Partial Differential Equations/Optimal Control

# Controllability of the Ginzburg–Landau equation

Lionel Rosier [a,b], Bing-Yu Zhang [c]

[a] *Centro de Modelamiento Matemático (CMM) and Departamento de Ingeniería Matemática, Universidad de Chile (UMI CNRS 2807), Avenida Blanco Encalada 2120, Casilla 170-3, Correo 3, Santiago, Chile*

[b] *Institut Élie-Cartan, UMR 7502 UHP/CNRS/INRIA, B.P. 239, 54506 Vandoeuvre-lès-Nancy cedex, France*

[c] *Department of Mathematical Sciences, University of Cincinnati,Cincinnati, OH 45221, USA*

**Abstract**

This Note investigates the boundary controllability, as well as the internal controllability, of the complex Ginzburg–Landau equation. Null-controllability results are derived from a Carleman estimate and an analysis based upon the theory of sectorial operators.

**Résumé**

**Contrôlabilité de l'équation de Ginzburg–Landau.** Cette Note est dévolue à l'étude de la contrôlabilité frontière, ou interne, de l'équation complexe de Ginzburg–Landau. Des résultats de contrôlabilité à zéro sont obtenus au moyen d'une inégalité de Carleman et d'une analyse basée sur la théorie des opérateurs sectoriels.

## Version française abrégée

On considère le problème de la contrôlabilité pour une équation non-linéaire de type Ginzburg–Landau. Soient $\Omega \subset \mathbb{R}^N$ un ouvert borné de classe $C^2$, $\Gamma_0 \subset \partial\Omega$ un ouvert arbitraire, $\alpha$ un nombre réel, et $f : \mathbb{C} \to \mathbb{C}$ une fonction continue, nulle en 0, et dérivable en 0 au sens complexe. Soit $X = \{u_0 \in C(\overline{\Omega}); u = 0 \text{ sur } \partial\Omega \setminus \Gamma_0\}$. On étudie le problème de contrôle frontière suivant :

$$\partial_t u = (1 + i\alpha)\Delta u + f(u) \quad \text{dans } \Omega, \tag{1}$$

$$u = 1_{\Gamma_0} h \quad \text{sur } \partial\Omega, \tag{2}$$

$$u(0) = u_0, \tag{3}$$

où $1_{\Gamma_0}(x) = 1$ si $x \in \Gamma_0$, 0 sinon.

Le résultat principal de cette Note est le suivant :

*E-mail addresses:* rosier@iecn.u-nancy.fr (L. Rosier), bzhang@math.uc.edu (B.-Y. Zhang).

**Théorème 0.1.** *Pour tout $T > 0$, le système* (1)–(3) *est localement contrôlable à zéro dans $X$. Plus précisément, il existe un nombre $R > 0$ tel que pour toute donnée intiale $u_0 \in X$ vérifiant $\|u_0\|_{L^\infty(\Omega)} < R$, il existe une fonction de contrôle $h \in C(\partial\Omega \times [0,T])$ avec supp $(h) \subset \Gamma_0 \times [0,T]$ telle que le système* (8)–(10) *admette une solution*

$$u \in L^2\big(0,T;H^1(\Omega)\big) \cap C\big([0,T];X\big)$$

*vérifiant $u(T) = 0$. De plus, la solution $u$ et le contrôle $h$ vérifient $\sqrt{t}u \in L^2(0,T;H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(0,T;L^2(\Omega))$, $\sqrt{t}h \in L^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$ et pour tout $\nu < 2$, $u \in C((0,T];C^\nu(\Omega))$ et $h \in C((0,T];C^\nu(\Omega))$.*

Le Théorème 0.1 se prouve au moyen d'une nouvelle inégalité de Carleman globale, et d'un argument de point fixe mettant en jeu les effets régularisants liés au caractère sectoriel de l'opérateur de Ginzburg–Landau.

Passons à présent à la contrôlabilité interne. On se donne un ouvert arbitraire $\omega \subset \Omega$ et on considère le système

$$\partial_t u = (1+i\alpha)\Delta u + f(u) + 1_\omega h \quad \text{dans } \Omega, \tag{4}$$

$$u = 0 \quad \text{sur } \partial\Omega, \tag{5}$$

$$u(0) = u_0. \tag{6}$$

Le résultat de contrôlabilité à zéro qui suit peut être déduit du Théorème 0.1 comme dans [2] :

**Théorème 0.2.** *Pour tout $T > 0$, le système* (4)–(6) *est localement contrôlable à zéro dans $C_0(\Omega)$. Plus précisément, il existe un nombre $R > 0$ tel que pour tout $u_0 \in C_0(\Omega)$ avec $\|u_0\|_{L^\infty(\Omega)} < R$, il existe un contrôle $h \in C(\overline{\Omega} \times [0,T])$ avec supp $h \subset \omega \times [0,T]$ tel que le système* (11)–(13) *admette une solution $u \in L^2(0,T;H^1(\Omega)) \cap C([0,T];C_0(\Omega))$ vérifiant $u(T) = 0$. De plus, $\sqrt{t}u \in L^2(0,T;H^2(\Omega)$, $\sqrt{t}\partial_t u \in L^2(0,T;L^2(\Omega))$, $\sqrt{t}h \in L^2(0,T;H^1(\Omega))$, et pour tout $\nu < 2$, $u \in C((0,T];C^\nu(\overline{\Omega}))$ et $h \in C((0,T];C^{\nu-1}(\overline{\Omega}))$.*

Les preuves détaillées des résultats de cette Note figurent dans [9].

## 1. Introduction

The classical cubic Ginzburg–Landau (GL) [6–8] equation

$$\partial_t u = (1+i\alpha)\Delta u + Ru - (1+i\beta)|u|^2 u, \quad t > 0,\ x \in \Omega \subset \mathbb{R}^N \tag{7}$$

plays an important role in the theory of amplitude equations, and provides a simple model of turbulence.

The Carleman inequalities constitute the main tool to investigate the controllability of semilinear parabolic equations [4,2]. In [3], Fu established a Carleman inequality for the operator $(a+ib)\partial_t + \sum_{j,k}\partial_k(a^{jk}\partial_j)$ ($(a^{jk})$ denoting a smooth uniformly elliptic matrix) and deduced a zero-controllability result for a *linear* PDE of GL type with an internal control.

The aim of this Note is twofold. First, we show that the Carleman estimate proved in [3] may be improved by a control in some weighted space of the time-derivative and of the Laplacian of the function, as for the heat equation. Secondly, we extend the null-controllability result in [3] to the nonlinear GL equation with a quite general nonlinear term.

The fixed-point argument applied here proves to be more tricky than for the heat equation, as many classical properties of the heat equation (comparison principle, maximum principle, etc.) fail for GL. The smoothing effect needed to apply Schauder fixed-point theorem is carefully proved with the aid of the theory of sectorial operators [5] in all the spaces $L^p(\Omega)$, $N < p < \infty$. Finally, the use of that theory allows to improve the results in [2], as far as the regularity of the trajectories is concerned.

## 2. Main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a $C^2$ boundary $\partial\Omega$. Let $\Gamma_0 \subset \partial\Omega$ denote an arbitrary open set. We introduce the spaces

$$C_0(\Omega) = \big\{u_0 \in C(\overline{\Omega});\, u = 0 \text{ on } \partial\Omega\big\}$$

and

$$X = \{u_0 \in C(\overline{\Omega});\ u = 0 \text{ on } \partial\Omega \setminus \Gamma_0\}.$$

Let $\alpha \in \mathbb{R}$, and let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function such that $f(0) = 0$ and $\lim_{z\to 0} f(z)/z = l$ exists (i.e., $f$ is differentiable at 0 in the complex sense). Clearly, $f(z) = Rz - (1 + i\beta)|z|^2 z$ (resp. $f(z) = Rz - (1 + i\beta)|z|^4 z$) are concerned.

We will consider first the following boundary control system

$$\partial_t u = (1 + i\alpha)\Delta u + f(u) \quad \text{in } \Omega, \tag{8}$$

$$u = 1_{\Gamma_0} h \quad \text{on } \partial\Omega, \tag{9}$$

$$u(0) = u_0, \tag{10}$$

where $1_{\Gamma_0}(x) = 1$ if $x \in \Gamma_0$, 0 otherwise.

One of the main contributions of this Note is to show that this boundary control system is locally null controllable.

**Theorem 1.** *For any $T > 0$, the system* (8)–(10) *is locally null controllable in $X$. More precisely, there exists a number $R > 0$ such that for any $u_0 \in X$ with $\|u_0\|_{L^\infty(\Omega)} < R$, there exists a control input $h \in C(\partial\Omega \times [0, T])$ with* $\operatorname{supp}(h) \subset \Gamma_0 \times [0, T]$ *such that the system* (8)–(10) *admits a solution*

$$u \in L^2(0, T; H^1(\Omega)) \cap C([0, T]; X)$$

*satisfying*

$$u(T) = 0.$$

*Moreover, the solution $u$ and the control $h$ satisfy $\sqrt{t}u \in L^2(0, T; H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(0, T; L^2(\Omega))$, $\sqrt{t}h \in L^2(0, T; H^{\frac{3}{2}}(\partial\Omega))$ and for all $\nu < 2$, $u \in C((0, T]; C^\nu(\Omega))$ and $h \in C((0, T]; C^\nu(\Omega))$.*

It is easy to see that the uniqueness of the solution of (8)–(10) in the above class holds provided that $f$ is locally Lipschitz continuous.

The following result is a direct consequence of Theorem 1 and of [6, Theorem 4]:

**Corollary 2.** *Assume that $f(z) = Rz + \mu|z|^{2\sigma} z$ with $\sigma \in \mathbb{R}^{+*}$, $R \in \mathbb{R}$ and $\mu \in \mathbb{C}$. Then the system* (8)–(10) *is locally null controllable in the space $L^p(\Omega)$ for any $p > \sigma N$, and in the Sobolev space $H^q(\Omega)$ for any $q > \frac{N}{2} - \frac{1}{\sigma}$ provided that $2\sigma \geqslant 1$ and $N\sigma \geqslant 1$.*

The space $H^q(\Omega)$ for $q < 0$ is defined as the dual space of the space $D((-\Delta_D)^{\frac{|q|}{2}})$ with respect to the pivot space $L^2(\Omega)$.

Following [2], one can also deduce:

**Corollary 3.** *Assume that $\lim_{|z|\to\infty} f(z)/(z \ln |z|) = 0$. Then the system* (8)–(10) *is globally null controllable in $X$, i.e., $R$ may be given any value in Theorem* 1.

We now proceed to the internal controllability. Assume given an open set $\omega \subset \Omega$. We investigate the control properties of the following forced initial-value problem.

$$\partial_t u = (1 + i\alpha)\Delta u + f(u) + 1_\omega h \quad \text{in } \Omega, \tag{11}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{12}$$

$$u(0) = u_0. \tag{13}$$

The following null controllability result may be deduced from Theorem 1 as in [2]:

**Theorem 4.** *For any $T > 0$, the system* (11)–(13) *is locally null controllable in $C_0(\Omega)$. More precisely, there exists a number $R > 0$ such that for any $u_0 \in C_0(\Omega)$ with $\|u_0\|_{L^\infty(\Omega)} < R$, there exists a control input $h \in C(\overline{\Omega} \times [0, T])$ with supp $h \subset \omega \times [0, T]$ such that the system* (11)–(13) *admits a solution*

$$u \in L^2\big(0, T; H^1(\Omega)\big) \cap C\big([0, T]; C_0(\Omega)\big)$$

*satisfying*

$$u(T) = 0.$$

*Moreover, $\sqrt{t}u \in L^2(0, T; H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(0, T; L^2(\Omega))$, $\sqrt{t}h \in L^2(0, T; H^1(\Omega))$, and for all $\nu < 2$, $u \in C((0, T]; C^\nu(\overline{\Omega}))$ and $h \in C((0, T]; C^{\nu-1}(\overline{\Omega}))$.*

**Corollary 5.** *Assume that $f(z) = Rz + \mu|z|^{2\sigma}z$ with $\sigma \in \mathbb{R}^{+*}$, $R \in \mathbb{R}$ and $\mu \in \mathbb{C}$. Then the system* (11)–(13) *is locally null controllable in the space $L^p(\Omega)$ for any $p > \sigma N$, and in the Sobolev space $H^q(\Omega)$ for any $q > \frac{N}{2} - \frac{1}{\sigma}$ provided that $2\sigma \geqslant 1$, $N\sigma \geqslant 1$.*

**Corollary 6.** *Assume that $\lim_{|z|\to\infty} f(z)/(z \ln|z|) = 0$. Then the system* (11)–(13) *is globally null controllable in $C_0(\Omega)$, i.e., $R$ may be given any value in Theorem* 4.

The key tool to prove Theorem 1 is a Carleman inequality for the Ginzburg–Landau operator $\partial_t + (a + ib)\Delta$. It is given in the next section.

## 3. Carleman inequalities

Let $S^-$ denote any open subset of $\partial\Omega$, and let $S^+ = \partial\Omega \setminus S^-$. Let $n$ denote the unit outward normal vector to $\partial\Omega$, and $\partial_n u = \partial u/\partial n$. According to [4], we may pick a function $\psi \in C^4(\overline{\Omega})$ such that $\psi > 0$ and $\nabla\psi \neq 0$ on $\overline{\Omega}$ and $\partial_n\psi \leqslant 0$ on $S^-$. Clearly, we may as well assume that

$$\psi(x) > \frac{3}{4}\|\psi\|_{L^\infty(\Omega)} \quad \forall x \in \Omega. \tag{14}$$

Let $C_\psi = (3/2)\|\psi\|_{L^\infty(\Omega)}$ and

$$\theta(x,t) := \frac{e^{\lambda\psi(x)}}{t(T-t)}, \quad \varphi(x,t) := \frac{e^{\lambda C_\psi} - e^{\lambda\psi(x)}}{t(T-t)}, \quad \forall (x,t) \in \Omega \times (0,T) \tag{15}$$

where $\lambda$ denotes some positive number whose range will be specified later. We also introduce the set $\mathcal{Z} := \{q \in C^{2,1}(\overline{\Omega} \times [0, T]);\ q = 0 \text{ on } \partial\Omega \times [0, T]\}$.

The following result gives a Carleman estimate for the Ginzburg–Landau equation, which improves that given in [3]:

**Proposition 3.1.** *Let $a > 0$ and $b \in \mathbb{R}$. Then there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$, and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ and all $q \in \mathcal{Z}$ it holds*

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \big[(s\theta)^{-1}\big(|\partial_t q|^2 + |\Delta q|^2\big) + \lambda^2(s\theta)|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\big]e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{S^-} \lambda(s\theta)|\partial_n\psi||\partial_n q|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant C_0\left(\int_0^T\!\!\int_\Omega \big|\partial_t q + (a+ib)\Delta q\big|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{S^+} \lambda(s\theta)|\partial_n\psi||\partial_n q|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned} \tag{16}$$

To deal with the controllability of the nonlinear GL equation, the following form of the Carleman inequality is needed:

**Corollary 7.** *Let $a > 0$, $b \in \mathbb{R}$ and $R > 0$. Introduce the set $\mathcal{Z}^+ := \{q \in \mathcal{Z};\ \partial_n q = 0$ on $S^+ \times (0,T)\}$. Then there exist some numbers $\lambda_0 = \lambda_0(\Omega, T)$, $C_0 = C_0(\Omega, T)$, and $s_0 = s_0(\Omega, T, R)$ with $s_0 \leqslant C(1 + R^{2/3})$ where $C = C(\Omega, T)$, such that for all $\lambda \geqslant \lambda_0$, all $s \geqslant s_0$, all $d \in L^\infty(\Omega \times (0,T))$ with $\|d\|_{L^\infty(\Omega\times(0,T))} \leqslant R$, and all $q \in \mathcal{Z}^+$, it holds*

$$\int_0^T\int_\Omega \big[(s\theta)^{-1}\big(|\partial_t q|^2 + |\Delta q|^2\big) + \lambda^2(s\theta)|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\big]e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_0 \int_0^T\int_\Omega \big|\partial_t q + (a+ib)\Delta q + dq\big|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t. \tag{17}$$

Notice that an *internal* Carleman estimate may be derived along the same lines. Let $\omega$ be any given open subset of $\Omega$, and let $\psi$ denote now some function of class $C^4$ on $\overline{\Omega}$ such that (14) holds true, $\partial_n \psi \leqslant 0$ on $\partial\Omega$, and $\nabla\psi \neq 0$ on $\overline{\Omega} \setminus \omega$ (see [4]). The functions $\theta$ and $\varphi$ being defined as in (15), we have the following Carleman estimate:

**Proposition 3.2.** *Let $a > 0$ and $b \in \mathbb{R}$. Then there exist some constants $\lambda_0 \geqslant 1$, $s_0 \geqslant 1$, and $C_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, $s \geqslant s_0$ and all $q \in \mathcal{Z}$ it holds*

$$\int_0^T\int_\Omega \big[(s\theta)^{-1}\big(|\partial_t q|^2 + |\Delta q|^2\big) + \lambda^2(s\theta)|\nabla q|^2 + \lambda^4(s\theta)^3|q|^2\big]e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega} \lambda(s\theta)|\partial_n\psi||\partial_n q|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_0\left(\int_0^T\int_\Omega \big|\partial_t q + (a+ib)\Delta q\big|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\omega \lambda^4(s\theta)^3|q|^2 e^{-2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\right). \tag{18}$$

## 4. Sketch of the proof of Theorem 1

STEP 1. NULL CONTROLLABILITY OF A LINEARIZED EQUATION. Introduce the space $V := C([0,T]; X)$ endowed with the uniform norm. Let $g \in C^0(\mathbb{C};\mathbb{C})$ be defined by $g(z) = -f(z)/z$ for $z \neq 0$. Pick an initial state $u_0 \in X$, a function $z \in V$, and set $d := g(z)$. Using Corollary 7 and following a classical method due to Fursikov–Imanuvilov (see [4]), we can assign a solution $\{u, h\}$ to the null controllability problem

$$\begin{cases} \partial_t u - (1+i\alpha)\Delta u + du = 0 & \text{in } \Omega \times (0,T),\\ u = 1_{\Gamma_0} h & \text{on } \partial\Omega \times (0,T),\\ u(0) = u_0, \quad u(T) = 0, \end{cases}$$

in such a way that $u \in V \cap L^2(0,T;H^1(\Omega))$, $h = u_{|\partial\Omega} \in C(\partial\Omega \times [0,T])$ with $\operatorname{supp} h \subset \Gamma_0 \times [0,T]$, $\sqrt{t}u \in L^2(0,T;H^2(\Omega))$, $\sqrt{t}\partial_t u \in L^2(0,T;L^2(\Omega))$, $\sqrt{t}h \in L^2(0,T;H^{\frac{3}{2}}(\partial\Omega))$, and for any $\nu < 2$, $u \in C((0,T];C^\nu(\overline{\Omega}))$ and $h \in C((0,T];C^\nu(\partial\Omega))$. To prove the above regularity properties, we use a bootstrap argument and the fact that the operator $A_p u := -(1+i\alpha)\Delta u$ with domain $\mathcal{D}(A_p) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset L^p(\Omega)$ is sectorial for any $p \in [2,\infty)$.
STEP 2. THE FIXED POINT ARGUMENT. Define a map $\Lambda : V \to V$ by $\Lambda(z) = u$. It may be seen that $\Lambda$ is continuous. Pick any number $r > 0$. For $\|u_0\|_{L^\infty(\Omega)}$ small enough, it may be proved that $\Lambda$ maps the closed ball $B_r(0) \subset V$ into itself, and that $\Lambda(B_r(0))$ is relatively compact in $V$. By virtue of Schauder fixed-point theorem, $\Lambda$ has a fixed point $z = u$ in $B_r(0)$. □

The reader is referred to [9] for the details of the proof.

## Acknowledgements

This work was partially supported by the Charles Phelps Taft Memorial Fund at the University of Cincinnati. It was begun when the first author was visiting the University of Cincinnati, and finished when the first author was visiting the CMM, UMI CNRS 2807. The first author thanks both institutions for their hospitality.

## Note added in proof

After [9] was submitted, the authors learned that similar results were obtained by Boldrini, Fernandez-Cara and Guerrero in [1].

## References

[1] J.L. Boldrini, E. Fernandez-Cara, S. Guerrero, On the controllability of the Ginzburg–Landau equation, in preparation.
[2] E. Fernandez-Cara, Null controllability of the semilinear heat equation, ESAIM Control Optim. Calc. Var. 2 (1997) 87–103.
[3] X. Fu, A weighted identity for partial differential operators of second order and its applications, C. R. Math. Acad. Sci. Paris, Ser. I 342 (2006) 579–584.
[4] A.V. Fursikov, O.Y. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Ser., vol. 34, Research Institute of Mathematics, Seoul National University, Seoul, Korea, 1996.
[5] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.
[6] C.D. Levermore, M. Oliver, Distribution-valued initial data for the complex Ginzburg–Landau equation, Comm. Partial Differential Equations 22 (1997) 39–48.
[7] C.D. Levermore, M. Oliver, in: The complex Ginzburg–Landau equation as a model problem, Dynamical Systems and Probabilistic Methods in Partial Differential Equations, Berkeley, CA, 1994, Lectures in Appl. Math., vol. 31, Amer. Math. Soc., Providence, RI, 1996, pp. 141–190.
[8] A. Mielke, The complex Ginzburg–Landau equation on large and unbounded domains: Sharper bounds and attractors, Nonlinearity 10 (1997) 199–222.
[9] L. Rosier, B.-Y. Zhang, Null controllability of the complex Ginzburg–Landau equation, submitted for publication.