RECIPROCAL UPPER SEMICONTINUITY AND BETTER REPLY SECURE GAMES: A COMMENT

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A convex, compact, and possibly discontinuous better reply secure game has a Nash equilibrium. We introduce a very weak notion of continuity that can be used to establish that a game is better reply secure and we show that this notion of continuity is satisfied by a large class of games.

1. INTRODUCTION

THE CLASS OF BETTER REPLY SECURE GAMES was introduced by Reny (1999) who showed the existence of a pure Nash equilibrium for games in this class. Furthermore, Reny provided two conditions that, when combined, become sufficient for a game to be better reply secure. The first condition is payoff security, which means that given a joint strategy x, every player can find a strategy that yields almost the same payoff at x, even when the other players slightly deviate from x. The second condition is *reciprocal upper semicontinuity* (rusc). This condition, roughly speaking, requires the payoff of one of the players to jump up whenever the payoff of another player jumps down. We introduce the notion of weak reciprocal upper semicontinuity (wrusc), a strict weakening of rusc, and prove that a game that is wrusc and payoff secure is better reply secure, and therefore has a pure strategy Nash equilibrium.

2. PRELIMINARIES

We consider a game G denoted by $(X_i, u_i)^N$. This game consists of N players and each player $i = 1, \ldots, N$ has a compact and convex strategy set $X_i \subset U$, where U is a metric space. Each player has a bounded payoff function $u_i : X \to \mathbb{R}$, where $X = \prod_{i \in N} X_i$. We use the standard notation $X_{-i} = \prod_{k \neq i}^{N} X_k$ and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \in X_{-i}$. We further assume that for every $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is quasiconcave. We call such a game a compact and quasiconcave game. The *graph* of the game is $\Gamma = \{(x, u) \in X \times \mathbb{R}^N | u_i(x) = u_i, \forall i\}$. The closure of Γ in $X \times \mathbb{R}^n$ is denoted by $\bar{\Gamma}$. The *frontier* of Γ, which is the sets of points that are in $\bar{\Gamma}$ but not in Γ, is denoted by $\operatorname{Fr} \Gamma$. We now review some of the basic definitions introduced in Reny (1999).

DEFINITION 1: Player i can *secure a payoff* $\alpha \in \mathbb{R}$ at the point $x \in X$, if $\exists \hat{x}_i \in X_i$ and $\exists V(x_{-i})$ is a neighborhood of x_{-i} such that, for all $x'_{-i} \in V(x_{-i})$, $u_i(\hat{x}_i, x'_{-i}) \geq \alpha$.

DEFINITION 2: A game $(X_i, u_i)^N$ is *better reply secure* if for every $(x^*, u) \in \bar{\Gamma}$, the point x^* is not an equilibrium implies that some player i can secure a payoff strictly above u_i at x^*.

Theorem 3.1 in Reny (1999) shows that a game that is compact, quasiconcave, and better reply secure must have a pure strategy Nash equilibrium. One can directly verify better reply security by checking the conditions of Definition 2. In most applications, however, it is more convenient to verify the following conditions:

DEFINITION 3: A game is *reciprocally upper semicontinuous* (rusc), if whenever (x, u) is in the closure of Γ and $u_i(x) \leq u_i$ for all i, then $u_i(x) = u_i$ for all i.

DEFINITION 4: A game $(X_i, u_i)^N$ is *payoff secure* if the following holds for every player i: for all $x \in X$, $\forall \varepsilon > 0$, $\exists \hat{x}_i \in X_i$, $\exists V(x_{-i})$ is a neighborhood of x_{-i}, such that for all $x'_{-i} \in V(x_{-i})$, $u_i(\hat{x}_i, x'_{-i}) \geq u_i(x) - \varepsilon$.

If a game $(X_i, u_i)^N$ is payoff secure and rusc, then it is better reply secure (Proposition 3.2 in Reny (1999)). However, the converse is not always true.

3. WEAK UPPER SEMICONTINUITY

We now introduce a modified definition of reciprocal upper semicontinuity.

DEFINITION 5: A game is *weakly reciprocal upper semicontinuous* (wrusc), if for any $(x, \alpha) \in \text{Fr} \, \Gamma$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > \alpha_i$.

Unlike rusc, the preceding condition allows for the payoff functions of all the players to jump down at some point x as long as the payoff of some player jumps up, relative to his old payoff at x, *somewhere* in X. In some sense, the relationship between wrusc and rusc is similar to the relationship between payoff security and lower semicontinuity. As we will demonstrate in Example 2, a classic Bertrand price competition game with producers facing a lower semicontinuous demand will be wrusc but not rusc (lower semicontinuous demand functions are very common; consider all the items priced at \$9.99). Furthermore, both rusc and wrusc are ordinal in the sense that they are preserved under any continuous and strictly monotone transformation of the utility of each player. In this note, we will use examples from timing games on the square to illustrate the properties of weak reciprocal upper semicontinuity. This concept of semicontinuity, however, is relevant to other types of discontinuous games.

Our first proposition shows that wrusc and payoff security are sufficient to make the game better reply secure.

PROPOSITION 1: *If the game is payoff secure and wrusc, then it is better reply secure.*

PROOF: Suppose $(x^*, \alpha) \in \bar{\Gamma}$ and assume that x^* is not an equilibrium. Assume first that $(x^*, \alpha) \in \Gamma$. Then, for all i, $u_i(x^*) = \alpha_i$. Whereas x^* is not an equilibrium, we must have that for some i, there exists $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > u_i(x^*) = \alpha_i$. Payoff security implies that this player can secure a payoff strictly higher than $u_i(x^*) = \alpha_i$. If $(x^*, \alpha) \in \bar{\Gamma} \setminus \Gamma$, then wrusc implies that there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x^*_{-i}) > \alpha_i$, and again the condition of payoff security implies that this player can secure a payoff strictly higher than α_i. Q.E.D.

PROPOSITION 2: *Reciprocal upper semicontinuity implies weak reciprocal upper semicontinuity.*

PROOF: Let $(x, \alpha) \in \bar{\Gamma} \setminus \Gamma$. Suppose that $\forall i$, $u_i(x) \leq \alpha_i$. Then, rusc implies $u_i(x) = \alpha_i$, which contradicts the fact that $(x, \alpha) \notin \Gamma$. Hence, there is a player i such that $u_i(x) > \alpha_i$ and the game is wrusc. Q.E.D.

Note that to verify wrusc, we only need to check the frontier of the graph of the game, which in general is a very small set. Consider, for example, a quasi-concave game where the tie-breaking rule is the only source of discontinuity. Such a game is wrusc as long as the tie-breaking rule is increasing on the diagonal. This is true no matter what type of discontinuity the tie-breaking rule exhibits.

The following example shows that the condition of wrusc is strictly weaker than rusc.

EXAMPLE 1: The following game is a special case of a class of timing games on the unit square that was considered by Reny (1999). For $i = 1, 2$ and $0 \leq t_i \leq 1$, let the payoff functions for the players be given by

$$u_i(t_i, t_{-i}) = \begin{cases} l_i(t_i) = 10 & \text{when } t_i < t_{-i}, \\ \varphi_i(t) & \text{when } t_i = t_{-i} = t, \\ m_i(t_{-i}) = -10 & \text{when } t_i > t_{-i}. \end{cases}$$

Let φ_i be such that $\varphi_1 = \varphi_2 = 1$ when $x_1 = x_2$ and $x_i < 0.5$, and $\varphi_1 = \varphi_2 = 0$ when $x_1 = x_2$ and $x_i \geq 0.5$. Now consider the point $(x_1^*, x_2^*, u_1^*, u_2^*) = (0.5, 0.5, 1, 1)$. This point is in the closure of the graph of the game. However, $u_i(0.5, 0.5) < 1$ for both players and so, contrary to the claim in Reny (1999), the game is not rusc. This is due to the fact that both functions φ_i jump down in the same direction at some point on the diagonal.[1] To show that this game

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[1] This game is rusc if $\forall i$, $-10 \leq \varphi_i \leq 10$ and φ_i is upper semicontinuous.

is wrusc, consider a point $(x_1, x_2, \alpha_1, \alpha_2) \in \bar{\Gamma} \backslash \Gamma$. Because l_i and m_i are continuous and $(x_1, x_2, \alpha_1, \alpha_2)$ is not in Γ, (x_1, x_2) has to be on the diagonal with $x_2 = x_1$. Moreover, whereas $(x_1, x_2, \alpha_1, \alpha_2) \in \bar{\Gamma} \backslash \Gamma$, $\exists (x_1^n, x_2^n) \to (x_1, x_2)$ such that $\lim_n u_i(x_i^n, x_{-i}^n) = \alpha_i$ for all i. If this sequence is approaching (x_1, x_2) from a direction off the diagonal, then we must have $\alpha_i = -10$ for some i. On the other hand, if the sequence is approaching (x_1, x_2) along the lower half of the diagonal, we must have $\alpha_1 = \alpha_2 = 1$. In the first case, for at least one i, one can take $0 < \hat{x}_i < x_i$ and obtain $u_i(\hat{x}_i, 0.5) = 10 > \alpha_i = -10$. In the second case, and for any i, we can take $\hat{x}_i = 0.25$ and obtain $u_i(\hat{x}_i, 0.5) = 10 > \alpha_i = 1$. Note that our sequence is not allowed to approach the point (x_1, x_2) along the upper half of the diagonal because this would, contrary to our assumption, force the point $(x_1, x_2, \alpha_1, \alpha_2)$ to be in Γ.

This game can also be easily shown to be payoff secure. Therefore, it is better reply secure by Proposition 1.

The continuity assumptions on l_i and m_i in the previous example can be weakened, yet the resulting game can still be wrusc and, in fact, better reply secure. This point is illustrated in the next example.

EXAMPLE 2: Consider a two-player Bertrand price competition game on the square $[0, 4] \times [0, 4]$. Assume that the demand function is discontinuous and is given by

$$D(p) = \begin{cases} 8 - p & \text{if } 0 \leq p < 2, \\ 4 & \text{when } p = 2, \\ 4 - p & \text{when } 2 < p \leq 4. \end{cases}$$

The discontinuity in demand can result from nonconvex preferences, bandwagon effects, network effects, or from a variety of other reasons. For more details and examples on discontinuities in demand functions, see Baye and Morgan (2002). The monopoly profits are given by

$$\pi(p) = \begin{cases} p(8 - p) & \text{if } 0 \leq p < 2, \\ 8 & \text{when } p = 2, \\ p(4 - p) & \text{when } 2 < p \leq 4 \end{cases}$$

and the payoff function of player i is

$$u_i(p_i, p_{-i}) = \begin{cases} \pi(p) & \text{when } p_i < p_{-i}, \\ \dfrac{1}{2}\pi(p) & \text{when } p_i = p_{-i} = p, \\ 0 & \text{when } p_i > p_{-i}. \end{cases}$$

This game is not rusc because the payoffs of both players jump down at $(2, 2)$ when we approach this point along the lower half of the diagonal. However,

this game is wrusc. Consider a point $(p_1, p_2, \alpha_1, \alpha_2) \in \bar{\Gamma} \backslash \Gamma$. Because this point is not in Γ and because the payoffs are continuous at any point with a zero price, p_i must be strictly positive for all i. If $p_1 < p_2$, then $\alpha_2 = 0$ and any $\hat{p}_2$ such $0 < \hat{p}_2 < p_1$ will give us $u_2(\hat{p}_2, p_1) > \alpha_2$. A similar argument works if $p_2 < p_1$ or if $p_1 = p_2 \neq 2$ (again keeping in mind that $(p_1, p_2, \alpha_1, \alpha_2) \notin \Gamma$). If $p_1 = p_2 = 2$ and $\alpha_i = 0$ for some i, then we can still take any $0 < \hat{p}_i < 2$ and obtain $u_i(\hat{p}_i, 2) > 0$. If $p_1 = p_2 = 2$ and $\alpha_i \neq 0$ for all i, then for every i, we either have $\alpha_i = 6$ or $\alpha_i = 2$. In either case, we can take $\hat{p}_i = 1$ and we will have $u_i(1, 2) = 7 > \alpha_i$.

Furthermore, this game is payoff secure because each player can secure his payoff at a given nonzero price vector by slightly lowering his own price. If the initial price of a player was zero, then his payoff of zero can be secured by keeping his initial price. Using Proposition 1, we conclude that this game is actually better reply secure.

Better reply security is more permissive than the combination of payoff security and wrusc (rusc). Therefore, it is *always* possible to show that a game is better reply secure without using Proposition 1, and for that matter without using Proposition 3.2. of Reny (1999). However, there are games where using Proposition 1 provides a more systematic and a more convenient approach for proving better reply security than a direct proof based on the definition. The following example illustrates this point.

EXAMPLE 3: Consider another variation on our initial timing game on the unit square $[0, 1] \times [0, 1]$: The payoff of player i is given by the functions (for $i = 1, 2$)

$$u_i(x_i, x_{-i}) = \begin{cases} \beta_i & \text{when } x_i = x_{-i} = 0, \\ \dfrac{f_i(x_i)}{f_i(x_i) + f_{-i}(x_{-i})} - g_i(x_i) & \text{otherwise.} \end{cases}$$

This game can be interpreted as a rent seeking competition between two players, where x_i represents the effort exerted or resources expended by player i to win a prize that is normalized to 1. The function $(f_i(x_i))/(f_i(x_i) + f_{-i}(x_{-i}))$ represents the probability that player i wins the prize, and $g_i(x_i)$ represents the cost of the effort exerted by this player. The variable β_i represents the share of the prize player i can obtain when both players do not exert any effort. This type of rent seeking game is often used to represent political contests such as running for an office, lobbying, or military conflicts. It is also used to model recruitment of talent, lotteries, advertisement, and patent races (Tullock (1980), Nitzan (1994), and Paul and Wilhite (1990)). Traditionally, the existence of a pure strategy Nash equilibrium in these applications is established using first-order conditions under the appropriate differentiability assumptions.

Rather than assuming that u_i is differentiable, we will assume the following conditions for every i:

(a) The function u_i is bounded on $[0, 1] \times [0, 1]$, and $u_i(\cdot, x_{-i})$ is quasi-concave. Moreover, u_i is continuous except possibly at the origin, $f_i(0) = 0$, f_i is strictly increasing, and g_i is continuous at 0 with $g_i(0) = 0$.

(b) We have $\beta_i < 1$ and $\beta_i + \beta_{-i} \le 1$.

The requirement that u_i is continuous in the interior of the unit square can be weakened by allowing the "right" type of discontinuities as we did in Example 2. In many applications, it is natural to assume that $\beta_1 + \beta_2$ is strictly less than 1. This means that, in the case where both players do not exert any effort, a fraction of the prize will be divided between the players in some fashion while the rest of the prize will be lost. We finally note that conditions (a) and (b) are both satisfied by the original rent seeking game suggested by Tullock (1980) as well as by many of the more recent variations on it.

The foregoing game is payoff secure. The payoff functions are continuous everywhere except at the origin. Furthermore, at the origin, and for every i, assumptions (a) and (b) imply that $\exists \tilde{x}_i \in (0, 1]$ such that $u_i(\tilde{x}_i, 0) = 1 - g_i(\tilde{x}_i) > \beta_i$. Moreover, u_i is continuous at $(\tilde{x}_i, 0)$. Hence, for any $\varepsilon > 0$, $\exists V$ is a neighborhood of 0 such that $u_i(\tilde{x}_i, x'_{-i}) > u_i(0, 0) = \beta_i$ for all $x'_{-i} \in V$ and, therefore, the game is payoff secure at the origin.

If $\beta_i < \frac{1}{2}$ for both players, then the preceding game is not necessarily rusc. This is due to the fact that $\lim_{x \to 0} u_i(x)$ can be anything between 0 and 1, depending on the exact forms of f_i and g_i as well as on the direction we use to approach $(0, 0)$. Take for example the case where $f_i = f_{-i}$ and $\beta_i = \beta_{-i} = 0$, and consider a sequence that approaches the origin along the diagonal of unit square. Following such a sequence, the payoffs of both players will jump down at the origin from $\frac{1}{2}$ to 0, and the game is not rusc. This preceding game, however, is always wrusc under assumptions (a) and (b). Because the origin is the only point of discontinuity, any point in $\bar{\Gamma} \backslash \Gamma$ has to be of the form $(0, 0, \alpha_1, \alpha_2)$, where for some $x^n \to 0$ and for every i, we have $\lim u_i(x^n) = \alpha_i$. Whereas $\alpha_1 + \alpha_2 = 1$, we must have $\alpha_i < 1$ for some i. Without loss of generality, suppose $\alpha_2 < 1$. Because $\lim_{x_2 \to 0} u_2(x_2, 0) = 1$, there exists $0 < \hat{x}_2 < 1$ such that $u_2(\hat{x}_2, 0) > \alpha_2$.

The combination of wrusc with payoff security gives us better reply security.

Using simple examples of timing games on the square, one can show that the conditions of payoff security and wrusc are independent in the sense that a compact and quasiconcave game can satisfy one without satisfying the other. Finally, the concepts of better reply security, payoff security, and rusc can be generalized in a natural way to extended games (games with mixed strategies). Moreover, every game that is rusc and payoff secure in the extended sense has a mixed strategy Nash equilibrium, even if the payoff functions are not quasi-concave (Corollary 5.1 in Reny (1999)). Similarly, the definition of wrusc can be easily generalized to extended games. Moreover, the combination of extended wrusc and extended payoff security implies the existence of a Nash equilibrium in mixed strategies without requiring the payoff functions to be quasiconcave.

BETTER REPLY SECURE GAMES

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