$$P_\alpha^t \cdot J_D \cdot P_\alpha.$$

Proof The induced polarization on A^α is given by the line bundle $L|_{A^\alpha}$. Its corresponding isogeny $\phi_{L|_{A^\alpha}} : A^\alpha \to \widehat{A}^\alpha$ is the composition

$$\phi_{L|_{A^\alpha}} = \widehat{i}_\alpha \circ \phi_L \circ i_\alpha.$$

The product $P_\alpha^t \cdot J_D \cdot P_\alpha$ is just the matrix version of this composition with respect to the chosen bases. This gives the assertion. □

We summarize the method to compute the induced representation of A^α in the following five steps:

(1) Compute the rational representation $\rho_r : G \to \mathrm{GL}(\Lambda \otimes \mathbb{Q})$.
(2) Determine a symplectic basis β of Λ. As outlined above, ρ_r is a representation with values in $\mathrm{Sp}_{2g}^D(\mathbb{Z})$ with respect to this basis.
(3) Determine a basis β^α of the lattice Λ^α. For this take the rational vector space generated by the columns of $\rho_r(\alpha)$ and intersect it with the $\mathbb{Z}$-module generated by the columns of $\rho_r(1)$. The elements of β^α will be given as linear combinations of the elements of β.
(4) Compute the product $P_\alpha^t \cdot J_D \cdot P_\alpha$ where P_α is the matrix whose columns are the coordinates of the elements of β^α with respect to the basis β.
(5) Apply the Frobenius algorithm ([9, VI.3. Lemma 1]) to compute the elementary divisors of this alternating matrix.

Steps (1) and (2) are certainly the difficult part of the computation. In the next proposition we outline a class of abelian varieties of type D with group action where this can be done. In the case of a principal polarization this was given in [3]. Denote by $\mathcal{H}^n$ the Siegel upper half-space of degree n, $\mathcal{H} := \mathcal{H}^1$ and for any $\tau \in \mathcal{H}$ by E_τ the elliptic curve defined by τ.

Proposition 2.5 *Let G be a finite group and $D = (d_1, \ldots, d_n)$ a tuple of positive integers with $d_i | d_{i+1}$. Given a faithful representation $\rho : G \to \mathrm{GL}_n(\mathbb{Z})$, a G-invariant real inner product B on $\mathbb{Z}^n$ and an element $\tau \in \mathcal{H}$, there is an abelian variety*

$$A_D = A_D(\rho, B, \tau)$$

of dimension n and a polarization L of type D on A_D such that G acts faithfully on (A_D, L). If B has rational values on $\mathbb{Z}^n$, the abelian variety A_D is isogenous to E_τ^n.

Proof Recall from [2, Section 8.1] that for any $Z \in \mathcal{H}^n$ the matrix (D, Z) is the period matrix of a polarized abelian variety (A, H) of type D and dimension n, where the hermitian form H is given by the matrix $(\mathrm{Im}\, Z)^{-1}$ with respect to the canonical basis of $\mathbb{C}^n$. In fact,

$$A = \mathbb{C}^n / \Lambda_D \quad \text{with} \quad \Lambda_D := \begin{pmatrix} D & 0 \\ 0 & Z \end{pmatrix} \mathbb{Z}^{2n}.$$

The group

$$G_D := \left\{ M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Q}) \mid M^t \Lambda_D \subset \Lambda_D \right\}$$

acts on $\mathcal{H}^n$ by $M(Z) = (\alpha + Z\gamma)^{-1}(\beta + Z\delta)$. In particular M defines an automorphism of (A, H) if and only if $M(Z) = Z$.

For (ρ, B, τ) as in the proposition we define $A_D = A_D(\rho, B, \tau)$ by the period matrix

$$\Gamma := (D, \tau B^{-1}).$$

Clearly the matrix

$$M(g) := \begin{pmatrix} \rho(g) & 0 \\ 0 & (\rho(g)^t)^{-1} \end{pmatrix}$$

is contained in G_D for any $g \in G$. The G-invariance of B implies

$$\rho(g)^{-1} B^{-1} (\rho(g)^t)^{-1} = B^{-1}.$$

This gives

$$M(g)(\tau B^{-1}) = \rho(g)^{-1} \tau B^{-1} ((\rho(g)^t)^{-1}) = \tau B^{-1}$$

for all $g \in G$. Hence $(A_D, \frac{1}{\operatorname{Im} \tau} B)$ is a polarized abelian variety of type D. Since G acts faithfully on the lattice Λ_D, it acts faithfully on the tangent space of A_D and thus on A_D itself. The action respects the polarization, since B is G-invariant.

For the last assertion note that

$$(D, \tau B^{-1}) \begin{pmatrix} D^{-1} & 0 \\ 0 & B \end{pmatrix} = (\mathbf{1}_n, \tau \mathbf{1}_n).$$

If $B \in \mathrm{GL}_n(\mathbb{Q})$, choose a positive integer m such that mB and mD^{-1} are integral matrices. Then the above equation implies that $m \cdot \begin{pmatrix} D^{-1} & 0 \\ 0 & B \end{pmatrix}$ gives an isogeny between A_D and E_τ^n. □

The following direct consequence of Proposition 2.5 is perhaps worth mentioning.

Corollary 2.6 *For any finite group G and any faithful integral representation ρ of G of degree n, there is a one-dimensional family of polarized abelian varieties of dimension n of any given type D such that G acts faithfully on each element of the family, with analytic representation determined by ρ.*

3 Abelian varieties of dimension n with a D_n-action

Consider the Riemann matrices Z of the following form: For $n = 2m - 1,\ m \geq 2$:

$$Z = \begin{pmatrix} z_1 & z_2 & \cdots & z_m & z_m & z_{m-1} & \cdots & & z_2 \\ & z_1 & z_2 & \cdots & z_m & z_m & z_{m-1} & \cdots & z_3 \\ & & \ddots & \ddots & & & & & \vdots \\ & & & & & & & z_1 & z_2 \\ & & & & & & & & z_1 \end{pmatrix}, \tag{3.1}$$

and for $n = 2m,\ m \geq 1$:

$$Z = \begin{pmatrix} z_1 & z_2 & \cdots & z_{m+1} & z_m & z_{m-1} & \cdots & & z_2 \\ & z_1 & z_2 & \cdots & z_{m+1} & z_m & z_{m-1} & \cdots & z_3 \\ & & \ddots & \ddots & & & & & \vdots \\ & & & & & & & z_1 & z_2 \\ & & & & & & & & z_1 \end{pmatrix}. \tag{3.2}$$

In both cases Z is symmetric and symmetric with respect to the antidiagonal.

Proposition 3.1 *The principally polarized abelian varieties $A = A(Z)$ with period matrix $(\mathbf{1}_n, Z)$ form an m-dimensional, respectively $(m+1)$-dimensional, family $\mathcal{A}_n$ for $n = 2m-1$, respectively $n = 2m$.*

Proof Suppose $n = 2m - 1$. The period matrices of the form $(\mathbf{1}_n, Z)$ form an open set in the m-dimensional complex vector space $\mathbb{C}^m$ with coordinate functions $z_1, \dots, z_m$. This set is non-empty, since the matrices with $z_2 = \dots = z_m = 0$ and with z_1 such that the imaginary part $\Im(z_1)$ is positive are contained in it. The proof for $n = 2m$ is similar. □

Let $n \geq 2$ be an integer, and consider the $n \times n$ integral matrices R and S given by

$$R = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & 0 & & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{3.3}$$

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & 0 & \iddots & \vdots & 0 \\ 0 & 1 & & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

and observe that

$$R^n = 1,\, S^2 = 1,\, (RS)^2 = 1;$$

that is, the group generated by R and S is the dihedral group D_n of order $2n$.

Proposition 3.2 *The abelian varieties $A \in \mathcal{A}_n$ admit an action of the group D_n with analytic representation given by* (3.3) *and* (3.4).

Proof Define the rational representation $\rho_r : D_n \to \mathrm{Sp}(2n, \mathbb{Z})$ of the group $D_n = \langle R, S\rangle$ by

$$\rho_r(R) = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \quad \text{and} \quad \rho_r(S) = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}. \tag{3.5}$$

Note that the matrices are contained in $\mathrm{Sp}(2n, \mathbb{Z})$, since the transpose inverse of R coincides with R: $(R^t)^{-1} = R$ and similarly for S. We have to check for $M = R$ and S that

$$M(\mathbf{1}_n, Z) = (\mathbf{1}_n, Z)\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \tag{3.6}$$

which is equivalent to $MZ = ZM$. But this is an easy computation. □

Remark 3.3 If we require the rational representation to be given by (3.5), then the form of the matrices Z is completely determined by the subgroup $\langle R\rangle$; that is, one can show that these are all the Riemann matrices of size n satisfying (3.6) with $M = R$, and then that they also satisfy (3.6) with $M = S$.

4 Action of D_p for p an odd prime

4.1 Notation and induced polarization on A_0 and A_1

Let $n = p = 2m - 1$ be an odd prime, and denote

$$G = D_p = \langle r, s : r^p = s^2 = (rs)^2 = 1\rangle.$$

Observe that G has three irreducible rational representations: the trivial one W_0, another one of degree 1, and W_1 of degree $p-1$, which over $\mathbb{C}$ decomposes as the sum of the degree two representations of G given by

$$V_j : r \to \begin{pmatrix} w_p^j & 0 \\ 0 & w_p^{-j} \end{pmatrix}, \ s \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where w_p denotes a primitive p-th root of unity and $1 \leq j \leq \frac{p-1}{2}$. Note that the representation of G given by R and S is equivalent (over $\mathbb{Q}$) to $W_0 \oplus W_1$.

The corresponding central idempotents are given by

$$e_0 = \frac{1}{2p} \sum_{g \in G} g \quad \text{and} \quad e_1 = \frac{1}{p} \left((p-1)\, 1_G - \sum_{j=1}^{p-1} r^j \right) \tag{4.1}$$

Note that e_0 is primitive in $\mathbb{Q}[G]$, but e_1 is not, being the sum of two (not uniquely determined) primitive idempotents in $\mathbb{Q}[G]$ (by (2.3)). So by (2.4) we have isogenies

$$A \sim A_0 \times A_1 \sim A_0 \times B_1^2$$

with an elliptic curve A_0 and abelian subvarieties A_1 and B_1 of A of dimensions $p-1$ and $\frac{p-1}{2}$, respectively. Note that if $A = V/\Lambda$, then $A_j = V_j/\Lambda_j$, with $V_j = \langle \rho_r(e_j)\rangle_{\mathbb{C}}$ and $\Lambda_j = \langle \rho_r(e_j)\rangle_{\Lambda}$ (see the notation in Sect. 1).

We will now explicitly compute A_0 and A_1, i.e. their lattices and induced polarizations. Let $\{\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p\}$ denote the symplectic basis of Λ with respect to which the matrices $\rho_r(R)$ and $\rho_r(S)$ are given.

Proposition 4.1 *The abelian subvariety A_0 associated to W_0 is of type (p), and its lattice Λ_0 is given by*

$$\Lambda_0 = \langle \alpha_1 + \alpha_2 + \cdots + \alpha_p, \beta_1 + \beta_2 + \cdots + \beta_p \rangle_{\mathbb{Z}}.$$

Proof The symmetric idempotent associated to W_0 is e_0. This gives the assertion concerning the basis of Λ_0. So with respect to the basis $\{\alpha_1, \dots, \beta_p\}$ the embedding $A_0 \hookrightarrow A$ is given by the matrix P_0 with

$$P_0^t = \begin{pmatrix} 1 \cdots 1 & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 \end{pmatrix}.$$

This implies the assertion, since

$$P_0^t \begin{pmatrix} 0 & \mathbf{1}_p \\ -\mathbf{1}_p & 0 \end{pmatrix} P_0 = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}.$$

□

Proposition 4.2 *The abelian subvariety A_1 of A associated to W_1 is of type* $(1, \dots, 1, p)$.

Proof A_0 and A_1 are complementary abelian subvarieties in the principally polarized abelian variety (A, L). So the assertion follows from Proposition 4.1 and [2, Corollary 12.1.5]. □

4.2 Two abelian subvarieties and the lattice of A_1

According to (4.1) and the fact that $\rho_a(R)$ is given by (3.3) we have

$$\rho_r(e_1) = \frac{1}{p}\begin{pmatrix} p-1 & -1 & \dots & & -1 & & & & & \\ -1 & p-1 & -1 & \dots & -1 & & & & & \\ & & \ddots & & & & & 0_{p\times p} & & \\ -1 & \dots & & & p-1 & & & & & \\ & & & & & p-1 & -1 & \dots & & -1 \\ & & & & & -1 & p-1 & -1 & \dots & -1 \\ & & 0_{p\times p} & & & & & \ddots & & \\ & & & & & -1 & \dots & & & p-1 \end{pmatrix}$$

Denote by c_j the j-th column of $\rho_r(e_1)$, identified with the corresponding element of $\Lambda \otimes \mathbb{Q}$. For instance,

$$c_1 = \frac{p-1}{p}\alpha_1 - \frac{1}{p}\alpha_2 - \dots - \frac{1}{p}\alpha_p.$$

Then

$$\Lambda_1 = \langle \rho_r(e_1)\rangle_\Lambda = \langle c_1, \dots, c_{2p}\rangle_\Lambda.$$

Lemma 4.3

$$\Lambda_1 = \langle \alpha_1 - \alpha_p, \alpha_2 - \alpha_p, \dots, \alpha_{p-1} - \alpha_p, \beta_1 - \beta_p, \beta_2 - \beta_p, \dots, \beta_{p-1} - \beta_p\rangle_\mathbb{Z}.$$

Moreover, the elements of the right hand side form a basis of Λ_1.

Note that Λ_1 is not the same as

$$\langle pc_1, \dots, pc_{2p}\rangle_\mathbb{Z}$$

since for instance $c_i - c_j = \alpha_i - \alpha_j$ belongs to Λ_1 but NOT to the last lattice.

Proof First note that the right-hand side is a sublattice of Λ_1, of the same rank as Λ_1, since

$$\alpha_j - \alpha_k = c_j - c_k \quad \text{for} \quad 1 \le j, k \le p,$$
$$\beta_j - \beta_k = c_{p+j} - c_{p+k} \quad \text{for} \quad 1 \le j, k \le p.$$

Hence it suffices to show that the intersection matrix for the proposed basis has determinant p^2, and therefore the two lattices coincide. The intersection product for the proposed basis is given by

$$\langle \alpha_j - \alpha_p, \beta_k - \beta_p\rangle = \delta_{jk} + 1.$$

Therefore the intersection matrix has the form

$$\begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}$$

where B is the size $p-1$ matrix with 2 in the diagonal coefficients and 1 otherwise. The determinant of B is a particular case of a well known determinant

$$D_n(a,b) = \det \begin{pmatrix} a & b & \dots & & b \\ b & a & b & \dots & b \\ & & \ddots & & \\ & & & & \\ b & \dots & & & a \end{pmatrix}_{n\times n} . \tag{4.2}$$

To compute it, add all the remaining rows to the first one, pull out its coefficient, which is $[a+(n-1)b]$, then subtract to each row b times the new first row, starting from the second row. At the end, the determinant is $[a+(n-1)b](a-b)^{(n-1)}$. In our case that is p, hence the determinant for the intersection matrix is p^2. □

Consider the following sublattices of Λ,

$$\Gamma_1 = \langle \alpha_1 - \alpha_p, \alpha_2 - \alpha_{p-1}, \dots, \alpha_{\frac{p-1}{2}} - \alpha_{\frac{p+3}{2}}, \beta_1 - \beta_p, \beta_2 - \beta_{p-1}, \dots, \beta_{\frac{p-1}{2}} - \beta_{\frac{p+3}{2}} \rangle_{\mathbb{Z}},$$
$$\Gamma_2 = \langle \alpha_2 - \alpha_p, \alpha_3 - \alpha_{p-1}, \dots, \alpha_{\frac{p+1}{2}} - \alpha_{\frac{p+3}{2}}, \beta_2 - \beta_p, \beta_3 - \beta_{p-1}, \dots, \beta_{\frac{p+1}{2}} - \beta_{\frac{p+3}{2}} \rangle_{\mathbb{Z}}.$$

Proposition 4.4 *For* $j=1$ *and* 2

$$P_j := (\Gamma_j \otimes \mathbb{C})/\Gamma_j$$

is an abelian subvariety of A_1, *of dimension* $\frac{p-1}{2}$ *with induced polarization of type* $(2,\dots,2)$.

Proof Note first that Γ_1 and Γ_2 are sublattices of Λ_1 of rank $p-1$, since

$$\alpha_j - \alpha_k = c_j - c_k \ \text{ for } 1 \le j,k \le p, \quad \text{and} \quad \beta_j - \beta_k = c_{p+j} - c_{p+k} \ \text{ for } 1 \le j,k \le p.$$

Now recall (from (3.1) with $Z = (z_{j,k})$) that

$$\beta_j = \sum_{k=1}^{p} z_{j,k}\alpha_k$$

In particular,

$$z_{j,\frac{p+1}{2}} = z_{p+1-j,\frac{p+1}{2}} \quad \text{and} \quad z_{j,k} = z_{p+1-j,p+1-k} \quad \text{for } 1 \le k < \frac{p+1}{2}.$$

Therefore for $1 \le j \le \dfrac{p-1}{2}$ we obtain

$$\beta_j - \beta_{p+1-j} = \sum_{k=1}^{p}(z_{j,k} - z_{p+1-j,k})\alpha_k = \sum_{k=1}^{\frac{p-1}{2}}(z_{j,k} - z_{p+1-j,k})(\alpha_k - \alpha_{p+1-k}),$$

which proves that P_1 is an abelian subvariety of dimension $\frac{p-1}{2}$. The assertion about the induced polarization follows from the following equation for the intersection product

$$(\alpha_i - \alpha_{p+1-i}, \beta_j - \beta_{p+1-j}) = 2\delta_{ij}.$$

The proof for P_2 is analogous. □

Proposition 4.5 *Consider the $\mathbb{Z}$-module $\Gamma_1 + \Gamma_2$ with basis*

$$\{\alpha_1 - \alpha_p, \alpha_2 - \alpha_{p-1}, \ldots, \alpha_{\frac{p-1}{2}} - \alpha_{\frac{p+3}{2}}, \alpha_2 - \alpha_p, \alpha_3 - \alpha_{p-1}, \ldots, \alpha_{\frac{p+1}{2}} - \alpha_{\frac{p+3}{2}},$$
$$\beta_1 - \beta_p, \beta_2 - \beta_{p-1}, \ldots, \beta_{\frac{p-1}{2}} - \beta_{\frac{p+3}{2}}, \beta_2 - \beta_p, \beta_3 - \beta_{p-1}, \ldots, \beta_{\frac{p+1}{2}} - \beta_{\frac{p+3}{2}}\}.$$

Its intersection matrix is

$$J = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} \quad \textit{with} \quad \Omega = \begin{pmatrix} 2 \cdot \mathbf{1}_{\frac{p-1}{2}} & N \\ N^t & 2 \cdot \mathbf{1}_{\frac{p-1}{2}} \end{pmatrix},$$

where N is the square matrix of size $\frac{p-1}{2}$

$$N = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 1 \end{pmatrix}.$$

Furthermore,

$$\det(J) = p^2$$

and therefore $\Gamma_1 + \Gamma_2 = \Lambda_1$.

Proof The first assertion follows from the fact that the α_i, β_j are a symplectic basis. The second assertion is a consequence of Lemma 4.6.

Finally, note that certainly $\Gamma_1 + \Gamma_2$ is a sublattice of Λ_1 of finite index. So the induced map $V_1/(\Gamma_1 + \Gamma_2) \to V_1/L_1 = A_1$ is an isogeny. The pullback polarization of the polarization of Proposition 4.2 is given by the matrix J. Since both polarizations are of degree p, the induced map is an isomorphism which implies the last assertion. □

Lemma 4.6 *For any odd positive integer m denote*

$$\Omega = \begin{pmatrix} 2 \cdot \mathbf{1}_{\frac{m-1}{2}} & N_m \\ N_m^t & 2 \cdot \mathbf{1}_{\frac{m-1}{2}} \end{pmatrix}$$

with the square matrix N_m of size $\frac{m-1}{2}$ of the above form (with $m = p$). Then we have

$$\det \Omega = m.$$

Proof The blocks $2 \cdot \mathbf{1}_{\frac{m-1}{2}}$ and N_m of the matrix Ω commute. Hence by [8] we have

$$\det \Omega = \det(2 \cdot \mathbf{1}_{\frac{m-1}{2}} \cdot 2 \cdot \mathbf{1}_{\frac{m-1}{2}} - N_m \cdot N_m^t) = \det(\widetilde{\Omega}_{ij})$$

with

$$\widetilde{\Omega}_{ij} = \begin{cases} 3 & \qquad i = j = 1, \\ 2 & for\ i = j = 2, \ldots, \frac{m-1}{2}, \\ -1 & \qquad i = j + 1 \text{ or } j = i + 1, \\ 0 & \qquad \text{otherwise}. \end{cases}$$

We claim that by admissible row operations (without changing the determinant) we can transform the matrix $\widetilde{\Omega}$ into the upper triangular matrix Ω' with diagonal elements $\frac{2k+1}{2k-1}$ for $k = 1, \ldots, \frac{m-1}{2}$.

This implies

$$\det \Omega = \det \Omega' = \prod_{k=1}^{\frac{m-1}{2}} \frac{2k+1}{2k-1} = m.$$

The assertion is trivial for $k = 1$. So suppose it is proven for some $1 \leq k \leq \frac{m-3}{2}$. Adding the $\frac{2k-1}{2k+1}$-fold of the k-th row to the $(k+1)$-th row we get $\Omega'_{k+1,k} = 0$ and

$$\Omega'_{k+1,k+1} = 2 - \frac{2k-1}{2k+1} = \frac{2k+3}{2k+1}.$$

This completes the proof of the lemma. □

Remark 4.7 A consequence of Proposition 4.5 is that the bases β^1 of Lemma 4.3 and β^2 of Lemma 4.5 are equivalent over $\mathbb{Z}$. The change of basis from β^1 to β^2 is given by the matrix

$$M^{\beta^1}_{\beta^2} = \begin{pmatrix} \mathbf{1}_{\frac{\mathbf{p-1}}{\mathbf{2}}} & A \\ B & C \end{pmatrix},$$

where A, B, C are of size $(p-1)/2$ of the following form:

$$A = \begin{pmatrix} 0\,0\,0 & 0 & 0 \\ 1\,0\,\ldots & 0 & 0 \\ 0\,1\,\ddots & 0 & 0 \\ 0\,0\,\ddots & \ddots & 0 \\ 0\,0\,0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0\,0 & 0 & 0 & 0 \\ 0\,0 & 0 & \ldots & -1 \\ 0\,\vdots & 0 & \vdots & 0 \\ 0\,0 & -1 & & 0 \\ 0\,-1 & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0\,0 & 0 & 0 & 1 \\ 0\,0 & 0 & \ldots & -1 \\ 0\,\vdots & 0 & \vdots & 0 \\ 0\,0 & -1 & & 0 \\ 0\,-1 & 0 & 0 & 0 \end{pmatrix}.$$

4.3 The main result for the action of D_p

For every involution sr^{k-1} in D_p, $1 \leq k \leq p$ we consider the subgroup

$$H_k = \langle sr^{k-1} \rangle$$

of order 2 of D_p with associated idempotent e_{H_k}. According to Remark 2.1, the idempotents

$$f_k = e_1\, e_{H_k} \quad \text{and} \quad e_1 - f_k$$

are primitive in $\mathbb{Q}[G]$. We denote by

$$B_k = \operatorname{Im} f_k \quad \text{and} \quad P_k = \operatorname{Im}(e_1 - f_k)$$

the corresponding abelian subvarieties of A_1. By definition B_k and P_k are a pair of complementary abelian subvarieties of A_1.

Theorem 4.8 (a) *The abelian subvariety P_k of A_1 is of dimension $\frac{p-1}{2}$, with induced polarization of type* $(2, \ldots, 2)$. *For $k = 1$ and 2, P_k coincides with the abelian subvariety P_k of Proposition 4.4;*

(b) *The abelian subvariety B_k of A_1 is of dimension $\frac{p-1}{2}$, with induced polarization of type* $(2, \ldots, 2, 2p)$*;*

(c) *for each k, the addition map induces an isogeny* $\mu : B_k \times P_k \to A_1$ *of degree* 2^{p-1}*;*
(d) *for each* $1 \leq j \neq k \leq p$*, the natural map* $P_j \times P_k \to A_1$ *is an isomorphism of complex tori.*

Proof Since r acts on A_1 (with action given by W_1), it is enough to prove our assertions for $k = 1$ and $j = 2$ say. First we compute

$$\rho_r(e_1 - f_1) = \rho_r(e_1)\,\rho_r\left(\frac{1}{2}(1_G - s)\right) = \frac{1}{2}\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$$

with

$$M = \mathbf{1}_p - \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.$$

Hence the lattice $\langle \rho_r(e_1 - f_1) \rangle_\Lambda$ for P_1 is precisely the lattice Γ_1 given in Proposition 4.4. So this P_1 coincides with the P_1 of Proposition 4.4. Similarly for P_2. This completes the proof of (a). □

For the proof of (b) we need the following lemma.

Lemma 4.9 *A basis for the lattice of* B_1 *is given by*

$$\omega_i^\alpha = (\alpha_i - \alpha_p) + (\alpha_{p+1-i} - \alpha_p) - 2(\alpha_{\frac{p+1}{2}} - \alpha_p) \;\; \textit{and}$$
$$\omega_i^\beta = (\beta_i - \beta_p) + (\beta_{p+1-i} - \beta_p) - 2(\beta_{\frac{p+1}{2}} - \beta_p)$$

for $i = 1, \ldots, \frac{p-1}{2}$.

Proof A basis for the lattice Λ_1 of A_1 is given in Lemma 4.3 and a basis for the lattice Γ_1 of P_1 is given just before Proposition 4.4. Since B_1 is the orthogonal complement of P_1 in A_1, we have that the lattice of B_1 is

$$\Delta_1 = \{\omega \in \Lambda_1 \mid (\omega, \ell) = 0 \quad \text{for all } \ell \in \Gamma_1\}.$$

So we look for elements

$$\omega^\alpha = \sum_{i=1}^{p-1} a_i(\alpha_i - \alpha_p)$$

with integer coefficients a_i satisfying

$$(\omega^\alpha, (\beta_j - \beta_{p+j-1})) = 0 \quad \text{for} \quad j = 1, \ldots, p-1$$

and similarly ω^β for the elements $(\alpha_j - \alpha_{p+j-1})$.

For $j = 1$ this gives

$$2a_1 + a_2 + \cdots + a_{p-1} = 0$$

and for $j = 2, \ldots, \frac{p-1}{2}$ we get

$$a_{p+1-j} = a_j.$$

Inserting this into the equation for $j = 1$ we get the assertion for the ω_i^α's. The proof for the ω_i^β's is similar. □

Proof of (b): The assertion for the dimension is clear, since B_1 is the complementary abelian subvariety of the $\frac{p-1}{2}$-dimensional abelian subvariety P_1 in the $(p-1)$-dimensional variety A_1. The intersection matrix of the basis of Lemma 4.9 is

$$\begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$$

with

$$\Omega = \left((\omega_i^\alpha, \omega_j^\beta)_{i,j=1}^{\frac{p-1}{2}}\right) = \left((2\delta_{ij} + 4)_{i,j=1}^{\frac{p-1}{2}}\right).$$

So $\Omega = D_{\frac{p-1}{2}}(6, 4)$ as defined in (4.2). As we noted in the proof of Lemma 4.3, this gives

$$\det \Omega = \left[6 + 4\frac{p-3}{2}\right](6-4)^{\frac{p-3}{2}} = 2^{\frac{p-1}{2}} p.$$

So the induced polarization on B_1 is of degree $2^{\frac{p-1}{2}} p$. On the other hand it is of exponent (see Sect. 2.4) $2p$, since its idempotent f_1 is. So the only possibility for its type is $(2, \ldots, 2p)$.

Proof of (c) According to [10, Lemma 2.2] we have for the degree of the isogeny μ,

$$\deg \mu = |B_k \cap P_k| = \frac{\deg(L|_{B_k}) \cdot \deg(L|_{P_k})}{\deg(L)} = \frac{2^{\frac{p-1}{2}} p \cdot 2^{\frac{p-1}{2}}}{p} = 2^{p-1}.$$

Proof of (d) In Proposition 4.5 we saw that the lattices Γ_1 of P_1 and Γ_2 of P_2 add up to Λ_1, the lattice of A_1. Since the basis of $\Gamma_1 + \Gamma_2$ given in Proposition 4.5 is just the disjoint union of the bases of Γ_1 and Γ_2 given just before Proposition 4.4, this implies that the addition map $P_1 \times P_2 \to A_1$ is an isomorphism. This completes the proof of the theorem. □

5 Jacobians in the family

In this section we will see that there is exactly a one-dimensional family of Jacobians contained in our $\frac{p+1}{2}$-dimensional family $\mathcal{A}_p$ of abelian varieties with D_p-action of the last section, and study their abelian subvarieties.

Proposition 5.1 *There is at most a one-dimensional irreducible family of curves of genus p with an action of D_p in our family $\mathcal{A}_p$.*

Proof Suppose C is a curve of genus p with an action of D_p such that the induced action on its Jacobian is given by the matrices R and S of the last section and with rational representation given by (3.5). We denote the corresponding action on C by the same letters R and S.

Since the eigenvalue 1 of R has multiplicity 1 and R is of order p, there is a cyclic covering $\mu : C \to E$ of degree p of an elliptic curve E. According to Riemann–Hurwitz, μ is totally ramified at exactly 2 points. We claim that S interchanges the 2 ramification points.

In Theorem 4.8 we saw that the abelian varieties P_i are of dimension $\frac{p-1}{2}$ with induced polarization of type $(2, \ldots, 2)$. The Theorem of Welters [13] implies that there is a fixed-point free involution on C whose Prym variety is P_i. Hence the elements of order 2 of D_p act without fixed points, which implies the assertion. Since there is at most a one-dimensional irreducible family of such coverings $\mu : C \to E$, the result follows. □

Conversely we have,

Proposition 5.2 *The curves*

$$C_t : y^2 = x(x^p - t^p)(x^p + t^{-p}) \tag{5.1}$$

for $t \in \mathbb{C}, t \neq 0, t^{2p} \neq -1$, *have the required action, with*

$$R(x, y) = (w_p x, w_p^{\frac{p+1}{2}} y)\ ,\ S(x, y) = (-\frac{1}{x}, \pm\frac{y}{x^{p+1}}).$$

Here ω_p *is a primitive* p*-th root of unity, and the adequate sign in* $\pm\frac{y}{x^{p+1}}$ *for* S *is such that* S *has no fixed points in* C_t*; that is* $\frac{y}{x^{p+1}}$ *if* $p + 1 \equiv 2(4)$*, and* $-\frac{y}{x^{p+1}}$ *if* $p + 1 \equiv 0(4)$.

Proof Note first that R and S generate the dihedral group of order $2p$ and R has exactly 2 fixed points, (0, 0) and ∞, which are interchanged by S. As we saw in the last section (see Remark 3.3), in this case the action of S is implied by the action of R; meaning that the abelian varieties having the action of R already have the action of S. Hence it suffices to show that R acts on the holomorphic differentials of C_t by a matrix equivalent to the matrix R of the last section.

A basis of the holomorphic differentials of C_t is

$$\left\{\frac{dx}{y}, x\frac{dx}{y}, \ldots, x^{p-1}\frac{dx}{y}\right\}.$$

Clearly the basis elements are eigenvectors for R and it is easy to see that the eigenvalues are exactly all p-th roots of unity. Hence the analytic representation of R is the regular representation of the cyclic group $\langle R\rangle$ and thus equivalent to (2.1). □

Corollary 5.3 *The curves of Proposition 5.2 are exactly the Jacobians in our family of principally polarized abelian varieties.*

Proof The Jacobians of Proposition 5.2 are a one-dimensional family of such Jacobians. Since it is closed in the moduli space of smooth curves of genus p, this implies the assertion. □

5.1 The case $p = 5$

By setting $t = t_0 = w_5 + w_5^4$ in (5.1), we obtain the curve

$$y^2 = x(x^{10} + 11x^5 - 1),$$

which according to Klein (see [6, Section II,13]) admits as full group of automorphisms the icosahedral group

$$A_5 \times \mathbb{Z}/2\mathbb{Z} = \langle(1, 2, 3, 5, 4), (1, 3)(2, 4)\rangle \times \langle j\rangle$$

with j the hyperelliptic involution. It is called Klein's icosahedral curve. In order to determine a period matrix for the Jacobian of C_{t_0}, we need the following proposition.

Proposition 5.4 *The principally polarized abelian varieties of dimension* 5 *admitting an action of the icosahedral group which restricts to our action of* D_5 *form a one-dimensional family, given by the Riemann matrices* $(\mathbf{1}, Z_\tau)$ *where*

$$Z_\tau = \frac{1}{6}\begin{pmatrix} 2\tau+6 & \tau-3 & \tau & \tau & \tau-3 \\ \tau-3 & 2\tau+6 & \tau-3 & \tau & \tau \\ \tau & \tau-3 & 2\tau+6 & \tau-3 & \tau \\ \tau & \tau & \tau-3 & 2\tau+6 & \tau-3 \\ \tau-3 & \tau & \tau & \tau-3 & 2\tau+6 \end{pmatrix} \tag{5.2}$$

with $\tau \in \mathcal{H}$.

Proof First, note that the eigenvalues of Z_τ are

$$\tau,\ \frac{5}{4}+\frac{\tau}{6}+\frac{\sqrt{5}}{4},\ \frac{5}{4}+\frac{\tau}{6}-\frac{\sqrt{5}}{4},\ \frac{5}{4}+\frac{\tau}{6}+\frac{\sqrt{5}}{4},\ \frac{5}{4}+\frac{\tau}{6}-\frac{\sqrt{5}}{4},$$

which implies that $\operatorname{Im} Z_\tau > 0$ is equivalent to $\operatorname{Im} \tau > 0$.

Using the algorithm developed in [1] for the action of the icosahedral group on C_{t_0}, we find that symplectic generators for the group are:

$$(1,4,3,5,2)j \mapsto x_1 = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$(1,5,4) \mapsto x_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$x_1 x_2^{-1} x_1 = \rho_r(R) \quad \text{and } x_2^{-1} x_1^4 = \rho_r(S)$$

and this representation of the icosahedral group restricts to the given one for D_5.

According to our convention in Sect. 2 (see the proof of Proposition 3.2) the action of $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2p, \mathbb{Z})$ is given by

$$Z \mapsto (A + ZC)^{-1}(B + ZD).$$

So the fixed points are given by $C = 0$ and the solutions of the equation $B + ZD = AZ$. Now a straightforward computation gives that the fixed points of the above action are just given by (5.2) which completes the proof of the proposition. □

Proposition 5.5 *Let $\mathcal{A}_{Z_\tau}$ be the principally polarized abelian variety with period matrix Z_τ of* (5.2). *The group algebra decomposition of $\mathcal{A}_{Z_\tau}$ with respect to the icosahedral group is given by*

$$\mathcal{A}_{Z_\tau} \sim E_{Z_\tau}^5$$

where E_{Z_τ} is an elliptic curve which is the connected component containing zero of the fixed point variety of any automorphism of order 5 *of A_{Z_τ}.*

Proof The icosahedral group has a unique faithful absolutely irreducible rational representation W of degree five, and our symplectic representation is isomorphic to $2W$. Therefore the analytic representation of $\mathcal{A}_{Z_\tau}$ is W. So the group algebra decomposition tells us that

$$\mathcal{A}_{Z_\tau} \sim E_{Z_\tau}^5$$

where E_{Z_τ} is any elliptic curve lying on $\mathcal{A}_{Z_\tau}$. Since the connected component containing zero of the fixed point variety of any automorphism of order 5 of A_{Z_τ} is an elliptic curve, this completes the proof of the proposition. □

Now consider again the curve C_t of Proposition 5.2 which for $p = 5$ is given by

$$C_t : y^2 = x(x^5 - t^5)(x^5 + t^{-5})$$

with $t \in \mathbb{C}$ and $t^{10} \neq 0, -1$ with its automorphism R of order 5. Let $\mu_t : C_t \to E_t$ denote the corresponding covering onto the elliptic curve $E_t := C_t/\langle R\rangle$.

Proposition 5.6 *The j-invariant of E_t is*

$$j(E_t) = 256\frac{(1 + t^{10} + t^{20})}{t^{20}(1 + t^{10})^2}.$$

Proof Consider the curve

$$D_t : y^2 = x^5(x^5 - t^5)(x^5 + t^{-5})$$

for $t \in \mathbb{C}$ with $t^{10} \neq 0, -1$. Then

$$\phi : D_t \to C_t, \quad (x, y) \mapsto (x, \frac{y}{x^2})$$

induces an isomorphism onto the normalization of C_t which we denote with the same symbol. Via the isomorphism ϕ the automorphism R corresponds to the automorphism r of D_t given by

$$r(x, y) = (\omega_5 x, y)$$

whose quotient $\pi : D_t \to F_t := D_t/\langle r\rangle$ is given by $\pi(x, y) = (x^5, y) =: (u, v)$. We obtain the elliptic curve

$$F_t : v^2 = u(u - t^5)(u + t^{-5}).$$

This gives $j(E_t) = j(F_t)$ and hence the assertion. □

Corollary 5.7 *The Jacobian of Klein's icosahedral curve C_{t_0}, $t_0 = \omega_5 + \omega_5^4$ is isogenous to $E_{t_0}^5$ where E_{t_0} is the elliptic curve with j-invariant*

$$j(E_{t_0}) = \frac{2^{14}(31)^3}{5^3}.$$

The Jacobian $J(C_{t_0})$ is isomorphic (unpolarized) to a product of elliptic curves which are isogenous to E_{t_0}.

Proof According to Propositions 5.4 and 5.5, $J(C_{t_0})$ is isogenous to $E_{t_0}^5$. So the first assertion follows from Proposition 5.6. The last assertion is a consequence of [2, Exercise 10.8.5]. □

According to Proposition 5.4 the Jacobian $J(C_{t_0})$ has a period matrix (5.2) with some $\tau = \tau_{t_0} \in \mathcal{H}$.

Theorem 5.8 *The period matrix of the Jacobian of Klein's icosahedral curve is $Z_{\tau_{t_0}}$ as given in (5.2) where $\tau_{t_0} \in \mathcal{H}$ is any element with*

$$j(\tau_{t_0}) = \frac{2^{14}(31)^3}{5^3}.$$

Proof Certainly $J(C_{t_0})$ is contained in the family of Proposition 5.4. Let $\tau_{t_0} \in \mathcal{H}$ be a value such that $Z_{\tau_{t_0}}$ is a period matrix of $J(C_{t_0})$.

For the subgroup $H = \langle R \rangle$ of the icosahedral group we have $\langle W, \rho_{\langle R \rangle} \rangle = 1$. Therefore $e_H e_W(J(C_{t_0}))$ is an elliptic curve on $J(C_{t_0})$. As above, let $\alpha_1, \dots, \alpha_5, \beta_1, \dots, \beta_5$ denote the basis of the lattice Λ defining the period matrix (5.2) of $J(C_{t_0})$. Then we have

$$\Lambda^{e_H e_W} = \langle \alpha_1 + \alpha_2 + \cdots + \alpha_5, \beta_1 + \beta_2 + \cdots + \beta_5 \rangle_{\mathbb{Z}}$$

and therefore the modulus μ of $e_H e_W(A_\tau)$ (mod SL(2, $\mathbb{Z}$)) is given by

$$\beta_1 + \beta_2 + \cdots + \beta_5 = \mu\, (\alpha_1 + \alpha_2 + \cdots + \alpha_5)\,.$$

Now from (5.2) we see that

$$\begin{aligned}
\beta_1 &= (2\tau_{t_0} + 6)\alpha_1 + (\tau_{t_0} - 3)\alpha_2 + \tau_{t_0}\alpha_3 + \tau_{t_0}\alpha_4 + (\tau_{t_0} - 3)\alpha_1 \\
&\ \ \vdots \\
\beta_5 &= (\tau_{t_0} - 3)\alpha_1 + \tau_{t_0}\alpha_2 + \tau_{t_0}\alpha_3 + (\tau_{t_0} - 3)\alpha_4 + (2\tau_{t_0} + 6)\alpha_1
\end{aligned}$$

and therefore

$$\beta_1 + \beta_2 + \cdots + \beta_5 = \tau_{t_0}\, (\alpha_1 + \alpha_2 + \cdots + \alpha_5)\,.$$

That is, τ_{t_0} is the modulus of $e_H e_W(A_\tau)$. Let E_{t_0} be the elliptic curve of Corollary 5.7. Since $C_{t_0} \to E_{t_0}$ is ramified, E_{t_0} embeds into $J(C_{t_0})$, and its image coincides with $e_H e_W(J(C_{t_0}))$.

This implies

$$j(\tau_{t_0}) = j(E_{t_0}).$$

So Corollary 5.7 completes the proof of the theorem. □

6 Action of D_{2p} for an odd prime p

6.1 Notation and induced polarization on A_1 and A_4

Let $n = 2p$ with an odd prime p and consider the group

$$G = D_{2p} = \langle r, s \ :\ r^{2p} = s^2 = (rs)^2 = 1 \rangle.$$

The group G has 6 rational irreducible representations, 4 of them of dimension 1, namely W_i with character χ_i defined by the following table

	r	s
χ_1	1	1
χ_2	1	-1
χ_3	-1	1
χ_4	-1	-1

and 2 of degree $p-1$ defined as follows: Define for $i = 1, \dots, p-1$ the complex irreducible representation V_i by

$$V_i : r \to \begin{pmatrix} w_{2p}^i & 0 \\ 0 & w_{2p}^{-i} \end{pmatrix}, \ s \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$W_5 = \bigoplus_{i=1}^{\frac{p-1}{2}} V_{2i} \quad \text{and} \quad W_6 = \bigoplus_{i=1}^{\frac{p-1}{2}} V_{2i-1}$$

are (the complexification of) irreducible rational representations.

Now let A be an abelian variety of Proposition 3.2 with an action of D_{2p} and analytic representation given by (3.3) and (3.4). According to (2.3) and (2.4) the group algebra decomposition of A is of the form

$$A \sim \times_{i=1}^{6} A_i \quad \text{with} \quad A_i \sim \begin{cases} B_i & for \ i = 1, \dots, 4, \\ B_i^2 & for \quad i = 5, 6. \end{cases}$$

where the factor A_i corresponds to χ_i for $i = 1, \dots, 4$ and for $i = 5$ and 6 to W_5 and W_6. If V_i denotes a complex irreducible representation contained in W_i, the dimension of B_i is given by the following formula (see [12, Equation (5.4)])

$$\dim B_i = \frac{1}{2} m_i [K_{V_i} : \mathbb{Q}] \langle \rho_r \otimes \mathbb{C}, V_i \rangle \tag{6.1}$$

where m_i is the Schur index and K_{V_i} the character field of V_i and $\langle \cdot, \cdot \rangle$ denotes the character product.

Proposition 6.1 *We have* $\dim A_i = 0$ *for* $i = 2$ *and* 3 *and* $\dim A_i = 1$ *for* $i = 1$ *and* 4. *The induced polarization on* A_1 *and* A_4 *is of type* $(2p)$.

Proof The assertion on the dimension is an easy computation using (6.1) and $A_i = B_i$ for $i = 1, \dots, 4$. The proof for the induced polarization on A_1 is similar as the proof of Proposition 4.1. The symmetric idempotent associated to W_4 is $e_4 = \frac{1}{4p} \sum_{g \in G} \chi_4(g^{-1}) g$. So with respect to the basis $\{\alpha_1, \dots, \beta_{2p}\}$ the embedding $A_4 \hookrightarrow A$ is given by the matrix P_4 with

$$P_4^t = \begin{pmatrix} 1 & -1 & \cdots & 1 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -1 & \cdots & 1 & -1 \end{pmatrix}.$$

Now the proof works in a similar way as for A_1. □

6.2 Induced polarization on A_5

For the type of the induced polarization on A_5 we first consider the abelian subvariety $A_{\langle r^p \rangle}$ associated to the subgroup $\langle r^p \rangle$.

Lemma 6.2 *The abelian subvariety* $A_{\langle r^p \rangle}$ *is of dimension* p *with induced polarization of type* $(2, \dots, 2)$.

Proof The symmetric idempotent associated to the subgroup $\langle r^p \rangle$ is $e_{\langle r^p \rangle} = \frac{1}{2}(\mathbf{1} + r^p)$ which gives

$$\rho_a(e_{\langle r^p \rangle}) = \frac{1}{2} \begin{pmatrix} \mathbf{1}_p & \mathbf{1}_p \\ \mathbf{1}_p & \mathbf{1}_p \end{pmatrix}.$$

It is of rank p which gives the dimension of $A_{\langle r^p \rangle}$. A basis for the primitive lattice generated by its columns is

$$\{\alpha_i + \alpha_{p+i}, \beta_i + \beta_{p+i} \mid i = 1, \dots, p\}.$$

Since $\langle \alpha_i + \alpha_{p+i}, \beta_j + \beta_{p+j} \rangle = 2\delta_{ij}$, this gives the assertion on the type of the induced polarization. $\square$

Lemma 6.3 *The subvarieties A_1 and A_5 are a pair of complementary abelian subvarieties of $A_{\langle r^p \rangle}$.*

Proof It suffices to show that

$$\rho_a(e_{\langle r^p \rangle}) = \rho_a(e_1 e_{\langle r^p \rangle}) + \rho_a(e_5 e_{\langle r^p \rangle}).$$

The character field of W_5 is $K_5 = \mathbb{Q}(\omega_p)$. So using (2.2) we compute

$$e_5 = \frac{2}{4p}\left[(p-1)\mathbf{1} - \sum_{j=1}^{p-1} r^j + (p-1)r^p - \sum_{j=p+1}^{2p-1} r^j\right]. \tag{6.2}$$

With this and $e_1 = \frac{1}{2p}(\sum_{g \in D_{2p}} g)$ one easily checks $e_5 e_{\langle r^p \rangle} = e_5$ and $e_1 e_{\langle r^p \rangle} = e_1$ and moreover

$$e_{\langle r^p \rangle} - e_5 = \frac{1}{2p}\sum_{j=0}^{2p-1} r^j.$$

Since we have $\rho_a(\frac{1}{2p}\sum_{j=0}^{2p-1} r^j) = \rho_a(e_1)$, this implies the assertion. $\square$

Using this we can show

Proposition 6.4 *The subvariety A_5 is of dimension $p-1$ with induced polarization of type* $(2, \dots, 2, 2p)$.

Proof According to Lemma 6.2 the abelian variety $A_{\langle r^p \rangle}$ admits a principal polarization, the double of which is the induced polarization. According to Proposition 6.1 the induced polarization of this principal polarization on the elliptic curve A_1 is of type (p). So by [2, Proposition 12.1.5] and Lemma 6.3 the induced polarization of this principal polarization on A_5 is of type $(1, \dots, 1, p)$ which implies the assertion for the double of this polarization. Clearly A_5 is of dimension $p-1$. $\square$

Corollary 6.5 *A basis of the lattice Λ_5 of A_5 is*

$$\{\alpha_i - \alpha_p + \alpha_{p+i} - \alpha_{2p}, \beta_i - \beta_p + \beta_{p+i} - \beta_{2p} \mid i = 1, \dots, p-1\}.$$

Proof According to (6.2) and the definition of R and S we have

$$\rho_a(e_5) = \frac{1}{2p}\begin{pmatrix} M & M \\ M & M \end{pmatrix} \quad \text{with} \quad M = p\mathbf{1}_p - (1)_{i,j=1}^{p}.$$

The lattice of A_5 is $\langle \rho_r(e_5) \rangle_\Lambda$. Denote by c_j the j-th column of the matrix $\rho_r(e_5)$ and note that

$$\frac{1}{2}(\alpha_i - \alpha_p + \alpha_{p+i} - \alpha_{2p}) = c_i - c_p \quad \text{and} \quad \frac{1}{2}(\beta_i - \beta_p + \beta_{p+i} - \beta_{2p}) = c_{p+i} - c_{2p}$$

for $i = 1, \dots, p-1$. Therefore $\alpha_i - \alpha_p + \alpha_{p+i} - \alpha_{2p}$ and $\beta_i - \beta_p + \beta_{p+i} - \beta_{2p}$ are contained in the lattice $\langle \rho_r(e_5)\rangle_\Lambda$. The intersection matrix of these elements is

$$E_5 := \begin{pmatrix} 0 & D_{p-1}(4,2) \\ -D_{p-1}(4,2) & 0 \end{pmatrix}$$

with $D_{p-1}(4,2)$ as defined in (4.2). Hence they generate a sublattice of $\langle \rho_r(e_5)\rangle_\Lambda$ the degree of which is the square root of

$$\det E_5 = [[4 + (p-1-1)\cdot 2](4-2)^{p-2}]^2 = (2^{p-1}p)^2.$$

Furthermore, by [2, Proposition 12.1.1] A_5 is of exponent $2p$ in A (see Sect. 2.4). Hence the induced polarization on the abelian subvariety defined by $\langle \rho_r(e_5)\rangle_\Lambda$ is of type $(2, \dots, 2, 2p)$ which implies that this abelian subvariety coincides with A_5. □

Consider the following sublattice of Λ (the analogue of the lattice Γ_1 of Proposition 4.4),

$$\Gamma_5 = \left\langle \alpha_i - \alpha_{p+1-i} + \alpha_{p+i} - \alpha_{2p+1-i}, \beta_i - \beta_{p+1-i} + \beta_{p+i} - \beta_{2p+1-i} \mid i = 1, \dots, \frac{p-1}{2} \right\rangle$$

and the abelian subvarieties B_5 and P_5 of A_5 defined by the idempotents $e_5 e_{\langle s\rangle}$ and $e_5 - e_5 e_{\langle s\rangle}$ [the analogues of the abelian subvarieties B_1 and P_1 of Theorem 4.8 (a) and (b)].

Proposition 6.6 *The subvarieties P_5 and B_5 are of dimension $\frac{p-1}{2}$ with induced polarization of type $(4, \dots, 4)$ and $(4, \dots, 4, 4p)$, respectively.*

Proof Note first that Γ_5 is a sublattice of the lattice Λ_5 of A_5, because its elements can be combined from the elements of the basis of Corollary 6.5. Now the proof is analogous to the proofs of Proposition 4.4 and Theorem 4.8 (a) and (b). □

6.3 Induced polarization on A_6

The proofs in this case are very similar to the proofs of the previous subsection for A_5. We only give the results.

Recall that $A_6 = \mathrm{Im}(e_6)$ and $A_6 \sim B_6^2$ where B_6 is a not uniquely determined abelian subvariety. We may choose $B_6 = \mathrm{Im}(e_6 e_{\langle s\rangle})$. Let $P_6 = \mathrm{Im}(e_6 - e_6 e_{\langle s\rangle})$ its complement in A_6. Then we have

Theorem 6.7 (a) *The abelian subvariety A_6 is of dimension $p-1$ with induced polarization of type $(2, \dots, 2, 2p)$. A basis of the sublattice of Λ defining A_6 is given by*

$$\{\alpha_i - \alpha_{p+i} + (-1)^i(\alpha_p - \alpha_{2p}), \beta_i - \beta_{p+i} + (-1)^i(\beta_p - \beta_{2p}) \mid i = 1, \dots, p-1\}.$$

(b) *The subvarieties B_6 and P_6 are of dimension $\frac{p-1}{2}$ with induced polarization of type $(4, \dots, 4)$ and $(4, \dots, 4, 4p)$ respectively.*

For the proof we only note that

$$e_6 = \frac{2}{4p}\left[(p-1)\mathbf{1} - \sum_{j=1}^{p-1} r^{2j} + \sum_{j=1}^{\frac{p-1}{2}} r^{2j-1} - (p-1)r^p + \sum_{j=\frac{p+3}{2}}^{p} r^{2j-1}\right]$$

which implies that

$$\rho_a(e_6) = \begin{pmatrix} N & -N \\ -N & N \end{pmatrix} \quad with \quad N = p\mathbf{1}_p - ((-1)^{i+j})_{i,j=1}^p.$$

For A_5 we worked with the abelian subvariety associated to the subgroup $\langle r^p \rangle$. Here we have to choose a different subgroup, since $A_6 \not\subset A_{\langle r^p \rangle}$. We work instead with the abelian subvariety $A_{\langle s \rangle}$ which is of dimension p with induced polarization of type $(2, \dots, 2)$.

Remark 6.8 Comparing Theorem 6.7 with Proposition 6.6, one notes that the types of the B's and P's are exchanged. If one chooses instead of the involution s in Theorem 6.7 the involution sr, then one has for the associated abelian subvarieties (with the obvious notation): The induced polarization on $P_{\langle sr \rangle}$ respectively $B_{\langle sr \rangle}$ is of type $(4, \dots, 4)$ respectively $(4, \dots, 4, 4p)$.

6.4 Jacobians in the family

In the case of D_{2p} we have the following fact which is different from the D_p-case.

Proposition 6.9 *The $(p+1)$-dimensional family $\mathcal{A}_{2p}$ of abelian varieties as in Proposition 3.1 with D_{2p}-action contains no Jacobian.*

Proof Suppose C is a smooth projective curve whose Jacobian is in the family. By the Torelli theorem the group D_{2p} acts faithfully on C. Then the analytic representation of rs is given by the size $2p$ matrix

$$\rho_a(rs) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & \iddots & 1 \\ & & \iddots & \\ 0 & 1 & & 0 \end{pmatrix}$$

and hence has $p+1$ eigenvalues equal to one. This implies that the quotient curve $C/\langle rs \rangle$ should have genus $p+1$. But this contradicts the Hurwitz formula. $\square$

7 Action of D_4

For the sake of completeness we also include (without proofs) the result for the group $G = D_4$. It has five irreducible rational representations, four of degree one, namely $\chi_1, \dots, \chi_4$ defined as in Sect. 6.1 and one of degree 2, defined by

$$\chi_5(r) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \chi_5(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let A be an abelian variety as in Proposition 3.2 with an action of D_4 and analytic representation given by (3.3) and (3.2). Denote by A_i the abelian subvariety associated to the representation χ_i.

Theorem 7.1 *(a) The isotypical decomposition of A is*

$$A \sim A_1 \times A_4 \times A_5$$

with elliptic curves A_1 and A_4 and an abelian surface A_5. A basis for the lattice of A_1 is $\{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \beta_1 + \beta_2 + \beta_3 + \beta_4\}$, of A_4 is $\{\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4, \beta_1 - \beta_2 + \beta_3 - \beta_4\}$ and of A_5 is $\{\alpha_1 - \alpha_3, \alpha_2 - \alpha_4, \beta_1 - \beta_3, \beta_2 - \beta_4\}$. So the induced polarizations on A_1 and A_4 are of type (4) and on A_5 of type $(2, 2)$.

(b) Let $B_5 := \mathrm{Im}(e_{\langle sr\rangle}e_5)$ *and* $P_5 := \mathrm{Im}(e_5 - e_{\langle sr\rangle}e_5)$ *its complement in* A_5*. The induced polarizations on* B_5 *and* P_5 *are of type* (2) *and the natural map*

$$B_5 \times P_5 \to A_5$$

is an isomorphism of polarized abelian varieties.

According to Proposition 3.1 the family of these abelian varieties is of dimension three. The same argument as for Proposition 6.9 shows that there is no Jacobian in this family.

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