Decomposition and reconstruction of multidimensional signals using polyharmonic pre-wavelets

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Abstract

In this paper, we build a multidimensional wavelet decomposition based on polyharmonic B-splines. The pre-wavelets are polyharmonic splines and so not tensor products of univariate wavelets. Explicit construction of the filters specified by the classical dyadic scaling relations is given and the decay of the functions and the filters is shown. We then design the decomposition/recomposition algorithm by means of downsampling/upsampling and convolution products.

Keywords: Pre-wavelet; Multiresolution; Polyharmonic splines; Multidimensional

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1. Introduction

The theory and practice of wavelet decomposition of signals and functions is a particularly attractive research area in approximation theory and in signal processing. Their applications range from transmitting or filtering signals to the numerical solution of partial differential equations (see, e.g., [4,8]). The most used wavelets bases for multiresolution analysis (MRA) in multiple dimension are obtained through tensor products of one-dimensional functions (see, e.g., [5,6]). Quite from the beginning, a more general multi-dimensional approach was given by the pioneers of the MRA (see [15,16,21]); for the actual construction of multivariate pre-wavelets, see [7].

In this context, polyharmonic functions have been considered very often in literature. Indeed, the possibility of doing MRA generated by classes of polyharmonic splines have been studied in some detail as far back as the works [15] and [17], where the scaling function generates orthogonal Riesz basis, and many others authors continued this development [14,16,18,20]. In these works, in particular we find the class of the so-called polyharmonic B-splines. As is well known, in [22,23] polyharmonic B-splines were introduced as a finite linear combination of translates—actually a discretization of the iterated Laplacean operator Δ^m—of the fundamental solution of Δ^m. These basis functions were earlier considered in [10], then in [11] and [12] for improving the condition number of linear systems involved for thin plate spline interpolation, and were used for cardinal interpolation (see, e.g., [3]). Despite the fact that the polyharmonic B-splines violate the rapid-decay requirements of classical wavelet theory (they typically algebraically decay), they generate Riesz bases and are perfectly scaling functions for $m > s/2$. This was proved for instance by Micchelli et al. [20] for a wider class of refinable functions, or by Madych [18], cf. pp. 274–276, with a different approach.

As it appears from the cited works, the theory of polyharmonic MRA have been well designed; whereas, the aspects connected with the actual construction of the related wavelet decomposition has not been addressed all that often in the applied literature, especially in dimension greater than two. For instance, one can find a numerical application in [19] where the refinement equation of the Lagrangean polyharmonic splines is used to recover a surface by means of the Fourier transform and the usual discrete convolution product; and very recently (actually later than, but independently from this work), Van De Vill et al. [26] defined a particular polyharmonic B-spline, which they called "isotropic polyharmonic B-spline" to build a specific bi-dimensional MRA, in order to process a signal.

The aim of this paper is to provide a very explicit construction of a polyharmonic pre-wavelet decomposition, which gives possible direct implementation of the involved filters in all dimensions; the scaling function is in the class of polyharmonic B-splines with centers over the lattice Z^s and the pre-wavelets are polyharmonic splines with centers over the fine lattice $2^{-1}Z^s$. In addition, we clarify some theoretical features, specific to this decomposition, such as the rate of decay of the filters. Explicit formulae for deriving the filters and functions involved in the polyharmonic B-spline wavelet transform are here provided. Our scheme works in the spatial domain. The filters are computable once for all, getting simple procedures for code's implementation. The wavelet decomposition/recomposition algorithm results computationally efficient since formulae involve upsampling/downsampling and convolutions, exactly as in the one-dimensional case. This offers the reader an easy use of the given wavelet decomposition, which may be useful in the applications [1,2]; in this sense this work improves the related earlier works.

The paper is organized as follows. In Section 2 we give the definitions and some properties. We define some classes of functions and vectors, which are absolutely bounded by a radial algebraically decaying function or vector, respectively, and we prove some of their properties that we use later on. In particular, we extend Wiener's lemma (see Lemma 2). The scaling function and its most important features are recalled and an estimate of the Riesz constants is given. Then we present the pre-wavelets, and the filters defined by the classical dyadic scaling relations. In Section 3 we provide the formulae for deriving the filters, and we show their algebraical decay. The algorithm for decomposing and recomposing a multidimensional signal is given in Section 4.

2. Preliminary results

2.1. Basic notation, concepts

Throughout this paper, s is the dimension of the space and m is an integer such that $m > s/2$.

Without further words, a "function" is a function from R^s to R and a "vector" has index in Z^s and its components are in R; multiintegers are in N^s and are denoted by Greek letters.

$\| \bullet \|$ denotes the euclidean norm on R^s, while $\| \bullet \|_p$ denotes the usual norm on $\ell^p(Z^s)$ or in $L^p(R^s)$ $(1 \leqslant p \leqslant \infty)$.

We use standard notation for the inner product on $L^2(R^s)$, i.e. $(f, g) := \int_{R^s} f(x)g(x)\,\mathrm{d}x$.

$*$ denotes the convolution products for all functions f and g in $L^1(R^s)$ and all vectors u and v in $\ell^1(Z^s)$

$$f * g := \int_{R^s} f(x)g(\bullet - x)\,\mathrm{d}x, \qquad u * v := \sum_{j \in Z^s} u_j v_{\bullet - j}, \qquad u * f := \sum_{j \in Z^s} u_j f(\bullet - j).$$

$\wedge$ is the Fourier transform, i.e. for any function f in $L^1(R^s)$ and any vector $u \in \ell^1(Z^s)$,

$$\hat{f}(\omega) := \int_{R^s} f(x)e^{-i\omega x}\,\mathrm{d}x, \quad \hat{u}(\omega) := \sum_{j \in Z^s} u_j\, e^{-ij\omega}, \quad \omega \in R^s.$$

Thus u is the vector of the Fourier coefficients of $\hat{u}$. The map $\wedge : u \to \hat{u}$ is linear and continuous (with norm 1) from $l^1(Z^s)$ into $C(T^s)$ (continuous functions over T^s), where $T^s = [-\pi, \pi)^s$, is the s-dimensional torus. We also use $\wedge$ for the Fourier transform of distributions.

$e_1, \ldots, e_s \in R^s$ are the coordinate vectors $(e_j)_k := \delta_{jk}$, $1 \leqslant j, k \leqslant s$.

Dirac denotes the usual Dirac distribution and Δ is the Laplacean operator ($\delta := \sum_{i=1}^s \partial^2/(\partial x_i^2)$). Δ^m is the mth iterated Laplacean operator ($\Delta^k = \Delta \cdot \Delta^{k-1}$).

Δ_1 is the discrete version of Δ, defined for any function f by

$$\Delta_1 f = \sum_{j=1}^s \big(f(\bullet - e_j) - 2f + f(\bullet + e_j)\big).$$

Δ_1^m is the discrete version of Δ^m ($\Delta_1^k = \Delta_1 \cdot \Delta_1^{k-1}$). A direct calculation provides

$$\widehat{\Delta_1 f}(\omega) = \left(-4\sum_{i=1}^s \sin^2\frac{\omega_i}{2}\right)\hat{f}(\omega), \qquad \widehat{\Delta_1^m f}(\omega) = \left(-4\sum_{i=1}^s \sin^2\frac{\omega_i}{2}\right)^m \hat{f}(\omega). \tag{1}$$

Let γ and γ^m be the vectors defined by

$$\gamma_j := \begin{cases} -2s, & \text{if } j = 0, \\ 1, & \text{if } \|j\| = 1, \ \ j \in Z^s, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

$$\gamma^k := \gamma * \gamma^{k-1}, \quad k \in N_*.$$

So we can express $\Delta_1^m f$ as a convolution

$$\Delta_1^m f = \sum_{j\in Z^s} \gamma_j^m f(\cdot - j) =: \gamma^m * f. \tag{3}$$

We say that a vector v is *symmetric* if $v_{-k} = v_k$ for all $k \in Z^s$.

We recall the multiresolution setup (see, for instance, [21]). Given a function $\phi \in L^2(R^s)$ which satisfies the stability condition

$$A \sum_{k\in Z^s} |\lambda_k|^2 \leqslant \left\| \sum_{k\in Z^s} \lambda_k \phi(\bullet - k) \right\|_2^2 \leqslant B \sum_{k\in Z^s} |\lambda_k|^2, \tag{4}$$

valid for all $\lambda = \{\lambda_k\}_{k\in Z^s} \in l^2(Z^s)$, where $A,\ B$ are constants such that $0 < A \leqslant B$. We associate with ϕ an infinite sequence of closed subspaces $\{V_n\}_{n\in Z}$ of L^2 defined for all $n \in Z$ as the one spanned by the family of functions $\{\phi(2^n \cdot -k)\}_{k\in Z^s}$, namely $V_n := \{\sum_{k\in Z^s} \lambda_k \phi(2^n \bullet - k)\colon\ \lambda \in l^2(Z^s)\}$. We say that ϕ admits a multiresolution, or that the sequence $\{V_n\}_{n\in Z}$ form a MRA of $L^2(R^s)$ with scaling function ϕ, provided that, in addition to (4), we have

$$V_n \subseteq V_{n+1}, \quad n \in Z,\ \overline{\left(\bigcup_{n\in Z} V_n\right)} = L^2\big(R^s\big),\ \bigcap_{n\in Z} V_n = \{0\}$$

where the overline denotes closure. We also say that n is the scale (or resolution) level.

We need the following concepts to express the algebraic decay of the scaling function, the pre-wavelets and the filters of this note.

Definition 1. Let $\alpha \in R$, $\alpha > s$. A function $\phi(x)$ is in the class $\mathcal{RB}(\alpha)$ if there exists some positive constant C such that

$$\big|\phi(x)\big| \leqslant \frac{C}{(1 + \|x\|)^\alpha}, \quad x \in R^s.$$

A vector $v = \{v_k\}_{k\in Z^s}$ is in the class $\mathcal{B}(\alpha)$ if there exists some positive constant C such that

$$|v_k| \leqslant \frac{C}{(1 + \|k\|)^\alpha}, \quad k \in Z^s.$$

From $\alpha > s$ we easily get $\mathcal{RB}(\alpha) \subset L^1(R^s)$ and $\mathcal{B}(\alpha) \subset l^1(Z^s)$. Let us show the following results.

Lemma 1. *The class $\mathcal{B}(\alpha)$ is an algebra with respect to the convolution product.*

Proof. Suppose that x and y belong to $\mathcal{B}(\alpha)$ and let $z = x * y$. Then $|(x * y)_n| \leqslant \sum_{k\in Z^s} |x_{n-k} y_k| = \sum_{k\in Z^s} |x_{n-k}||y_k|$. Let $\|n\| = \ell$ and separate the sum $\sum_{k\in Z^s} |x_{n-k}||y_k|$ in two parts: in one part $\|k\| \leqslant \ell/2$, in the other $\|k\| > \ell/2$. We have

$$\sum_{\|k\|\leqslant \ell/2} |x_{n-k}||y_k| \leqslant C_1 \sum_{\|k\|\leqslant \ell/2} \big(1+\|n-k\|\big)^{-\alpha}|y_k|$$
$$\leqslant C_1(1+\ell/2)^{-\alpha} \sum_{\|k\|\leqslant \ell/2} |y_k| \leqslant C_1\|y\|_1(1+\ell/2)^{-\alpha},$$
$$\sum_{\|k\|> \ell/2} |x_{n-k}||y_k| \leqslant C_2 \sum_{\|k\|> \ell/2} |x_{n-k}|\big(1+\|k\|\big)^{-\alpha}$$
$$\leqslant C_2(1+\ell/2)^{-\alpha} \sum_{\|k\|> \ell/2} |x_{n-k}| \leqslant \|x\|_1(1+\ell/2)^{-\alpha},$$

and so $|z_n| \leqslant (C_1\|y\|_1 + C_2\|x\|_2)(1+\|n\|/2)^{-\alpha}$, which implies $z \in \mathcal{B}(\alpha)$. □

Moreover, every element v in $\mathcal{B}(\alpha)$ with $\hat{v}(\omega) \neq 0$, $\omega \in R^s$, has an inverse according to the following lemma, which extends a well-known result due to N. Wiener.

Lemma 2. *Let $v \in \mathcal{B}(\alpha)$ such that $\hat{v}(\omega) \neq 0$, $\omega \in R^s$. Then there exists a vector $u \in \mathcal{B}(\alpha)$ such that $v * u = \delta$.*

Proof. The existence of the vector $u \in l^1(Z^s)$ is assured by the Wiener's lemma (cf. [24, p. 226]). Let now U and V be the infinite matrix defined by $U_{ij} = u_{i-j}$ and $V_{ij} = v_{i-j}$. Let I be the infinite identity matrix ($I_{ij} = \delta_{i-j}$). Then $(U \cdot V)_{ij} = \sum_{k\in Z^s} U_{ik}V_{kj} = \sum_{k\in Z^s} u_{i-k}v_{k-j} = \sum_{k\in Z^s} u_k v_{i-k-j} = \delta_{i-j}$. Thus $UV = I$, that is, U is the inverse of an infinite matrix V which meets $|V_{ij}| < C_1(1+\|i-j\|)^{-\alpha}$, all $i, j \in Z^d$, with $\alpha > d$. Then (cf. [13]), U has the same order of decay of V, i.e. u meets $|u_{i-j}| = |U_{ij}| < C_2(1+\|i-j\|)^{-\alpha}$, all $i, j \in Z^d$, that is $u \in \mathcal{B}(\alpha)$. □

Finally we prove the following

Lemma 3. *Let $f \in \mathcal{RB}(\alpha)$ and $y \in \mathcal{B}(\alpha)$, then $g := y * f \in \mathcal{RB}(\alpha)$.*

Proof. Let $x \in R^s$ be fixed with $\|x\| = \ell$ and let $v_k := f(x-k)$, $k \in Z^s$. We have $v \in \mathcal{B}(\alpha)$. As in the proof of Lemma 1, separate the sum $\sum_{k\in Z^s} |y_k f(x-k)|$ in two parts: in one part $\|k\| \leqslant \ell/2$, in the other $\|k\| > \ell/2$. We have

$$\sum_{\|k\|\leqslant \ell/2} \big(1+\|x-k\|\big)^{-\alpha}|y_k| \leqslant (1+\ell/2)^{-\alpha} \sum_{\|k\|\leqslant \ell/2} |y_k| \leqslant \|y\|_1(1+\ell/2)^{-\alpha},$$
$$\sum_{\|k\|> \ell/2} \big|f(x-k)\big|\big(1+\|k\|\big)^{-\alpha} \leqslant (1+\ell/2)^{-\alpha} \sum_{\|k\|> \ell/2} \big|f(x-k)\big| \leqslant \|v\|_1(1+\ell/2)^{-\alpha}$$

and so $|g(x)| \leqslant C(1+\|x\|)^{-\alpha}$. □

2.2. Polyharmonic B-spline as scaling function

According to Duchon [9], we recall that a polyharmonic spline is defined through an interpolation problem associated to a certain set of knots: let A be a discrete set in R^s, and suppose $m > s/2$; the

m-harmonic spline interpolating f in A is the solution (unique if A contains a P_{m-1}-unisolvent subset) of the problem of minimizing the seminorm

$$\|D^m u\|_{L^2(R^s)} := \left(\int\limits_{R^s} \sum_{\alpha\in N^s,\,|\alpha|=m} \frac{m!}{\alpha!}\left(\frac{\partial^\alpha u}{\partial x^\alpha}(x)\right)^2 \mathrm{d}x \right)^{1/2}$$

amongst all the functions u in the Sobolev space $H^m(R^s)$ (the space $H^m(R^s)$ is the space of all functions whose mth derivatives, in the sense of distributions, are in $L^2(R^s)$) and which are equal to f in A. All m-harmonic splines associated to a given set A form a vectorial space. In this paper we consider the case of a cardinal mesh.

More precisely, for any $m \in N$, $m > s/2$, let V_0 be the space of the m-harmonic splines associated to the set $A = Z^s$. Rabut [22,23] showed that V_0 is generated by the family of functions $\{\phi_m(\cdot - k)\}_{k\in Z^s}$, where ϕ_m is a basic m-harmonic spline named polyharmonic B-spline, which is defined as follows. For every $m \in N$, $m > s/2$, let

$$v_m := \begin{cases} \frac{1}{2^{2m}\pi^{s/2}} \frac{(-1)^{m-s/2+1}}{(m-s/2)!\Gamma(m)} \|\bullet\|^{2m-s} \ln\|\bullet\|^2, & s \text{ even}, \\ \frac{1}{2^{2m}\pi^{s/2}} \frac{(-1)^m \Gamma(d/2-m)}{\Gamma(m)} \|\bullet\|^{2m-s}, & s \text{ odd}. \end{cases} \tag{5}$$

This function is the Green's function for the iterated Laplacean operator Δ^m. Indeed it has been shown [25, p. 257] that

$$\widehat{v_m}(\omega) = (-1)^m \|\omega\|^{-2m}, \tag{6}$$

and then

$$\Delta^m v_m = \text{Dirac}. \tag{7}$$

The polyharmonic B-spline is then defined as

$$\phi_m := \Delta_1^m v_m. \tag{8}$$

According to (3) ϕ_m can be fairy efficiently computed via a convolution product

$$\phi_m = \gamma^m * v_m. \tag{9}$$

Note that γ^m has a finite support, and so the convolution in (9) is exactly computed with a finite number of operations.

The function ϕ_m is a multivariate extension of the odd degree, equidistant knots polynomial B-splines (however ϕ_m is not positive for all x in R^s).

We list now some properties of the polyharmonic B-splines, which will be useful in the wavelet decomposition (one can find more extensive proofs of points (iv) to (viii) in [22,23]).

(i) Symmetry.

$$\phi_m(-x) = \phi_m(x), \quad x \in R^s, \tag{10}$$

since $\gamma_{-k} = \gamma_k$ and v_m is a radial function.

(ii) Fourier transform.

From (8), (1), and (3) we get

$$\widehat{\phi_m}(\omega) = \frac{(\sum_{i=1}^s \sin^2 \omega_i/2)^m}{\|\omega/2\|^{2m}}, \quad \text{a.e. } \omega \in R^s. \tag{11}$$

(iii) The class of the polyharmonic B-splines is closed with respect to convolution

$$\phi_s * \phi_r = \phi_{r+s}, \tag{12}$$

that is an direct consequence of (11).

(iv) Shape of the Fourier transform.

From a Taylor expansion of $\sin \omega_i/2$ centered in $\omega = 0$, it follows:

$$\widehat{\phi_m}(\omega) - 1 = O\left(\|\omega\|^2\right), \quad \|\omega\| \to 0. \tag{13}$$

Making partial derivative with respect to a variable, the order of the zero as $\|\omega\| \to 0$ decreases of one unity; thus

$$\widehat{\phi_m}(0) = 1 \quad \text{and} \quad |\alpha| = 1 \Rightarrow D^\alpha \widehat{\phi_m}(0) = 0, \tag{14}$$

and $\widehat{\phi_m} \in C^2(R^s)$; being also $2m \geqslant s + 1$, then

$$\widehat{\phi_m} \in C^2\left(R^s\right) \cap L^1\left(R^s\right).$$

(v) Decay at infinity: ϕ_m belongs to the class $\mathcal{RB}(s + 2)$.

Let us take $\alpha \in N^s$; using the previous arguments, we can show that $|D^\alpha \widehat{\phi_m}|$ is locally summable for any α such that $|\alpha| \leqslant k$ with the condition $2 - k \geqslant -s + 1$. But $D^\alpha \widehat{\phi_m}(\omega) = O(\|\omega\|^{-2m})$, $\|\omega\| \to \infty$, and $2m > s$; then $|D^\alpha \widehat{\phi_m}|$ is always summable in a neighborhood of infinity. We can then conclude that $|D^\alpha \widehat{\phi_m}| \in L^1(R^s)$ for $|\alpha| \leqslant s + 1$, and from this it follows, cf. [27, p. 26], that $\phi_m(x) = o(\|x\|^{-s-1})$, $\|x\| \to \infty$. Finally, since ϕ_m admits a series expansion out of a certain neighborhood of the origin (see the next observation), then

$$\phi_m(x) = O\left(\|x\|^{-s-2}\right), \quad \|x\| \to \infty. \tag{15}$$

(vi) Regularity.

Since ϕ_m is a finite sum of integer translates of functions $v_m \in C^{2m-s-1}(R^s) \cap C^\infty(R^s \setminus \{0\})$, then (we remind the reader that $2m - s \geqslant 1$)

$$\phi_m \in C^{2m-s-1}\left(R^s\right) \cap C^\infty\left(R^s \setminus \mathcal{U}_m(0)\right),$$

where $\mathcal{U}_m(0)$ is a neighborhood of zero. Thus, according to (15)

$$\phi_m \in C^{2m-s-1}\left(R^s\right) \cap L^1\left(R^s\right) \subset L^\infty\left(R^s\right) \cap L^2\left(R^s\right).$$

(vii) Partition of unity.

Using (11) and (14), we get for all $\omega \in R^s$, $\sum_{k \in Z^s} \widehat{\phi_m}(2\pi k) e^{i2\pi k\omega} = \widehat{\phi_m}(0) = 1$ and applying the Poisson summation formula we get

$$\sum_{k \in Z^s} \phi_m(\bullet - k) = 1.$$

(viii) Reproduction of polynomials of degree one.

For $|\alpha| = 1$, $\sum_{k \in Z^s} D^\alpha \widehat{\varphi_m}(2\pi k) e^{i2\pi k\omega} = D^\alpha \widehat{\varphi_m}(0) = 0$ which implies for any x in R^s $\sum_{k \in Z^s}(x - k)\varphi_m(x - k) = 0$, which is $\sum_{k \in Z^s} k\varphi_m(x - k) = x \sum_{k \in Z^s} \varphi_m(x - k) = x$. Note that all summations are absolutely convergent thanks to the decay of ϕ_m given in (15).

(ix) Stability condition.

The family of functions $\{\phi_m(\bullet - k)\}_{k\in Z^s}$ is a Riesz basis of $V_0 \subset L^2(R^s)$, i.e. ϕ_m satisfies the stability condition (4) which is equivalent to

$$A \leqslant \sum_{k\in Z^s} \left|\widehat{\phi_m}(\omega + 2k\pi)\right|^2 \leqslant B, \quad \text{a.e. } \omega \in R^s, \tag{16}$$

with the same constants A and B. Let us give an estimate of these constants. Considering the first inequality in (16), let us take the term when $k = 0$. For all $x \in [-\pi/2, \pi/2]$, $\sin^2 x \geqslant ((2/\pi)x)^2$, so $\sum_{i=1}^s \sin^2 \omega_i/2 \geqslant (2/\pi)^2 \|\omega/2\|^2$ and so we have

$$\sum_{k\in Z^s} \left|\widehat{\phi_m}(\omega + 2k\pi)\right|^2 \geqslant \left(\frac{\sum_{i=1}^s \sin^2 \omega_i/2}{\|\omega/2\|^2}\right)^{2m} \geqslant \left(\frac{2}{\pi}\right)^{4m} := A.$$

Let us now consider the second inequality in (16). If $s = 1$

$$\sum_{k\in Z} \left|\widehat{\phi_m}(\omega + 2k\pi)\right|^2 = \sum_{k\in Z}\left(\frac{\sin^2 \omega/2}{(\omega/2 + k\pi)^2}\right)^{2m} \leqslant \left(\frac{\sin^2 \omega/2}{(\omega/2)^2}\right)^{2m} \leqslant 1,$$

and $B = 1$. Now, if $s \geqslant 2$

$$\sum_{k\in Z^s} \left|\widehat{\phi_m}(\omega + 2k\pi)\right|^2 = \sum_{k\in Z^s}\left(\frac{\sum_{i=1}^s \sin^2 \omega_i/2}{\|\omega/2 + k\pi\|^2}\right)^{2m} \leqslant 1 + \sum_{\substack{k\in Z^s\\ \|k\|\neq 0}} \frac{s^{2m}}{\|\omega/2 + k\pi\|^{4m}}.$$

Since the series is periodic of period $\{2\pi\}^s$, we can limit the study to the case $\omega \in [-\pi, \pi]^s$. Let $\omega/2 \in [-\pi/2, \pi/2]^s$ and let $j = \|k\|_\infty \neq 0$. Then

$$\left\|\frac{\omega}{2} + k\pi\right\| \geqslant \left\|\frac{\omega}{2} + k\pi\right\|_\infty \geqslant \left|\left\|\frac{\omega}{2}\right\|_\infty - \|k\pi\|_\infty\right| \geqslant \left(\|k\| - \frac{1}{2}\right)\pi = \left(j - \frac{1}{2}\right)\pi \geqslant \frac{j\pi}{2}.$$

Since $4m > 2m > s$ the last series is absolutely convergent and we can change the order of the terms. For any $j \geqslant 1$ let $D_j = \{k \in Z^s \mid \|k\|_\infty = j\}$ and let $|D_j|$ be the numbers of the elements of D_j. Then, being $|D_j| \leqslant 2^{2s-1} j^{s-1}$, we get

$$\sum_{k\in Z^s} \left|\widehat{\phi_m}(\omega + 2k\pi)\right|^2 \leqslant 1 + s^{2m} \sum_{j=1}^{\infty} \frac{|D_j|}{((j/2)\pi)^{4m}} \leqslant 1 + s^{2m} \frac{2^{2s-1+4m}}{\pi^{4m}} \zeta(4m - s + 1) =: B,$$

where $\zeta(k) = \sum_{j=1}^\infty j^{-k}$ is the usual special function. Note that the value of B can be made smaller since the bound $(j - 1/2)\pi \geqslant j\pi/2$ is not a sharp one. As an example, if $s = 2$ and $m = 2$ we find the values $A \cong 0.27$ and $B \cong 3.48$.

2.3. Pre-wavelets and multiresolution

Let $m \in N$, and let us consider the harmonic B-spline ϕ_m defined by (8). In the previous section it is shown that $\phi_m \in C^{2m-s-1}(R^s) \cap L^1(R^s)$ and it satisfies the stability condition (4). For the use of the reader, in this section we present the decomposition theorem and the related definitions. The results here stated are proved by Miccelli et al. [20] in a more general context, where multiresolution analysis, stationary subdivision, and pre-wavelet decomposition of $L^2(R^s)$ are provided, based on functions in a specific class, called $R_{r,p}$. $R_{r,p}$ is defined as the set of functions ϕ whose Fourier transform is of the form $\hat{\phi} = T/q$, where T is a trigonometric polynomial such that $T(\omega) = 0,\ \|\omega\|_\infty \leqslant \pi$ implies $\omega = 0$, and q is an elliptic (i.e. $q(\omega) = 0$ implies $\omega = 0$) and homogeneous polynomial of degree r with $r > s/2$; moreover there exists an integer p such that $T(\omega) - q(\omega) = O(\|\omega\|_\infty^{r+1+p})$, $\omega \to 0$. From (11) and (13) it follows that $\phi_m \in R_{2m,1}$ and we get the case of polyharmonic B-splines.

Now, let us define the infinite sequence of closed subspaces $\{V_n\}_{n\in Z}$ of L^2 by

$$V_n := \left\{ \sum_{k\in Z^s} \lambda_k \phi_m\big(2^n \bullet - k\big)\colon\ \lambda \in l^2\big(Z^s\big) \right\}.$$

So V_n is the space of m-harmonic splines with knots in $2^{-n}Z^s$ [22]. In order to define the associated pre-wavelet, let L_{2m} denote the Lagrangean $2m$-harmonic spline, which is by definition the $2m$-harmonic spline in $H^{2m}(R^s)$ interpolating the data $(j, \delta_j)_{j\in Z^s}$, i.e. L_{2m} satisfies

$$L_{2m}(j) = \begin{cases} 1, & j = 0, \\ 0, & j \in Z^s \setminus \{0\}. \end{cases}$$

Let E and E' be

$$E = [0,1]^s \cap Z^s \quad \text{and} \quad E' = E \setminus \{0\}^s$$

and define

$$\psi_m := \big(\Delta^m L_{2m}\big)(2\bullet), \qquad \psi_m^e := \psi_m\left(\bullet + \frac{e}{2}\right), \quad e \in E. \tag{17}$$

The functions ψ_m and ψ_m^e, $e \in E$, are m-harmonic splines with knots in $Z^s/2$.

For any $n \in Z$ and $e' \in E'$ let $W_n^{e'}$ denote the space spanned by the family $\{\psi_m^{e'}(2^n\bullet - j)\}_{j\in Z^s}$ and let W_n be the one spanned by $\{\psi_m^{e'}(2^n\bullet - j)\}_{j\in Z^s,\, e'\in E'}$. Hence

$$W_n = \biguplus_{e'\in E'} W_n^{e'}, \quad n \in Z.$$

Now we can state the following theorem.

Theorem 1. *For each $m \in Z,\ m > s/2$, the polyharmonic B-spline ϕ_m is a scaling function that forms a MRA $\{V_n\}_{n\in Z}$ of $L^2(R^s)$ and the sequence $\{W_n\}_{n\in Z}$ provides an orthogonal decomposition of $L^2(R^s)$ with pre-wavelets $\{\psi_m^{e'}\}'_{e'\in E'}$. Moreover*

$$V_n = V_{n-1} \oplus W_{n-1}, \quad n \in Z.$$

In particular for each $n \in Z$, V_n is orthogonal to W_n, namely

$$\big(\phi_m\big(2^n\bullet - j\big), \psi_m^{e'}\big(2^n\bullet\big)\big) = 0 \quad \text{for all } e' \in E' \text{ and } j \in Z^s. \tag{18}$$

Note that, if $r < n$, then W_r is a subspace of V_n, which is orthogonal to W_n, so each W_n is orthogonal to all the others W_r. Hence the functions of the family $\bigcup_{n\in Z}\{\psi_m^{e'}(2^n\bullet - k)\}_{e'\in E',\, k\in Z^s}$ are orthogonal on different scales. However, for given n, functions of W_n are usually not orthogonal to other functions of same W_n. That in general the orthogonality fails if we take functions of W_n, i.e. with the same scale level n (i.e. in general if $n \in Z,\ e', \bar{e}' \in E',\ j, k \in Z$, then $(\psi_m^{e'}(2^n\bullet - j), \psi_m^{\bar{e}'}(2^n\bullet - k)) \neq 0$). This is the reason why these functions $\psi_m^{e'}(2^n \bullet - j)$ are called *pre*-wavelets, instead of wavelets which usually form an orthogonal basis of W_n.

Since both the scaling function $\phi_m \in V_0$ and the wavelet $\psi_m \in W_0$ are in V_1, and since V_1 is generated by $\{\phi_m(2\bullet - j)\}_{k\in Z^s}$, there exist two (unique) vectors, named *recomposition filters*, $c = (c_j)_{j\in Z^s}$ and $d = (d_j)_{j\in Z} \in l^2(Z^s)$ such that the two-scale relations of the scaling function and wavelet

$$\phi_m = \sum_{j\in Z^s} c_j \phi_m(2\bullet - j) = (c * \phi_m)(2\bullet), \tag{19}$$

$$\psi_m = \sum_{j\in Z} d_j \phi_m(2\bullet - j) = (d * \phi_m)(2\bullet), \tag{20}$$

are satisfied. By (17) it follows:

$$\psi_m^e = \sum_{j\in Z} d_j \phi_m(2\bullet + e - j) = (d * \phi_m)(2\bullet + e), \quad e \in E. \tag{21}$$

On the other hand, since all the functions $\{\phi_m(2\bullet + e)\}_{e\in E}$ are in $V_1 = V_0 \oplus W_0$, and the family of functions $\{\phi_m(\bullet - k)\}_{k\in Z^s}$ and $\{\psi_m^{e'}(\bullet - k)\}_{k\in Z^s, e'\in E'}$ are basis of V_0 and W_0, respectively. So, for any $e \in E$ there exist (unique) vectors in $l^2(Z^s)$, which we denote by p^e and $q^{e,e'}$, $e' \in E'$ such that

$$\phi_m(2\bullet + e) = p^e * \phi_m + \sum_{e'\in E'} q^{e,e'} * \psi_m^{e'}, \quad e \in E. \tag{22}$$

The so-called *decomposition filters* p and $q^{e'}$, $e' \in E'$ are defined for all $j \in Z^s$ and $e \in E$ by

$$p_{2j+e} = p_j^e \quad \text{and} \quad q_{2j+e}^{e'} = q_j^{e,e'}, \quad e' \in E'. \tag{23}$$

3. The filters

In this section we give formulae for deriving the filters c, d, p, and $q^{e'}$ ($e' \in E'$). Moreover, we show that the filters belong to $\mathcal{B}(s+2)$, and therefore they belong to $l^1(Z^s)$.

Lemma 4. *For all $m \in N$, $m > s/2$, let $\beta^m = (\beta_k^m)_{k\in Z^s}$ be the vector defined by*

$$\beta_k^m = \phi_m(k), \quad k \in Z^s. \tag{24}$$

Then $\beta^m \in \mathcal{B}(s+2)$, for all $\omega \in R^s$, $\widehat{\beta^m}(\omega) \neq 0$, and there exists a symmetric vector $a^m \in \mathcal{B}(s+2)$ meeting

$$a^m * \beta^m = \delta. \tag{25}$$

Proof. First note that, from (15), β^m belongs to $\mathcal{B}(s+2)$. By Poisson summation formulae $\sum_{k\in Z^s} \widehat{\phi_m}(\omega + 2\pi k) = \sum_{j\in Z^s} \phi_m(j) e^{-ij\omega} = \sum_{j\in Z^s} \beta_j^m e^{-ij\omega} = \widehat{\beta^m}(\omega)$ for any $\omega \in T^s$. Now, using $\sin^2 x > (2/\pi x)^2$, $x \in [-\pi/2, \pi/2]$ and taking only the term with $k = 0$, we get

$$\left|\widehat{\beta^m}(\omega)\right| = \sum_{k\in Z^s} \left(\frac{\sum_{i=1}^d \sin^2(\omega_i/2 + k\pi)}{\|\omega/2 + k\pi\|^2}\right)^m \geqslant \left(\frac{\sum_{i=1}^d \sin^2 \omega_i/2}{\|\omega/2\|^2}\right)^m \geqslant (2/\pi)^{2m}.$$

Thus $\widehat{\beta^m}(\omega) \neq 0$, $\omega \in R^s$ and by Lemma 2 there exists a vector $a^m \in \mathcal{B}(s+2)$ meeting (25). Besides a^m is symmetric since β^m is. □

We can now give explicit formulae for the filters.

Theorem 2. *Let c be the recomposition filter defined by the two-scale equation* (19)*, and let $b^m = (b_k^m)_{k\in Z^s}$ be defined by*

$$b_k^m = \phi_m\left(\frac{k}{2}\right), \quad k \in Z^s, \tag{26}$$

and let a^m be the vector defined by (25)*. Then*

$$c = a^m * b^m, \tag{27}$$

and $c \in \mathcal{B}(s+2)$.

Proof. Let L be the function defined by

$$L = a^m * \phi_m = \sum_{j\in Z^s} a_j^m \phi_m(\bullet - j). \tag{28}$$

Since by (25), for all k in Z^s, $L(k) = \sum_{j\in Z^s} a_j^m \phi_m(k-j) = \sum_{j\in Z^s} a_j^m \beta_{k-j}^m = \delta_k$, L is the fundamental Lagrangean m-harmonic spline (interpolating the data $(k, \delta_k)_{k\in Z^s}$). Now, the family of functions $\{L(\bullet - j)\}_{j\in Z^s}$ is a basis for V_0, and $V_{-1} \subset V_0$. Thus we can uniquely write, by using (26),

$$\phi_m(\bullet/2) = \sum_{j\in Z^s} b_j^m L(\bullet - j). \tag{29}$$

Then we write $\phi_m(\bullet/2)$ in two different ways, according to (19) and (29), and let us use (28):

$$\phi_m(\bullet/2) = c * \phi_m = b^m * L = b^m * \left(a^m * \phi_m\right) = \left(b^m * a^m\right) * \phi_m,$$

where in the last equality we changed the order in the summation, being the series absolutely convergent. As there is a sole decomposition of the function $\phi_m(\bullet/2)$ in the basis $\{\phi_m(\bullet - j)\}_{j\in Z^s}$, we get (27). Finally $c \in \mathcal{B}(s+2)$ by Lemma 1. □

Let now $\beta^{2m} = (\beta_k^{2m})_{k\in Z^s}$ be the vector defined as follows:

$$\beta_k^{2m} = \phi_{2m}(k), \quad k \in Z^s, \tag{30}$$

where ϕ_{2m} is the $2m$-harmonic B-spline which, according to (8) and (9), is expressed by $\phi_{2m} = \Delta_1^{2m} v_{2m} = \gamma^{2m} * v_{2m}$. From Lemma 4, $\beta^{2m} \in \mathcal{B}(s+2)$, $\widehat{\beta^{2m}}(\omega) \neq 0$, $\omega \in R^s$, and there exists a symmetric vector $a^{2m} \in \mathcal{B}(s+2)$ meeting

$$a^{2m} * \beta^{2m} = \delta. \tag{31}$$

Theorem 3. *Let d be the recomposition filter as defined by* (20) *and let γ^m and a^{2m} be the vectors defined by* (2) *and* (31)*. Then*

$$d = a^{2m} * \gamma^m, \tag{32}$$

and $d \in \mathcal{B}(s+2)$.

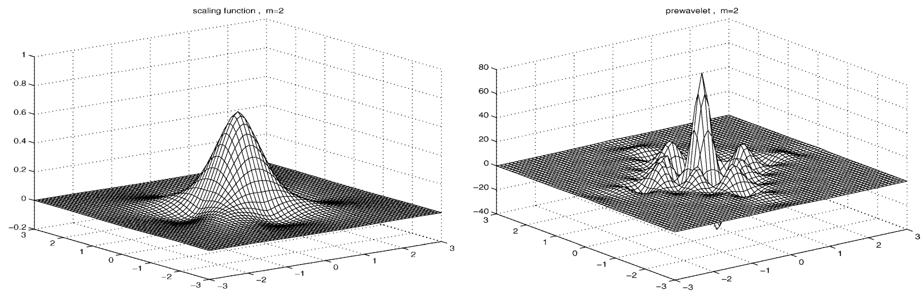

Fig. 1. Scaling function ϕ_m (on the left) and pre-wavelet generator ψ_m (on the right), $s = 2$, $m = 2$.

Proof. From (6) it follows that $v_{2m} = v_m * v_m$. Then by (7) $\Delta^m v_{2m} = \Delta^m(v_m * v_m) = (\Delta^m v_m) * v_m =$ Dirac $* v_m = v_m$. Using the linearity of the operator Δ^m, $\gamma^{2m} = \gamma^m * \gamma^m$, and the fact that the support of γ^m is finite, we get

$$\begin{aligned}\Delta^m \phi_{2m} &= \Delta^m\big(\gamma^{2m} * v_{2m}\big) = \gamma^{2m} * \Delta^m v_{2m} = \gamma^{2m} * v_m = \big(\gamma^m * \gamma^m\big) * v_m \\ &= \gamma^m * \big(\gamma^m * v_m\big) = \gamma^m * \phi_m. \end{aligned} \tag{33}$$

Now note that the $2m$-harmonic Lagrangean spline L_{2m} can be written as $L_{2m} = a^{2m} * \phi_{2m}$. Then write $\psi_m(\bullet/2)$ in two different ways, as (20), and according to (17), then by using the linearity of Δ^m and (33) we obtain

$$\begin{aligned}\psi_m(\bullet/2) &= d * \phi_m = \Delta^m L_{2m} = \Delta^m\big(a^{2m} * \phi_{2m}\big) = a^{2m} * \Delta^m \phi_{2m} = a^{2m} * \big(\gamma^m * \phi_m\big) \\ &= \big(a^{2m} * \gamma^m\big) * \phi_m\end{aligned}$$

and since there is a sole decomposition of the function $\psi_m(\bullet/2)$ in the basis $(\phi_m(\bullet - j))_{j\in Z^s}$ we get (32). Finally $d \in \mathcal{B}(s+2)$ by Lemma 1. $\square$

Note. Since d belongs to the class $\mathcal{B}(s+2)$ and ϕ_m belongs to the class $\mathcal{RB}(s+2)$, by Lemma 3 it follows that ψ_m belongs to the class $\mathcal{RB}(s+2)$. Moreover, since $\phi_m \in C^{2m-s-1}(R^s)$ and $d \in l^1(Z^s)$ then $\psi_m \in C^{2m-s-1}(R^s)$. Since ϕ_m is symmetric with respects to its central point, and d is symmetric as a convolution between symmetric vectors, then ψ_m is symmetric too with respects to its central point.

By (32) one can compute the filter d. We can solve Eq. (31) by means of classical use of direct and inverse Fourier transform. Alternative, an efficient algorithm is given in [1], to compute the inverse in the convolution. Then, knowing the filter d, one can easily compute the multidimensional pre-wavelets by using (20). The scaling function ϕ_m is computed according to (9).

Figure 1 shows the functions ψ_m and ϕ_m for the bivariate bi-harmonic case ($m = 2$).

Let us now consider the decomposition filter p as defined by (23) by means of p^e, $e \in E$. The following theorem gives the formulae for deriving p^e from the vectors c, β^{2m} and a^{2m}.

Theorem 4. *Let β^{2m}, a^{2m}, c, and p^e be defined by* (30), (31), (19), *and* (22). *For any $e \in E$, let $\alpha^e = (\alpha^e_j)_{j\in Z^s}$ be defined by*

$$\alpha^e_j = \big(\phi_m(2\bullet + e), \phi_m(\bullet - j)\big). \tag{34}$$

Then for any $e \in E$

$$\alpha_j^e = 2^{-s}\big(c * \beta^{2m}\big)_{2j+e}, \quad j \in Z^s, \tag{35}$$

$$p^e = \alpha^e * a^{2m}, \tag{36}$$

and the filter p defined by (23) *belongs to* $\mathcal{B}(s+2)$.

Proof. According to (12) $\phi_{2m} = \phi_m * \phi_m$, and due to the symmetry (10) of ϕ_m and ϕ_{2m} we get

$$\big(\phi_m(\bullet + j), \phi_m\big) = \beta_{-j}^{2m} = \beta_j^{2m}. \tag{37}$$

To prove (35) let us compute the inner product (34) by using (19), the change of variables $2\bullet - 2j + k$ into $\bullet$, and (37). For all $j \in Z^s$ and $e \in E$ we get

$$\begin{aligned}
\alpha_j^e &= \big(\phi_m(2\bullet + e), \phi_m(\bullet - j)\big) = \big(\phi_m(2\bullet + e), \phi_m(j - \bullet)\big) \\
&= \Big(\phi_m(2\bullet + e), \sum_{k\in Z^s} c_k \phi_m\big(2(j-\bullet) - k\big)\Big) = \sum_{k\in Z^s} c_k\big(\phi_m(2\bullet + e), \phi_m(2j - 2\bullet - k)\big) \\
&= 2^{-s}\sum_{k\in Z^s} c_k\big(\phi_m(\bullet + 2j + e - k), \phi_m\big) = 2^{-s}\sum_{k\in Z^s} c_k \beta_{2j+e-k}^{2m} = 2^{-s}\big(c * \beta^{2m}\big)_{2j+e}.
\end{aligned}$$

To prove (36) let us compute the inner product (34) by using (22), the orthogonality (18), the change of variable $\bullet - j = \bullet$ and (37): for any $j \in Z^s$ and $e \in E$ we get

$$\begin{aligned}
\alpha_j^e &= \big(\phi_m(2\bullet + e), \phi_m(\bullet - j)\big) = \sum_{k\in Z^s} p_k^e\big(\phi_m(\bullet - k), \phi_m(\bullet - j)\big) \\
&= \sum_{k\in Z^s} p_k^e\big(\phi_m(\bullet + j - k), \phi_m\big) = \sum_{k\in Z^s} p_k^e \beta_{j-k}^{2m} = \big(p^e * \beta^{2m}\big)_j,
\end{aligned}$$

thus $\alpha^e = p^e * \beta^{2m}$, and by (31) $\alpha^e * a^{2m} = p^e * \beta^{2m} * a^{2m} = p^e$. Finally $p \in \mathcal{B}(s+2)$ by Lemma 1. □

Let us now consider the decomposition filters $\{q^{e'}\}_{e'\in E'}$. For any $e' \in E'$, the filter $q^{e'}$ is defined by (23) by means of the vectors $q^{e,e'}$, $e \in E$. The following theorem shows that for all $e \in E$ the vectors $(q^{e,e'})_{e'\in E'}$ satisfy the following linear system (38) where the unknowns are $(q_k^{e,\tilde{e}'})_{k\in Z^s, \tilde{e}'\in E'}$. Solving each linear system obtained for each e in E, the filters $q^{e'}$, $e' \in E'$ can be derived by using (23).

Theorem 5. *Let d be the recomposition filter as defined by* (20) *and let γ^m be defined by* (2). *For any $e \in E$ the vectors $(q^{e,e'})_{e'\in E'}$ defined by* (22) *meet the linear system*

$$\sum_{\substack{k\in Z^s\\ \tilde{e}'\in E'}} q_k^{e,\tilde{e}'}\big(d * \gamma^m\big)_{\tilde{e}'-e'+2j-2k} = \gamma_{2j+e-e'}^m, \quad j \in Z^s,\ e' \in E'. \tag{38}$$

Proof. Let us define the vectors $\xi^{e,e'} = (\xi_j^{e,e'})_{j\in Z^s}$, $e \in E$, $e' \in E'$ as follows: for any $j \in Z^s$, $e \in E$, $e' \in E'$

$$\xi_j^{e,e'} := \big(\phi_m(2\bullet + e), \psi_m^{e'}(\bullet - j)\big).$$

For any $e \in E$, $e' \in E'$ let us compute $\xi_j^{e,e'}$ in two different ways. First, we use (21), (37), (32), (31), and the symmetry of γ^m, getting for any $j \in Z^s$

$$\begin{aligned}
\xi_j^{e,e'} &= \left(\phi_m(2\bullet+e)\,,\,\psi_m^{e'}(\bullet-j)\right) = \left(\phi_m(2\bullet+e),\sum_{k\in Z^s} d_k\phi_m\left(2(\bullet-j)+e'-k\right)\right)\\
&= \left(\phi_m(2\bullet+e),\sum_{k\in Z^s} d_k\phi_m(2\bullet-2j+e'-k)\right) = 2^{-s}\sum_{k\in Z^s} d_k\left(\phi_m,\phi_m(\bullet-e-2j+e'-k)\right)\\
&= 2^{-s}\sum_{k\in Z^s} d_k\beta^{2m}_{-e-2j+e'-k} = 2^{-s}\left(d*\beta^{2m}\right)_{-e-2j+e'} = 2^{-s}\left(\gamma^m*a^{2m}*\beta^{2m}\right)_{-e-2j+e'}\\
&= 2^{-s}\gamma^m_{2j+e-e'}.
\end{aligned}$$

Second, we compute $\xi_j^{e,e'}$ by using (22), the orthogonality (18), then (17), (21), (37), and finally (32), and (31):

$$\begin{aligned}
\xi_j^{e,e'} &= \sum_{\substack{k\in Z\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(\psi_m^{\tilde e'}(\bullet-k),\psi_m^{e'}(\bullet-j)\right) = \sum_{\substack{k\in Z\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(\psi_m\left(\bullet+\frac{\tilde e'}{2}-k\right),\psi_m\left(\bullet+\frac{e'}{2}-j\right)\right)\\
&= \sum_{\substack{k\in Z\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(\psi_m\left(\bullet+\frac{\tilde e'-e'}{2}+j-k\right),\psi_m\right)\\
&= \sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(\sum_{i\in Z^s} d_i\phi_m(2\bullet+\tilde e'-e'+2j-2k-i),\sum_{l\in Z^s} d_l\phi_m(2\bullet-l)\right)\\
&= \sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} 2^{-s} q_k^{e,\tilde e'}\sum_{i,l\in Z^s} d_i d_l\phi_{2m}(\tilde e'-e'+2j-2k+l-i)\\
&= 2^{-s}\sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\sum_{i,h\in Z^s} d_i d_{i-h}\phi_{2m}(\tilde e'-e'+2j-2k-h)\\
&= 2^{-s}\sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\sum_{h\in Z^s}(d*d)_h\beta^{2m}_{\tilde e'-e'+2j-2k-h} = 2^{-s}\sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(d*d*\beta^{2m}\right)_{\tilde e'-e'+2j-2k}\\
&= 2^{-s}\sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(d*\gamma^m*a^{2m}*\beta^{2m}\right)_{\tilde e'-e'+2j-2k} = 2^{-s}\sum_{\substack{k\in Z^s\\ \tilde e'\in E'}} q_k^{e,\tilde e'}\left(d*\gamma^m\right)_{\tilde e'-e'+2j-2k}.
\end{aligned}$$

Comparing the expressions we get (38). □

Theorem 6. *The filters* $\{q^{e'}\}_{e'\in E'}$ *belong to the class* $\mathcal{B}(s+2)$.

Proof. By (23) it is sufficient to show, for all $e\in E$, that the vectors $(q^{e,e'})_{e'\in E'}$ belong to $\mathcal{B}(s+2)$. For all $e\in E$, we can express the system (38) as

$$CQ^e = B^e,$$

where for all $j,k\in Z^s,\ e',\tilde e'\in E'$

$$C_{2j-e',2k-\tilde e'} = \left(\psi_m^{\tilde e'}(\bullet-k),\psi_m^{e'}(\bullet-j)\right) = \left(d*\gamma^m\right)_{(2j-e')-(2k-\tilde e')},$$

$$B^e_{2j-e} = \gamma^m_{e+2j-e},\quad \text{and}\quad Q^e_{2k-\tilde e'} = q_k^{e,e'}.$$

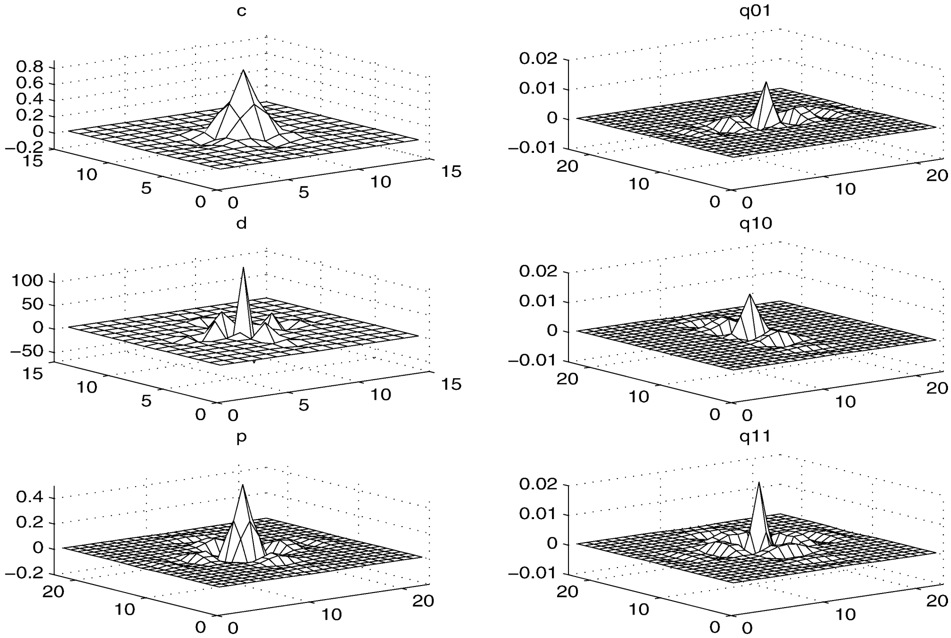

Fig. 2. Recomposition filters c and d and the decomposition filter p (on the left), recomposition filters q^{01}, q^{10}, q^{11} (on the right), $s = 2$, $m = 2$.

Let $T = \{2j - e'\}_{j \in Z^s, e' \in E'}$. The coefficient matrix $C = [C_{t,r}]_{t,r \in T}$ is a real symmetric Toeplitz infinite matrix which is non singular since the family $\{\psi_m^{e'}(\bullet - j)\}_{j \in Z^s, e' \in E'}$ is a basis of W_0. Since $d * \gamma^m$ belongs to the class $\mathcal{B}(s+2)$ then C is an infinite nonsingular matrix which satisfies the property $|C_{t,r}| < h(1 + \|t - r\|)^{-s-2}$, $t, r \in T$, and then (cf. [15, Proposition 3]) its inverse matrix C^{-1} has the same decay property, with the same order $s + 2$. Finally since $S = \text{support}(B^e)$ is finite there exists $\bar{t} \in S$ such that $\|r - t\| \geqslant \|r - \bar{t}\|$, for all $t \in S$, and let $M = \max_{t \in S} |B_t^e|$, then $Q^e \in \mathcal{B}(s+2)$:

$$\left|Q_r^e\right| = \left|\sum_{t \in S} C_{r,t}^{-1} B_t^e\right| \leqslant \left|\sum_{t \in S} \frac{h}{(1 + \|r - t\|)^{s+2}} M\right| \leqslant \left|\sum_{t \in S} \frac{hM}{(1 + \|r - \bar{t}\|)^{s+2}}\right| = \frac{h'}{(1 + \|r - \bar{t}\|)^{s+2}}.$$

□

A graph of the recomposition filters is given in Fig. 2, together with the decomposition filters in the bi-harmonic bivariate case.

4. Decomposition and recomposition algorithm

According to Theorem 1, for every $n \in Z$ each signal $\sigma_{n+1} \in V_{n+1}$ admits a (unique) representation in $V_n + W_n$, namely,

$$\sigma_{n+1} = \sigma_n + \sigma_n^w, \tag{39}$$

σ_n is the of approximation function of σ_{n+1} in V_n and σ_n^w is the detail function of σ_{n+1}, in W_n.

For every $n \in Z$, and $e' \in E'$, we call approximation coefficients and detail coefficients, the vectors λ^n and $\mu^{n,e'}$, $e' \in E'$ respectively, which are defined by

$$\sigma_n = \sum_{j \in Z^s} \lambda_j^n \phi_m\big(2^n \bullet - j\big), \qquad \sigma_n^w = \sum_{e' \in E'} \sum_{j \in Z^s} \mu_j^{n,e'} \psi_m^{e'}\big(2^n \bullet - j\big). \tag{40}$$

As usual we say that a signal $\sigma_0 \in V_0$ is decomposed at the lth level ($l \in N$) if and only if it is expressed in the (unique) form

$$\sigma_0 = \sigma_{-l} + \sigma_{-l}^w + \sigma_{-l+1}^w + \cdots + \sigma_{-1}^w,$$

where $\sigma_{-l} \in V_{-l}$ and $\sigma_{-i}^w \in W_{-i}$, $i = 1, 2, \ldots, l$. Note that σ_{-l} is a signal (approximation) with knots in $2^l Z^s$, and so it has less information than σ_0 has (the complementary information being in $(\sigma_{-i}^w)_{i=1,\ldots,l}$ (coarser and coarser details)).

Deriving σ_{-l} and $(\sigma_{-i}^w)_{i=1,\ldots,l}$ from σ_0, is called decomposition of the signal at the lth level. Deriving σ_0 from σ_{-l} and $(\sigma_{-i}^w)_{i=1,\ldots,l}$ is called reconstruction of the signal. These processes are realized by using decomposition and recomposition formulae involving the approximation and detail coefficients, which we provide in this section.

Decomposition algorithm. For every n in Z and $e' \in E'$, let λ^{n+1}, λ^n, and $\mu^{n,e'}$ be the approximation and detail coefficients, as defined by (41). Let p and, for every $e' \in E'$, $q^{e'}$ be the decomposition filters, as defined by (22) and (23). Then the following relation holds for all $j \in Z^s$:

$$\lambda_j^n = \big(\lambda^{n+1} * p\big)_{2j}, \qquad \mu_j^{n,e'} = \big(\lambda^{n+1} * q^{e'}\big)_{2j}, \quad e' \in E'. \tag{41}$$

Indeed, using (40), (41), and (22) we get

$$\begin{aligned}
\sigma_{n+1} = \sigma_n + \sigma_n^w &= \sum_{j \in Z^s} \lambda_j^n \phi_m\big(2^n \bullet - j\big) + \sum_{\substack{j \in Z^s \\ e' \in E'}} \mu_j^{n,e'} \psi_m^{e'}\big(2^n \bullet - j\big) \\
&= \sum_{k \in Z^s} \lambda_k^{n+1} \phi_m\big(2^{n+1} \bullet - k\big) = \sum_{\substack{k \in Z^s \\ e \in E}} \lambda_{2k-e}^{n+1} \phi_m\big(2^{n+1} \bullet - 2k + e\big) \\
&= \sum_{e \in E} \sum_{k \in Z^s} \lambda_{2k-e}^{n+1} \Bigg(\sum_{j \in Z^s} p_j^e \phi_m\big(2^n \bullet - k - j\big) + \sum_{\substack{j \in Z^s \\ e' \in E'}} q_j^{e,e'} \psi_m^{e'}\big(2^n \bullet - k - j\big) \Bigg) \\
&= \sum_{e \in E} \Bigg(\sum_{k \in Z^s} \sum_{j \in Z^s} \lambda_{2k-e}^{n+1} p_j^e \phi_m\big(2^n \bullet - k - j\big) + \sum_{k \in Z^s} \sum_{\substack{j \in Z^s \\ e' \in E'}} \lambda_{2k-e}^{n+1} q_j^{e,e'} \psi_m^{e'}\big(2^n \bullet - k - j\big) \Bigg),
\end{aligned}$$

which is, by changing j into $j - k$,

$$\begin{aligned}
\sigma_{n+1} &= \sum_{e \in E} \Bigg(\sum_{k \in Z^s} \sum_{j \in Z^s} \lambda_{2k-e}^{n+1} p_{j-k}^e \phi_m\big(2^n \bullet - j\big) + \sum_{k \in Z^s} \sum_{\substack{j \in Z^s \\ e' \in E'}} \lambda_{2k-e}^{n+1} q_{j-k}^{e,e'} \psi_m^{e'}\big(2^n \bullet - j\big) \Bigg) \\
&= \sum_{j \in Z^s} \Bigg(\sum_{\substack{k \in Z^s \\ e \in E}} \lambda_{2k-e}^{n+1} p_{j-k}^e \Bigg) \phi_m\big(2^n \bullet - j\big) + \sum_{\substack{j \in Z^s \\ e' \in E'}} \Bigg(\sum_{\substack{k \in Z^s \\ e \in E}} \lambda_{2k-e}^{n+1} q_{j-k}^{e,e'} \Bigg) \psi_m^{e'}\big(2^n \bullet - j\big).
\end{aligned}$$

So, from the unique representation of $\sigma_{n+1} \in V_{n+1} = V_n \oplus W_n$ in the bases $\{\phi_m(2^n\bullet - j)\}_{j\in Z^s}$ and $\{\psi_m^{e'}(2^n\bullet - j)\}_{j\in Z^s,\, e'\in E'}$, we get

$$\lambda_j^n = \sum_{\substack{k\in Z^s\\ e\in E}} \lambda_{2k-e}^{n+1} p_{j-k}^e \quad \text{and} \quad \mu_j^{n,e'} = \sum_{\substack{k\in Z^s\\ e\in E}} \lambda_{2k-e}^{n+1} q_{j-k}^{e,e'};$$

the recursive formulae (42) can be obtained from the previous one by using (23).

Note that formulae (42) only involve convolutions and downsampling.

Recomposition algorithm. For every n in Z, and any $e' \in E'$, let λ^{n+1}, λ^n, and $\mu^{n,e'}$ be the approximation and the detail coefficients, as defined by (41), and let c and d be the recomposition filters as defined by (19) and (20). Then the following relation holds for all $j \in Z^s$:

$$\lambda_j^{n+1} = \sum_{k\in Z^s} \lambda_k^n\, c_{j-2k} + \sum_{\substack{k\in Z^s\\ e'\in E'}} \mu_k^{n,e'} d_{j+e'-2k}. \tag{42}$$

Indeed, using (40), (41), (19), and (20) we get (we can change the orders of the sums since the series are absolutely convergent):

$$\begin{aligned}
\sigma_{n+1} &= \sum_{j\in Z} \lambda_j^{n+1} \phi_m\left(2^{n+1}\bullet - j\right) = \sigma_n + \sigma_n^w = \sum_{k\in Z} \lambda_k^n \phi_m\left(2^n\bullet - k\right) + \sum_{\substack{k\in Z^s\\ e'\in E'}} \mu_k^{n,e'} \psi_m^{e'}\left(2^n\bullet - k\right)\\
&= \sum_{k\in Z^s} \lambda_k^n \sum_{j\in Z^s} c_j \phi_m\left(2^{n+1}\bullet - 2k - j\right) + \sum_{\substack{k\in Z^s\\ e'\in E'}} \mu_k^{n,e'} \sum_{j\in Z^s} d_j \phi_m\left(2^{n+1}\bullet - 2k + e' - j\right)\\
&= \sum_{k\in Z^s} \lambda_k^n \sum_{j\in Z^s} c_{j-2k} \phi_m\left(2^{n+1}\bullet - j\right) + \sum_{\substack{k\in Z^s\\ e'\in E'}} \mu_k^{n,e'} \sum_{j\in Z} d_{j+e'-2k} \phi_m\left(2^{n+1}\bullet - j\right)\\
&= \sum_{j\in Z^s} \left(\sum_{k\in Z^s} \lambda_k^n c_{j-2k} + \sum_{\substack{k\in Z^s\\ e'\in E'}} \mu_k^{n,e'} d_{j+e'-2k} \right) \phi_m\left(2^{n+1}\bullet - j\right)
\end{aligned}$$

and from the unique representation of $\sigma_{n+1} \in V_{n+1}$ in the bases $\{\phi_m(2^{n+1}\bullet - j)\}_{j\in Z^s}$, we get (43).

Note that formulae (43) only involve upsampling and convolutions.

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