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Rieffel Deformation and Twisted Crossed Products

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To a continuous action of a vector group on a C^*-algebra, twisted by the imaginary exponential of a symplectic form, one associates a Rieffel deformed algebra as well as a twisted crossed product. We show that the second one is isomorphic to the tensor product of the first one with the C^*-algebra of compact operators in a separable Hilbert space and we indicate some applications.

1 Introduction

In order to provide a unified framework for a large class of examples in deformation quantization, Rieffel [15] significantly extended the basic part of the Weyl pseudodifferential calculus. Rieffel's calculus starts from the action Θ of a finite-dimensional vector space Ξ on a C^*-algebra $\mathcal{A}$, together with a skew-symmetric linear operator $J : \Xi \to \Xi$ that serves to twist the product on $\mathcal{A}$. Using J, one defines first a new composition law # on the set of smooth elements of $\mathcal{A}$ under the action and then a completion is

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taken in a suitable C^*-norm. The outcome is a new C^*-algebra $\mathfrak{A}$, also endowed with an action of the vector space Ξ. The corresponding subspaces of smooth vectors under the two actions, $\mathcal{A}^\infty$ and $\mathfrak{A}^\infty$, respectively, coincide. In [15], the functorial properties of the correspondence $\mathcal{A} \mapsto \mathfrak{A}$ are studied in detail and many examples are given. It is also shown that one gets a strict deformation quantization of a natural Poisson structure defined on $\mathcal{A}^\infty$ by the couple (Θ, J).

Assuming J nondegenerate (so it defines a symplectic form on Ξ), one gets a twisted action (Θ, κ) of Ξ on the C^*-algebra $\mathcal{A}$, where κ is the 2-cocycle on Ξ given by $(X, Y) \mapsto \kappa(X, Y) := \exp(iX \cdot JY)$. To such a twisted action, one associates canonically [11, 12] a twisted crossed product C^*-algebra $\mathcal{A} \rtimes_\Theta^\kappa \Xi$, whose representations are determined by covariant representations of the quadruplet $(\mathcal{A}, \Theta, \kappa, \Xi)$.

In the present article, we are going to show that the two C^*-algebras $\mathfrak{A}$ and $\mathcal{A} \rtimes_\Theta^\kappa \Xi$ that can be constructed from the data $(\mathcal{A}, \Theta, \kappa, \Xi)$ are actually stably isomorphic. This happens in a particularly precise way: one has an isomorphism (called the canonical mapping) $M : \mathscr{K} \otimes \mathfrak{A} \to \mathcal{A} \rtimes_\Theta^\kappa \Xi$, where $\mathscr{K}$ is an elementary C^*-algebra, that is, it is faithfully represented as the ideal of all compact operators in a separable Hilbert space. The mapping M is naturally defined first between convenient Fréchet subalgebras (vector-valued Schwartz spaces); the extension to a C^*-isomorphism needs a nontrivial isometry argument.

Such a stable isomorphism has standard consequences [14]: the (closed, bi-sided self-adjoint) ideals of the two algebras $\mathfrak{A}$ and $\mathcal{A} \rtimes_\Theta^\kappa \Xi$ are in one-to-one correspondence, the spaces of primitive ideals are homeomorphic, and the two representation theories are identical. By using basic information about the twisted crossed product, we also get a simple proof of the known fact [8, 17] that the K-groups of the Rieffel deformed algebra $\mathfrak{A}$ are the same as those of the initial algebra $\mathcal{A}$. A covariant morphism $\mathcal{R} : (\mathcal{A}^1, \Theta^1) \to (\mathcal{A}^2, \Theta^2)$ can be raised both to a morphism $\mathfrak{R} : \mathfrak{A}^1 \to \mathfrak{A}^2$ and to a morphism $\mathcal{R}^\rtimes : \mathcal{A}^1 \rtimes_{\Theta^1}^\kappa \Xi \to \mathcal{A}^2 \rtimes_{\Theta^2}^\kappa \Xi$. The canonical mappings M^1, M^2 have the intertweening property $\mathcal{R}^\rtimes \circ M^1 = M^2 \circ (\mathrm{id} \otimes \mathfrak{R})$.

When the initial algebra $\mathcal{A}$ is commutative, it is associated by Gelfand theory with a locally compact topological dynamical system (Σ, Θ, Ξ). Under some assumptions on this system, one can get information on the primitive ideal space of the C^*-algebra $\mathfrak{A}$. A choice of an invariant measure on Σ leads to L^2-orthogonality relations for the canonical mapping M. We hope to continue to investigate the canonical mappings in the commutative case (the one closest in spirit with traditional pseudodifferential theory), having in view a more detailed study of representations, modulation spaces, and applications to spectral analysis [9].

2 Involutive Algebras Associated to a Twisted C^*-Dynamical System

We shall recall briefly, in a particular setting, some constructions and results concerning twisted crossed products algebras and Rieffel's pseudodifferential calculus.

The common starting point is a $2n$-dimensional real vector space Ξ endowed with a symplectic form $[\![\cdot,\cdot]\!]$. When needed we are going to suppose that $\Xi = \mathscr{X} \times \mathscr{X}^*$, with $\mathscr{X}^*$ the dual of the n-dimensional vector space $\mathscr{X}$, and that, for $X := (x, \xi)$, $Y := (y, \eta) \in \Xi$, the symplectic form reads $[\![X, Y]\!] := y \cdot \xi - x \cdot \eta$.

An action Θ of Ξ by automorphisms of a (maybe noncommutative) C^*-algebra $\mathcal{A}$ is also given. For $(f, X) \in \mathcal{A} \times \Xi$, we are going to use the notation $\Theta(f, X) = \Theta_X(f) = \Theta_f(X) \in \mathcal{A}$ for the X-transform of the element f. This action is assumed strongly continuous, that is, for any $f \in \mathcal{A}$ the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is continuous. The initial object, containing *the classical data*, is a quadruplet $(\mathcal{A}, \Theta, \Xi, [\![\cdot,\cdot,]\!])$ with the properties defined above.

To arrive at twisted crossed products, we define

$$\kappa : \Xi \times \Xi \to \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}, \quad \kappa(X, Y) := \exp\left(-\frac{i}{2}[\![X, Y]\!]\right), \tag{1}$$

and note that it is a group 2-cocycle, that is, for all $X, Y, Z \in \Xi$, one has

$$\kappa(X, Y)\kappa(X + Y, Z) = \kappa(Y, Z)\kappa(X, Y + Z), \quad \kappa(X, 0) = 1 = \kappa(0, X).$$

Thus, the classical data is converted into $(\mathcal{A}, \Theta, \Xi, \kappa)$, a very particular case of *twisted C^*-dynamical system* [11, 12]. To any twisted C^*-dynamical system, one associates canonically a C^*-algebra $\mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$ (called *twisted crossed product*). This is the enveloping C^*-algebra of the Banach *-algebra $(L^1(\Xi; \mathcal{A}), \diamond, ^\diamond, \|\cdot\|_1)$, where

$$\| G \|_1 := \int_\Xi dX \, \| G(X) \|_{\mathcal{A}}, \quad G^\diamond(X) := G(-X)^*,$$

and (symmetrized version of the standard form; cf. Remark 4.3)

$$(G_1 \diamond G_2)(X) := \int_\Xi dY \kappa(X, Y) \Theta_{(Y-X)/2}[G_1(Y)] \Theta_{Y/2}[G_2(X - Y)]. \tag{2}$$

In [11, 12] $\mathcal{A}$ is supposed separable; since our cocycle is explicit and very simple, this will not be needed here.

We turn now to *Rieffel deformation* [15, 16]. Let us denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞. It is a dense *-algebra of $\mathcal{A}$ and also a Fréchet algebra with the family of seminorms

$$|f|^k_{\mathcal{A}} := \sum_{|\alpha|=k} \frac{1}{\alpha!} \parallel \partial^\alpha_X [\Theta_X(f)]_{X=0} \parallel_{\mathcal{A}} \equiv \sum_{|\alpha|=k} \frac{1}{\alpha!} \parallel \delta^\alpha(f) \parallel_{\mathcal{A}} \quad k \in \mathbb{N}. \tag{3}$$

To quantize the above structure, one keeps the involution but introduces $\mathcal{A}^\infty$ the product

$$f \# g := 2^{2n} \int_\Xi \int_\Xi \mathrm{d}Y \, \mathrm{d}Z \, \mathrm{e}^{2\mathrm{i}[\![Y,Z]\!]} \Theta_Y(f) \Theta_Z(g), \tag{4}$$

suitably defined by oscillatory integral techniques. Thus, one gets a *-algebra $(\mathcal{A}^\infty, \#, ^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\| \cdot \|_{\mathfrak{A}}$ defined by Hilbert module techniques; we are going to call $\mathfrak{A}$ *the R-deformation of* $\mathcal{A}$. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends to a strongly continuous action on the C^*-algebra $\mathfrak{A}$, which will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincide with $\mathcal{A}^\infty$, even topologically, that is, the family (3) on $\mathcal{A}^\infty = \mathfrak{A}^\infty$ is equivalent to the family of seminorms

$$|f|^k_{\mathfrak{A}} := \sum_{|\alpha|=k} \frac{1}{\alpha!} \parallel \partial^\alpha_X [\Theta_X(f)]_{X=0} \parallel_{\mathfrak{A}} \equiv \sum_{|\alpha|=k} \frac{1}{\alpha!} \parallel \delta^\alpha(f) \parallel_{\mathfrak{A}}, \quad k \in \mathbb{N}. \tag{5}$$

An important particular case is obtained when $\mathcal{A}$ is the C^*-algebra $\mathrm{BC_u}(\Xi)$ of bounded uniformly continuous functions on the group Ξ, which is invariant under translations, that is, if $a \in \mathcal{A}$ and $X \in \Xi$, then $[\mathcal{T}_X(a)](\cdot) := a(\cdot - X) \in \mathcal{A}$. Note that the *-algebra of smooth vectors coincides with $\mathrm{BC}^\infty(\Xi)$, the space of all smooth complex functions on Ξ with bounded derivatives of every order. In this case, Rieffel's construction, done for $\Theta = \mathcal{T}$, reproduces essentially the standard Weyl calculus; we are going to use the special notations $\sharp$ (instead of #) for the corresponding composition law and $\mathscr{B}(\Xi)$ for the R-deformation of $\mathrm{BC_u}(\Xi)$.

One can also consider C^*-subalgebras $\mathcal{A}$ of $\mathrm{BC_u}(\Xi)$ that are invariant under translations. An important one is $C_0(\Xi)$, formed of all the complex continuous functions on Ξ that decay at infinity. Its Rieffel deformation will be denoted by $\mathscr{K}(\Xi)$; it contains the Schwartz space $\mathscr{S}(\Xi)$ densely. By Rieffel [15, Example 10.1 and Proposition 5.2], it is elementary, that is, isomorphic to the C^*-algebra of all compact operators in a separable Hilbert space.

Following [15], we introduce the Fréchet space $\mathscr{S}(\Xi; \mathfrak{A}^\infty)$ composed of smooth functions $F : \Xi \to \mathcal{A}^\infty = \mathfrak{A}^\infty$ with derivatives that decay rapidly with respect to all $| \cdot |^k_{\mathfrak{A}}$.

The relevant seminorms on the space $\mathscr{S}(\Xi;\mathfrak{A}^\infty)$ are $\{\|\cdot\|_{\mathfrak{A}}^{k,\beta,N} \mid k, N\in\mathbb{N},\ \beta\in\mathbb{N}^{2n}\}$ where

$$\| F \|_{\mathfrak{A}}^{k,\beta,N} := \sup_{X\in\Xi}\{(1+|X|)^N |(\partial^\beta F)(X)|_{\mathfrak{A}}^k\}, \tag{6}$$

and the index $\mathfrak{A}$ can be replaced by $\mathcal{A}$, by the argument above. We are going to use repeatedly the identification of $\mathscr{S}(\Xi;\mathfrak{A}^\infty)$ with the topological tensor product $\mathscr{S}(\Xi)\hat{\otimes}\mathfrak{A}^\infty$ (recall that the Fréchet space $\mathscr{S}(\Xi)$ is nuclear). On it (and on many other larger spaces), one can define obvious actions $\mathfrak{T}:=\mathcal{T}\otimes 1$ and $\mathcal{T}\otimes\Theta$ of the vector spaces Ξ and $\Xi\times\Xi$, respectively. Explicitly, for all $A, Y, X\in\Xi$, one sets $[\mathfrak{T}_A(F)](X):=F(X-A)$ and $[(\mathcal{T}_A\otimes\Theta_Y)F](X):=\Theta_Y[F(X-A)]$. Then on $\mathscr{S}(\Xi;\mathfrak{A}^\infty)$ one can introduce the composition law

$$(F_1\Box F_2)(X) = 2^{2n}\int_\Xi\int_\Xi \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{-2\mathrm{i}[\![A,B]\!]}[\mathfrak{T}_A(F_1)](X)\#[\mathfrak{T}_B(F_2)](X) \tag{7}$$

$$= 2^{4n}\int_\Xi\int_\Xi\int_\Xi\int_\Xi \mathrm{d}A\,\mathrm{d}B\,\mathrm{d}Y\,\mathrm{d}Z\,\mathrm{e}^{-2\mathrm{i}[\![A,B]\!]}\,\mathrm{e}^{2\mathrm{i}[\![Y,Z]\!]} \tag{8}$$

$$[(\mathcal{T}_A\otimes\Theta_Y)(F_1)](X)[(\mathcal{T}_B\otimes\Theta_Z)(F_2)](X).$$

Note that the last expression should be interpreted as an oscillatory integral [15] and that it involves the multiplication in the C^*-algebra $\mathcal{A}$. If the involution is given by $F^{\Box}(X):=F(X)^*,\ \forall X\in\Xi$, it can be shown that one gets a Fréchet *-algebra.

Remark 2.1. We recall that $\mathcal{A}^\infty=\mathfrak{A}^\infty$, even topologically, but the algebraic structures are different. When the forthcoming arguments will involve the composition #, in order to be more suggestive, we will use the notation $\mathscr{S}(\Xi;\mathfrak{A}^\infty)$. In other situations, the notation $\mathscr{S}(\Xi;\mathcal{A}^\infty)$ will be more natural. For instance, it is easy to check that $\mathscr{S}(\Xi;\mathcal{A}^\infty)$ is a (dense) *-subalgebra of the Banach *-algebra $(L^1(\Xi;\mathcal{A}),\diamond,^\diamond,\|\cdot\|_1)$, which is defined in terms of the product $\cdot$ on $\mathcal{A}$ and has a priori nothing to do with the composition law #. Proposition 4.2 is a good illustration for this distinction. □

Remark 2.2. One can also consider $\mathrm{BC_u}(\Xi;\mathfrak{A})$, the C^*-algebra of all bounded and uniformly continuous functions $F:\Xi\to\mathfrak{A}$. Rieffel deformation can also be applied to the new classical data $(\mathrm{BC_u}(\Xi;\mathfrak{A}),\mathfrak{T},\Xi,-[\![\cdot,\cdot]\!])$, getting essentially (7) as the corresponding composition law. By using the second part (8) of the formula, this can also be regarded as the Rieffel composition constructed from the extended twisted C^*-dynamical system $(\mathrm{BC_u}(\Xi;\mathcal{A}),\mathcal{T}\otimes\Theta,\ \Xi\times\Xi,\ \bar\kappa\otimes\kappa)$. This will not be needed in this form. But we are going to use below the fact that, for elements $f,g\in\mathfrak{A}^\infty$, $a,b\in\mathscr{S}(\Xi)$, one has

$(a\otimes f)\Box(b\otimes g)=(b\sharp a)\otimes(f\#g)$, so $\Box$ can be seen as the tensor product between # and the law opposite to $\sharp$. By Rieffel [16, Proposition 2.1], one can identify $\mathscr{K}(\Xi)\otimes\mathfrak{A}$ with the R-deformation of $C_0(\Xi)\otimes\mathcal{A}\equiv C_0(\Xi;\mathcal{A})$ and $\mathscr{B}(\Xi)\otimes\mathfrak{A}$ with the R-deformation of $\mathrm{BC_u}(\Xi)\otimes\mathcal{A}$. $\Box$

3 The Schrödinger Representation

We are going to denote by $\mathbb{B}(\mathcal{M},\mathcal{N})$ the space of all linear continuous operators acting between the topological vector spaces $\mathcal{M}$ and $\mathcal{N}$ and use the abbreviation $\mathbb{B}(\mathcal{M})$ for $\mathbb{B}(\mathcal{M},\mathcal{M})$.

Let us recall that $\mathscr{X}$ is a finite-dimensional vector space. The corresponding *Heisenberg algebra* $\mathfrak{h}_{\mathscr{X}}=\mathscr{X}\times\mathscr{X}^*\times\mathbb{R}$ is the Lie algebra with the bracket

$$[(x,\xi,t),(y,\eta,s)]:=(0,0,y\cdot\xi-x\cdot\eta).$$

We use notations as $\bar{X}=(x,\xi,t)$ and $X=(x,\xi)$. The *Heisenberg group* $\mathbb{H}_{\mathscr{X}}$ is just $\mathfrak{h}_{\mathscr{X}}$, thought of as a group with the multiplication $*$ defined by

$$\bar{X}*\bar{Y}=\bar{X}+\bar{Y}+\tfrac{1}{2}[\bar{X},\bar{Y}],\quad \bar{X},\bar{Y}\in\mathbb{H}_{\mathscr{X}}.$$

The unit element is $0\in\mathbb{H}_{\mathscr{X}}$ and the inversion mapping given by $\bar{X}^{-1}:=-\bar{X}$.

The Schrödinger representation is the unitary representation $\Pi:\mathbb{H}_{\mathscr{X}}\to\mathbb{B}(\mathcal{L})$ in the Hilbert space $\mathcal{L}:=L^2(\mathscr{X})$, defined by

$$[\Pi(\bar{X})u](y)=[\Pi(x,\xi,t)u](y)=\mathrm{e}^{\mathrm{i}(y\cdot\xi+\frac{1}{2}x\cdot\xi+t)}u(y+x)\quad\text{for a.e. } y\in\mathscr{X},\tag{9}$$

for arbitrary $u\in L^2(\mathscr{X})$ and $\bar{X}=(x,\xi,t)\in\mathbb{H}_{\mathscr{X}}$. When restricted to $\Xi=\mathscr{X}\times\mathscr{X}^*$ (which is not a subgroup and should be regarded as a quotient of $\mathbb{H}_{\mathscr{X}}$), Π becomes a projective representation that will be denoted by π: it satisfies

$$\pi(X)\pi(Y)=\kappa(X,Y)\pi(X+Y),\quad \forall X,Y\in\Xi.$$

The Wigner distributions defined by π are given by

$$\mathscr{W}(u,v):=\mathcal{F}(\langle u,\pi(\cdot)v\rangle),\quad u,v\in\mathcal{L}.$$

We used *the symplectic Fourier transform*

$$(\mathcal{F}a)(X):=\int_{\Xi}\mathrm{d}Y\,\mathrm{e}^{-\mathrm{i}[\![X,Y]\!]}a(Y),$$

and forced it to be L^2-unitary and satisfy $\mathcal{F}^2 = \mathrm{id}$, by a suitable choice of Lebesgue measure $\mathrm{d}Y$ on Ξ. Recall that $\mathscr{W}(u, v) \in \mathscr{S}(\Xi)$ when $u, v \in \mathscr{S}(\mathscr{X})$, $\mathscr{W} : \mathcal{L} \times \mathcal{L} \to L^2(\Xi)$ is an isometry and extends to a unitary mapping $\mathscr{W} : L^2(\mathscr{X}) \otimes \overline{L^2(\mathscr{X})} \to L^2(\Xi)$. *The Weyl pseudodifferential calculus* is then a linear isomorphism

$$\mathrm{Op} : \mathscr{S}'(\Xi) \to \mathbb{B}[\mathscr{S}(\mathscr{X}), \mathscr{S}'(\mathscr{X})], \quad \langle v, \mathrm{Op}(a)u\rangle = \langle \overline{\mathscr{W}(v, u)}, a\rangle. \tag{10}$$

Recall also that $\mathrm{Op}[\mathscr{W}(u, v)] = \langle \cdot \mid v\rangle u$ for all $u, v \in \mathcal{L}$, and $\mathrm{Op} : L^2(\Xi) \to \mathfrak{S}_2(\mathcal{L})$ (Hilbert–Schmidt operators) is unitary. For $a, b \in \mathscr{S}'(\Xi)$, $a \sharp b$ is the symbol of the operator $\mathrm{Op}(a)\,\mathrm{Op}(b)$ whenever this is well defined and continuous from $\mathscr{S}(\mathscr{X})$ to $\mathscr{S}'(\mathscr{X})$. Of course, the symbol $\sharp$ is an extension of the one used in the previous section. The action of $\mathrm{Op}(a)$ on $\mathscr{S}(\mathscr{X})$ or $\mathcal{L} := L^2(\mathscr{X})$ (under various assumptions on the symbol a and with various interpretations) is given by

$$[\mathrm{Op}(a)v](x) := \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi}\, a\left(\frac{x+y}{2}, \xi\right) v(y). \tag{11}$$

Consider next the space of operators

$$\mathbb{B}_{\mathrm{u}}(\mathcal{L}) = \{T \in \mathbb{B}(\mathcal{L}) \mid \Xi \ni X \to \pi(X)T\pi(-X) \in \mathbb{B}(\mathcal{L}) \text{ is norm continuous}\}.$$

Then $\mathbb{B}_{\mathrm{u}}(\mathcal{L})$ is a proper C^*-subalgebra of $\mathbb{B}(\mathcal{L})$ with the norm given by the operator norm and involution given by Hilbert space adjoint, and it contains the ideal $\mathbb{K}(\mathcal{L})$ of compact operators on $\mathcal{L}$ (see [6, Theorem 1.1]). The representation

$$\pi \otimes \bar{\pi} : \Xi \to \mathbb{B}[\mathbb{B}_{\mathrm{u}}(\mathcal{L})], \quad (\pi \otimes \bar{\pi})(X)T = \pi(X)T\pi(-X)$$

is then strongly continuous. Let $\mathbb{B}_{\mathrm{u}}^{\infty}(\mathcal{L})$ be the space of smooth vectors for this representation. Then $\mathbb{B}_{\mathrm{u}}^{\infty}(\mathcal{L})$ is dense in $\mathbb{B}_{\mathrm{u}}(\mathcal{L})$ [6, Theorem 1.1], and consists precisely of those Weyl pseudodifferential operators with symbols in $BC^{\infty}(\Xi)$ [6, Theorem 1.2, 7, Theorem 2.3.7].

Lemma 3.1. The Weyl calculus Op realizes an isomorphism between $\mathscr{B}(\Xi)$ (the R-deformation of $BC_{\mathrm{u}}(\Xi)$) and $\mathbb{B}_{\mathrm{u}}(\mathcal{L})$. The image through Op of $\mathscr{K}(\Xi)$ is precisely $\mathbb{K}(\mathcal{L})$. □

Proof. Indeed, recall that when $a \in \mathrm{BC}^\infty(\Xi)$, the norm $\| a \|_{\mathscr{B}(\Xi)}$ of a in the Rieffel algebra is given by the norm of the operator $L_a : \mathscr{S}(\Xi) \to \mathscr{S}(\Xi)$, $L_a(b) = a\#b$, where on $\mathscr{S}(\Xi)$ one considers the L^2-norm (a particular case of Rieffel [15, Proposition 4.15]). Taking $b = \mathscr{W}(u, v)$, with $u, v \in \mathscr{S}(\mathscr{X})$, one obtains that $a\#\mathscr{W}(u, v) = \mathscr{W}(\mathrm{Op}(a)u, v)$, hence

$$\|L_a[\mathscr{W}(u, v)]\|_{L^2(\Xi)} = \|v\| \, \|\mathrm{Op}(a)u\|, \quad u, v \in \mathscr{S}(\mathscr{X}).$$

Thus, $\|\mathrm{Op}(a)\|_{\mathbb{B}(\mathcal{L})} \le \|a\|_{\mathscr{B}(\Xi)}$. On the other hand, denoting by $\| \cdot \|_{\mathfrak{S}_2(\mathcal{L})}$ the Hilbert–Schmidt norm, one has

$$\|L_a(b)\|_{L^2(\Xi)} = \|\mathrm{Op}(a\#b)\|_{\mathfrak{S}_2(\mathcal{L})} \le \|\mathrm{Op}(a)\|_{\mathbb{B}(\mathcal{L})} \|\mathrm{Op}(b)\|_{\mathfrak{S}_2(\mathcal{L})} = \|\mathrm{Op}(a)\|_{\mathbb{B}(\mathcal{L})} \|b\|_{L^2(\Xi)},$$

hence $\|\mathrm{Op}(a)\|_{\mathbb{B}(\mathcal{L})} \ge \|a\|_{\mathscr{B}(\Xi)}$. It follows that the norm of the operator L_a is in fact equal to the norm of $\mathrm{Op}(a)$ in $\mathbb{B}(\mathcal{L})$. The Rieffel algebra $\mathscr{B}(\Xi)$ is the closure of $\mathrm{BC}^\infty(\Xi)$ in the norm $a \to \|a\|_{\mathscr{B}(\Xi)} = \|L_a\|_{\mathbb{B}[L^2(\Xi)]}$, hence it is isomorphic to $\mathbb{B}_{\mathrm{u}}(\mathcal{L})$, the closure of $\mathbb{B}_{\mathrm{u}}^\infty(\mathcal{L}) = \mathrm{Op}[\mathrm{BC}^\infty(\Xi)]$, as stated.

Now the last statement of the lemma is trivial if we recall that $\mathrm{Op}[\mathscr{S}(\Xi)] \subset \mathbb{K}(\mathcal{L})$. ■

4 The Canonical Mappings

Definition 4.1. On $\mathscr{S}(\Xi; \mathfrak{A}^\infty)$ we introduce the canonical mappings

$$[M(F)](X) := \int_\Xi \mathrm{d}Y \, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]} \Theta_Y[F(Y)] \tag{12}$$

and

$$[M^{-1}(G)](X) := \int_\Xi \mathrm{d}Y \, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]} \Theta_{-X}[G(Y)]. \tag{13}$$

□

To give a precise meaning to these relations, use *the* (*symplectic*) *partial Fourier transform*

$$\mathfrak{F} \equiv \mathcal{F} \otimes 1 : \mathscr{S}(\Xi; \mathfrak{A}^\infty) \to \mathscr{S}(\Xi; \mathcal{A}^\infty), \quad (\mathfrak{F}F)(X) := \int_\Xi \mathrm{d}Y \, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]} F(Y).$$

Defining also C by $[C(F)](X) := \Theta_X[F(X)]$, we have $M = \mathfrak{F} \circ C$ and $M^{-1} = C^{-1} \circ \mathfrak{F}$.

Proposition 4.2. The mapping $M:(\mathscr{S}(\Xi;\mathfrak{A}^\infty),\Box,{}^\Box)\to(\mathscr{S}(\Xi;\mathcal{A}^\infty),\diamond,{}^\diamond)$ is an isomorphism of Fréchet *-algebras and M^{-1} is its inverse. □

Proof. The partial Fourier transform is an isomorphism. One also checks that C is an isomorphism of $\mathscr{S}(\Xi;\mathfrak{A}^\infty)$; this follows from the explicit form of the seminorms on $\mathcal{S}(\Xi;\,\mathfrak{A}^\infty)$, from the fact that Θ_X is isometric, and from the formula

$$\partial^\beta[\Theta_X(F(X))]=\sum_{\gamma\le\beta}C_{\beta\gamma}\Theta_X\{\delta^\gamma[(\partial^{\beta-\gamma}F)(X)]\}.$$

With these remarks, we conclude that $M=\mathfrak{F}\circ C$ and $M^{-1}=C^{-1}\circ\mathfrak{F}$ are reciprocal topological linear isomorphisms.

We still need to show that M is a *-morphism. *For the involution*:

$$[M(F)]^\diamond(X)=\left\{\int_\Xi \mathrm{d}Y\,\mathrm{e}^{\mathrm{i}[\![X,Y]\!]}\Theta_Y[F(Y)]\right\}^*\qquad=\int_\Xi \mathrm{d}Y\,\mathrm{e}^{-\mathrm{i}[\![X,Y]\!]}\Theta_Y[F(Y)^*]=[M(F^\Box)](X).$$

For the product: it is enough to show that $M^{-1}[M(F)\diamond M(G)]=F\Box G$ for all $F,G\in\mathscr{S}(\Xi;\mathfrak{A}^\infty)$. One has (iterated integrals):

$$\begin{aligned}
(M^{-1}[MF\diamond MG])(X)&=\int_\Xi \mathrm{d}Y_1\,\mathrm{e}^{-\mathrm{i}[\![X,Y_1]\!]}\Theta_{-X}\{[MF\diamond MG](Y_1)\}\\
&=\int_\Xi \mathrm{d}Y_1\,\mathrm{e}^{-\mathrm{i}[\![X,Y_1]\!]}\Theta_{-X}\left\{\int_\Xi \mathrm{d}Y_2\,\mathrm{e}^{-\frac{\mathrm{i}}{2}[\![Y_1,Y_2]\!]}\Theta_{(Y_2-Y_1)/2}[(MF)(Y_2)]\Theta_{Y_2/2}[(MG)(Y_1-Y_2)]\right\}\\
&=\int_\Xi \mathrm{d}Y_1\int_\Xi \mathrm{d}Y_2\,\mathrm{e}^{-\mathrm{i}[\![X,Y_1]\!]}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}[\![Y_1,Y_2]\!]}\Theta_{-X}\{\Theta_{(Y_2-Y_1)/2}[(MF)(Y_2)]\Theta_{Y_2/2}[(MG)(Y_1-Y_2)]\}\\
&=\int_\Xi \mathrm{d}Y_1\int_\Xi \mathrm{d}Y_2\,\mathrm{e}^{-\mathrm{i}[\![X,Y_1]\!]}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}[\![Y_1,Y_2]\!]}\cdot\Theta_{(Y_2-Y_1)/2-X}\left\{\int_\Xi \mathrm{d}Y_3\,\mathrm{e}^{-\mathrm{i}[\![Y_2,Y_3]\!]}\Theta_{Y_3}[F(Y_3)]\right\}\\
&\quad\cdot\Theta_{Y_2/2-X}\left\{\int_\Xi \mathrm{d}Y_4\,\mathrm{e}^{-\mathrm{i}[\![Y_1-Y_2,Y_4]\!]}\Theta_{Y_4}[G(Y_4)]\right\}\\
&=\int_\Xi \mathrm{d}Y_1\int_\Xi \mathrm{d}Y_2\int_\Xi \mathrm{d}Y_3\int_\Xi \mathrm{d}Y_4\,\mathrm{e}^{-\mathrm{i}[\![X,Y_1]\!]}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}[\![Y_1,Y_2]\!]}\,\mathrm{e}^{-\mathrm{i}[\![Y_2,Y_3]\!]}\,\mathrm{e}^{-\mathrm{i}[\![Y_1-Y_2,Y_4]\!]}\\
&\quad\cdot\Theta_{Y_3+(Y_2-Y_1)/2-X}[F(Y_3)]\Theta_{Y_4+Y_2/2-X}[G(Y_4])\\
&=2^{4n}\int_\Xi \mathrm{d}Y\int_\Xi \mathrm{d}Z\int_\Xi \mathrm{d}Y_3\int_\Xi \mathrm{d}Y_4\,\mathrm{e}^{-2\mathrm{i}[\![X,Y_3-Y_4]\!]}\,\mathrm{e}^{2\mathrm{i}[\![Y,Z]\!]}\,\mathrm{e}^{-2\mathrm{i}[\![Y_3,Y_4]\!]}\Theta_Y[F(Y_3)]\Theta_Z[G(Y_4)].
\end{aligned}$$

For the last equality, we made the substitution $Y = Y_3 + \frac{1}{2}(Y_2 - Y_1) - X,\ Z = Y_4 + \frac{1}{2}Y_2 - X$. Finally, setting $Y_3 = X - A,\ Y_4 = X - B$, we get

$$(M^{-1}[MF \diamond MG])(X) = [F\square G](X)$$
$$= 2^{4n}\int_{\Xi} \mathrm{d}Y \int_{\Xi} \mathrm{d}Z \int_{\Xi} \mathrm{d}A \int_{\Xi} \mathrm{d}B\, \mathrm{e}^{-2\mathrm{i}[\![A,B]\!]}\, \mathrm{e}^{2\mathrm{i}[\![Y,Z]\!]}\, \Theta_Y[F(X-A)]\Theta_Z[G(X-B)]. \qquad \blacksquare$$

Remark 4.3. Let us make some comments about how one could modify the definitions above. We are going to need the notation $[C_\alpha(F)](X) := \Theta_{\alpha X}[F(X)]$, where $X \in \Xi,\ F \in \mathscr{S}(\Xi; \mathcal{A}^\infty)$ (or $F \in L^1(\Xi; \mathcal{A})$) and α is a real number. All these operations are isomorphisms and our previous transformation C coincides with C_1. The traditional composition law in the twisted crossed product is not (2), but

$$(G_1 \diamond' G_2)(X) := \int_{\Xi} \mathrm{d}Y \kappa(X, Y) G_1(Y) \Theta_Y[G_2(X - Y)].$$

The distinction is mainly an ordering matter and it corresponds to the distinction between the Weyl and the Kohn–Nirenberg forms of pseudodifferential theory. Applying $C_{1/2}$ leads to an isomorphism between the two algebraic structures. So, if we want to use this second realization, we should replace $M = \mathfrak{F}C_1$ with $M' := C_{1/2}\mathfrak{F}C_1$, leading explicitly to

$$[M'(F)](X) := \int_{\Xi} \mathrm{d}Y\, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]}\, \Theta_{Y+X/2}[F(Y)]. \qquad \square$$

5 The C^*-Algebraic Isomorphism

We recall that $\mathscr{K}(\Xi)$, with multiplication $\sharp$, has been defined as the R-deformation of the commutative C^*-algebra $C_0(\Xi)$ on which Ξ acts by translations. Then $\mathscr{K}(\Xi)$ is an elementary (hence nuclear) C^*-subalgebra of $\mathscr{B}(\Xi)$, and $\mathscr{S}(\Xi)$ is dense in $\mathscr{K}(\Xi)$. The Fréchet *-algebra $\mathscr{S}(\Xi; \mathfrak{A}^\infty) \equiv \mathscr{S}(\Xi)\hat{\otimes}\mathfrak{A}^\infty$ with the composition law $\square$ given in (8) is dense in the C^*-algebra $\mathscr{K}(\Xi) \otimes \mathfrak{A}$, which can be viewed (see Remark 2.2 and [16, Proposition 2.1]) as the R-deformation of $C_0(\Xi) \otimes \mathcal{A}$ with respect to the action of $\Xi \times \Xi$ composed of translations in the first variable and the initial action Θ in the second.

This section is mainly dedicated to the proof of the next result.

Theorem 5.1. The mapping M extends to a C^*-isomorphism : $\mathscr{K}(\Xi) \otimes \mathfrak{A} \to \mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$. $\square$

The following definition (see [18, Definition 1.2] and the concept of *differential seminorm* in [5, Definition 3.1]) isolates a situation in which any injective morphism between a dense *-subalgebra of a C^*-algebra and another C^*-algebra can be extended to a C^*-algebraic monomorphism.

Definition 5.2. Let $\mathscr{F}$ be a dense Fréchet subalgebra of a C^*-algebra $\mathscr{C}$. We say that $\mathscr{F}$ *satisfies the Blackadar–Cuntz condition in* $\mathscr{C}$ if the topology on $\mathscr{F}$ is given by a family of seminorms $\{p_k\}_{k\geq 0}$ such that p_0 is the C^*-norm giving the topology on $\mathscr{C}$ and

$$p_k(ab) \leq \sum_{i+j=k} p_i(a)\, p_j(b), \qquad a, b \in \mathscr{F}.$$

□

Tracing back through [5], one realizes that if $\mathscr{F}$ satisfies the Blackadar–Cuntz condition in $\mathscr{C}$, it is a smooth algebra in the sense of Blackadar and Cuntz [5, Definition 6.6]. Actually, the more general concept of *derived seminorm* [5, Definition 5.1] involved in the definition of a smooth algebra is meant to model quotients of differential seminorms. Therefore, the following result is in fact a particular case of Blackadar and Cuntz [5, Proposition 6.8].

Proposition 5.3. Assume that $\mathscr{F}$ is a dense Fréchet subalgebra of a C^*-algebra $\mathscr{C}$ and satisfies the Blackadar–Cuntz condition in $\mathscr{C}$. Then if $\mathscr{D}$ is another C^*-algebra and $\Phi : \mathscr{F} \mapsto \mathscr{D}$ is an injective *-morphism, then Φ is isometric for the C^*-norm on $\mathscr{C}$. □

We now prove Theorem 5.1.

Proof. The algebra $\mathscr{S}(\Xi)\hat{\otimes}\mathfrak{A}^\infty$ is a dense subalgebra of $\mathscr{K}(\Xi)\otimes\mathfrak{A}$. As mentioned before, it can be identified to $\mathscr{S}(\Xi;\mathfrak{A}^\infty)$. Proposition 4.2 gives an injective *-morphism $M : \mathscr{S}(\Xi;\mathfrak{A}^\infty) \to \mathcal{A} \rtimes^\kappa_\Theta \Xi$ with dense range. If one proves that $\mathscr{S}(\Xi)\hat{\otimes}\mathfrak{A}^\infty$ satisfies Blackadar–Cuntz condition in $\mathscr{K}(\Xi)\otimes\mathfrak{A}$, Proposition 5.3 shows that M is isometric for the C^*-norm on $\mathscr{K}(\Xi)\otimes\mathfrak{A}$, so it extends to an isomorphism : $\mathscr{K}(\Xi)\otimes\mathfrak{A} \to \mathcal{A} \rtimes^\kappa_\Theta \Xi$.

To show the Blackadar–Cuntz condition for $\mathscr{S}(\Xi)\hat{\otimes}\mathfrak{A}^\infty$, we are going to express it as the space of smooth vectors for a continuous group action in $\mathscr{K}(\Xi)\otimes\mathfrak{A}$.

Using the Schrödinger representation (9) of the Heisenberg group $\mathbb{H}_{\mathscr{X}}$ in $\mathcal{L} = L^2(\mathscr{X})$, we consider the strongly continuous representation (by Banach space isomorphisms)

$$\Delta : \mathbb{H}_{\mathscr{X}} \times \mathbb{H}_{\mathscr{X}} \to \mathbb{B}[\mathbb{K}(\mathcal{L})],$$

$$\Delta(\bar{X}, \bar{Y})T = \Pi(\bar{X})T\Pi(-\bar{Y}), \qquad \bar{X}, \bar{Y} \in \mathbb{H}_{\mathscr{X}}.$$

Note that $\mathbb{K}(\mathcal{L})$ is an admissible ideal in $\mathbb{B}(\mathcal{L})$, as in [2, Definition 3.8]. Also recall that the Weyl–Pedersen calculus for general nilpotent Lie groups G, introduced in [13] and developed in [1–3], particularizes to the usual Weyl calculus if $G=\mathbb{H}_{\mathscr{X}}$ is the Heisenberg group; therefore one can use the results of these papers. It follows from [2, Theorems 4.6 and 3.13] (see also [3, 13, Theorem 4.1.4]) that the space $\mathbb{K}(\mathcal{L})^\infty$ of smooth vectors of the representation $\varDelta$ is precisely $\mathrm{Op}[\mathscr{S}(\varXi)]$ and that $\mathrm{Op}:\mathscr{S}(\varXi)\to\mathbb{K}(\mathcal{L})^\infty$ is a topological isomorphism of Fréchet spaces (a restriction of the isomorphism given by Lemma 3.1). Hence, $\mathrm{Op}\hat{\otimes}\,\mathrm{id}:\mathscr{S}(\varXi)\hat{\otimes}\,\mathfrak{A}^\infty\to\mathbb{K}(\mathcal{L})^\infty\hat{\otimes}\,\mathfrak{A}^\infty$ is also an isomorphism of Fréchet spaces. Thus, to complete the proof, it will be enough to show that $\mathbb{K}(\mathcal{L})^\infty\hat{\otimes}\,\mathfrak{A}^\infty$ satisfies the Blackadar–Cuntz condition in $\mathbb{K}(\mathcal{L})\otimes\mathfrak{A}$.

We set

$$\varOmega:\mathbb{H}_{\mathscr{X}}\times\mathbb{H}_{\mathscr{X}}\times\varXi\to\mathbb{B}[\mathbb{K}(\mathcal{L})\otimes\mathfrak{A}],$$

$$\varOmega(\bar{X},\bar{Y},Z)=\varDelta(\bar{X},\bar{Y})\otimes\varTheta_Z,\quad \bar{X},\bar{Y}\in\mathbb{H}_{\mathscr{X}},\ Z\in\varXi.$$

It is easy to check that $\varOmega$ is a strongly continuous representation and that its space of smooth vectors $[\mathbb{K}(\mathcal{L})\otimes\mathfrak{A}]^\infty$ coincides with the (unique) topological tensor product $\mathbb{K}(\mathcal{L})^\infty\hat{\otimes}\,\mathfrak{A}^\infty$. The transformations $\varOmega(\bar{X},\bar{Y},Z)$ are isometric.

It follows that the topology of the tensor product $\mathbb{K}(\mathcal{L})^\infty\hat{\otimes}\,\mathfrak{A}^\infty$ is also given by the countable family of seminorms

$$p_k(\varPhi):=\sum_{|\alpha|+|\beta|\le k}\frac{1}{\alpha!\beta!}\|\partial_{\bar{X}}^{\alpha_1}\partial_{\bar{Y}}^{\alpha_2}\partial_Z^{\beta}[\varOmega(\bar{X},\bar{Y},Z)\varPhi]_{\bar{X}=\bar{Y}=Z=0}\|_{\mathbb{K}(\mathcal{L})\otimes\mathfrak{A}},$$

where $\alpha=(\alpha_1,\alpha_2)\in\mathbb{N}^{4n+2}$ and $\beta\in\mathbb{N}^{2n}$. When computing on products $\varPhi\circ\varPsi$, one has to face the fact that the action $\varDelta$ is not automorphic on $\mathbb{K}(\mathcal{L})$. Note, however, that when S, $T\in\mathbb{K}(\mathcal{L})$, we have

$$\varDelta(\bar{X},\bar{Y})(ST)=[\varDelta(\bar{X},0)S][\varDelta(0,\bar{Y})T],$$

implying for all $\varPhi,\varPsi\in\mathbb{K}(\mathcal{L})\otimes\mathfrak{A}$ and all $(\bar{X},\bar{Y},Z)\in\mathbb{H}_{\mathscr{X}}\times\mathbb{H}_{\mathscr{X}}\times\varXi$,

$$\varOmega(\bar{X},\bar{Y},Z)(\varPhi\circ\varPsi)=[\varOmega(\bar{X},0,Z)\varPhi]\circ[\varOmega(0,\bar{Y},Z)\varPsi].$$

Then a simple calculation shows that

$$p_k(\varPhi\circ\varPsi)\le\sum_{i+j=k}p_i(\varPhi)\,p_j(\varPsi),$$

for all $\varPhi,\varPsi\in\mathbb{K}(\mathcal{L})^\infty\hat{\otimes}\,\mathfrak{A}^\infty$. One also has $p_0(\varPhi)=\|\varPhi\|_{\mathbb{K}(\mathcal{L})\otimes\mathfrak{A}}$, so the proof is completed. ∎

Remark 5.4. Let us consider the continuous action $\beta : \Xi \to \mathrm{Aut}(\mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi)$ given for $G \in L^1(\Xi; \mathcal{A})$ by

$$[\beta_Z(G)](X) := \mathrm{e}^{\mathrm{i}[\![X,Z]\!]} G(X), \quad X, Z \in \Xi$$

(this is the dual action in disguise). Then a short computation gives, for any $Z \in \Xi$,

$$M \circ (\mathcal{T}_{-Z} \otimes \Theta_Z) = \beta_Z \circ M, \tag{14}$$

and so actually M can be seen as an isomorphism of C^*-dynamical systems. Thus, the twisted crossed product $\mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$ endowed with the action β can be seen as (an isomorphic copy of) the Rieffel deformation of the C^*-algebra $C_0(\Xi; \mathcal{A})$. □

6 Applications

One can rephrase Theorem 5.1 by saying that $\mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$ is (isomorphic to) the stable algebra of $\mathfrak{A}$. In particular, $\mathfrak{A}$ and $\mathcal{A} \rtimes_{\Theta}^{\kappa} \Xi$ are stably isomorphic. Therefore, they have identical representation theories (indexed by covariant representations of the system $(\mathcal{A}, \Theta, \kappa, \Xi)$), isomorphic ideal lattices, and there are canonical homeomorphisms between the corresponding spaces of primitive ideals [14].

We investigate now the interplay between the canonical maps and Ξ-morphisms. Let $(\mathcal{A}^j, \Theta^j, \Xi, \kappa)$, $j = 1, 2$, be two sets of classical data and let $\mathcal{R} : \mathcal{A}^1 \to \mathcal{A}^2$ be a Ξ-morphism, that is, a C^*-morphism intertwining the two actions Θ^1, Θ^2. Then $\mathcal{R}$ acts coherently on C^∞-vectors ($\mathcal{R}[\mathcal{A}^{1,\infty}] \subset \mathcal{A}^{2,\infty}$) and extends to a morphism $\mathfrak{R} : \mathfrak{A}^1 \to \mathfrak{A}^2$ of the R-quantized C^*-algebras that also intertwines the corresponding actions (see [15]). On the other hand [11, 12], another C^*-morphism $\mathcal{R}^{\rtimes} : \mathcal{A}^1 \rtimes_{\Theta^1}^{\kappa} \Xi \to \mathcal{A}^2 \rtimes_{\Theta^2}^{\kappa} \Xi$ is assigned canonically to $\mathcal{R}$, uniquely defined by

$$[\mathcal{R}^{\rtimes}(F)](X) := \mathcal{R}[F(X)], \quad \forall\, F \in L^1(\Xi; \mathcal{A}^1).$$

Proposition 6.1. Denoting by id the identical map on $\mathscr{K}(\Xi)$ and by M^j the canonical map for the data $(\mathcal{A}^j, \Theta^j, \kappa, \Xi)$, one has

$$\mathcal{R}^{\rtimes} \circ M^1 = M^2 \circ (\mathrm{id} \otimes \mathfrak{R}). \tag{15}$$

□

Proof. It is enough to compute on $F \in \mathscr{S}(\Xi;\mathfrak{A}^\infty)$:

$$
\begin{aligned}
[(\mathcal{R}^{\rtimes} \circ M^1)(F)](X) &= \mathcal{R}[(M^1 F)(X)] \\
&= \mathcal{R}\left[\int_\Xi \mathrm{d}Y\, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]}\Theta^1_Y(F(Y))\right] \\
&= \int_\Xi \mathrm{d}Y\, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]}\mathcal{R}[\Theta^1_Y(F(Y))] \\
&= \int_\Xi \mathrm{d}Y\, \mathrm{e}^{-\mathrm{i}[\![X,Y]\!]}\Theta^2_Y[\mathcal{R}(F(Y))] \\
&= M^2[(\mathrm{id}\otimes\mathfrak{R})F](X).
\end{aligned}
$$
■

The next two results have been proved in [17] without asking the skew-symmetric operator J to be nondegenerate (see also [8]). The proofs relying on Theorem 5.1 are very simple.

Corollary 6.2. The C^*-algebras $\mathcal{A}$ and $\mathfrak{A}$ have the same K-groups. □

Proof. Since $\mathfrak{A}$ and $\mathcal{A} \rtimes^\kappa_\Theta \Xi$ are stably isomorphic, they have the same K-theory [4]. On the other hand, by 'the stabilization trick' [11], $\mathcal{A} \rtimes^\kappa_\Theta \Xi$ is stably isomorphic to a usual (untwisted) crossed product $(\mathcal{A}\otimes\mathscr{K}) \rtimes_\Gamma \Xi$ associated to an action Γ of Ξ on the tensor product of $\mathcal{A}$ with an elementary algebra $\mathscr{K}$. The vector space Ξ has even dimension, and hence, by Connes' Thom isomorphism [4], the K-groups of the crossed product coincide with the K-groups of $\mathcal{A}\otimes\mathscr{K}$, that is, with those of $\mathcal{A}$. ■

Corollary 6.3. The C^*-algebras $\mathcal{A}$ and $\mathfrak{A}$ are simultaneously nuclear. □

Proof. The argument is analogous to the previous one. One must also recall [4, Theorem 15.8.2] that nuclearity is preserved under stable isomorphism and that the crossed product with a commutative group of a C^*-algebra $\mathcal{B}$ is nuclear iff $\mathcal{B}$ is nuclear (for the converse use Takai duality). ■

If $\mathcal{A}$ is commutative, by Gelfand theory, it is isomorphic (and will be identified) to $C_0(\Sigma)$, the C^*-algebra of all complex continuous functions on the locally compact space Σ which converge to zero at infinity. The space Σ is a homeomorphic copy of the Gelfand spectrum of $\mathcal{A}$ and it is compact iff $\mathcal{A}$ is unital. Then the group Θ of automorphisms is induced by an action (also called Θ) of Ξ by homeomorphisms of Σ. We are going to use the convention

$$[\Theta_X(f)](\sigma) := f[\Theta_X(\sigma)], \quad \forall\, \sigma \in \Sigma,\ X \in \Xi,\ f \in \mathcal{A},$$

as well as the notation $\Theta_X(\sigma) = \Theta(X, \sigma)$ for the X-transform of the point σ. Let us set $\mathfrak{A} =: \mathfrak{C}(\Sigma)$ for the (noncommutative) Rieffel C^*-algebra associated to $C_0(\Sigma)$ by deformation and $\mathfrak{C}^\infty(\Sigma)$ for the space of smooth vectors under the action Θ.

Remark 6.4. A rather surprising picture follows from a symmetry argument. Using [15, Theorem 6.5], one gets easily an isomorphism $\mathscr{K}(\Xi) \otimes \mathcal{A} \cong \mathfrak{A} \rtimes_{\Theta}^{\bar{\kappa}} \Xi$. The complex conjugated cocycle $\bar{\kappa}$ is defined by the symplectic form $-[\![\cdot,\cdot]\!]$. One usually thinks of $\mathcal{A}$ as a rather simple C^*-algebra, giving after deformation a more complicated one $\mathfrak{A}$. In the commutative case, for instance, we might be surprised that the twisted crossed product $\mathfrak{C}(\Sigma) \rtimes_{\Theta}^{\bar{\kappa}} \Xi$ decomposes as $C_0(\Sigma; \mathscr{K}(\Xi))$. □

Remark 6.5. Twisted crossed products with commutative C^*-algebras are discussed in [10]. In some situations, their primitive ideal space is understood (as a topological space) and this can be transferred by our stable isomorphism to the level of $\mathfrak{C}(\Sigma)$. By Packer [10, Example 4.3], for instance, if the action Θ is free (all the isotropy groups are trivial), then Prim$[\mathfrak{C}(\Sigma)]$ is homeomorphic to the quasiorbit space $Q^\Theta(\Sigma)$. If, in addition, Θ is minimal, $\mathfrak{C}(\Sigma)$ will be a simple C^*-algebra. If Θ is minimal without being free, the situation is described in [10, Example 4.11]. □

Remark 6.6. Assume that Ξ act freely on Σ. By Theorem 5.1 and [10, Theorem 4.5], $\mathfrak{C}(\Sigma)$ is a continuous trace C^*-algebra if and only if the action Θ is proper. □

We discuss briefly orthogonality matters. On Σ, we pick a Θ-invariant measure $d\sigma$. The relationship between the spaces $\mathscr{S}(\Xi; \mathcal{A}^\infty)$ and $L^2(\Xi \times \Sigma)$ depends on the assumptions we impose on $(\Sigma, \mathrm{d}\sigma)$. If $\mathrm{d}\sigma$ is a finite measure, for instance, one has $\mathscr{S}(\Xi; \mathcal{A}^\infty) \subset L^2(\Xi \times \Sigma)$. Anyhow, the canonical map can be defined independently on $L^2(\Xi \times \Sigma)$.

Proposition 6.7. One has the orthogonality relations valid for $F, G \in L^2(\Sigma \times \Xi)$:

$$\langle \overline{M(F)}, M(G) \rangle_{\Xi \times \Sigma} = \langle \bar{F}, G \rangle_{\Xi \times \Sigma}. \tag{16}$$

Thus, the operator $M : L^2(\Xi \times \Sigma) \to L^2(\Xi \times \Sigma)$ is unitary. □

Proof. It is enough to note that $M = \mathfrak{F} \circ C$ and to use the fact that $\mathfrak{F}$ and C are isomorphisms of $L^2(\Sigma \times \Xi)$ if $d\sigma$ is Θ-invariant. ■

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