Delay Perturbed Sweeping Process

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Abstract This paper is devoted to the study of a nonconvex perturbed sweeping process with time delay in the infinite dimensional setting. On the one hand, the moving subset involved is assumed to be *prox-regular* and to move in an *absolutely continuous way*. On the other hand, the perturbation which contains the delay is single-valued, separately measurable, and separately Lipschitz. We prove, without any compactness assumption, that the problem has one and only one solution.

Key words sweeping process · differential inclusion · normal cone · prox-regular set · delay · perturbation · absolutely continuous map · set-valued map.

1. Introduction

In this paper we are interested in the existence of solutions for a delay perturbed sweeping process in an infinite dimensional Hilbert space. The problem is the following: Let H be a real Hilbert space, $T > 0$, $C\colon [0,T] \rightrightarrows H$ a set-valued map with nonempty closed values. Given a finite delay $\rho \geqslant 0$, one considers the space $\mathcal{C}_0 := \mathcal{C}_H([-\rho, 0])$ endowed with the norm of the uniform convergence $\|\cdot\|_{\mathcal{C}_0}$. With each $t \in [0,T]$, one associates a map $\tau(t)$ from $\mathcal{C}_H([-\rho, t])$ into $\mathcal{C}_H([-\rho, 0])$ defined, for all $u(\cdot) \in C_H([-\rho, t])$, by

$$(\tau(t)u(\cdot))(s) := u(t+s) \quad \text{for all} \quad s \in [-\rho, 0].$$

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Let $f\colon [0,T]\times \mathcal{C}_H([-\rho,0]) \to H$ be a single-valued map and let φ be a fixed member of $\mathcal{C}_H([-\rho,0])$ such that $\varphi(0)\in C(0)$. Then, we investigate the existence of solutions for the following perturbed sweeping process

$$\begin{cases} -\dot{u}(t) \in N(C(t),u(t)) + f(t,\tau(t)u(\cdot)) \text{ a.e. } t\in[0,T], \\ u(s)=\varphi(s)\ \forall s\in[-\rho,0]. \end{cases} \tag{0.1}$$

We need an existence result for this problem in order to study, in the infinite dimensional setting, an optimal control problem whose dynamic is given by a delay perturbed sweeping process. Indeed, using the result of this paper, we prove, under a classical convexity assumption, the existence of a solution for an optimal control problem of the type

$$\inf_{\substack{\zeta(\cdot)\\ \zeta(t)\in\Gamma(t)}} \quad L(u^{\zeta}(T)) + \int_0^T J(t,u^{\zeta}(t),\zeta(t))dt,$$

where $u^{\zeta}(\cdot)$ is the unique solution of the delay perturbed sweeping process

$$\begin{cases} -\dot{u}(t) \in N(C(t),u(t)) + g(t,\tau(t)u(\cdot),\zeta(t)) \text{ a.e. } t\in[0,T], \\ u(s)=\varphi(s)\ \forall s\in[-\rho,0]. \end{cases}$$

This result will be published in a forthcoming paper.

While the differential inclusions of the type (0.1) encompass the differential equations (the case $C(t):=H$ for all $t\in[0,T]$), they are necessary to study some systems. They are used, particularly, to describe mechanical systems with inelastic shocks (see [16, 19], and [20]), which explains, besides mathematical motivations, the interest for optimal control problems governed by such dynamics.

The problem (0.1) is a particular case of the more general one obtained by replacing f by a set-valued map $G\colon [0,T]\times\mathcal{C}_H([-\rho,0]) \rightrightarrows H$, that is,

$$\begin{cases} -\dot{u}(t) \in N(C(t),u(t)) + G(t,\tau(t)u(\cdot)) \text{ a.e. } t\in[0,T], \\ u(s)=\varphi(s)\ \forall s\in[-\rho,0]. \end{cases} \tag{0.2}$$

It is worth noting that those problems are extensions of the following one

$$\begin{cases} -\dot{u}(t) \in N(C(t),u(t)) \text{ a.e. } t\in[0,T], \\ u(0)\in C(0), \end{cases} \tag{0.3}$$

which was introduced and thoroughly studied by Moreau (see [17, 18] and the references therein) with $C(t)$ convex for all t and moving in an *absolutely continuous way*. In this case, $N(C(t),u(t))$ is the normal cone to $C(t)$ at $u(t)$ in the sense of the convex analysis. Other references concerning the problem (0.3) are [1, 5, 6, 12], and [23].

The problem (0.2) has been solved by Castaing and Monteiro Marques [7] under some conditions. Among others, G has all its values included in a fixed bounded set and C is Lipschitz and takes convex compact values. On the other hand, Thibault [22] proved, in the finite dimensional context, the existence of solutions for general subsets $C(t)$ and for G satisfying

$$G(t,\phi(\cdot)) \subset \beta(t)\mathbb{B}$$

for all $(t, \phi(\cdot)) \in [0, T] \times \mathcal{C}_H([-\rho, 0])$, where $\beta(\cdot) \in L^1([0, T], \mathbb{R}^+)$. More recently, still in the finite-dimensional setting, Castaing et al. [8] proved the same result for sets $C(t)$ that are bounded and *r-prox-regular*, with G satisfying a more general growth condition of the type

$$G(t, \phi(\cdot)) \subset \beta(t)(1 + \|\phi(0)\|)\mathbb{B} \tag{0.4}$$

for all $(t, \phi(\cdot)) \in [0, T] \times \mathcal{C}_H([-\rho, 0])$. Later, Bounkhel and Yarou [4] proved in the infinite-dimensional setting the existence of solutions in the case the set-valued map G has all its valued contained in a fixed compact set. More recently, we proved in [13] a more general result where G satisfies (0.4) with $\mathbb{B}$ replaced by a fixed compact set.

In infinite-dimensional Hilbert spaces, unless appropriate compactness assumptions on the sets $C(t)$, the problem (0.2) with the condition (0.4) may have no solution.

In this paper we address, in the infinite-dimensional setting, the case where G is a single-valued map. We establish an existence result without any compactness assumption. More precisely, we prove that the problem (0.1) has one and only one solution if the sets $C(t)$ are r-prox-regular (not necessarily bounded), the map f is measurable with respect to the first argument and Lipschitz with respect to the second one, and

$$\|f(t, \phi(\cdot))\| \leqslant \beta(t)(1 + \|\phi(\cdot)\|_{\mathcal{C}_0})$$

for all $(t, \phi(\cdot)) \in [0, T] \times \mathcal{C}_H([-\rho, 0])$. Note that this growth condition involves $\|\phi\|_{\mathcal{C}_0}$ instead of $\|\phi(0)\|$. This condition is weaker than (0.4) when G is single-valued. Whereas it is more natural as a growth condition, it is more difficult to deal with.

To our knowledge, up to now, even in the case the sets $C(t)$ are convex, there is no existence result for (0.1) without compactness assumptions on the sets $C(t)$. Such assumptions guarantee, for any bounded sequence of continuous maps $u\colon [0, T] \to H$ such that $u(t) \in C(t)$ for each t, the existence of a convergent subsequence. But, in our setting, a priori, such a subsequence does not exist. Therefore, to obtain convergence results for a sequence, not only that must be constructed carefully, but also some effort is required.

Our existence result is obtained thanks to the one proved recently in [14] concerning perturbed sweeping processes without delay. We proceed as follows: We consider, for each $n \in \mathbb{N}$, a partition of the interval $[0, T]$ given by $t_j^n := \frac{jT}{n}$ $(j = 0, \cdots, n)$. Then, on each subinterval $\left[t_j^n, t_{j+1}^n\right]$, we replace f by the map $f_j^n\colon \left[t_j^n, t_{j+1}^n\right] \times H \to H$ defined by $f_j^n(t, x) := f(t, \tau(t)h_j^n(\cdot, x))$, where

$$h_0^n(t, x) := \begin{cases} \varphi(t) & \text{if} \quad t \in [-\rho, 0], \\ \varphi(0) + \dfrac{n}{T}t(x - \varphi(0)) & \text{if} \quad t \in \left[0, t_1^n\right] \end{cases}$$

and $h_j^n(\cdot, \cdot)$ $(j \geqslant 1)$ are defined in a quasi similar way. Doing so, we obtain a perturbed sweeping process without delay for which our result in [14] insures the existence of a solution $u_n(\cdot)$. This approach is slightly different from the classic idea in that, in our definition of f_j^n, we allow the second argument to depend on each $t \in \left[t_j^n, t_{j+1}^n\right]$. In addition to other techniques used to overcome the absence of compactness, this adaptation enables the proof of the convergence of the sequence (u_n) to a solution of the original problem.

The paper is organized as follows. In Section 2, we recall some notions which are used throughout the paper. In Section 3 are summarized some results concerning perturbed sweeping processes without delay. Finally, Section 4, which is the most important, is devoted to the existence result for the delay perturbed sweeping process.

2. Preliminaries

In all the paper $I := [0, T]$ ($T > 0$) is an interval of $\mathbb{R}$ and H is a real Hilbert space whose scalar product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|$.

NOTATION 1.1. We will use the following notations.

The closed unit ball of H will be denoted by $\mathbb{B}$.

For $\eta > 0$, one denotes by $B[0, \eta]$ the closed ball of radius η centered at 0. For any subset S of H, $\overline{\text{co}}S$ stands for the closed convex hull of S, and $\sigma(S, \cdot)$ represents the support function of S, that is, for all $\zeta \in H$,

$$\sigma(S, \zeta) := \sup_{x \in S} \langle \zeta, x \rangle.$$

We will denote by $\mathcal{C}(I, H)$ or $\mathcal{C}_H(I)$ the set of all continuous maps from I to H. The norm of the uniform convergence on $\mathcal{C}(I, H)$ will be denoted by $\| \cdot \|_\infty$. The Lebesgue measure is denoted by λ.

For any $p \in [1, +\infty]$, we denote by $L^p(I, H)$ the quotient space of all λ-Bochner measurable maps $g(\cdot) \colon I \to H$ such that $\|g(\cdot)\|$ belongs to $L^p(I, \mathbb{R})$.

For the following concepts, the reader is referred to Clarke et al. [10, 11] and Poliquin et al. [21].

Let S be a nonempty closed subset of H and $x \in H$. The distance of x to S, denoted by $d_S(x)$ or $d(x, S)$, is defined by

$$d_S(x) := \inf\{\|x - u\| : u \in S\}.$$

One defines the (possibly empty) set of nearest points of x in S by

$$\text{proj}_S\{x\} := \{u \in S : d_S(x) = \|x - u\|\}.$$

If $u \in \text{proj}_S\{x\}$ and $\alpha \geqslant 0$, then the vector $\alpha(x - u)$ is called a *proximal normal* to S at u. The set of all vectors obtainable in this manner is a cone termed the *proximal normal cone* to S at u. It is denoted by $N_S^P(u)$.

One also defines the *limiting normal cone* and the *Clarke normal cone*, respectively, by

$$N_S^L(u) := \left\{ \zeta \in H \colon \zeta_n \overset{w}{\rightharpoonup} \zeta, \zeta_n \in N_S^P(u_n), u_n \overset{S}{\to} u \right\}$$

and

$$N_S^C(u) := \overline{\text{co}} N_S^L(u).$$

Here, $\zeta_n \overset{w}{\rightharpoonup} \zeta$ signifies that the sequence ζ_n converges weakly to ζ, and $u_n \overset{S}{\to} u$ means that $u_n \to u$ with $u_n \in S$ for all n.

For a fixed $r > 0$, the set S is said to be *r-prox-regular* (or *uniformly prox-regular with constant* $\frac{1}{r}$) if, for any $u \in S$ and any $\zeta \in N_S^L(u)$ such that $\|\zeta\| < 1$, one has $\{u\} = \operatorname{proj}_S\{u + r\zeta\}$. Equivalently, S is *r-prox-regular* if and only if (see [21]) every nonzero proximal normal to S at any point $u \in S$ can be realized by an r-ball, that is, for all $u \in S$ and all $\zeta \in N_S^P(u)$,

$$\langle \zeta, y - u \rangle \leqslant \frac{\|\zeta\|}{2r} \|y - u\|^2 \text{ for all } y \in S. \tag{1.1}$$

Another characterization (see [21]) is the following *hypomonotonicity* property: For any $u_i \in S$ ($i = 1, 2$), the inequality

$$\langle \zeta_1 - \zeta_2, u_1 - u_2 \rangle \geqslant -\|u_1 - u_2\|^2$$

holds whenever $\zeta_i \in N_S^L(u_i) \cap B(0, r)$, where $B(0, r)$ stands for the open ball of radius r centered at 0.

If S is r-prox-regular, then the following holds (see [21]):

- for any $u \in S$, all the cones defined above coincide and will be denoted by $N_S(u)$ or $N(S, u)$;
- for any $x \in H$ such that $d_S(x) < r$, the set $\operatorname{proj}_S\{x\}$ is a singleton.

In the other hand, let $f : H \to \mathbb{R}$ be Lipschitz near $x \in H$. One defines the Clarke directional derivative of f at $x \in H$ in the direction $u \in H$ by (see Clarke [9])

$$f^\circ(x; u) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tu) - f(y)}{t}.$$

The Clarke subdifferential of f at x is then defined by

$$\partial^C f(x) := \{\zeta \in H : \langle \zeta, u \rangle \leqslant f^\circ(x; u) \forall u \in H\}.$$

We also recall the definition of the proximal subdifferential of f at $x \in H$ denoted by $\partial^P f(x)$. One says that $\zeta \in H$ belongs to $\partial^P f(x)$ (see, e.g., Clarke et al. [10]) if there exist positive numbers α and $M > 0$ such that

$$f(y) - f(x) + M\|y - x\|^2 \geqslant \langle \zeta, y - x \rangle \forall y \in B(x, \alpha).$$

Obviously, the inclusion $\partial^P f(x) \subset \partial^C f(x)$ holds for all $x \in H$. There are some links between the cones and the subdifferentials defined above (see [2] and [10]): For any nonempty closed subset S of H and $x \in S$, the following relations hold true

$$\partial^P d_S(x) = N_S^P(x) \cap \mathbb{B} \tag{1.2}$$

and

$$\partial^C d_S(x) \subset N_S^C(x) \cap \mathbb{B}. \tag{1.3}$$

Remark 1.1. If S is r-prox-regular, by (1.2), (1.3), and the equality between the proximal and Clarke normal cones, one has

$$\partial^P d_S(x) = \partial^C d_S(x)$$

whenever $x \in S$.

Let $r > 0$. In all the paper a set-valued map $C(\cdot)$ from I to H will be involved. It is required to satisfy the following assumptions:

(H_1) For each $t \in I$, $C(t)$ is a nonempty closed subset of H which is r-prox-regular;
(H_2) $C(t)$ varies in an *absolutely continuous way*, that is, there exists an absolutely continuous function $v(\cdot)\colon I \to \mathbb{R}$ such that, for any $y \in H$ and $s, t \in I$,

$$|d(y, C(t)) - d(y, C(s))| \leqslant |v(t) - v(s)|.$$

We will use the following result which is a straightforward consequence of Gronwall's lemma.

LEMMA 1.1. *Let $I = [T_0, T]$ and let $(x_n(\cdot))$ be a sequence of non-negative continuous functions define on I, (α_n) a sequence of real numbers, and $\beta(\cdot) \in L^1(I, \mathbb{R}^+)$. Assume that* $\lim_n \alpha_n = 0$ *and, for all n,*

$$x_n(t) \leqslant \int_{T_0}^{t} \beta(s) x_n(s)\, ds + \alpha_n. \tag{1.4}$$

Then, for all $t \in [T_0, T]$,

$$\lim_n x_n(t) = 0.$$

Proof. Fix any $t \in I$. Mutiplying both sides of (1.4) by $\beta(t)$, we obtain

$$\beta(t) x_n(t) \leqslant \beta(t) \int_{T_0}^{t} \beta(s) x_n(s)\, ds + \alpha_n \beta(t).$$

According to Gronwall's lemma, this entails that

$$\int_{T_0}^{t} \beta(s) x_n(s)\, ds \leqslant \alpha_n \int_{T_0}^{t} \beta(u) \exp\{\int_{u}^{t} \beta(s)\, ds\}\, du$$

and then

$$\lim_n \int_{T_0}^{t} \beta(s) x_n(s) = 0.$$

Taking (1.4) into account, we deduce that $\lim_n x_n(t) = 0$. □

3. Perturbation without Delay

In this section we summarize two results concerning perturbed sweeping processes. They will be used in the sequel.

PROPOSITION 2.1. *Let H be a real Hilbert space. Assume that $C(\cdot)$ satisfies (H_1) and (H_2). Let $h\colon [T_0, T] \to H$ be a λ-integrable map. Then, for any $x_0 \in C(T_0)$, the sweeping process with perturbation*

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) + h(t) \text{ a.e. } t \in [T_0, T], \\ u(T_0) = x_0 \end{cases} \tag{2.1}$$

has one and only one absolutely continuous solution $u(\cdot)$. Moreover, the following inequality holds true

$$\|\dot{u}(t) + h(t)\| \leqslant \|h(t)\| + |\dot{v}(t)| \text{ a.e. } t \in [T_0, T].$$

Proof. We use a classical transformation. For each $t \in [T_0, T]$, let us set

$$\psi(t) := \int_{T_0}^{t} h(s)\, ds \text{ and } D(t) := C(t) + \psi(t).$$

Obviously, the set-valued map $D(\cdot)$ satisfies (H_1). Now, let $y \in H$ and $t, s \in [T_0, T]$. One has

$$\begin{aligned} |d(y, D(t)) - d(y, D(s))| &\leqslant |d(y - \psi(t), C(t)) - d(y - \psi(s), C(s))| \\ &\leqslant \|\psi(t) - \psi(s)\| + |v(t) - v(s)| \\ &\leqslant |V(t) - V(s)|, \end{aligned}$$

where

$$V(t) := \int_{T_0}^{t} (|\dot{v}(s)| + \|h(s)\|)\, ds.$$

Hence $D(\cdot)$ satisfies also (H_2) with the absolutely continuous function $V(\cdot)$. As $x_0 \in C(T_0) = D(T_0)$, from [3] (or [15]) we know that the following sweeping process

$$\begin{cases} -\dot{y}(t) \in N(D(t), y(t)) \text{ a.e. } t \in [T_0, T], \\ y(T_0) = x_0 \end{cases}$$

has an absolutely continuous solution $y(\cdot)$. According to [22], the map $y(\cdot)$ satisfies also the inclusion

$$-\dot{y}(t) \in \dot{V}(t)\partial d_{D(t)}(y(t)) \text{ a.e. } t \in [T_0, T].$$

Thus,

$$\|\dot{y}(t)\| \leqslant |\dot{V}(t)| = |\dot{v}(t)| + \|h(t)\| \text{ a.e. } t \in [T_0, T]. \tag{2.2}$$

Futhermore, the map $u(\cdot)$ defined by $u(t) := y(t) - \psi(t)$ is clearly an absolutely continuous solution of (2.1). Finally, by (2.2), we obtain the estimation

$$\|\dot{u}(t) + h(t)\| \leqslant \|h(t)\| + |\dot{v}(t)| \text{ a.e. } t \in [T_0, T].$$

Now, we turn to the uniqueness. If $u_1(\cdot)$ and $u_2(\cdot)$ are two solutions, the hypomonotonicity property of the normal cone yields, for almost all $t \in I$,

$$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t)\rangle \leqslant \frac{1}{r}(\|\dot{u}_1(t)\| + \|\dot{u}_2(t)\| + \|h(t)\|)\|u_1(t) - u_2(t)\|^2$$

and then

$$\frac{d}{dt}(\|u_1(t) - u_2(t)\|^2) \leqslant \frac{2}{r}(\|\dot{u}_1(t)\| + \|\dot{u}_2(t)\| + \|h(t)\|)\|u_1(t) - u_2(t)\|^2.$$

It follows from Gronwall's lemma that $u_1(\cdot) = u_2(\cdot)$. The proof is then complete. □

We will need also the following theorem which is proved in [14].

THEOREM 2.1. *Let H be a Hilbert space. Assume that $C(\cdot)$ satisfies (H_1) and (H_2). Let $f: I \times H \to H$ be a map, which is measurable with respect to the first argument, such that*

(a) *for every $\eta > 0$ there exists a non-negative function $k_\eta(\cdot) \in L^1(I, \mathbb{R})$ such that for all $t \in I$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,*

$$\|f(t, x) - f(t, y)\| \leqslant k_\eta(t)\|x - y\|;$$

(b) *there exists a non-negative function $\beta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $t \in I$ and for all $x \in \bigcup_{s \in I} C(s)$, $\|f(t, x)\| \leqslant \beta(t)(1 + \|x\|)$.*

Then, for any $x_0 \in C(T_0)$, the following perturbed sweeping process

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) + f(t, u(t)) \text{ a.e. } t \in I, \\ u(T_0) = x_0 \end{cases} \tag{SPP}$$

has one and only one absolutely continuous solution $u(\cdot)$.

4. Perturbation with Delay

This section constitutes the most important part of the paper. It is devoted to the study of a perturbed sweeping process whose perturbation is single-valued and contains a delay.

In the following, I is the interval $[0, T]$, that is, $T_0 = 0$.

Let $\rho \geqslant 0$. Consider the space $\mathcal{C}_0 := \mathcal{C}_H([-\rho, 0])$ endowed with the uniform convergence norm denoted by $\|\cdot\|_{\mathcal{C}_0}$. With each $t \in [0, T]$, one associates a map

$$\tau(t): \mathcal{C}_H([-\rho, t]) \to \mathcal{C}_0$$

defined, for all $u(\cdot) \in C_H([-\rho, t])$ by

$$(\tau(t)u(\cdot))(s) := u(t + s) \text{ for all } s \in [-\rho, 0].$$

Let $C(\cdot): [0, T] \rightrightarrows H$ be a set-valued map and $f: I \times \mathcal{C}_0 \to H$ a single-valued map. Let φ be a fixed member of $\mathcal{C}_0$ such that $\varphi(0) \in C(0)$. We are going to investigate the existence of solutions for the following problem

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) + f(t, \tau(t)u(\cdot)) \text{ a.e.} t \in [0, T], \\ u(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases} \tag{P$_\varphi$}$$

One calls *solution* of (P_φ) any map $u(\cdot)\colon [-\rho, T] \to H$ such that

(a) for any $s \in [-\rho, 0]$, one has $u(s) = \varphi(s)$;
(b) the restriction $u|_{[0,T]}(\cdot)$ of $u(\cdot)$ is absolutely continuous and its derivative, denoted by $\dot{u}(\cdot)$, satisfies the inclusion

$$-\dot{u}(t) \in N(C(t), u(t)) + f(t, \tau(t)u(\cdot)) \text{ a.e.} t \in [0, T].$$

Now we are going to state and prove our existence result concerning the problem (P_φ).

THEOREM 3.1. *Let H be a Hilbert space. Assume that $C(\cdot)$ satisfies (H_1), (H_2). Let $f\colon I \times \mathcal{C}_0 \to H$ be a map satisfying:*

(i) *for any $\phi \in C_0$, $f(\cdot, \phi)$ is measurable;*
(ii) *for any $\eta > 0$, there exists a non-negative function $k_\eta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $\phi_1, \phi_2 \in C_0$ with $\|\phi_i\|_{\mathcal{C}_0} \leqslant \eta$ $(i = 1, 2)$ and for all $t \in I$,*

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leqslant k_\eta(t)\|\phi_1 - \phi_2\|_{\mathcal{C}_0}$$

(iii) *there exists a non-negative function $\beta(\cdot) \in L^1(I, \mathbb{R})$ such that, for all $t \in I$ and for all $\phi \in \mathcal{C}_0$,*

$$\|f(t, \phi)\| \leqslant \beta(t)(1 + \|\phi\|_{\mathcal{C}_0}).$$

Then, for any $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in C(0)$, the problem (P_φ) has one and only one solution.

Proof. I – Assume that

$$\int_0^T \beta(s)\, ds < \frac{1}{4}. \tag{3.1}$$

We are going to construct a sequence of maps $(u_n(\cdot))$ in $\mathcal{C}_H([-\rho, T])$ which converges uniformly on $[-\rho, T]$ to a solution of (P_φ).

A) Construction of the sequence $(u_n(\cdot))$.

We will introduce a discretization, being inspired by the one used in [8].

For each $n \geqslant 1$, consider the partition of $[0, T]$ defined by the points $t_j^n := \frac{jT}{n}$ $(j = 0, \cdots, n)$. Define on $[-\rho, t_1^n] \times H$ the map $h_0^n(\cdot, \cdot)$ by

$$h_0^n(t, x) := \begin{cases} \varphi(t) & \text{if} \quad t \in [-\rho, 0] \\ \varphi(0) + \frac{n}{T}t(x - \varphi(0)) & \text{if} \quad t \in [0, t_1^n]. \end{cases}$$

Let us consider the map $f_0^n\colon [0, t_1^n] \times H \to H$ defined by

$$f_0^n(t, x) := f(t, \tau(t)h_0^n(\cdot, x)).$$

We have, for any $t \in [0, t_1^n]$ and for any $x, y \in H$,

$$\begin{aligned}\|\tau(t)h_0^n(\cdot, x) - \tau(t)h_0^n(\cdot, y)\|_{\mathcal{C}_0} &= \sup_{s\in[-\rho,0]} \|h_0^n(t+s, x) - h_0^n(t+s, y)\| \\ &= \sup_{s\in[-\rho+t,t]} \|h_0^n(s, x) - h_0^n(s, y)\| \\ &\leqslant \sup_{s\in[0,t]} \|h_0^n(s, x) - h_0^n(s, y)\| \\ &\leqslant \sup_{s\in[0,t]} \tfrac{n}{T}s\|x - y\| \\ &\leqslant \|x - y\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left\|\tau(t)h_0^n(\cdot, x)\right\|_{\mathcal{C}_0} &= \sup_{s\in[-\rho+t,t]} \left\|h_0^n(s, x)\right\| \\ &\leqslant \max\{\|\varphi\|_{\mathcal{C}_0}, \sup_{s\in[0,t]} \left\|\varphi(0) + \tfrac{n}{T}s(x - \varphi(0))\right\|\} \\ &\leqslant \max\{\|\varphi\|_{\mathcal{C}_0}, \sup_{s\in[0,t]} \left(\left(1 - \tfrac{n}{T}s\right) \|\varphi(0)\| + \tfrac{n}{T}s \|x\|\right\} \\ &\leqslant \max\{\|\varphi\|_{\mathcal{C}_0}, \|\varphi(0)\| + \|x\|\}.\end{aligned}$$

Then, according to (ii), for any $\eta > 0$, there exists a non-negative function $k_\eta(\cdot) \in L^1(I, \mathbb{R})$ such that for all $t \in [0, t_1^n]$ and for any $(x, y) \in B[0, \eta] \times B[0, \eta]$,

$$\| f_0^n(t, x) - f_0^n(t, y)\| \leqslant k_\eta(t)\|x - y\|.$$

Moreover, thanks to (iii), for all $(t, x) \in [0, t_1^n] \times H$,

$$f_0^n(t, x) \leqslant \beta(t)(1 + \|\varphi\|_{\mathcal{C}_0} + \|x\|) \leqslant (1 + \|\varphi\|_{\mathcal{C}_0})\beta(t)(1 + \|x\|).$$

Note also that, due to the fact that $h_0^n(\cdot, x)$ is uniformly continuous on $[0, t_1^n]$, the map $t \mapsto \tau(t)h_0^n(\cdot, x)$ is continuous from $[0, t_1^n]$ into $(\mathcal{C}_0, \|\cdot\|_{\mathcal{C}_0})$ and hence $f_0^n(\cdot, x)$ is measurable. Consequently, according to Theorem 2.1, there exists one and only one absolutely continuous map $u_0^n(\cdot)\colon [0, t_1^n] \to H$ such that $u_0^n(0) = \varphi(0)$ and, for almost all $t \in [0, t_1^n]$,

$$\dot{u}_0^n(t) + f_0^n\left(t, u_0^n(t)\right) \in -N\left(C(t), u_0^n(t)\right) \text{ a.e. } t \in \left[0, t_1^n\right],$$

and Proposition 2.1 yields

$$\left\|\dot{u}_0^n(t) + f_0^n\left(t, u_0^n(t)\right)\right\| \leqslant \left\| f_0^n\left(t, u_0^n(t)\right)\right\| + |\dot{v}(t)| \text{ a.e. } t \in \left[0, t_1^n\right].$$

Now, define $h_1^n\colon \left[-\rho, t_2^n\right] \times H \to H$ with

$$h_1^n(t, x) := \begin{cases} \varphi(t) & \text{if} \quad t \in [-\rho, 0], \\ u_0^n(t) & \text{if} \quad t \in [0, t_1^n], \\ u_0^n(t_1^n) + \dfrac{n}{T}(t - t_1^n)\left(x - u_0^n(t_1^n)\right) & \text{if} \quad t \in \left[t_1^n, t_2^n\right]. \end{cases}$$

As previously, we show that, for any $t \in \left[0, t_2^n\right]$, the map $x \mapsto \tau(t)h_1^n(\cdot, x)$ is 1-Lipschitz and

$$\left\|\tau(t)h_1^n(\cdot, x)\right\|_{\mathcal{C}_0} \leqslant \max\left\{\|\varphi\|_{\mathcal{C}_0}, \sup_{s\in\left[0,t_1^n\right]} \left\|u_0^n(s)\right\|\right\} + \|x\|.$$

Therefore, the map f_1^n: $\left[t_1^n, t_2^n\right] \times H \to H$ defined by

$$f_1^n(t,x) := f\left(t, \tau(t)h_1^n(\cdot,x)\right)$$

satisfies the assumptions of Theorem 2.1, and hence there exists one and only one absolutely continuous map $u_1^n(\cdot)$: $[t_1^n, t_2^n] \to H$ such that $u_1^n(t_1^n) = u_0^n(t_1^n)$,

$$\dot{u}_1^n(t) + f_1^n\left(t, u_1^n(t)\right) \in -N\left(C(t), u_1^n(t)\right) \text{ a.e. } t \in \left[t_1^n, t_2^n\right],$$

and

$$\left\|\dot{u}_1^n(t) + f_1^n\left(t, u_1^n(t)\right)\right\| \leqslant \left\| f_1^n\left(t, u_1^n(t)\right)\right\| + |\dot{v}(t)| \text{ a.e. } t \in \left[t_1^n, t_2^n\right].$$

Now, suppose that $u_0^n(\cdot), \cdots, u_{j-1}^n(\cdot)$ $(1 \leqslant j \leqslant n-1)$ are defined similarly. Let us define h_j^n: $[-\rho, t_{j+1}^n] \times H \to H$ by

$$h_j^n(t,x) := \begin{cases} \varphi(t) & \text{if} \quad t \in [-\rho, 0], \\ u_i^n(t) & \text{if} \quad t \in \left[t_i^n, t_{i+1}^n\right], i \in \{0, \cdots, j-1\}, \\ u_{j-1}^n(t_j^n) + \dfrac{n}{T}\left(t - t_j^n\right)\left(x - u_{j-1}^n\left(t_j^n\right)\right) & \text{if} \quad t \in \left[t_j^n, t_{j+1}^n\right] \end{cases}$$

and let us consider the map f_j^n: $[t_j^n, t_{j+1}^n] \times H \to H$ with

$$f_j^n(t,x) := f(t, \tau(t)h_j^n(\cdot,x)).$$

As above, it is not difficult to prove that, for all $t \in [t_j^n, t_{j+1}^n]$ and $x, y \in H$,

$$\|\tau(t)h_j^n(\cdot,x) - \tau(t)h_j^n(\cdot,y)\|_{\mathcal{C}_0} \leqslant \|x - y\|$$

and

$$\left\|\tau(t)h_j^n(\cdot,x)\right\|_{\mathcal{C}_0} \leqslant A_j^n + \|x\|, \tag{3.2}$$

where

$$A_j^n := \max\left\{\|\varphi\|_{\mathcal{C}_0}, \max_{0 \leqslant i \leqslant j-1} \sup_{s \in [t_i^n, t_{i+1}^n]} \left\|u_i^n(s)\right\|\right\}.$$

It results that the map $f_j^n(\cdot,\cdot)$ complies with the assumptions of Theorem 2.1. Thus, there exists one and only one absolutely continuous map $u_j^n(\cdot)$: $[t_j^n, t_{j+1}^n] \to H$ such that $u_j^n(t_j^n) = u_{j-1}^n(t_j^n)$,

$$\dot{u}_j^n(t) + f_j^n\left(t, u_j^n(t)\right) \in -N\left(C(t), u_j^n(t)\right) \text{ a.e. } t \in \left[t_j^n, t_{j+1}^n\right],$$

and

$$\left\|\dot{u}_j^n(t) + f_j^n\left(t, u_j^n(t)\right)\right\| \leqslant \left\| f_j^n\left(t, u_j^n(t)\right)\right\| + |\dot{v}(t)| \text{ a.e. } t \in \left[t_j^n, t_{j+1}^n\right].$$

In this way, we define $u_0^n(\cdot), \cdots, u_{n-1}^n(\cdot)$ such that, for each $i \in \{0, \cdots, n-1\}$, $u_i^n(\cdot)$ is absolutely continuous on $[t_i^n, t_{i+1}^n]$, $u_i^n(t_i^n) = u_{i-1}^n(t_i^n)$ (with the convention $u_{-1}^n(0) := \varphi(0)$),

$$\dot{u}_i^n(t) + f_i^n\left(t, u_i^n(t)\right) \in -N\left(C(t), u_i^n(t)\right) \text{ a.e. } t \in \left[t_i^n, t_{i+1}^n\right],$$

and

$$\left\|\dot{u}_i^n(t) + f_i^n\left(t, u_i^n(t)\right)\right\| \leqslant \left\| f_i^n\left(t, u_i^n(t)\right)\right\| + |\dot{v}(t)| \text{ a.e. } t \in \left[t_i^n, t_{i+1}^n\right].$$

Let us define $u_n(\cdot)\colon [-\rho, T] \to H$ by

$$u_n(t) := \begin{cases} \varphi(t) & \text{if} \quad t \in [-\rho, 0] \\ u_i^n(t) & \text{if} \quad t \in \left[t_i^n, t_{i+1}^n\right], i \in \{0, \cdots, n-1\}. \end{cases}$$

Then, for each $i \in \{0, \cdots, n-1\}$,

$$h_i^n(t, x) = \begin{cases} u_n(t) & \text{if} \quad t \in \left[-\rho, t_i^n\right] \\ u_n\left(t_i^n\right) + \dfrac{n}{T}\left(t - t_i^n\right)\left(x - u_n(t_i^n)\right) & \text{if} \quad t \in \left[t_i^n, t_{i+1}^n\right]. \end{cases} \tag{3.3}$$

Put

$$\theta_n(t) := \begin{cases} 0 & \text{if} \quad t = 0, \\ t_i^n & \text{if} \quad t \in \left]t_i^n, t_{i+1}^n\right], i \in \{0, \cdots, n-1\}. \end{cases}$$

One has, by construction, $u_n(0) = \varphi(0)$ and, for almost all $t \in I$,

$$\dot{u}_n(t) + f\left(t, \tau(t) h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right) \in -N(C(t), u_n(t)), \tag{3.4}$$

$$\left\|\dot{u}_n(t) + f\left(t, \tau(t) h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right)\right\| \leqslant \left\| f(t, \tau(t) h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t)))\right\| + |\dot{v}(t)|, \tag{3.5}$$

and

$$u_n(s) = \varphi(s) \text{ for all } s \in [-\rho, 0].$$

Thanks to (3.2), we have

$$\left\|\tau(t) h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right\|_{\mathcal{C}_0} \leqslant 2\left\|u_n(\cdot)\right\|_{\mathcal{C}_H([-\rho, T])}. \tag{3.6}$$

This, along with (iii), implies

$$\left\| f\left(t, \tau(t) h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right)\right\| \leqslant \beta(t)(1 + 2\left\|u_n(\cdot)\right\|_{\mathcal{C}_H([-\rho, T])}) \text{ a.e. } t \in I. \tag{3.7}$$

B) We are going to prove that $(u_n(\cdot))$ converges uniformly in $\mathcal{C}_H([-\rho, T])$.
As $u_n(\cdot)$ is absolutely continuous on $[0, T]$, it follows from (3.5) and (3.7) that, for any $t \in [0, T]$,

$$\|u_n(t)\| \leqslant \|\varphi(0)\| + \int_0^T |\dot{v}(s)| ds + 2(1 + 2\|u_n(\cdot)\|_{\mathcal{C}_H([-\rho, T])}) \int_0^T \beta(s) ds$$

and hence

$$\|u_n(\cdot)\|_{\mathcal{C}_H([-\rho, T])} \leqslant \|\varphi\|_{\mathcal{C}_0} + \int_0^T |\dot{v}(s)| ds + 2(1 + 2\|u_n(\cdot)\|_{\mathcal{C}_H([-\rho, T])}) \int_0^T \beta(s) ds$$

Taking (3.1) into account, it follows that

$$\|u_n(\cdot)\|_{\mathcal{C}_H([-\rho, T])} \leqslant \frac{M}{2}, \tag{3.8}$$

where

$$M := \frac{2}{1 - 4\int_0^T \beta(s)ds} \left(\|\varphi\|_{\mathcal{C}_0} + \frac{1}{2} + \int_0^T |\dot{v}(s)|ds \right).$$

By (3.5) and (3.7) we have

$$\left\| \dot{u}_n(t) + f\left(t, \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right) \right\| \leqslant \gamma(t) \text{ a.e. } t \in I, \tag{3.9}$$

where

$$\gamma(t) := |\dot{v}(t)| + (1 + M)\beta(t).$$

One has also

$$\|\dot{u}_n(t)\| \leqslant \alpha(t) := |\dot{v}(t)| + 2(1 + M)\beta(t) \text{ a.e. } t \in I. \tag{3.10}$$

Now, we proceed to prove that $(u_n(\cdot))$ is a Cauchy sequence in $\mathcal{C}_H([0, T])$. Thanks to (3.4), (3.9), and the hypomonotonicity property of the normal cone, for $m, n \geqslant 1$ and for almost all $t \in I$, we have

$$\langle \dot{u}_n(t) + z_n(t) - \dot{u}_m(t) - z_m(t), u_n(t) - u_m(t) \rangle \leqslant 1/r\, \gamma(t) \|u_n(t) - u_m(t)\|^2,$$

where

$$z_n(t) := f(t, \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))).$$

Hence,

$$\begin{aligned} \langle \dot{u}_n(t) - \dot{u}_m(t), u_n(t) - u_m(t) \rangle &\leqslant 1/r\, \gamma(t)\, \|u_n(t) - u_m(t)\|^2 \\ &+ \|u_n(t) - u_m(t)\| \left\| f(t, \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))) - f\left(t, \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot, u_m(t))\right) \right\| \end{aligned}$$

and then

$$\frac{1}{2}\frac{d}{dt}(\|u_n(t) - u_m(t)\|^2) \leqslant \frac{1}{r}\gamma(t)\|u_n(t) - u_m(t)\|^2 + B_{n,m}(t)\|u_n(t) - u_m(t)\|, \tag{3.11}$$

where

$$B_{n,m}(t) := \left\| f(t, \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))) - f\left(t, \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot, u_m(t))\right) \right\|.$$

According to (ii), (3.6), and (3.8), we have, for some non-negative function $k_M(\cdot) \in L^1(I, \mathbb{R})$ and for all $t \in I$

$$B_{n,m}(t) \leqslant k_M(t) \left\| \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t)) - \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot, u_m(t)) \right\|_{\mathcal{C}_0}.$$

Then,

$$\begin{aligned} B_{n,m}(t) \leqslant k_M(t) &\left\| \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t)) - \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_m(t)) \right\|_{\mathcal{C}_0} + \\ &k_M(t) \left\| \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_m(t)) - \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot, u_m(t)) \right\|_{\mathcal{C}_0}. \end{aligned}$$

The map $x \mapsto \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, x)$ being 1-Lipschitz, one has

$$B_{n,m}(t) \leqslant k_M(t)\,\|u_n(t) - u_m(t)\| + k_M(t)\left\|\tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_m(t)) - \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot, u_m(t))\right\|_{\mathcal{C}_0}. \qquad (3.12)$$

Let $i \in \{0, \cdots, n-1\}$ and $j \in \{0, \cdots, m-1\}$ such that $t \in]t^n_i, t^n_{i+1}]$ and $t \in]t^n_j, t^n_{j+1}]$. Then,

$$\left\|\tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_m(t)) - \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot, u_m(t))\right\|_{\mathcal{C}_0} = \sup_{s\in[-\rho+t,t]} \left\|h^n_i(s, u_m(t)) - h^m_j(s, u_m(t))\right\| \leqslant \sup_{s\in[0,t]} \left\|h^n_i(s, u_m(t)) - h^m_j(s, u_m(t))\right\|.$$

In the case $t^n_i \leqslant t^m_j$ one has

$$\sup_{s\in[0,t]} \left\|h^n_i(s, u_m(t)) - h^m_j(s, u_m(t))\right\| = \max\{A^1_{n,m}(t), A^2_{n,m}(t), A^3_{n,m}(t)\},$$

with

$$A^1_{n,m}(t) := \sup_{s\in[0,t^n_i]} \|u_n(s) - u_m(s)\|,$$

$$A^2_{n,m}(t) := \sup_{s\in\left[t^n_i, t^m_j\right]} \left\|u_n\left(t^n_i\right) + \frac{n}{T}\left(s - t^n_i\right)\left(u_m(t) - u_n\left(t^n_i\right)\right) - u_m(s)\right\|,$$

and

$$A^3_{n,m}(t) := \sup_{s\in[t^m_j,t]} \Big\|u_n\left(t^n_i\right) + \frac{n}{T}\left(s - t^n_i\right)\left(u_m(t) - u_n\left(t^n_i\right)\right) - u_m\left(t^m_j\right) - \frac{m}{T}\left(s - t^m_j\right)\left(u_m(t) - u_m\left(t^m_j\right)\right)\Big\|.$$

We have

$$A^2_{n,m}(t) \leqslant \sup_{s\in\left[t^n_i, t^m_j\right]} \left\|u_n\left(t^n_i\right) - u_n(s)\right\| + \|u_n(s) - u_m(s)\| + \frac{n}{T}\left(s - t^n_i\right)\left(\|u_m(t) - u_n(t)\| + \left\|u_n(t) - u_n\left(t^n_i\right)\right\|\right).$$

Taking (3.10) into account, it follows that

$$A^2_{n,m}(t) \leqslant \sup_{s\in[t^n_i,t^m_j]} \left\{\int_{t^n_i}^t \alpha(\tau)d\tau + \|u_n(s) - u_m(s)\| + \|u_m(t) - u_n(t)\| + \int_{t^n_i}^t \alpha(\tau)d\tau\right\}$$

$$\leqslant 2\int_{t^n_i}^t \alpha(\tau)d\tau + \sup_{s\in[t^n_i,t]} \|u_n(s) - u_m(s)\|.$$

$$
\begin{aligned}
A^3_{n,m}(t) \leqslant \sup_{s\in[t_j^m,t]} \Big\{ &\left\|u_n\left(t_i^n\right)-u_n(t)\right\| + \|u_n(t)-u_m(t)\| + \left\|u_m(t)-u_m\left(t_j^m\right)\right\| \\
&+\frac{n}{T}\left(s-t_i^n\right)\left(\|u_m(t)-u_n(t)\| + \left\|u_n(t)-u_n\left(t_i^n\right)\right\|\right) \\
&+\frac{m}{T}\left(s-t_j^m\right)\left\|u_m(t)-u_m\left(t_j^m\right)\right\| \Big\} \\
\leqslant \int_{t_i^n}^t \alpha(\tau)d\tau &+ \|u_n(t)-u_m(t)\| + \int_{t_j^m}^t \alpha(\tau)d\tau \\
+ \|u_m(t)-u_n(t)\| &+ \int_{t_i^n}^t \alpha(\tau)d\tau + \int_{t_j^m}^t \alpha(\tau)d\tau \\
\leqslant 2\left(\int_{t_i^n}^t \alpha(\tau)d\tau + \int_{t_j^m}^t \alpha(\tau)d\tau\right) &+ 2\|u_n(t)-u_m(t)\|.
\end{aligned}
$$

Thus, if $t_i^n \leqslant t_j^m$, we have

$$
\sup_{s\in[0,t]} \left\|h_i^n\left(s,u_m(t)\right) - h_j^m(s,u_m(t)\right)\| \leqslant
$$

$$
\begin{aligned}
\max &\left\{\sup_{s\in[0,t_i^n]} \|u_n(s)-u_m(s)\|, \sup_{s\in[t_i^n,t]} \|u_n(s)-u_m(s)\|, 2\|u_n(t)-u_m(t)\|\right\} \\
&+2\left(\int_{t_i^n}^t \alpha(\tau)d\tau + \int_{t_j^m}^t \alpha(\tau)d\tau\right) \\
&\leqslant 2\|u_n(\cdot)-u_m(\cdot)\|_{\mathcal{C}_H([0,t])} + 2\left(\int_{t_i^n}^t \alpha(\tau)d\tau + \int_{t_j^m}^t \alpha(\tau)d\tau\right).
\end{aligned}
$$

Likewise, if $t_j^m \leqslant t_i^n$, interchanging t_j^m and t_i^n, we obtain the same previous inequality. Therefore, for any $t \in [-\rho, T]$, we get

$$
\begin{aligned}
\left\|\tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot,u_n(t)) \; - \; \tau(t)h^m_{\frac{m}{T}\theta_m(t)}(\cdot,u_m(t))\right\|_{\mathcal{C}_0} &\leqslant 2\|u_n(\cdot)-u_m(\cdot)\|_{\mathcal{C}_H([0,t])} \\
&+ 2\left(\int_{\theta_n(t)}^t \alpha(\tau)d\tau + \int_{\theta_m(t)}^t \alpha(\tau)d\tau\right).
\end{aligned}
$$

Coming back to (3.12), we obtain

$$
B_{n,m}(t) \leqslant 3k_M(t)\|u_n(\cdot)-u_m(\cdot)\|_{\mathcal{C}_H([0,t])} + 2k_M(t)\left(\int_{\theta_n(t)}^t \alpha(\tau)d\tau + \int_{\theta_m(t)}^t \alpha(\tau)d\tau\right).
$$

Taking (3.11) into account, it follows that, for almost all $t \in I$,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u_n(t) \; - \; u_m(t)\|^2\right) &\leqslant \left(\frac{1}{r}\gamma(t) + 3k_M(t)\right)\|u_n(\cdot)-u_m(\cdot)\|^2_{\mathcal{C}_H([0,t])} \\
&+ 2\|u_n(\cdot)-u_m(\cdot)\|_{\mathcal{C}_H([0,t])}k_M(t)\left(\int_{\theta_n(t)}^t \alpha(\tau)d\tau + \int_{\theta_m(t)}^t \alpha(\tau)d\tau\right)
\end{aligned}
$$

and, using (3.8), it results that

$$\frac{1}{2}\frac{d}{dt}\left(\|u_n(t) - u_m(t)\|^2\right) \leqslant \left(\frac{1}{r}\gamma(t) + 3k_M(t)\right)\|u_n(\cdot) - u_m(\cdot)\|^2_{\mathcal{C}_H([0,t])}$$
$$+ 2Mk_M(t)\left(\int_{\theta_n(t)}^t \alpha(\tau)d\tau + \int_{\theta_m(t)}^t \alpha(\tau)d\tau\right). \tag{3.13}$$

In the following we use the fact that the map $t \mapsto \|u_n(\cdot) - u_m(\cdot)\|_{\mathcal{C}_H([0,t])}$ is continuous. Integrating on $[0, t]$, one has

$$\frac{1}{2}\|u_n(t) - u_m(t)\|^2 \leqslant \int_0^t \left(\frac{1}{r}\gamma(s) + 3k_M(s)\right)\|u_n(\cdot) - u_m(\cdot)\|^2_{\mathcal{C}_H([0,s])}ds$$
$$+ 2M\int_0^t k_M(s)\left(\int_{\theta_n(s)}^s \alpha(\tau)d\tau + \int_{\theta_m(s)}^s \alpha(\tau)d\tau\right)ds.$$

The above inequality being true for any $t \in [0, T]$, it follows that

$$\|u_n(\cdot) - u_m(\cdot)\|^2_{\mathcal{C}_H([0,t])} \leqslant a_{n,m}$$
$$+ 2\int_0^t \left(\frac{1}{r}\gamma(s) + 3k_M(s)\right)\|u_n(\cdot) - u_m(\cdot)\|_{\mathcal{C}_H([0,s])}ds, \tag{3.14}$$

where

$$a_{n,m} := 4M\int_0^T k_M(s)\left(\int_{\theta_n(s)}^s \alpha(\tau)d\tau + \int_{\theta_m(s)}^s \alpha(\tau)d\tau\right)ds.$$

Note that $\lim_n \theta_n(t) = t$ for any t and then $\lim_n \int_{\theta_n(t)}^t \alpha(\tau)d\tau = 0$. Therefore, by the dominated convergence theorem we get $\lim_{n,m} a_{n,m} = 0$ and, according to Lemma 1.1,

$$\lim_{n,m} \|u_n(\cdot) - u_m(\cdot)\|_\infty = 0,$$

which proves that the sequence $(u_n(\cdot))$ converges uniformly in $\mathcal{C}([-\rho, T], H)$ to some map $u(\cdot) \in \mathcal{C}([-\rho, T], H)$ with $u(s) = \varphi(s)$ for all $s \in [-\rho, 0]$. Moreover, thanks to (3.10), we may suppose that $(\dot{u}_n(\cdot))$ converges weakly in $L^1(I, H)$ to some map $g(\cdot) \in L^1(I, H)$. It results that, for all $t \in [0, T]$, $u(t) = \varphi(0) + \int_0^t g(s)ds$ and hence $u(\cdot)$ is absolutely continuous on $[0, T]$ with $\dot{u}(t) = g(t)$ for almost all $t \in [0, T]$. Consequently,

$$\dot{u}_n(\cdot) \to \dot{u}(\cdot) \text{ weakly in } L^1(I, H). \tag{3.15}$$

C) Now, we aim at proving that $u(\cdot)$ is a solution of (P_φ).
First, let us prove that, for any $t \in]0, T]$, one has

$$\lim_n f\left(t, \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right) = f(t, \tau(t)u(\cdot)).$$

Fix $t \in]0, T]$. For each $n \geqslant 1$, there exists $j \in \{0, \cdots, n-1\}$ such that $t \in]t^n_j, t^n_{j+1}]$ and thus $\theta_n(t) = t^n_j$. Then,

$$\begin{aligned}\|\tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t)) - \tau(t)u(\cdot)\|_{\mathcal{C}_0} &= \sup_{s\in[-\rho,0]}\left\|h^n_j(t+s, u_n(t)) - u(t+s)\right\| \\ &= \sup_{s\in[-\rho+t,t]}\left\|h^n_j(s, u_n(t)) - u(s)\right\| \\ &\leqslant \max\left\{\sup_{s\in[0,t^n_j]}\|u_n(s) - u(s)\|, B^1_{n,m}(t)\right\},\end{aligned}$$

where

$$B^1_{n,m}(t) := \sup_{s\in[t^n_j,t]} \left\| u_n\left(t^n_j\right) + \frac{n}{T}\left(s - t^n_j\right)\left(u_n(t) - u_n\left(t^n_j\right)\right) - u(s) \right\|.$$

We have

$$\begin{aligned} B^1_{n,m}(t) &\leqslant \sup_{s\in\left[t^n_j,t\right]} \left(\left\| u_n(t^n_j) - u(s) \right\| + \left\| u_n(t) - u_n\left(t^n_j\right) \right\| \right) \\ &\leqslant \sup_{s\in\left[t^n_j,t\right]} \left(\left\| u_n(t^n_j) - u_n(s) \right\| + \| u_n(s) - u(s) \| + \left\| u_n(t) - u_n\left(t^n_j\right) \right\| \right). \end{aligned}$$

It follows from (3.10) that

$$B^1_{n,m}(t) \leqslant \sup_{s\in[t^n_j,t]} \|u_n(s) - u(s)\| + 2\int_{\theta_n(t)}^t \alpha(\tau)d\tau.$$

As a result,

$$\left\| \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t)) - \tau(t)u(\cdot) \right\|_{\mathcal{C}_0} \leqslant \|u_n(\cdot) - u(\cdot)\|_\infty + 2\int_{\theta_n(t)}^t \alpha(\tau)d\tau$$

and thus

$$\left\| \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t)) - \tau(t)u(\cdot) \right\|_{\mathcal{C}_0} \to 0.$$

Due to the continuity of the map $f(t, \cdot)$, we have

$$f\left(t, \tau(t)h^n_{\frac{n}{T}\theta_n(t)}(\cdot, u_n(t))\right) \to f(t, \tau(t)u(\cdot)). \tag{3.16}$$

Now, we are going to prove that

$$\dot{u}(t) + f(t, \tau(t)u(\cdot)) \in -N(C(t), u(t)) \text{ a.e. } t \in I.$$

Thanks to (3.15) and (3.16), by Mazur's lemma, there exists a sequence $(\zeta_n(\cdot))$ which converges strongly in $L^1(I, H)$ to the map $t \mapsto \dot{u}(t) + f(t, \tau(t)u(\cdot))$ with

$$\zeta_n(t) \in \mathrm{co}\left\{ \dot{u}_k(t) + f(t, \tau(t)h^k_{\frac{k}{T}\theta_k(t)}(\cdot, u_k(t)) : k \geqslant n \right\}$$

for each $n \geqslant 1$ and for all $t \in I$. Extracting a subsequence, we may suppose that,

$$\zeta_n(t) \to \dot{u}(t) + f(t, \tau(t)u(\cdot)) \text{ a.e. } t \in I.$$

Consequently, for almost all $t \in I$,

$$\dot{u}(t) + f(t, \tau(t)u(\cdot)) \in \bigcap_n \overline{\mathrm{co}} \left\{ \dot{u}_k(t) + f(t, \tau(t)h^k_{\frac{k}{T}\theta_k(t)}(\cdot, u_k(t)) : k \geqslant n \right\}.$$

It follows that, for some fixed negligable set $N_0 \subset [0, T]$, for all $t \notin N_0$, for any $\xi \in H$,

$$\langle \xi, \dot{u}(t) + f(t, \tau(t)u(\cdot)) \rangle \leqslant \inf_n \sup_{k\geqslant n} \left\langle \xi, \dot{u}_k(t) + f\left(t, \tau(t)h^k_{\frac{k}{T}\theta_k(t)}(\cdot, u_k(t))\right) \right\rangle.$$

By (3.4), (3.9) and (1.2), this entails that

$$\begin{aligned} \langle \xi, \dot{u}(t) + f(t, \tau(t)u(\cdot)) \rangle &\leqslant \alpha(t) \limsup_n \sigma(-\partial^P d_{C(t)}(u_n(t)), \xi) \\ &\leqslant \alpha(t) \limsup_n \sigma(-\partial^C d_{C(t)}(u_n(t)), \xi). \end{aligned}$$

As, for all $t \in I$, $\sigma(-\partial^C d_{C(t)}(\cdot), \xi)$ is upper semicontinuous on I, one has, for all $t \notin N_0$, for all $\xi \in H$,

$$\langle \xi, \dot{u}(t) + f(t, \tau(t)u(\cdot)) \rangle \leqslant \alpha(t)\sigma(-\partial^C d_{C(t)}(u(t)), \xi).$$

The Clarke subdifferential $\partial^C d_{C(t)}(u(t))$ being closed and convex for any $t \in I$, we deduce that

$$\dot{u}(t) + f(t, \tau(t)u(\cdot)) \in -\alpha(t)\partial^C d_{C(t)}(u(t)) \subset -N(C(t), u(t)) \text{ a.e. } t \in I,$$

the last inclusion coming from (1.3). Consequently, the map $u(\cdot)$ is a solution of (P_φ).

II – Now assume that $\int_0^T \beta(s)ds \geqslant \frac{1}{4}$.

Consider a partition $0 = T_0 < T_1 < \cdots < T_n = T$ of $[0, T]$ such that, for any $i \in \{0, \cdots, n-1\}$,

$$\int_{T_i}^{T_{i+1}} \beta(s)ds < \frac{1}{4}. \tag{3.17}$$

According to the part I, there exist a map $u_0(\cdot)\colon [-\rho, T_1] \to H$ absolutely continuous on $[0, T_1]$ such that

$$u_0(s) = \varphi(s) \text{ for all } s \in [-\rho, 0]$$

and

$$\dot{u}_0(t) + f(t, \tau(t)u_0(\cdot)) \in -N(C(t), u_0(t)) \text{ a.e. } t \in [0, T_1].$$

Assume that, for any $i \in \{0, \cdots, n-2\}$, there exists a map $u_i(\cdot)\colon [-\rho, T_{i+1}] \to H$ absolutely continuous on $[0, T_{i+1}]$ such that

$$u_i(s) = \varphi(s) \text{ for all } s \in [-\rho, 0] \tag{3.18}$$

and

$$\dot{u}_i(t) + f(t, \tau(t)u_i(\cdot)) \in -N(C(t), u_i(t)) \text{ a.e. } t \in [0, T_{i+1}]. \tag{3.19}$$

Let us define $\tilde{f}\colon [0, T_{i+2} - T_{i+1}] \times \mathcal{C}_0 \to H$, $\tilde{C}\colon [0, T_{i+2} - T_{i+1}] \rightrightarrows H$, and $\tilde{\varphi}(\cdot)\colon [-\rho, 0] \to H$ by

$$\tilde{f}(t, \phi) := f(t + T_{i+1}, \phi),\ \tilde{C}(t) := C(t + T_{i+1}), \tag{3.20}$$

and

$$\tilde{\varphi}(s) := u_i(s + T_{i+1}).$$

Define also $\tilde{\beta}(\cdot)\colon [0, T_{i+2} - T_{i+1}] \to \mathbb{R}$ by

$$\tilde{\beta}(t) := \beta(t + T_{i+1}).$$

Obviously, for all $t \in [0, T_{i+2} - T_{i+1}]$ and for all $\phi \in \mathcal{C}_0$

$$\|\tilde{f}(t, \phi)\| \leqslant (1 + \|\phi\|_{\mathcal{C}_0})\tilde{\beta}(t)$$

and, due to (3.17),

$$\int_0^{T_{i+2} - T_{i+1}} \tilde{\beta}(s)ds < \frac{1}{4}.$$

According to the part I again, there exist a map $\tilde{u}(\cdot)$: $[-\rho, T_{i+2} - T_{i+1}] \to H$ which is absolutely continuous on $[0, T_{i+2} - T_{i+1}]$ such that

$$\tilde{u}(s) = \tilde{\varphi}(s) \text{ for all } s \in [-\rho, 0] \tag{3.21}$$

and

$$\dot{\tilde{u}}(t) + \tilde{f}(t, \tau(t)\tilde{u}(\cdot)) \in -N(\tilde{C}(t), \tilde{u}(t)) \text{ a.e. } t \in [0, T_{i+2} - T_{i+1}]. \tag{3.22}$$

Consider the map $u_{i+1}(\cdot)$: $[-\rho, T_{i+2}] \to H$ defined by

$$u_{i+1}(t) := \begin{cases} u_i(t) & \text{if} \quad t \in [-\rho, T_{i+1}], \\ \tilde{u}(t - T_{i+1}) & \text{if} \quad t \in [T_{i+1}, T_{i+2}]. \end{cases}$$

It follows from (3.20) and (3.22) that

$$\dot{u}_{i+1}(t) + f(t, \tau(t)u_{i+1}(\cdot)) \in -N(C(t), u_{i+1}(t)) \text{ a.e. } t \in [T_{i+1}, T_{i+2}]. \tag{3.23}$$

Thanks to (3.18) and (3.19), along with (3.23), we obtain

$$u_{i+1}(s) = \varphi(s) \text{ for all } s \in [-\rho, 0]$$

and

$$\dot{u}_{i+1}(t) + f(t, \tau(t)u_{i+1}(\cdot)) \in -N(C(t), u_{i+1}(t)) \text{ a.e. } t \in [0, T_{i+2}].$$

By repeating the process we obtain a solution on the whole interval $[-\rho, T]$.
Now, we turn to the uniqueness part. Assume that $u_1(\cdot)$ and $u_2(\cdot)$ are two solutions of (P_φ). Let us set

$$\eta := \max(\|u_1(\cdot)\|_{\mathcal{C}_H([-\rho,T])}, \|u_2(\cdot)\|_{\mathcal{C}_H([-\rho,T])}).$$

One has, for $i = 1, 2$ and for all $t \in [0, T]$,

$$\|\tau(t)u_i(\cdot)\|_{\mathcal{C}_0} \leqslant \eta \tag{3.24}$$

and, due to (iii),

$$\|f(t, \tau(t)u_i(\cdot))\| \leqslant (1 + \eta)\beta(t). \tag{3.25}$$

It follows from proposition 2.1 that, for $i = 1, 2$,

$$\|\dot{u}_i(t) + f(t, \tau(t)u_i(\cdot))\| \leqslant m(t) := |\dot{v}(t)| + (1 + \eta)\beta(t) \text{ a.e. } t \in [0, T].$$

The hypomonotonicity of the normal cone, along with the last inequality yields, for almost all $t \in [0, T]$,

$$\langle \dot{u}_1(t) + f(t, \tau(t)u_1(\cdot)) - \dot{u}_2(t) - f(t, \tau(t)u_2(\cdot)), u_1(t) - u_2(t)\rangle \leqslant \frac{1}{r}m(t)\|u_1(t) - u_2(t)\|^2$$

and then

$$\frac{1}{2}\frac{d}{dt}(\|u_1(t) - u_2(t)\|^2) \leqslant \frac{1}{r}m(t)\|u_1(t) - u_2(t)\|^2 + \|u_1(t) - u_2(t)\|\|f(t, \tau(t)u_1(\cdot)) - f(t, \tau(t)u_2(\cdot))\|.$$

From (ii) and (3.24), it results that, for some non-negative function $k_\eta(\cdot) \in L^1([0, T], \mathbb{R})$ and for almost all $t \in [0, T]$,

$$\frac{d}{dt}\left(\|u_1(t) - u_2(t)\|^2\right) \leqslant 2\left(\frac{1}{r}m(t)k_\eta(t)\right)\|u_1(\cdot) - u_2(\cdot)\|^2_{\mathcal{C}_H([0,t])}.$$

Integrating on $[0, t]$, one obtains

$$\|u_1(t) - u_2(t)\|^2 \leqslant \int_0^t 2\left(\frac{1}{r}m(s)k_\eta(s)\right)\|u_1(\cdot) - u_2(\cdot)\|^2_{\mathcal{C}_H([0,s])}ds.$$

This implies that, for all $t \in [0, T]$,

$$\|u_1(\cdot) - u_2(\cdot)\|^2_{\mathcal{C}_H([0,t])} \leqslant \int_0^t 2\left(\frac{1}{r}m(s)k_\eta(s)\right)\|u_1(\cdot) - u_2(\cdot)\|^2_{\mathcal{C}_H([0,s])}ds.$$

According to Gronwall's lemma, one has

$$\|u_1(\cdot) - u_2(\cdot)\|_{\mathcal{C}_H([0,T])} = 0,$$

which proves that $u_1(\cdot) = u_2(\cdot)$. The proof is then complete. □

The following proposition gives an estimation of the derivative of the solution of the problem (P_φ) depending only on $\varphi(\cdot)$, $\beta(\cdot)$, and $v(\cdot)$.

PROPOSITION 3.1. *Let $u(\cdot)$ be the unique solution of the problem (P_φ). For*

$$l := \|\varphi\|_{\mathcal{C}_0} + \exp\left\{2\int_0^T \beta(\tau)d\tau\right\}\int_0^T [2(1 + \|\varphi\|_{\mathcal{C}_0})\beta(s) + |\dot{v}(s)|]ds,$$

one has

$$\|\dot{u}(t) + f(t, \tau(t)u(\cdot))\| \leqslant (1 + l)\beta(t) + |\dot{v}(t)| \text{ a.e. } t \in I$$

and hence

$$\|\dot{u}(t)\| \leqslant 2(1 + l)\beta(t) + |\dot{v}(t)| \text{ a.e. } t \in [0, T].$$

Proof. Let $u(\cdot)$ be the unique solution of (P_φ). According to Proposition 2.1, one has

$$\|\dot{u}(t) + f(t, \tau(t)u(\cdot))\| \leqslant \|f(t, \tau(t)u(\cdot))\| + |\dot{v}(t)| \text{ a.e. } t \in [0, T]. \tag{3.26}$$

It follows that

$$\|\dot{u}(t)\| \leqslant 2\beta(t)(1 + \|\tau(t)u(\cdot)\|_{\mathcal{C}_0}) + |\dot{v}(t)| \text{ a.e. } t \in [0, T]$$

and hence

$$\|\dot{u}(t)\| \leqslant 2\beta(t)\left(1 + \max\left(\|\varphi\|_{\mathcal{C}_0}, \sup_{s\in[0,t]}\|u(s)\|\right)\right) + |\dot{v}(t)| \text{ a.e. } t \in [0, T].$$

This yields

$$\|\dot{u}(t)\| \leqslant 2\beta(t)\int_0^t \|\dot{u}(s)\|ds + 2(1 + \|\varphi\|_{\mathcal{C}_0})\beta(t) + |\dot{v}(t)| \text{ a.e. } t \in [0, T].$$

By Gronwall's lemma we obtain, for all $t \in [0, T]$,

$$\int_0^t \|\dot{u}(s)\|ds \leqslant \int_0^t \left[(2(1 + \|\varphi\|_{\mathcal{C}_0})\beta(s) + |\dot{v}(s)|)\exp\left\{2\int_s^t \beta(\tau)d\tau\right\}\right]ds.$$

As a result, for

$$l := \|\varphi\|_{\mathcal{C}_0} + \exp\left\{2\int_0^T \beta(\tau)d\tau\right\}\int_0^T [2(1 + \|\varphi\|_{\mathcal{C}_0})\beta(s) + |\dot{v}(s)|]ds,$$

one has

$$\|u(\cdot)\|_{\mathcal{C}_H([-\rho,T])} \leqslant l.$$

Consequently,

$$\| f(t, \tau(t)u(\cdot))\| \leqslant (1+l)\beta(t) \text{ a.e. } t \in [0, T]$$

and, from (3.26),

$$\|\dot{u}(t) + f(t, \tau(t)u(\cdot))\| \leqslant (1+l)\beta(t) + |\dot{v}(t)| \text{ a.e. } t \in [0, T].$$

The proof is then complete. □

As expected, the map $\varphi \mapsto u_\varphi(\cdot)$ which associates with each φ in the set $\mathcal{C} := \{\phi \in \mathcal{C}_H([-\rho, 0]) : \varphi(0) \in C(0)\}$ the unique solution of the problem (P_φ) is continuous. That is the object of the following result.

PROPOSITION 3.2. *Assume that the assumptions of Theorem 2.1 hold. For each $\varphi \in \mathcal{C}$, let $u_\varphi(\cdot)$ be the unique solution of the delay perturbed sweeping process*

$$\begin{cases} -\dot{u}(t) \in N(C(t), u(t)) + f(t, \tau(t)u(\cdot)) \text{ a.e.} t \in [0, T], \\ u(s) = \varphi(s) \ \forall s \in [-\rho, 0]. \end{cases}$$

Then, the map $\varphi \mapsto u_\varphi(\cdot)$ from $\mathcal{C}$ to the space $\mathcal{C}([-\rho, T], H)$ endowed with the uniform convergence norm is Lipschitz on any bounded subset of $\mathcal{C}$.

Proof. Let M be any fixed positive real number. We are going to prove that the map $\varphi \mapsto u_\varphi(\cdot)$ is Lipschitz on $\mathcal{C} \cap M\mathbb{B}_0$, where $\mathbb{B}_0$ is the unit ball of $\mathcal{C}_0 := \mathcal{C}_H([-\rho, 0])$.

According to Proposition 3.1, there exists a real number M_1 depending only on M such that, for all $\varphi \in \mathcal{C} \cap M\mathbb{B}_0$ and, for almost all $t \in [0, T]$,

$$\|\dot{u}_\varphi(t) + f(t, \tau(t)u_\varphi(\cdot))\| \leqslant \alpha(t) := (1 + M_1)\beta(t) + |\dot{v}(t)|$$

and

$$\|\dot{u}_\varphi(t)\| \leqslant 2(1 + M_1)\beta(t) + |\dot{v}(t)|.$$

Thanks to this last inequality, for some $\eta > 0$ depending only on M, for all $\varphi \in \mathcal{C} \cap M\mathbb{B}_0$ and for all $t \in [0, T]$,

$$\|u_\varphi(\cdot)\|_{\mathcal{C}_H([-\rho,T])} \leqslant \eta. \tag{3.27}$$

Fix any $\varphi_1, \varphi_2 \in \mathcal{C} \cap M\mathbb{B}_0$. By the hypomonotonicity property of the normal cone, we have, for almost all $t \in [0, T]$,

$$\begin{aligned} \langle \dot{u}_{\varphi_1}(t) + f(t, \tau(t)u_{\varphi_1}(\cdot)) - \dot{u}_{\varphi_2}(t) - f(t, \tau(t)u_{\varphi_2}(\cdot)), u_{\varphi_1}(t) - u_{\varphi_2}(t)\rangle \\ \leqslant \frac{\alpha(t)}{r} \|u_{\varphi_1}(t) - u_{\varphi_2}(t)\|^2 \end{aligned}$$

and then

$$\begin{aligned} \langle \dot{u}_{\varphi_1}(t) - \dot{u}_{\varphi_2}(t), u_{\varphi_1}(t) - u_{\varphi_2}(t)\rangle &\leqslant \frac{\alpha(t)}{r} \|u_{\varphi_1}(t) - u_{\varphi_2}(t)\|^2 \\ &+ \| f(t, \tau(t)u_{\varphi_1}(\cdot)) - f(t, \tau(t)u_{\varphi_2}(\cdot))\| \|u_{\varphi_1}(t) - u_{\varphi_2}(t)\|. \end{aligned}$$

Since, by assumptions, there is a non-negative function $k(\cdot) \in L^1([0, T], \mathbb{R})$ such that $f(t, \cdot)$ is $k(t)$-Lipschitz on $\eta\mathbb{B}_0$ (this function depends only on M), the above inequality, along with (3.27), entails that, for almost all $t \in [0, T]$,

$$\frac{d}{dt}(\|u_{\varphi_1}(t) - u_{\varphi_2}(t)\|^2) \leqslant 2\left(\frac{\alpha(t)}{r} + k(t)\right)\|u_{\varphi_1}(\cdot) - u_{\varphi_2}(\cdot)\|^2_{\mathcal{C}_H([-\rho,t])}.$$

Integrating on $[0, t]$, we deduce that

$$\|u_{\varphi_1}(\cdot) - u_{\varphi_2}(\cdot)\|^2_{\mathcal{C}_H([-\rho,t])} \leqslant \|\varphi_1(\cdot) - \varphi_2(\cdot)\|^2_{\mathcal{C}_0}$$
$$+2\int_0^t \left(\frac{\alpha(s)}{r} + k(s)\right)\|u_{\varphi_1}(\cdot) - u_{\varphi_2}(\cdot)\|^2_{\mathcal{C}_H([-\rho,s])}ds.$$

Via Gronwall's lemma, we obtain, for any $t \in [0, T]$,

$$\|u_{\varphi_1}(\cdot) - u_{\varphi_2}(\cdot)\|^2_{\mathcal{C}_H([-\rho,t])} \leqslant \|\varphi_1(\cdot) - \varphi_2(\cdot)\|^2_{\mathcal{C}_0}$$
$$+ 2\|\varphi_1(\cdot) - \varphi_2(\cdot)\|^2_{\mathcal{C}_0}\int_0^T\left(\left(\frac{\alpha(s)}{r} + k(s)\right)\exp\{2\int_0^T\left(\frac{\alpha(\tau)}{r} + k(\tau)\right)d\tau\}\right)ds.$$

Therefore,

$$\|u_{\varphi_1}(\cdot) - u_{\varphi_2}(\cdot)\|_{\mathcal{C}_H([-\rho,T])} \leqslant A\|\varphi_1 - \varphi_2\|_{\mathcal{C}_0},$$

where

$$A := \left(1 + 2\exp\left\{2\int_0^T\left(\frac{\alpha(\tau)}{r} + k(\tau)\right)d\tau\right\}\int_0^T\left(\frac{\alpha(s)}{r} + k(s)\right)ds\right)^{\frac{1}{2}}.$$

The proof is then complete. □

Remark 3.1. Note that in the proof above, unlike the construction in [8], the second argument of f in the definition of the maps f_j^n's depends not only on x but also on t.

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