Geometrical Mesh Improvement Properties of Delaunay Terminal Edge Refinement

Bruce Simpson[1] and Maria-Cecilia Rivara[2]

[1] David Cheriton School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1
rbsimpson@uwaterloo.ca
[2] Department of Computer Science, Universidad de Chile, Blanco Encalada 2120, Santiago, Chile
mcrivara@dcc.uchile.cl

Abstract. The use of edge based refinement in general, and Delaunay terminal edge refinement in particular are well established for adaptive meshing, but largely on a heuristic basis. In this paper, we present some theoretical results on geometric improvement, and it limitations, for these methods. Angle bounds for simple longest edge bisection are reviewed and extended. Terminal edges are local maximal edges in a mesh; two additional bounds that apply to simple bisection of terminal edges in Delaunay meshes are presented. The angle properties of Delaunay insertion of the midpoint of a terminal edge are described.

1 Introduction

Delaunay terminal edge refinement, specified in §2 below, is a member of the family of edge-based adaptive mesh refinement methods; references to which can be found in [1, 5, 12]. The use of edge based refinement in general, and Delaunay terminal edge refinement in particular are well established for planar meshing, but largely on a heuristic basis. In this paper, we present a series of theoretical results on the geometric mesh improvement properties of these methods. Iterative refinement methods for generating such meshes typically take a triangulation of D, M_0, as input. M_0 is not connected to any approximation task necessarily. It is simply a representation of D and may have arbitrarily small angles and/or edges. For some applications, the merit of an unstructured mesh for discretizing a domain D is influenced by the geometric quality of its triangles, e.g. Berzins, [2]. Part of the task of generating a mesh is to improve these measures, which typically involves better aspect ratios in the triangles. It is well known that for the goals of efficient and appropriate meshes for piecewise linear approximation, the measures of length and angles should be made in an error based metric, e.g. George and Borouchaki [4], or Simpson [12].

Strong mesh improvement properties for Delaunay circumcenter based refinement have been established by Chew, [3], Ruppert [6], and Shewchuk [9]. In particular, under appropriate conditions on D, the methods are guaranteed to produce meshes with the minimum angle larger that a specified angle tolerance.

Our discussion presents, in §2, an overview of triangle properties local to Delaunay terminal edge bisection, including some new results. We will usually shorten 'Delaunay terminal' to 'Deter' in the sequel. The terminal edge concept is explained in §2.2. In §3, we analyse the angle distribution in the mesh resulting from a Deter bisection and study a case of repeated Delaunay terminal edge refinement of a triangle with one small edge.

2 Local Features of Deter Edge Bisection

We first look at the properties of the angles produced in a simple longest edge bisection of t. We usually use 'LEBis' for this bisection. Individual properties have been reported in a variety of references, [7, 5, 8]. In §2.1, we believe we have included all previously published properties, provided some simpler proofs for some cases, and added new properties in Theorem 2.1 b) and c) and Theorem 2.2.

In §2.2, we explain the components of Deter refinement and our terminology for them. We then present two bounds on the angles in the pairs of triangles incident on a Delaunay terminal edge.

2.1 Basic Properties of LEBis

We introduce a standardized notation for this splitting by labeling the vertices of t as A, B, C and normalizing this labeling by requiring

$$|B - C| \ \leq \ |C - A| \, ; \ |C - A| \ \leq \ |B - A| \, ; \ M = (A + B)/2$$

where M is the where M is the midpoint of the longest edge that is to be split. The two new child triangles of t are labelled t_A and t_B, and the angles of t_A (t_B) are labelled α_j (β_j) ; for $j = 0, 1, 2$.

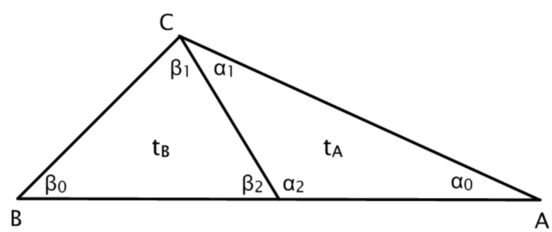

Fig. 1. Notation for longest edge bisection

The following lemma and theorems present some simple properties of a LEBis of any t.

Lemma 2.1. *Each of the assertions in the following groups is equivalent to any other in the group.*

a)	b)	c)
t is right angled	*t is acute*	*t is obtuse*
$\alpha_1 = \alpha_0$	$\alpha_1 < \alpha_0$	$\alpha_1 > \alpha_0$
$\beta_1 = \beta_0$	$\beta_1 < \beta_0$	$\beta_1 > \beta_0$
$\|A - M\| = \|C - M\|$	$\|A - M\| < \|C - M\|$	$\|A - M\| > \|C - M\|$

Theorem 2.1. *The following angle bounds apply*

$$a)\quad \alpha_1 \geq \alpha_0/2 \quad ; \quad b) \quad \beta_1 \geq \pi/6 \quad ; \quad c) \quad \beta_2 \geq 3\alpha_0/2$$

Theorem 2.2. *The following angle bounds apply conditionally*

a) if t is obtuse, then $\beta_2 \geq 2\alpha_0$
b) if $\alpha_0 < \pi/6$,then $\beta_1 > min(\beta_0, \beta_2)$
c) if $\alpha_0 > \arcsin(1/3) = 19.5°$, then $\beta_1 < \pi/2$, i.e. t_B is acute.

2.2 Deter Edge Bisection

One of the tactics in refinement for geometric improvement is to refine the largest triangles first. If the size measure for largest is the length of the longest edge, then refining terminal edges of the mesh is an approximation to using the longest first ordering. A terminal edge in a mesh is a local maximum edge length in a graph sense. Figure 2, a) shows edge AB as an example of an internal terminal edge with two neighbouring triangles, t_2, t_3 and b) shows edge CD as a boundary terminal edge with one neighbouring triangle. This, and the following, concepts were introduced and used in references [5, 8, 7].

We now explain the terminology of the paper. Terminal edge bisection is simple LEBis of each triangle incident on a terminal edge as described in §2.1 above. Delaunay terminal edge bisection is is a modification of terminal edge bisection in which the mesh being refined is Delaunay, or constrained Delaunay, and the insertion is a Delaunay point insertion. 'Lepp' is an acronym for ' the longest edge propagation path'. Given a triangle, t_0 that is to be refined, Lepp locates a terminal edge near t_0, in a graph sense. Figure 2 illustrates this process for the two triangles marked t_0 on the left and t_0^* on the right. Finally, Deter refinement of a triangle t will refer to finding a terminal edge associated with t using Lepp and performing Deter edge bisection on it. As these examples show, Deter refinement of t_0 may not modify t_0, in which case the process can be

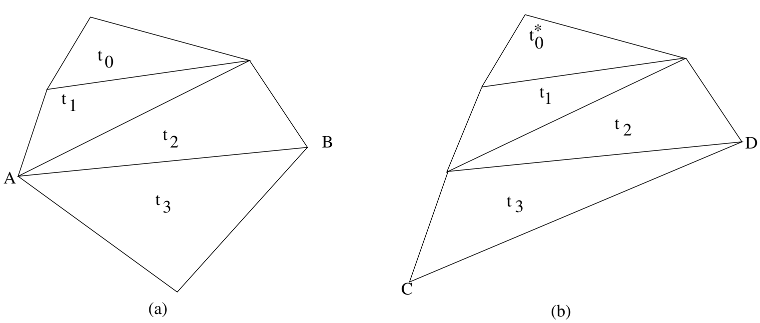

Fig. 2. (a) AB is an interior terminal edge shared by terminal triangles (t_2, t_3) associated to Lepp(t_0) =$\{t_0, t_1, t_2\}$; (b) CD is a boundary terminal edge with terminal triangle t_3 associated to Lepp$(t_0^* = \{t_0^*, t_1, t_2, t_3\}$

repeated in the refined mesh. Algorithmic details of Deter refinement, including repeated application to a given t, are given in [5, 8, 7, 13].

Our discussion of simple LEBis applies to terminal boundary edges of a mesh in full generality. However, special 'encroachment' rules are needed to ensure mesh improvement for refining boundary edges, [3, 6]. In this paper, we will restrict our attention to refinement of internal edges. The fact that the neighbouring triangles, t_1 and t_2 of an internal terminal edge are a Delaunay pair places significant restrictions their configuration. So, for an internal edge, Delaunay terminal edge bisection is more constrained than terminal edge bisection applied to two independent triangles separately. This can been seen in Figure 3.

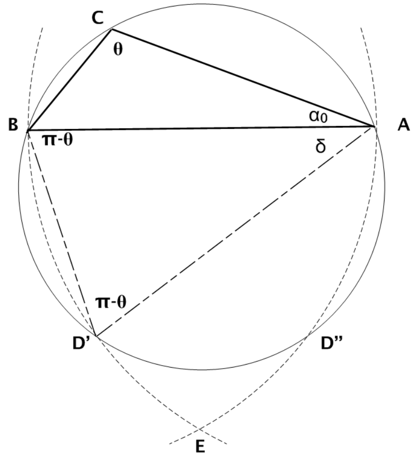

Fig. 3. The configuration of triangles at an internal terminal edge

The figure shows only t_1; we denote the vertex of t_2 opposite edge AB by D, which is not shown. The dashed circular arcs are part of the circles of radius $|B - A|$ centered at A and B respectively. We will use $CC(t)$ to denote 'the circumcircle of t'. $CC(t_1)$ is shown with a solid perimeter. Because edge AB is terminal and the triangles are Delaunay, D must lie in the small region at the bottom of the diagram below the short arc of $CC(t_1)$ from D' to D'' and inside the two dashed arcs that meet at E.

Simple implications of this diagram are the following lemma and corollary.

Lemma 2.2. *For any pair of Delaunay terminal-triangles t_1, t_2 sharing an internal terminal edge, largest angle(t_i) $\leq$ 2π/3 for $i = 1, 2$.*

Corollary 2.1. *For child triangle t_A of LEBis in a Delaunay mesh, if* $\max(\alpha_0, \alpha_1) < \pi/6$ *then edge CA is not a terminal edge.*

For a pair of triangles (t_1, t_2) sharing a Delaunay edge, the sum of the angles opposite the common edge cannot exceed π, Consequently, at most one of the t_k can be obtuse. The following theorem shows that if the edge is a terminal edge, then the more obtuse t_1 is, the larger the smallest angle of acute t_2 is.

Theorem 2.3. *Let t_1 and t_2 be incident on an internal terminal edge. Let t_1 be obtuse, with largest angle $\theta > \pi/2$ and let $\alpha_0(2)$ be the smallest angle of t_2. Then*

$$\alpha_0(2) \geq 2\theta - \pi$$

This theorem, and Figure 3 illustrate restrictions on the configuration of triangles that share a terminal edge, e.g. if $\theta = 7\pi/12$, then $\alpha_0(2) \geq \pi/6$.

Intuitively, as Figure 1 suggests, LEBis produces an improved triangle,t_B, and a triangle,t_A which is not improved. So mesh improvement can depend on subsequent processing of t_A. It may happen that the Delaunay insertion of M removes t_A from the mesh. The implications of of this possibility are discussed in the next section. If not, i.e. if edge AC is an internal Delaunay edge , it may not be a terminal edge. Intuitively, it would be expected that the configurations of the two triangles incident on edge AC would not commonly meet the conditions presented above for it to be a terminal edge, in general. Corollary 2.1 is a particular instance of this. So, in general, it would be expected that repeated Deter refinements of t_A would, sooner or later, result in edge AC being removed from the mesh by a Delaunay insertion following the bisection of some other nearby terminal edge.

3 Mesh Properties of Deter Edge Refinement

We now look at properties of Deter edge bisection associated with the region of the mesh affected by the refinement. We start, in §3.1, with a study of the angles in the updated mesh, assuming that t_A is not Delaunay in the mesh resulting from a LEBis of t. However, it may be that no update is necessary. , i.e. that t_A is already Delaunay in the refined mesh. The second subsection discusses a configuration that that could be applicable to repeated Deter refinement in this case.

3.1 Delaunay Insertion of Point M

We describe the Deter edge bisection process as the simple LEBis of each trian gle, t, incident on the terminal edge, which results in an updated mesh M_{SB}, followed by the conversion of M_{SB} to a Delaunay mesh, M_{CD}.

To describe the conversion of M_{SB}, we will use the terminology of George and Borouchaki ,[4]. The cavity of the vertex M in M_{SB} is the set of triangles, t, such that $M \in \mathrm{CC}(t)$. It has a polygonal boundary that is star-shaped with respect to vertex M. We will denote the boundary vertices by P_k for $k = 0$, to N in clockwise order about M starting with $P_0 = C$. Since A, B and C are on this boundary, $N \geq 2$. The result of the Delaunay insertion of vertex M is that the triangles in the cavity of M are removed from M_{SB} and triangles MP_kP_{k+1} replace them in M_{CD}.

We let NA be the index of A in the list of boundary vertices of the cavity of M i.e. $P_{NA} = A$. The subset of the cavity of M that is bounded by the first $NA+1$ vertices and the edges AM and MC will be referred to as the partial cavity

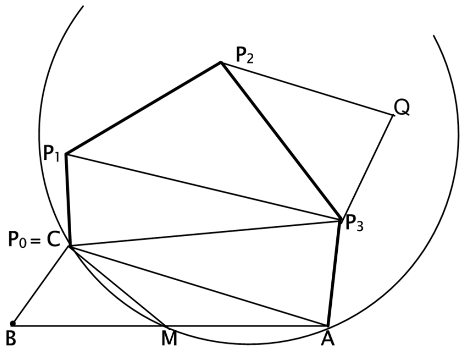

Fig. 4. Example of the partial cavity of vertex M in mesh M_{SB} with $NA = 4$

of M. An example is shown in Figure 4; this figure also shows CC(t_A) of child triangle $t_A = CMA$ with the P_k in its interior. This illustrates the statement of the following lemma. We have also shown a mesh vertex, Q , and triangle P_2QP_3 which are not in the cavity of M although they are in $CC(t_A)$. So the converse of the lemma is not true.

Lemma 3.1. *If $NA > 1$, P_k is in CC(t_A) for $1 < k < NA$.*

We will study the angles of the new triangles incident on M. Let $\alpha_{min}(M)$ be the the minimum angle of the triangles in the partial cavity of M excluding t_A Each triangle $t \in M_{SB}$ in the cavity of M, except t_A, has vertices, P_i, P_j, P_k for $i < j < k$. If t has an edge on the boundary of the cavity, then $i = j - 1$. In this case, $M \in \mathrm{CC}\,(t)$ implies that the angle at M in M_{CD} opposite edge P_{j-1}, P_j is larger than the angle opposite edge P_{j-1}, P_j in t. So, in particular, the angle at M is larger than $\alpha_{min}(M)$. Intuitively, we can see that the closer a cavity edge, $P_{j-1}P_j$, is to M the larger this angle improvement will be. Conversely, if CC(t) is very close to CC(t_A) then very little angle improvement can occur.

The following theorem details the worse case limits of angle improvement. Its proof provides insight into the mechanisms of angle improvement resulting from Delaunay insertion.

Theorem 3.1. *Angle $CP_1M \geq \alpha_0$ and the other two angles of triangle CP_1M exceed $\alpha_{min}(M)$.*
Angle $MP_{NA-1}A \geq \alpha_1$ and the other two angles of triangle $MP_{NA-1}A$ exceed $\alpha_{min}(M)$.
If $NA > 2$, then in the set of triangles $P_jP_{j+1}M$ for $1 \leq j \leq NA - 2$, every angle exceeds $\alpha_{min}(M)$.

Corollary 3.1. *If t is obtuse, then no new angles smaller than the existing ones in the unrefined mesh result from Deter bisection of t.*

3.2 A Case of Repeated Delaunay Refinement

In this section, we show that if t is shaped so that $|B - C| < |C - M|$, i.e. if $|B - C|$ is the shortest edge of t_B, then Delaunay insertion of M into the mesh can only produce new edges that are longer than $|B - C|$. We then look at repeated Deter refinement applied to a special case of t.

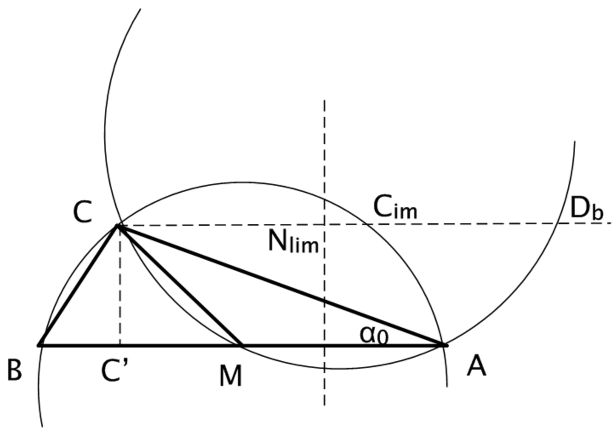

Fig. 5. Configuration of terminal triangle t and its neighbour

Figure 5 shows the terminal triangle $t = ABC$, and an arc of its circumcircle $CC(t)$. The point C' is the projection of C onto edge BA. The figure also shows the insertion point M, and an arc of $CC(t_A = CMA)$. We assume that $|C - M| > |A - M|$, and consequently, that t is acute and that $\alpha_1 < \alpha_0$. Other features of the figure are used in the proofs; see technical report [14].

Lemma 3.2. *If $|C - M| > |B - C|$, the circle of radius $|B - C|$ about M lies inside $CC(ABC)$.*

Corollary 3.2. *If $|B - C|$ is the shortest edge of t_B then Delaunay insertion of M into the current mesh can only produce edges longer than $|B - C|$.*

We now use this lemma in a theorem that demonstrates a special case of t for which we can prove that no new small edges are produced in repeated Deter refinements of t. Let D be the vertex of the triangle that shares edge CA with t. D must be outside $CC(ABC)$.

Theorem 3.2. *If $\alpha_0 \leq$ angle ACD ; $|B - C|$ is the shortest edge of t_B, and edge CA is not terminal as an edge of t_A, then the circle of radius $|B - C|$ about M is empty for repeated applications of Deter refinement to t_A.*

Corollary 3.3. *Under the conditions of Theorem 3.2, if t_B is acceptable, i.e. $\beta_2 \geq \theta_{tol}$, then no edge smaller than $|B - C|$ is introduced at M by Deter refinement of the mesh.*

4 Conclusions

Our motivation in this study has been to understand how, or in what circumstances, Deter refinement can produce a submesh near a small angled triangle t with improved triangles. There are two ways in which the method can create this submesh, either at some stage of repeated Deter refinement, the longest edge of t is a terminal edge, in which case its midpoint is Delaunay inserted into the mesh, or the longest edge is removed from the mesh by the Deter bisection of the terminal edge of Lepp(t). This paper addresses the first case. As mentioned in §2.2, simple terminal edge bisection of a small angled triangle, t, produces a demonstrably improved triangle, t_B and an unimproved triangle, t_A both incident on the new vertex M. Improvement of the submesh near t may come from the Delaunay insertion of M. Our analysis of §3.1 identifies the only circumstances under which this does not happen; i.e. t must be acute and the neighbour of t on side AC must have a special configuration. These circumstances do not preclude Deter refinement from successfully improving the mesh, of course, but suggests that it may not be possible to identify improvement on the basis of Deter refinement aplied to one small angled triangle. On the other hand, we demonstrate in §3.2, a special case in which it is possible to show at least non-degeneration of the local submesh based on properties of t and its neighbour.

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