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CONVERGENCE OF THE LAGRANGE–GALERKIN METHOD FOR THE EQUATIONS MODELLING THE MOTION OF A FLUID-RIGID SYSTEM*

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Abstract. In this paper, we consider a Lagrange–Galerkin scheme to approximate a two-dimensional fluid-rigid body problem. The equations of the system are the Navier–Stokes equations in the fluid part, coupled with ordinary differential equations for the dynamics of the rigid body. In this problem, the equations of the fluid are written in a domain whose variation is one of the unknowns. We introduce a numerical method based on the use of characteristics and on finite elements with a fixed mesh. Our main result asserts the convergence of this scheme.

Key words. fluid-structure interaction, incompressible Navier–Stokes equations, finite element method, Lagrange–Galerkin method

AMS subject classifications. 35Q30, 76D05, 65M12, 76M10

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1. Introduction.

The aim of this paper is to analyze a Lagrange–Galerkin approximation of the equations modelling the motion of a two-dimensional rigid body immersed in a fluid. We first briefly describe the equations modelling this system. Assume that the system fluid-rigid body occupies a bounded domain $\mathcal{O}$ in $\mathbb{R}^2$ with a regular boundary $\partial\mathcal{O}$. The solid is supposed to occupy at each instant t a closed connected subset $B(t) \subset \mathcal{O}$ which is surrounded by a viscous homogeneous fluid filling the domain $\Omega(t) = \mathcal{O}\backslash B(t)$.

The motion of the fluid is described by the classical Navier–Stokes equations, whereas the motion of the rigid body is governed by the balance equations for linear and angular momentum (Newton's laws). More precisely, we consider the following system coupling partial differential and ordinary differential equations:

$$\rho_f \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \rho_f(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \rho_f \mathbf{f}, \quad \mathbf{x} \in \Omega(t), \quad t \in [0,T], \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0, \quad \mathbf{x} \in \Omega(t), \quad t \in [0,T], \tag{1.2}$$

$$\mathbf{u} = 0, \quad \mathbf{x} \in \partial\mathcal{O}, \quad t \in [0,T], \tag{1.3}$$

$$\mathbf{u} = \boldsymbol{\zeta}'(t) + \omega(t)(\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp}, \quad \mathbf{x} \in \partial B(t), \quad t \in [0,T], \tag{1.4}$$

$$M\boldsymbol{\zeta}''(t) = -\int_{\partial B(t)} \boldsymbol{\sigma}\mathbf{n}\, \mathrm{d}\Gamma + \rho_s \int_{B(t)} \mathbf{f}(\mathbf{x},t)\, \mathrm{d}\mathbf{x}, \quad t \in [0,T], \tag{1.5}$$

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(1.6)
$$J\omega'(t) = -\int_{\partial B(t)} (\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \cdot \boldsymbol{\sigma}\mathbf{n} \, \mathrm{d}\Gamma + \rho_s \int_{B(t)} (\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \cdot \mathbf{f}(\mathbf{x}, t) \, \mathrm{d}\mathbf{x}, \quad t \in [0, T],$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0), \tag{1.7}$$

$$\boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0 \in \mathbb{R}^2, \quad \boldsymbol{\zeta}'(0) = \boldsymbol{\zeta}_1 \in \mathbb{R}^2, \quad \omega(0) = \omega_0 \in \mathbb{R}. \tag{1.8}$$

In the above equations the unknowns are $\mathbf{u}(\mathbf{x}, t)$ (the Eulerian velocity field of the fluid), $p(\mathbf{x}, t)$ (the pressure of the fluid), $\boldsymbol{\zeta}(t)$ (the position of the mass center of the rigid body), and $\omega(t)$ (the angular velocity of the rigid body). The domain $B(t)$ is defined by

$$B(t) = \{\mathbf{R}_{-\theta(t)}\mathbf{y} + \boldsymbol{\zeta}(t), \ \mathbf{y} \in B\},$$

where

$$\theta(t) = \int_0^t \omega(s) \, \mathrm{d}s, \tag{1.9}$$

$B = B(0)$, and $\mathbf{R}_\theta$ is the rotation matrix of angle θ. Moreover, we have denoted by $\partial B(t)$ the boundary of the rigid body at instant t and by $\mathbf{n}(\mathbf{x}, t)$ the unit normal to $\partial B(t)$ at the point $\mathbf{x}$ directed to the interior of the rigid body.

The constants ρ_f and ρ_s are, respectively, the density of the fluid and the density of the rigid body. In what follows, we assume that the densities of the fluid and of the solid are equal, that is

$$\rho_f = \rho_s = 1, \tag{1.10}$$

and that the rigid body is a ball in $\mathbb{R}^2$. Assumption (1.10) is clearly restrictive but it is important for the forthcoming analysis (see Remarks 2.1 and 2.4 below). On the contrary, the assumption that the rigid body is a ball is not essential but avoids some technicalities.

The constants M and J are the mass and the moment of inertia of the rigid body, and the positive constant ν is the viscosity of the fluid. Moreover, $\mathbf{f}(\mathbf{x}, t)$ is the applied force (per unit mass).

For all $\mathbf{x} = \binom{x_1}{x_2}$, we denote by $\mathbf{x}^{\perp}$ the vector $\mathbf{x}^{\perp} = \binom{x_2}{-x_1}$. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then $\mathbf{x} \cdot \mathbf{y}$ stands for the inner product of $\mathbf{x}$ and $\mathbf{y}$ and $|\mathbf{x}|$ stands for the corresponding norm. We have also denoted by w' and w'' the derivatives of a function w depending only on the time t.

Finally, the stress tensor (also called the Cauchy stress) is defined by

$$\boldsymbol{\sigma}(\mathbf{x}, t) = -p(\mathbf{x}, t)\mathbf{Id} + 2\nu\mathbf{D}(\mathbf{u}), \tag{1.11}$$

where $\mathbf{Id}$ is the identity matrix and $\mathbf{D}(\mathbf{u})$ is the tensor field defined by

$$D(u)_{k,l} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right).$$

The main difficulties of this problem are that

- the equations of the structure are coupled with those of the fluid,
- the domain of the fluid is variable, and it is one of the unknowns of the problem (we thus have a free boundary problem).

The well-posedness of this type of system has been recently studied in a large number of papers (see, for instance, Desjardins and Esteban [6], Gunzburger, Lee, and Seregin [17], San Martín, Starovoitov, and Tucsnak [25], Grandmont and Maday [16], Takahashi [27], and the references therein).

The literature on the numerical approximation of the solution of (1.1)–(1.8) also contains a large number of recent papers. Some of these papers are based on an arbitrary Lagrangian Eulerian (ALE) formulation; see, for example, Grandmont, Guimet, and Maday [15], Nobile [22], Maury and Glowinski [21], Maury [19], [20], Formaggia and Nobile [9], and Farhat, Geuzaine, and Grandmont [8]. In the ALE method, at each time step, the mesh is moved with an arbitrary velocity in the fluid to follow the motion of the rigid body. The stability of the ALE method is studied in [9] (in the case of the finite element context) and in [8] (in the case of the finite volume context). We also mention the work of Gastaldi [11], where, in the case of an advection-diffusion equation in a moving two-dimensional domain, a priori error estimates that are optimal both in space and time have been obtained.

Another approach, developed by Glowinski et al. [14], [13] is based on a fictitious domain formulation: the rigid bodies are filled by the surrounding fluid, and the constraint of rigid body motion is relaxed by introducing a distributed Lagrange multiplier.

As far as we know, the only proof of the convergence of one of these methods is given in [15] for a simplified problem in one space dimension. The main novelty brought in by our paper consists of the fact that we construct a new approximation method using a fixed mesh and that we prove a convergence result. This method is inspired by the Galerkin–Lagrange approximation, which is commonly used for Navier–Stokes equations (see Pironneau [23] and Süli [26]).

The remaining part of this paper is organized as follows. In section 2 we introduce some function spaces and semidiscretize our problem with respect to the time variable. In section 3 we give the full discretization of the problem and state the main result. Section 4 is devoted to the study of the finite element spaces that were introduced in the previous section. Section 5 is devoted to the study of a change of variables that plays an important role in the proof of the main result. In section 6 we prove the consistency of our scheme. Finally, in section 7 we give the proof of the main result.

2. Notation and preliminaries.

2.1. Notation and function spaces. Throughout this paper, we shall use the classical Sobolev spaces $H^s(\Omega)$, $H^s_0(\Omega)$, $H^{-s}(\Omega)$, $s \geqslant 0$, and the space of Lipschitz continuous functions $C^{0,1}(\overline{\Omega})$ on the closure of Ω. We also define

$$L^2_0(\Omega) = \left\{ f \in L^2(\Omega) \;\middle|\; \int_\Omega f \, \mathrm{d}\mathbf{x} = 0 \right\}$$

and denote by $\mathcal{L}^2_0(\Omega)$, $\mathcal{H}^s(\Omega)$, $\mathcal{H}^s_0(\Omega)$, $\mathcal{H}^{-s}(\Omega)$, $s \geqslant 0$, the spaces $\left[L^2_0(\Omega)\right]^2$, $[H^s(\Omega)]^2$, $[H^s_0(\Omega)]^2$, $[H^{-s}(\Omega)]^2$. The usual inner product in $\mathcal{L}^2(\mathcal{O})$ will be denoted by

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{L}^2(\mathcal{O}). \tag{2.1}$$

If $\mathbf{A}$ is a matrix, we denote by $\mathbf{A}^*$ its transpose. For any 2×2 matrices $\mathbf{A}, \mathbf{B} \in \mathcal{M}_{2\times 2}$, we denote by $\mathbf{A} : \mathbf{B}$ their inner product $\mathbf{A} : \mathbf{B} = \mathrm{Trace}(\mathbf{A}^*\mathbf{B})$, and by $|\mathbf{A}|$ the corresponding norm. For convenience, we use the same notation as in (2.1) for the

inner product in $L^2(\mathcal{O}, \mathcal{M}_{2\times 2})$, that is,

$$(\mathbf{A}, \mathbf{B}) = \int_{\mathcal{O}} \mathbf{A} : \mathbf{B} \, \mathrm{d}\mathbf{x} \quad \forall \mathbf{A}, \mathbf{B} \in L^2(\mathcal{O}, \mathcal{M}_{2\times 2}). \tag{2.2}$$

We also define the spaces

$$\mathcal{K}(\boldsymbol{\zeta}) = \{\mathbf{u} \in \mathcal{H}_0^1(\mathcal{O}) \mid \mathbf{D}(\mathbf{u}) = 0 \quad \text{in } B(\boldsymbol{\zeta})\} \tag{2.3}$$

and

$$\widehat{\mathcal{K}}(\boldsymbol{\zeta}) = \{\mathbf{u} \in \mathcal{H}_0^1(\mathcal{O}) \mid \operatorname{div} \mathbf{u} = 0 \text{ in } \mathcal{O}, \quad \mathbf{D}(\mathbf{u}) = 0 \text{ in } B(\boldsymbol{\zeta})\}, \tag{2.4}$$

where $\boldsymbol{\zeta} \in \mathbb{R}^2$ and $B(\boldsymbol{\zeta}) = \{\mathbf{x} \in \mathbb{R}^2, |\mathbf{x} - \boldsymbol{\zeta}| \leqslant 1\}$. According to Lemma 1.1 of [29, p. 18], for any $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta})$, there exist $\mathbf{l}_{\mathbf{u}} \in \mathbb{R}^2$ and $\omega_{\mathbf{u}} \in \mathbb{R}$ such that

$$\mathbf{u}(\mathbf{y}) = \mathbf{l}_{\mathbf{u}} + \omega_{\mathbf{u}} (\mathbf{y} - \boldsymbol{\zeta})^{\perp} \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}).$$

These spaces are specific to our problem. In fact, if the solution $\mathbf{u}$ of (1.1)–(1.8) is extended by

$$\mathbf{u}(\mathbf{x}, t) = \boldsymbol{\zeta}'(t) + \omega(t)(\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}(t)),$$

then we easily see that $\mathbf{u}(t) \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}(t))$.

In what follows, the solution $\mathbf{u}$ *of* (1.1)–(1.8) *will be extended as above.*

We also notice that, by using (1.10), for any $\mathbf{u}, \mathbf{v} \in \mathcal{K}(\boldsymbol{\zeta})$ we have

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathcal{O}\setminus B(\boldsymbol{\zeta})} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + M \mathbf{l}_{\mathbf{u}} \cdot \mathbf{l}_{\mathbf{v}} + J \omega_{\mathbf{u}} \, \omega_{\mathbf{v}}. \tag{2.5}$$

Remark 2.1. In the case of different densities $\rho_F \neq \rho_S$, the natural inner product to be used seems to be

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\zeta} = \rho_F \int_{\mathcal{O}\setminus B(\boldsymbol{\zeta})} \mathbf{u} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + M \mathbf{l}_{\mathbf{u}} \cdot \mathbf{l}_{\mathbf{v}} + J \omega_{\mathbf{u}} \, \omega_{\mathbf{v}},$$

which clearly depends on the position of the ball. This fact would considerably complicate the further analysis.

An important ingredient of the numerical method we use is given by the characteristic functions whose level lines are the integral curves of the velocity field. More precisely (see, for instance, [23], [26]), the characteristic function $\widetilde{\boldsymbol{\psi}} : [0, T]^2 \times \mathcal{O} \to \mathcal{O}$ is defined as the solution of the initial value problem

$$\begin{cases} \dfrac{d}{dt} \widetilde{\boldsymbol{\psi}}(t; s, \mathbf{x}) = \mathbf{u}(\widetilde{\boldsymbol{\psi}}(t; s, \mathbf{x}), t), \\ \widetilde{\boldsymbol{\psi}}(s; s, \mathbf{x}) = \mathbf{x}. \end{cases} \tag{2.6}$$

It is well known that the material derivative $D_t \mathbf{u} = \partial \mathbf{u}/\partial t + (\mathbf{u} \cdot \nabla)\mathbf{u}$ of $\mathbf{u}$ at instant t_0 satisfies

$$D_t \mathbf{u}(\mathbf{x}, t_0) = \frac{d}{dt} \left[\mathbf{u}(\widetilde{\boldsymbol{\psi}}(t; t_0, \mathbf{x}), t) \right]_{|_{t=t_0}}. \tag{2.7}$$

Remark 2.2. By using a classical result of Liouville (see, for instance, Arnold [1, p. 251]), if

$$\boldsymbol{\zeta} \in \mathcal{H}^2(0,T), \quad \omega \in H^1(0,T), \quad \mathbf{u} \in C([0,T];\widehat{\mathcal{K}}(\boldsymbol{\zeta}(t))),$$

then we have that

$$\det \mathbf{J}_{\widetilde{\boldsymbol{\psi}}} = 1, \tag{2.8}$$

where we have denoted by

$$\mathbf{J}_{\widetilde{\boldsymbol{\psi}}} = \left(\frac{\partial \widetilde{\psi}_i}{\partial y_j}\right)_{i,j}$$

the jacobian matrix of the transformation $\mathbf{y} \mapsto \widetilde{\boldsymbol{\psi}}(\mathbf{y})$.

2.2. Weak form and semidiscretization scheme. In this subsection we give a weak form of (1.1)–(1.8) which is then used to discretize the problem with respect to time.

The fact that (2.9) is called a "weak formulation" of the system (1.1)–(1.8) is justified by the following result.

LEMMA 2.3. *Assume that*

$$\mathbf{u} \in L^2(0,T;\mathcal{H}^2(\Omega(t))) \cap H^1(0,T;\mathcal{L}^2(\Omega(t))) \cap C([0,T];\mathcal{H}^1(\Omega(t))),$$
$$p \in L^2(0,T;H^1(\Omega(t))),$$
$$\boldsymbol{\zeta} \in \mathcal{H}^2(0,T), \quad \omega \in H^1(0,T)$$

and that $\mathbf{u}$ *is extended by*

$$\mathbf{u}(\mathbf{x},t) = \boldsymbol{\zeta}'(t) + \omega(t)(\mathbf{x} - \boldsymbol{\zeta}(t))^{\perp} \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}(t)).$$

Then $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ *is the solution of* (1.1)–(1.8) *if and only if* $\mathbf{u}(t) \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}(t))$ *for all* t *and* $(\mathbf{u}, p)$ *satisfies*

$$\left(\frac{d}{dt}\left[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}\right](t), \boldsymbol{\varphi}\right) + 2\nu\left(\mathbf{D}(\mathbf{u}(t)), \mathbf{D}(\boldsymbol{\varphi})\right) - \int_{\Omega(t)} (\operatorname{div}\boldsymbol{\varphi}) p(t)\, d\mathbf{x} = (\mathbf{f}(t), \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}(t)). \tag{2.9}$$

We skip the proof of Lemma 2.3 since it is similar to the proof of the corresponding result for the classical Navier–Stokes system (see, for instance, [24, Chap. 12]).

Remark 2.4. In the case of different densities $\rho_F \neq \rho_S$, a similar weak statement can be obtained (see, for instance, [5]). In this case $\mathbf{u}$ in the first term of (2.9) should be replaced by $\rho\mathbf{u}$, where $\rho = \rho_F$ in the fluid and $\rho = \rho_S$ in the moving solid. Thus ρ would depend on the time and a transport equation for ρ should be added to the system.

By using the weak formulation given above we can derive a semidiscrete version of our system. For $N \in \mathbb{N}^*$ we denote $\Delta t = T/N$ and $t_k = k\Delta t$ for $k = 0, \ldots, N$. Denote by $(\mathbf{u}^k, \boldsymbol{\zeta}^k) \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}^k) \times \mathbb{R}^2$ the approximation of the solution of (1.1)–(1.8) at the time $t = t_k$. We approximate the position of the rigid ball at instant t_{k+1} by $\boldsymbol{\zeta}^{k+1}$, which is defined by the relation

$$\boldsymbol{\zeta}^{k+1} = \boldsymbol{\zeta}^k + \mathbf{u}^k(\boldsymbol{\zeta}^k)\Delta t. \tag{2.10}$$

We then define characteristic function $\overline{\boldsymbol{\psi}}$ associated to the semidiscretized velocity field as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{x}) = \mathbf{u}^k(\overline{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{x})), \\ \overline{\boldsymbol{\psi}}(t_{k+1};t_{k+1},\mathbf{x}) = \mathbf{x}, \end{cases} \tag{2.11}$$

and we denote

$$\overline{\boldsymbol{X}}^{\boldsymbol{k}}(\mathbf{x}) = \overline{\boldsymbol{\psi}}(t_k;t_{k+1},\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}. \tag{2.12}$$

One can easily check that $\overline{\boldsymbol{X}}^{\boldsymbol{k}}(\mathcal{O}) = \mathcal{O}$.

We next define $\mathbf{u}^{k+1} \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}^{k+1})$ as the solution of the following Stokes type system:

$$\left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k \circ \overline{\boldsymbol{X}}^{\boldsymbol{k}}}{\Delta t}, \boldsymbol{\varphi}\right) + 2\nu\left(\mathbf{D}(\mathbf{u}^{k+1}), \mathbf{D}(\boldsymbol{\varphi})\right) = (\mathbf{f}^{k+1}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}^{k+1}), \tag{2.13}$$

where $\mathbf{f}^{k+1} = \mathbf{f}(t_{k+1})$.

The above equation can be rewritten by using a mixed formulation. To achieve this, we first define

$$M(\boldsymbol{\zeta}) = \left\{p \in L^2_0(\mathcal{O}) \mid p = 0 \text{ in } B(\boldsymbol{\zeta})\right\}, \tag{2.14}$$

$$a(\mathbf{u},\mathbf{v}) = 2\nu \int_{\mathcal{O}} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})\, d\mathbf{x} \quad \forall \mathbf{u},\mathbf{v} \in \mathcal{H}^1(\mathcal{O}), \tag{2.15}$$

$$b(\mathbf{u},p) = -\int_{\mathcal{O}} \operatorname{div}(\mathbf{u}) p\, d\mathbf{x} \quad \forall \mathbf{u} \in \mathcal{H}^1(\mathcal{O}), \quad \forall p \in L^2_0(\mathcal{O}). \tag{2.16}$$

With the above notation, it is clear that (2.13) is equivalent to the system

$$\left(\frac{\mathbf{u}^{k+1} - \mathbf{u}^k \circ \overline{\boldsymbol{X}}^{\boldsymbol{k}}}{\Delta t}, \boldsymbol{\varphi}\right) + a(\mathbf{u}^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p^{k+1}) = (\mathbf{f}^{k+1}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}^{k+1}), \tag{2.17}$$

$$b(\mathbf{u}^{k+1}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}^{k+1}) \tag{2.18}$$

of unknowns $(\mathbf{u}^{k+1}, p^{k+1}) \in \mathcal{K}(\boldsymbol{\zeta}^{k+1}) \times M(\boldsymbol{\zeta}^{k+1})$.

Remark 2.5. The requirement $p = 0$ in $B(\boldsymbol{\zeta})$ for the definition of $M(\boldsymbol{\zeta})$ allows us to define the form b on the whole domain $\mathcal{O}$. This extension does not affect the form b since $\operatorname{div}(\mathbf{u}) = 0$ in $B(\boldsymbol{\zeta})$ for all $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta})$.

It is well known (see, for example, [12, Corollary I.4.1., p. 61]) that the mixed formulation (2.17), (2.18) is a well-posed problem, provided that the spaces $\mathcal{K}(\boldsymbol{\zeta})$, $M(\boldsymbol{\zeta})$ and the bilinear form b satisfy an *inf-sup* condition. The fact that this *inf-sup* condition is satisfied in our case follows from the result below.

LEMMA 2.6. *Suppose that $\boldsymbol{\zeta} \in \mathcal{O}$ is such that $d(\boldsymbol{\zeta}, \partial\mathcal{O}) = 1 + \eta$, with $\eta > 0$. Then there exists a constant $\beta > 0$, depending only on η and on $\mathcal{O}$, such that for all $q \in M(\boldsymbol{\zeta})$ there exists $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta})$ with*

$$\int_{\mathcal{O}} \operatorname{div}(\mathbf{u})\, q\, d\mathbf{x} \geq \beta \|\mathbf{u}\|_{\mathcal{H}^1(\mathcal{O})} \|q\|_{L^2(\mathcal{O})}. \tag{2.19}$$

The proof of the above result can be obtained by slightly modifying the approach used for the mixed formulation of the standard Stokes system (see, for instance [12, p. 81]), so it is left to the reader.

3. Full discretization and statement of the main result. In order to discretize the problem (2.17), (2.18) with respect to the space variable we introduce two families of finite element spaces. We first define a family of finite element spaces that approximate the space $\mathcal{K}(\boldsymbol{\zeta})$ defined in (2.3). Let h denote a discretization parameter, $0 < h < 1$, and let P_1 be the space of all affine functions in $\mathbb{R}^2$.

Consider a quasi-uniform triangulation $\mathcal{T}_h$ of $\mathcal{O}$, as defined, for instance, in [2, p. 106] (this assumption will be accepted in the remainder of this paper and will allow us to make use of inverse estimates). If $T \in \mathcal{T}_h$ is a triangle of vertices $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, we denote by $\varphi_1(\mathbf{x})$, $\varphi_2(\mathbf{x})$, and $\varphi_3(\mathbf{x})$ the corresponding barycentric coordinates of $\mathbf{x} \in \mathbb{R}^2$ with respect to the vertices $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ (see, for instance, [4, p. 45] for the definition of barycentric coordinates). We associate to this triangulation two classical approximation spaces used in the mixed finite element methods for the Stokes system. The first space, classically used for the approximation of the velocity field in the mixed statement of the Stokes system, is denoted by $\mathcal{W}_h$ and is defined as the subspace of $\mathcal{H}_0^1(\mathcal{O})$ formed by the P_1-bubble finite elements associated to $\mathcal{T}_h$. More precisely, $\boldsymbol{\varphi} \in \mathcal{W}_h$ if and only if

$$\boldsymbol{\varphi}(\mathbf{x}) = \varphi_1(\mathbf{x})\boldsymbol{\alpha_1} + \varphi_2(\mathbf{x})\boldsymbol{\alpha_2} + \varphi_3(\mathbf{x})\boldsymbol{\alpha_3} + \frac{\varphi_1(\mathbf{x})\varphi_2(\mathbf{x})\varphi_3(\mathbf{x})}{\displaystyle\int_T \varphi_1\varphi_2\varphi_3 \, \mathrm{d}\mathbf{x}}\boldsymbol{\lambda} \qquad \forall\, \mathbf{x} \in T$$

for some constant vectors $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$, $\boldsymbol{\alpha_3}$, $\boldsymbol{\lambda} \in \mathbb{R}^2$. We may notice that all functions in $\mathcal{W}_h$ are continuous.

The second space, classically used for the approximation of the pressure in mixed statements of the Stokes system, is denoted by E_h and is defined by

$$E_h = \left\{ q \in C(\overline{\mathcal{O}}) \;\middle|\; q_{|T} \in P_1(T) \right\}. \tag{3.1}$$

For our problem we use two spaces that are related to the presence of the rigid body. The first one, which is used for the approximation of the velocity field, is denoted by $\mathcal{K}_h(\boldsymbol{\zeta})$ and defined by

$$\mathcal{K}_h(\boldsymbol{\zeta}) = \mathcal{W}_h \cap \mathcal{K}(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{O}.$$

The second one, which is used for the approximation of the pressure, is denoted by $M_h(\boldsymbol{\zeta})$ and defined by

$$M_h(\boldsymbol{\zeta}) = E_h \cap M(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathcal{O}.$$

We also define the finite element space (see [23])

$$\mathcal{R}_h = \{ \mathbf{rot}\ \varphi_h, \quad \varphi_h \in E_h, \quad \varphi_h = 0 \text{ on } \partial\mathcal{O} \}.$$

We denote by $\mathbf{P}$ the orthogonal projection from $\mathcal{L}^2$ onto $\mathcal{R}_h$. More precisely, if $\mathbf{u} \in \mathcal{L}^2(\mathcal{O})$, then $\mathbf{Pu} \in \mathcal{R}_h$ satisfies

$$(\mathbf{u} - \mathbf{Pu}, \mathbf{r}_h) = 0 \quad \forall \mathbf{r}_h \in \mathcal{R}_h.$$

Let N be a positive integer. We denote $\Delta t = T/N$ and $t_k = k\Delta t$. Assume that the approximate solution $(\mathbf{u}_h^k, p_h^k, \boldsymbol{\zeta}_h^k)$ of (1.1)–(1.8) at $t = t_k$ is known. We describe below the numerical scheme allowing us to determinate the approximate solution $(\mathbf{u}_h^{k+1}, p_h^{k+1}, \boldsymbol{\zeta}_h^{k+1})$ at $t = t_{k+1}$. First, we compute $\boldsymbol{\zeta}_h^{k+1} \in \mathbb{R}^2$ by

$$\boldsymbol{\zeta}_h^{k+1} = \boldsymbol{\zeta}_h^k + \mathbf{u}_h^k(\boldsymbol{\zeta}_h^k)\Delta t. \tag{3.2}$$

We denote by $\mathbf{P}\mathbf{u}_h^k$ the projection of $\mathbf{u}_h^k$ onto $\mathcal{R}_h$. Then we define the characteristic function $\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}$ associated to the fully discretized velocity field as the solution of

$$\begin{cases} \dfrac{d}{dt}\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(t;t_{k+1},\mathbf{x}) = \mathbf{P}\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(t;t_{k+1},\mathbf{x})), \\ \overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(t_{k+1};t_{k+1},\mathbf{x}) = \mathbf{x}. \end{cases} \tag{3.3}$$

We also define

$$\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}(\mathbf{x}) = \overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(t_k;t_{k+1},\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}, \tag{3.4}$$

and as for the problem (2.11), one can check that $\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}(\mathcal{O}) = \mathcal{O}$ (see Remark 3.1 below).

Then we define $(\mathbf{u}_h^{k+1}, p_h^{k+1}) \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}) \times M_h(\boldsymbol{\zeta}_h^{k+1})$ as the solution of the problem

$$\left(\frac{\mathbf{u}_h^{k+1} - \mathbf{u}_h^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}}{\Delta t}, \boldsymbol{\varphi}\right) + a(\mathbf{u}_h^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, p_h^{k+1}) = (\mathbf{f}_h^{k+1}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}), \tag{3.5}$$

$$b(\mathbf{u}_h^{k+1}, q) = 0 \quad \forall q \in M_h(\boldsymbol{\zeta}_h^{k+1}), \tag{3.6}$$

where $\mathbf{f}_h^{k+1}$ is the $\mathcal{L}^2$-projection of $\mathbf{f}^{k+1} = \mathbf{f}(t_{k+1})$ on $(E_h)^2$. We take $\boldsymbol{\zeta}_h^0 = \boldsymbol{\zeta}^0$, and the initial approximate velocity $\mathbf{u}_h^0$ is the $\mathcal{H}_0^1$-projection of $\mathbf{u}_0$ onto $\mathcal{K}_h(\boldsymbol{\zeta}_h^0)$.

Remark 3.1. In (3.3), we use the projection of $\mathbf{u}_h^k$ on $\mathcal{R}_h$ rather than the function $\mathbf{u}_h^k$ itself because $\operatorname{div}(\mathbf{P}\mathbf{u}_h^k) = 0$ in $\mathcal{O}$. By using a classical result of Liouville, this implies that $\det \mathbf{J}_{\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}} = 1$ and in particular that $\det \mathbf{J}_{\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}} = 1$. This property, combined with the fact that the velocity field $\mathbf{P}\mathbf{u}_h^k$ vanishes along the boundary $\partial\mathcal{O}$, entails the invariance property of the whole domain $\mathcal{O}$ through $\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}$, i.e., $\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}(\mathcal{O}) = \mathcal{O}$. Moreover, since $\mathbf{P}\mathbf{u}_h^k$ is constant in each triangle, the initial value problem (3.3) can be solved exactly.

In what follows, we suppose that

$$\begin{aligned} &\mathbf{f} \in C([0,T];\mathcal{H}^1(\mathcal{O})), \quad \mathbf{u}_0 \in \mathcal{H}^2(\Omega), \quad \operatorname{div}(\mathbf{u}_0) = 0 \quad \text{in } \Omega, \\ &\mathbf{u}_0 = 0 \quad \text{on } \partial\mathcal{O}, \quad \mathbf{u}_0(\mathbf{y}) = \boldsymbol{\zeta}_1 + \omega_0(\mathbf{y} - \boldsymbol{\zeta}_0)^\perp \quad \text{on } \partial B. \end{aligned} \tag{3.7}$$

The corresponding solution $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ of problem (1.1)–(1.8) will be assumed to satisfy the following regularity hypotheses:

$$\begin{cases} \mathbf{u} \in C([0,T];\mathcal{H}^2(\Omega(t))) \cap H^1(0,T;\mathcal{L}^2(\Omega(t))), \\ D_t^2\mathbf{u} \in L^2(0,T;\mathcal{L}^2(\Omega(t))), \quad \mathbf{u} \in C([0,T];\mathcal{C}^{0,1}(\overline{\mathcal{O}})) \\ p \in C([0,T];\mathcal{H}^1(\Omega(t))), \quad \boldsymbol{\zeta} \in \mathcal{H}^3(0,T), \quad \omega \in H^2(0,T). \end{cases} \tag{3.8}$$

Moreover, we assume that

$$\operatorname{dist}(B(t), \partial\mathcal{O}) > 0 \quad \forall t \in [0,T]. \tag{3.9}$$

The hypotheses (3.8) and (3.9) imply the existence of $\eta > 0$ such that

$$\operatorname{dist}(B(t), \partial\mathcal{O}) > 3\eta \quad \forall t \in [0,T]. \tag{3.10}$$

THEOREM 3.2. *Let $C_0 > 0$ be a fixed constant. Suppose that $\mathcal{O}$ is the interior of a convex polygon and that $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ is a solution of (1.1)–(1.8) satisfying (3.8) and (3.9). Moreover, assume that $\mathbf{f}$ and $\mathbf{u}_0$ satisfy (3.7). Consider the functions $\boldsymbol{\zeta}_h^k$, $\mathbf{u}_h^k$, and p_h^k defined in this section. Then there exist two positive constants C and τ^* not depending on h and Δt such that for all $0 < \Delta t \leqslant \tau^*$ and for all $h \leqslant C_0 (\Delta t)^2$ we have*

$$\sup_{1 \leqslant k \leqslant N} \left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}(t_k) - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} \right) \leqslant C \Delta t.$$

Remark 3.3. For the Navier–Stokes system, the same type of result is obtained in [23] for $h \leqslant C_0 \Delta t$ and in [26] for $h^2 \leqslant C_0 \Delta t \leqslant C_1 h^\sigma$ and $\sigma > 1/2$ (for h and Δt small enough).

Remark 3.4. It can be easily checked, by using the fact that $\det \mathbf{J}_{\overline{\boldsymbol{\psi}}_h^k} = 1$, that our method is unconditionally stable.

4. Some properties of the finite element spaces. Next we give some technical results on the finite element spaces introduced above. Throughout this section we consider $\boldsymbol{\zeta} \in \mathcal{O}$ such that $\operatorname{dist}(B(\boldsymbol{\zeta}), \partial\mathcal{O}) > 2\eta$ and we suppose that $h < \eta$. Therefore, we have that

$$\operatorname{dist}(B(\boldsymbol{\zeta}), \partial\mathcal{O}) > 2h. \tag{4.1}$$

Notice that, by definition, if $q \in M_h(\boldsymbol{\zeta})$, then $q = 0$ in $B(\boldsymbol{\zeta})$. Since q is a P_1 function in each triangle, it follows that $q_{|A_h} = 0$, where

$$A_h = \bigcup_{\substack{T \in \mathcal{T}_h \\ \overset{\circ}{T} \cap \overset{\circ}{B}(\boldsymbol{\zeta}) \neq \emptyset}} T.$$

Moreover, if we denote by Q_h the union of all triangles $T \in \mathcal{T}_h$ such that the three vertices of T are contained in $\overline{A_h}$, then, by using again the fact that q is a P_1 function in each triangle, it follows that

$$q_{|Q_h} = 0 \qquad \forall\, q \in M_h(\boldsymbol{\zeta}).$$

A similar argument shows that

$$\mathbf{D}(\mathbf{u})_{|A_h} = 0 \qquad \forall\, \mathbf{u} \in \mathcal{K}_h(\boldsymbol{\zeta}).$$

In order to study the properties of the spaces $\mathcal{K}_h(\boldsymbol{\zeta})$ and $M_h(\boldsymbol{\zeta})$ defined above we divide the triangles in $\mathcal{T}_h$ into four categories. These categories are defined as follows (see Figure 1):

- $\mathcal{F}_1$ is the subset of $\mathcal{T}_h$ formed by all triangles $T \in \mathcal{T}_h$ such that $\overline{T} \subset B(\boldsymbol{\zeta})$.
- $\mathcal{F}_2$ is the subset formed by all triangles $T \in \mathcal{T}_h \setminus \mathcal{F}_1$ such that $\overline{T} \subset \overline{Q}_h$.
- $\mathcal{F}_3$ is the subset formed by all triangles $T \in \mathcal{T}_h$ such that $\overline{T} \cap \overline{Q_h} \neq \emptyset$ and $T \not\subset \overline{Q_h}$.
- $\mathcal{F}_4 = \mathcal{T}_h \setminus (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$.

LEMMA 4.1. *There exists a positive constant C_1 (not depending on the position of $B(\boldsymbol{\zeta})$) such that*

$$\begin{aligned} \inf_{\mathbf{v}_h \in \mathcal{K}_h(\boldsymbol{\zeta})} \|\mathbf{v} - \mathbf{v}_h\|_{L^2(\mathcal{O})} &\leqslant C_1 h^{\frac{3}{2}} \left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))} \right), \\ \inf_{\mathbf{v}_h \in \mathcal{K}_h(\boldsymbol{\zeta})} \|\mathbf{v} - \mathbf{v}_h\|_{\mathcal{H}^1(\mathcal{O})} &\leqslant C_1 \sqrt{h} \left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))} \right), \end{aligned}$$

for all $\mathbf{v} \in \mathcal{K}(\boldsymbol{\zeta}) \cap \mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))$.

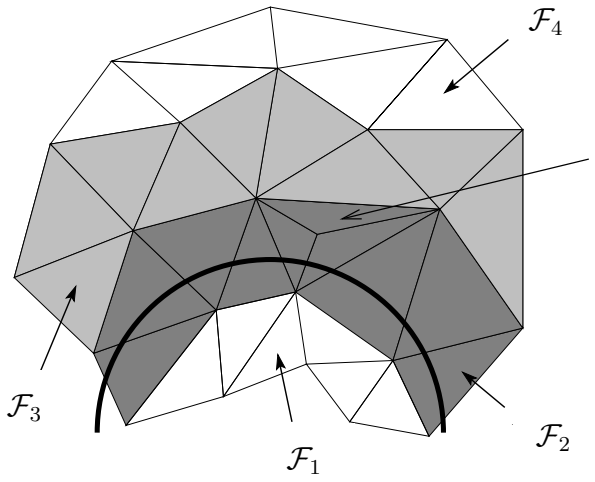

FIG. 1. *Splitting of the triangulation into four families of triangles.*

Proof. Let $\mathbf{v} \in \mathcal{K}(\boldsymbol{\zeta}) \cap \mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))$. This means, in particular, that

$$\mathbf{v}(\mathbf{x}) = \mathbf{l} + \omega \mathbf{x}^\perp \qquad \forall\, \mathbf{x} \in B(\boldsymbol{\zeta}),$$

for some $\mathbf{l} \in \mathbb{R}^2$ and $\omega \in \mathbb{R}$. In the remaining part of this section we denote

$$\mathbf{R}(\mathbf{x}) = \mathbf{l} + \omega \mathbf{x}^\perp \qquad \forall\, \mathbf{x} \in \mathbb{R}^2.$$

We denote by $\mathbf{v}_{Ih}$ the unique function in $(E_h)^2$ which agrees with $\mathbf{v}$ at every node $\mathbf{x}_j$ of the triangulation $\mathcal{T}_h$ (recall the definition of E_h in (3.1)). Then we consider the function $\mathbf{v}_h \in (E_h)^2$ whose value in a node $\mathbf{x_j}$ of $\mathcal{T}_h$ is defined by

$$\mathbf{v_h}(\mathbf{x_j}) = \begin{cases} \mathbf{R}(\mathbf{x_j}) & \text{if} \quad \mathbf{x_j} \in \overline{A_h}, \\ \mathbf{v}_{Ih}(\mathbf{x_j}) & \text{if} \quad \mathbf{x_j} \notin \overline{A_h}. \end{cases}$$

Since $\mathbf{v_h}$ is affine in each triangle, it follows that

$$\mathbf{v_h}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) \qquad \forall\, \mathbf{x} \in \overline{Q_h}. \tag{4.2}$$

We will show that there exists a positive constant C_1 (not depending on the position of $B(\boldsymbol{\zeta})$) such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C_1 h^{\frac{3}{2}} \left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))} \right), \tag{4.3}$$

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathcal{H}^1(\mathcal{O})} \leqslant C_1 \sqrt{h} \left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))} \right). \tag{4.4}$$

In order to prove the above inequalities, we divide the domain $\mathcal{O}$ into four parts:

$$\mathcal{O} = B(\boldsymbol{\zeta}) \cup (Q_h \setminus B(\boldsymbol{\zeta})) \cup \left(\bigcup_{T \in \mathcal{F}_3} T \right) \cup \left(\bigcup_{T \in \mathcal{F}_4} T \right).$$

Let us first remark that

$$\mathbf{v} = \mathbf{R} \ \text{ in } B(\boldsymbol{\zeta}). \tag{4.5}$$

On the other hand it is clear that Q_h is contained in the closed ball of center $\boldsymbol{\zeta}$ and radius $1+h$, denoted by $B_h(\boldsymbol{\zeta})$. Let us remark that the ball $B_h(\boldsymbol{\zeta})$ is included in the domain $\mathcal{O}$ due to condition (4.1). According to a classical result (see, for instance, Lemma 5.11 in Fujita and Sauer [10]), there exists a universal constant $C>0$, such that for all $\boldsymbol{\varphi} \in \mathcal{H}^1(\mathcal{O} \setminus B(\boldsymbol{\zeta}))$,

$$\|\boldsymbol{\varphi}\|_{\mathcal{L}^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))} \leq C\left(\sqrt{h}\|\boldsymbol{\varphi}\|_{\mathcal{L}^2(\partial B(\boldsymbol{\zeta}))} + h\|\nabla\boldsymbol{\varphi}\|_{[L^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))]^4}\right). \tag{4.6}$$

The above relation with $\boldsymbol{\varphi} = \mathbf{v} - \mathbf{R}$ and (4.5) imply that

$$\|\mathbf{v}-\mathbf{R}\|_{\mathcal{L}^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))} \leqslant Ch\,\|\nabla(\mathbf{v}-\mathbf{R})\|_{[L^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))]^4}. \tag{4.7}$$

By again applying Lemma 5.11 in [10] (this time for the function $\nabla(\mathbf{v}-\mathbf{R})$), we obtain that

$$\|\nabla(\mathbf{v}-\mathbf{R})\|_{[L^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))]^4} \leqslant C\Big(\sqrt{h}\,\|\nabla(\mathbf{v}-\mathbf{R})\|_{[L^2(\partial B(\boldsymbol{\zeta}))]^4} + h\,\|\nabla(\mathbf{v}-\mathbf{R})\|_{[H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))]^4}\Big).$$

The above inequality, combined with the trace theorem in Sobolev spaces, gives that

$$\|\nabla(\mathbf{v}-\mathbf{R})\|_{[L^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))]^4} \leq C\sqrt{h}\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.8}$$

From (4.7) and (4.8) it follows that

$$\|\mathbf{v}-\mathbf{R}\|_{\mathcal{L}^2(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))} \leqslant Ch^{\frac{3}{2}}\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.9}$$

The above relation implies, by using the fact that $Q_h \subset B_h(\boldsymbol{\zeta})$ and (4.2), that

$$\|\mathbf{v}-\mathbf{v_h}\|_{\mathcal{L}^2(Q_h\setminus B(\boldsymbol{\zeta}))} \leq Ch^{\frac{3}{2}}\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.10}$$

Consequently, we have that

$$\|\mathbf{v}-\mathbf{v}_h\|_{\mathcal{L}^2(Q_h\setminus B(\boldsymbol{\zeta}))} \leqslant C_1 h^{\frac{3}{2}}\left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))}\right). \tag{4.11}$$

On the other hand, (4.8) and (4.9) imply that

$$\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^1(B_h(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))} \leq Ch^{\frac{1}{2}}\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}.$$

The above relation implies, by using the fact that $Q_h \subset B_h(\boldsymbol{\zeta})$ and (4.2), that

$$\|\mathbf{v}-\mathbf{v_h}\|_{\mathcal{H}^1(Q_h\setminus B(\boldsymbol{\zeta}))} \leq Ch^{\frac{1}{2}}\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))},$$

which clearly implies

$$\|\mathbf{v}-\mathbf{v_h}\|_{\mathcal{H}^1(Q_h\setminus B(\boldsymbol{\zeta}))} \leq C\sqrt{h}\left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))}\right). \tag{4.12}$$

Let us now consider a triangle $T \in \mathcal{F}_3$. In order to estimate the restriction of $\mathbf{v}-\mathbf{v_h}$ to T we use the interpolating function $\mathbf{v}_{Ih}$. More precisely, we have

$$\|\mathbf{v}-\mathbf{v}_h\|_\alpha \leq \|\mathbf{v}-\mathbf{v}_{Ih}\|_\alpha + \|\mathbf{v}_{Ih}-\mathbf{v}_h\|_\alpha, \quad \alpha \in \{0,1\}, \tag{4.13}$$

where $\|\cdot\|_\alpha$ stands for the $\mathcal{L}^2$-norm or the $\mathcal{H}^1$-norm on T. We first estimate the second term in the right-hand side of (4.13). Since the function $\mathbf{v}_{Ih} - \mathbf{v}_h$ is affine in T, we have

$$\mathbf{v}_{Ih}(\mathbf{x}) - \mathbf{v}_h(\mathbf{x}) = \sum_{i=1}^{3} \left(\mathbf{v}_{Ih}(\mathbf{x}_i) - \mathbf{v}_h(\mathbf{x}_i)\right) \varphi_i(x),$$

where $(\mathbf{x}_i)$ are the nodes of T and (φ_i) are the corresponding Lagrange barycentric functions. We have

$$\|\mathbf{v}_{Ih} - \mathbf{v}_h\|_\alpha \leq \sum_{i=1}^{3} |\mathbf{v}_{Ih}(\mathbf{x}_i) - \mathbf{v}_h(\mathbf{x}_i)| \, \|\varphi_i\|_\alpha. \tag{4.14}$$

A simple calculation shows that

$$\|\varphi_i\|_{L^2(T)} \leq Ch \tag{4.15}$$

and

$$\|\nabla\varphi_i\|_{L^2(T)} \leq C. \tag{4.16}$$

Since the mesh is quasi-uniform, the constant C can be chosen independent of the triangle. We now estimate $|\mathbf{v}_{Ih}(\mathbf{x}_i) - \mathbf{v}_h(\mathbf{x}_i)|$. Since $T \not\subset Q_h$, it follows that T has at most two nodes in Q_h and, consequently, at least one node such that $\mathbf{v}_{Ih}(\mathbf{x}_i) - \mathbf{v}_h(\mathbf{x}_i) = 0$. Therefore we tackle only the nodes in Q_h. If $\mathbf{x}_i$ is a node in Q_h, then

$$|\mathbf{v}_{Ih}(\mathbf{x}_i) - \mathbf{v}_h(\mathbf{x}_i)| = |\mathbf{v}(\mathbf{x}_i) - \mathbf{R}(\mathbf{x}_i)|. \tag{4.17}$$

Relations (4.14), (4.15), and (4.17) imply that

$$\begin{aligned}\|\mathbf{v}_{Ih} - \mathbf{v}_h\|_{\mathcal{L}^2(T)} &\leqslant Ch \, \|\mathbf{v} - \mathbf{R}\|_{\mathcal{L}^\infty(T)} \\ &\leqslant Ch \left(\|\mathbf{v} - \mathbf{v}_{Ih}\|_{\mathcal{L}^\infty(T)} + \|\mathbf{v}_{Ih} - \mathbf{R}\|_{\mathcal{L}^\infty(T)}\right).\end{aligned}$$

By using a classical interpolation error (see, for example, [2, Corollary 4.4.7]) and an inverse estimate (see, for example, [2, Lemma 4.5.3]), the above inequality yields

$$\|\mathbf{v}_{Ih} - \mathbf{v}_h\|_{\mathcal{L}^2(T)} \leqslant Ch \left(h \, \|\mathbf{v}\|_{\mathcal{H}^2(T)} + h^{-1} \, \|\mathbf{v}_{Ih} - \mathbf{R}\|_{\mathcal{L}^2(T)}\right),$$

which implies that

$$\begin{aligned}\|\mathbf{v}_{Ih} - \mathbf{v}_h\|_{\mathcal{L}^2(T)} &\leqslant C \left(h^2 \, \|\mathbf{v}\|_{\mathcal{H}^2(T)} + \|\mathbf{v}_{Ih} - \mathbf{v}\|_{\mathcal{L}^2(T)} + \|\mathbf{v} - \mathbf{R}\|_{\mathcal{L}^2(T)}\right) \\ &\leqslant C \left(h^2 \, \|\mathbf{v}\|_{\mathcal{H}^2(T)} + \|\mathbf{v} - \mathbf{R}\|_{\mathcal{L}^2(T)}\right).\end{aligned}$$

Above we have used again a classical result on the interpolation error (see, for example, [2, Theorem 4.4.4]).

Now, summing up the above relation for all triangles $T \in \mathcal{F}_3$ we obtain that

$$\|\mathbf{v}_{Ih} - \mathbf{v}_h\|_{\mathcal{L}^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)} \leqslant C \left(h^2 \, \|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))} + \|\mathbf{v} - \mathbf{R}\|_{\mathcal{L}^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)}\right). \tag{4.18}$$

In order to estimate the last term in the right-hand side of (4.18) we proceed as previously by introducing the closed ball $B_{2h}(\boldsymbol{\zeta})$ of center $\boldsymbol{\zeta}$ and radius $1+2h$. This ball is included in $\mathcal{O}$ thanks to (4.1). It is clear that all triangles of $\mathcal{F}_3$ are contained in $B_{2h}(\boldsymbol{\zeta}) \setminus B(\boldsymbol{\zeta})$. Then we can once again use Lemma 5.11 in Fujita and Sauer [10] and prove an estimate similar to (4.9), namely,

$$\|\mathbf{v}-\mathbf{R}\|_{\mathcal{L}^2(B_{2h}(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))} \leqslant Ch^{\frac{3}{2}}\|\mathbf{v}-\mathbf{R}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.19}$$

From (4.18) and (4.19) we deduce that

$$\|\mathbf{v}_{Ih}-\mathbf{v}_h\|_{\mathcal{L}^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)} \leqslant Ch^{\frac{3}{2}}\left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}+\|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))}\right). \tag{4.20}$$

The above relation, combined with (4.13) and with an interpolation error estimate (see [2, Theorem 4.4.4]), implies that

$$\|\mathbf{v}-\mathbf{v}_h\|_{\mathcal{L}^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)} \leqslant Ch^{\frac{3}{2}}\left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}+\|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))}\right). \tag{4.21}$$

Now we turn to the H^1-estimate for the family $\mathcal{F}_3$ of triangles. From the usual inverse inequality (see [2, Lemma 4.5.3]) and the L^2-estimate (4.20) we obtain

$$\|\nabla(\mathbf{v}_{Ih}-\mathbf{v}_h)\|^4_{\left[L^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)\right]} \leq C_1h^{\frac{1}{2}}\left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}+\|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))}\right), \tag{4.22}$$

which implies, together with (4.13) and an interpolation error estimate (see [2, Theorem 4.4.4]), that

$$\|\nabla(\mathbf{v}-\mathbf{v}_h)\|^4_{\left[L^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)\right]} \leq C_1h^{\frac{1}{2}}\left(\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}+\|\mathbf{v}\|_{\mathcal{H}^2(B(\boldsymbol{\zeta}))}\right). \tag{4.23}$$

Finally, we consider the case of the triangle family $\mathcal{F}_4$. Interpolation error estimates lead to

$$\|\mathbf{v}-\mathbf{v}_h\|_{L^2\left(\bigcup_{T\in\mathcal{F}_4} T\right)} \leq C_1h^2\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))} \tag{4.24}$$

and

$$\|\nabla(\mathbf{v}-\mathbf{v}_h)\|_{L^2\left(\bigcup_{T\in\mathcal{F}_4} T\right)} \leq C_1h\|\mathbf{v}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.25}$$

Relations (4.11), (4.21), (4.24) and the fact that $\mathbf{v}=\mathbf{v}_h$ in $B(\boldsymbol{\zeta})$ imply (4.3). Moreover, (4.12), (4.23), (4.25) and the fact that $\mathbf{v}=\mathbf{v}_h$ in $B(\boldsymbol{\zeta})$ imply (4.4). ∎

LEMMA 4.2. *There exists a positive constant C_2 (independent of the position of $B(\boldsymbol{\zeta})$) such that*

$$\inf_{q_h\in M_h(\boldsymbol{\zeta})}\|q-q_h\|_{L^2(\mathcal{O})} \leqslant C_2h^{\frac{1}{2}}\|q\|_{H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))} \tag{4.26}$$

for all $q\in M(\boldsymbol{\zeta})\cap H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))$.

Proof. The proof of this lemma is similar to that of Lemma 4.1. Consider a function $q \in M(\boldsymbol{\zeta}) \cap H^1(\mathcal{O} \setminus B(\boldsymbol{\zeta}))$. According to a classical result (see, for example, [3, Theorem IX.7]), there exists $\widetilde{q} \in H^1(\mathcal{O})$ such that

$$\widetilde{q}_{|\mathcal{O}\setminus B(\boldsymbol{\zeta})} = q, \quad \|\widetilde{q}\|_{H^1(\mathcal{O})} \leq C\|q\|_{H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}, \tag{4.27}$$

and it can be proved that we can choose the constant C independent of the position of $B(\boldsymbol{\zeta})$. Moreover, by a classical interpolation argument (see, for example, [2, Theorem 4.4.4]), there exists $\widetilde{q}_h \in E_h$ such that

$$\|\widetilde{q} - \widetilde{q}_h\|_{L^2(\mathcal{O})} \leq Ch\|\widetilde{q}\|_{H^1(\mathcal{O})}.$$

The above relation and (4.27) clearly imply that there exists a constant $C > 0$ such that

$$\|q - \widetilde{q}_h\|_{L^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}))} \leq Ch\|q\|_{H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.28}$$

Denote by q_h the function in E_h satisfying the conditions

$$q_h(\mathbf{x}_i) = 0 \ \text{ if } \ \mathbf{x}_i \in \overline{A_h},$$

$$q_h(\mathbf{x}_i) = \widetilde{q}_h(\mathbf{x}_i) \ \text{ if } \ \mathbf{x}_i \in \mathcal{T}_h \setminus \overline{A_h}.$$

Then as in the proof of Lemma 4.1, we can show that

$$\|q - q_h\|_{L^2(\mathcal{O})} \leqslant C_2 h^{\frac{1}{2}} \|q\|_{H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \qquad \square$$

We next show that the finite element spaces $\mathcal{K}_h(\boldsymbol{\zeta})$, $M_h(\boldsymbol{\zeta})$ and the bilinear form b satisfy a discrete *inf-sup* condition. This proves in particular that the approximate problem (3.5)–(3.6) is well-posed (see [12, Theorem II.1.1., p. 114]). More precisely, the following result holds.

LEMMA 4.3. *There exists a constant $\beta^* > 0$ such that for all $q_h \in M_h(\boldsymbol{\zeta})$ there exists $\mathbf{u}_h \in \mathcal{K}_h(\boldsymbol{\zeta})$ with*

$$\int_{\mathcal{O}} \operatorname{div}(\mathbf{u}_h) q_h \, \mathrm{d}\mathbf{x} \geqslant \beta^* \|\mathbf{u}_h\|_{\mathcal{H}^1(\mathcal{O})} \|q_h\|_{L^2(\mathcal{O})}. \tag{4.29}$$

Proof. Let $q_h \in M_h(\boldsymbol{\zeta})$. Since $M_h(\boldsymbol{\zeta}) \subset M(\boldsymbol{\zeta})$, Lemma 2.6 yields the existence of $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta})$ such that

$$\int_{\mathcal{O}} \operatorname{div}(\mathbf{u}) q_h \, \mathrm{d}\mathbf{x} \geqslant \beta \|\mathbf{u}\|_{\mathcal{H}^1(\mathcal{O})} \|q_h\|_{L^2(\mathcal{O})},$$

with β independent of q_h. In order to prove the conclusion of the lemma it suffices to show the existence of $\mathbf{u}_h \in \mathcal{K}_h(\boldsymbol{\zeta})$ such that

$$\int_{\mathcal{O}} \operatorname{div}(\mathbf{u}_h) q_h \, \mathrm{d}\mathbf{x} = \int_{\mathcal{O}} \operatorname{div}(\mathbf{u}) q_h \, \mathrm{d}\mathbf{x}, \tag{4.30}$$

$$\|\mathbf{u}_h\|_{\mathcal{H}^1(\mathcal{O})} \leqslant C\|\mathbf{u}\|_{\mathcal{H}^1(\mathcal{O})}, \tag{4.31}$$

where C is a constant independent of q_h.

Note that (4.30) is equivalent to

$$\int_{\mathcal{O}} \mathbf{u}_h \cdot \nabla q_h \, \mathrm{d}\mathbf{x} = \int_{\mathcal{O}} \mathbf{u} \cdot \nabla q_h \, \mathrm{d}\mathbf{x}.$$

Since ∇q_h is constant in each triangle and vanishes in any triangle from $\mathcal{F}_1 \cup \mathcal{F}_2$, in order to check (4.30), it suffices to show that

$$\int_T \mathbf{u}_h \, \mathrm{d}\mathbf{x} = \int_T \mathbf{u} \, \mathrm{d}\mathbf{x} \qquad \forall\, T \in \mathcal{F}_3 \cup \mathcal{F}_4. \tag{4.32}$$

Note first that if $\mathbf{u}_h \in \mathcal{K}_h(\boldsymbol{\zeta})$, then for any triangle $T \in \mathcal{T}_h$ of vertices $\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3$ and of corresponding barycentric functions $\varphi_1,\ \varphi_2,\ \varphi_3$, we have

$$\mathbf{u}_h(\mathbf{x}) = \overline{\mathbf{u}}_h(x) + \frac{\varphi_1(\mathbf{x})\varphi_2(\mathbf{x})\varphi_3(\mathbf{x})}{\displaystyle\int_T \varphi_1\varphi_2\varphi_3 \, \mathrm{d}\mathbf{x}} \boldsymbol{\lambda} \qquad \forall\, \mathbf{x} \in T, \tag{4.33}$$

where $\overline{\mathbf{u}}_h \in \mathcal{C}(\overline{\mathcal{O}})$ satisfies

$$\overline{\mathbf{u}}_h(x) = \varphi_1(\mathbf{x})\boldsymbol{\alpha_1} + \varphi_2(\mathbf{x})\boldsymbol{\alpha_2} + \varphi_3(\mathbf{x})\boldsymbol{\alpha_3} \qquad \forall\, \mathbf{x} \in T, \tag{4.34}$$

for some constant vectors $\boldsymbol{\alpha_1},\ \boldsymbol{\alpha_2},\ \boldsymbol{\alpha_3},\ \boldsymbol{\lambda} \in \mathbb{R}^2$ (these constants depend on the triangle T). Notice that, since the restriction of $\mathbf{u}_h$ to triangles in $\mathcal{F}_1 \cup \mathcal{F}_2$ is a rigid velocity field, the constant $\boldsymbol{\lambda}$ in (4.33) is equal to zero for all triangles in $\mathcal{F}_1 \cup \mathcal{F}_2$. If $\overline{\mathbf{u}}_h$ satisfies (4.34) and $T \in \mathcal{F}_3 \cup \mathcal{F}_4$, then condition (4.32) holds provided that

$$\lambda = \int_T (\mathbf{u} - \overline{\mathbf{u}}_h) \, \mathrm{d}\mathbf{x} \qquad \forall\, T \in \mathcal{F}_3 \cup \mathcal{F}_4. \tag{4.35}$$

Some simple calculations show that there exists a constant $C > 0$ (independent of the triangle) such that

$$\left\| \frac{\varphi_1\varphi_2\varphi_3}{\displaystyle\int_T \varphi_1\varphi_2\varphi_3 \, \mathrm{d}\mathbf{x}} \right\|_{H^1(T)} \leqslant \frac{C}{h^2}. \tag{4.36}$$

Moreover, (4.35) and the Cauchy–Schwarz inequality imply that

$$|\lambda| \leqslant Ch \|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}^2(T)} \qquad \forall\, T \in \mathcal{F}_3 \cup \mathcal{F}_4, \tag{4.37}$$

for some constant C. From (4.33), (4.36), and (4.37) it follows that

$$\|\mathbf{u}_h\|_{\mathcal{H}^1(T)} \leqslant \|\overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{C}{h}\|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}^2(T)} \qquad \forall\, T \in \mathcal{F}_3 \cup \mathcal{F}_4. \tag{4.38}$$

The remaining part of the proof is devoted to the construction of $\overline{\mathbf{u}}_h$ such that $\mathbf{u}_h$ satisfies (4.31). According to a classical result (see, for instance, [12, Theorem I.A.2., p. 101]), there exists a function $\overline{\mathbf{u}}_h^c \in C(\overline{\mathcal{O}})$ which is affine in each triangle $T \in \mathcal{T}_h$ such that

$$\|\mathbf{u} - \overline{\mathbf{u}}_h^c\|_{\mathcal{L}^2(T)} \leq Ch\|\mathbf{u}\|_{\mathcal{H}^1(T)}, \tag{4.39}$$

$$\|\overline{\mathbf{u}}_h^c\|_{\mathcal{H}^1(T)} \leq C\|\mathbf{u}\|_{\mathcal{H}^1(T)}, \tag{4.40}$$

with the constant C independent of h. We are now in a position to define $\overline{\mathbf{u}}_h$. This function is defined by

$$\overline{\mathbf{u}_h}(\mathbf{x}) = \begin{cases} \overline{\mathbf{u}}_h^c(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \displaystyle\bigcup_{T\in\mathcal{F}_4} T, \\ \mathbf{R}(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \displaystyle\bigcup_{T\in\mathcal{F}_1\cup\mathcal{F}_2} T, \end{cases} \tag{4.41}$$

where $\mathbf{R}$ is the extension of $\mathbf{u}_{|B(\boldsymbol{\zeta})}$ (which is a rigid velocity field) to $\mathbb{R}^2$. We remark that relation (4.41) also defines the values of $\overline{\mathbf{u}}_h$ in the triangles of $\mathcal{F}_3$. Indeed, the vertices of each triangle in $\mathcal{F}_3$ are also vertices of a triangle in either $\mathcal{F}_2$ or in $\mathcal{F}_4$. In order to prove (4.31) we estimate the terms in the right-hand side of (4.38). We first consider a triangle $T \in \mathcal{F}_4$. By using the fact that $\overline{\mathbf{u}}_h = \overline{\mathbf{u}}_h^c$ in T, (4.39), and (4.40), we obtain that

$$\|\overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}^2(T)} \leq C\|\mathbf{u}\|_{\mathcal{H}^1(T)} \qquad \forall\, T \in \mathcal{F}_4, \tag{4.42}$$

with the constant C independent of $\mathbf{u}$. We next consider a triangle $T \in \mathcal{F}_3$. We first notice that

$$\begin{aligned} \|\overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}^2(T)} &\leq \|\overline{\mathbf{u}}_h^c\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\mathbf{u} - \overline{\mathbf{u}}_h^c\|_{\mathcal{L}^2(T)} \\ &\quad + \|\overline{\mathbf{u}}_h^c - \overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\overline{\mathbf{u}}_h - \overline{\mathbf{u}}_h^c\|_{\mathcal{L}^2(T)} \qquad \forall\, T \in \mathcal{F}_3. \end{aligned} \tag{4.43}$$

The first two terms in the right-hand side of (4.43) can be directly estimated by using (4.39) and (4.40). Moreover, by using inverse estimates (see, for example, [2, Lemma 4.5.3]), there exists a positive constant C independent of h such that

$$\|\overline{\mathbf{u}}_h^c - \overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\overline{\mathbf{u}}_h - \overline{\mathbf{u}}_h^c\|_{\mathcal{L}^2(T)} \leqslant C\,\|\overline{\mathbf{u}}_h^c - \overline{\mathbf{u}}_h\|_{\mathcal{L}^\infty(T)} \qquad \forall\, T \in \mathcal{F}_3.$$

The above relation and the fact that $\overline{\mathbf{u}}_h$ is equal either to $\mathbf{R}$ or to $\overline{\mathbf{u}}_h^c$ in the vertices of a triangle $T \in \mathcal{F}_3$ imply that

$$\|\overline{\mathbf{u}}_h^c - \overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\mathbf{u}_h - \overline{\mathbf{u}}_h^c\|_{\mathcal{L}^2(T)} \leq C\,\|\overline{\mathbf{u}}_h^c - \mathbf{R}\|_{\mathcal{L}^\infty(T)} \qquad \forall\, T \in \mathcal{F}_3.$$

The above inequality, combined once again with an inverse inequality, implies that

$$\|\overline{\mathbf{u}}_h^c - \overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\mathbf{u}_h - \overline{\mathbf{u}}_h^c\|_{\mathcal{L}^2(T)} \leq \frac{C}{h}\,\|\overline{\mathbf{u}}_h^c - \mathbf{R}\|_{\mathcal{L}^2(T)} \qquad \forall\, T \in \mathcal{F}_3. \tag{4.44}$$

On the other hand,

$$\|\overline{\mathbf{u}}_h^c - \mathbf{R}\|_{\mathcal{L}^2(T)} \leq \|\overline{\mathbf{u}}_h^c - \mathbf{u}\|_{\mathcal{L}^2(T)} + \|\mathbf{u} - \mathbf{R}\|_{\mathcal{L}^2(T)} \qquad \forall\, T \in \mathcal{F}_3. \tag{4.45}$$

Combining (4.39), (4.45), (4.44), and (4.43), we obtain

$$\|\overline{\mathbf{u}}_h\|_{\mathcal{H}^1(T)} + \frac{1}{h}\|\mathbf{u} - \overline{\mathbf{u}}_h\|_{\mathcal{L}^2(T)} \leq C\|\mathbf{u}\|_{\mathcal{H}^1(T)} + \frac{C}{h}\|\mathbf{u} - \mathbf{R}\|_{\mathcal{L}^2(T)} \qquad \forall\, T \in \mathcal{F}_3. \tag{4.46}$$

We recall that all triangles of $\mathcal{F}_3$ are contained in $B_{2h}(\boldsymbol{\zeta}) \setminus B(\boldsymbol{\zeta})$. Therefore, by taking the sum of the above relations for all $T \in \mathcal{F}_3$ and by using (4.6), combined with the fact that $\mathbf{u} = \mathbf{R}$ on $\partial B(\boldsymbol{\zeta})$, we obtain

$$\|\overline{\mathbf{u}}_h\|_{\mathcal{H}^1\left(\bigcup_{T\in\mathcal{F}_3} T\right)} + \frac{1}{h}\|\mathbf{u}-\overline{\mathbf{u}}_h\|_{\mathcal{L}^2\left(\bigcup_{T\in\mathcal{F}_3} T\right)} \leq C\|\mathbf{u}\|_{\mathcal{H}^1(B_{2h}(\boldsymbol{\zeta})\setminus B(\boldsymbol{\zeta}))}. \tag{4.47}$$

Now by combining (4.42) and (4.47) in (4.38), we obtain

$$\|\mathbf{u}_h\|_{\mathcal{H}^1\left(\bigcup_{T\in\mathcal{F}_3\cup\mathcal{F}_4} T\right)} \leq C\|\mathbf{u}\|_{\mathcal{H}^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}))}. \tag{4.48}$$

We next consider the triangles $T \in \mathcal{F}_1 \cup \mathcal{F}_2$. By using the fact that $\mathbf{u}_h = \overline{\mathbf{u}}_h = \mathbf{R}$ in T, we obtain that

$$\|\mathbf{u}_h\|_{\mathcal{H}^1\left(\bigcup_{T\in\mathcal{F}_1\cup\mathcal{F}_2} T\right)} = \|\mathbf{R}\|_{\mathcal{H}^1\left(\bigcup_{T\in\mathcal{F}_1\cup\mathcal{F}_2} T\right)}.$$

A simple calculation shows that the right-hand side of the above relation is bounded by $C\|\mathbf{u}\|_{\mathcal{H}^1(B(\boldsymbol{\zeta}))}$, where C is a constant independent of h. We thus obtain

$$\|\mathbf{u}_h\|_{\mathcal{H}^1\left(\bigcup_{T\in\mathcal{F}_1\cup\mathcal{F}_2} T\right)} \leq C\|\mathbf{u}\|_{\mathcal{H}^1(B(\boldsymbol{\zeta}))}. \tag{4.49}$$

If we join (4.48) and (4.49), we see that the function $\mathbf{u}_h$ satisfies (4.31). This concludes the proof of the lemma. ∎

Now, we are in position to introduce a projector in $\mathcal{K}_h(\boldsymbol{\zeta}) \times M_h(\boldsymbol{\zeta})$ that will be a key ingredient in the proof of the convergence result.

LEMMA 4.4. *Suppose that $\mathbf{V} \in \mathcal{K}(\boldsymbol{\zeta})$ and that $P \in M(\boldsymbol{\zeta})$. Then there exists a unique couple $(\mathbf{V}_h, P_h)$ in $\mathcal{K}_h(\boldsymbol{\zeta}) \times M_h(\boldsymbol{\zeta})$ such that*

$$\left\{\begin{array}{rcll} a(\mathbf{V}-\mathbf{V}_h, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P-P_h) &=& 0 & \forall\, \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}), \\ b(\mathbf{V}-\mathbf{V}_h, q) &=& 0 & \forall\, q \in M_h(\boldsymbol{\zeta}). \end{array}\right. \tag{4.50}$$

Moreover, if we suppose in addition that $\mathbf{V}_{|\mathcal{O}\setminus B(\boldsymbol{\zeta})} \in \mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}))$ and that $P_{|\mathcal{O}\setminus B(\boldsymbol{\zeta})} \in H^1(\mathcal{O} \setminus B(\boldsymbol{\zeta}))$, then there exists a positive constant C such that

$$\|\mathbf{V}-\mathbf{V}_h\|_{\mathcal{L}^2(\mathcal{O})} \leqslant Ch.$$

Proof. The result in Lemma 4.3 combined with Theorem 1.1 in [12, p. 114] implies the existence and uniqueness of $(\mathbf{V}_h, P_h)$ in $\mathcal{K}_h(\boldsymbol{\zeta}) \times M_h(\boldsymbol{\zeta})$, satisfying (4.50) together with

$$\|\mathbf{V}-\mathbf{V}_h\|_{\mathcal{H}^1(\mathcal{O})} + \|P-P_h\|_{L^2(\mathcal{O})} \leqslant C\left\{\inf_{\mathbf{v}\in\mathcal{K}_h(\boldsymbol{\zeta})} \|\mathbf{V}-\mathbf{v}\|_{\mathcal{H}^1(\mathcal{O})} + \inf_{q\in M_h(\boldsymbol{\zeta})} \|P-q\|_{L^2(\mathcal{O})}\right\}.$$

Using Lemmas 4.1 and 4.2, we obtain

$$\|\mathbf{V}-\mathbf{V}_h\|_{\mathcal{H}^1(\mathcal{O})} + \|P-P_h\|_{L^2(\mathcal{O})} \leqslant Ch^{1/2}\left\{\|\mathbf{V}\|_{\mathcal{H}^2(\mathcal{O}\setminus B)} + \|\mathbf{V}\|_{\mathcal{H}^2(B)} + \|P\|_{H^1(\mathcal{O})}\right\}.$$

Moreover, by applying the usual Aubin–Nitsche duality argument (see, for example, [12, p. 119]), one can easily prove

$$\|\mathbf{V}-\mathbf{V}_h\|_{\mathcal{L}^2(\mathcal{O})} \leqslant Ch\left\{\|\mathbf{V}\|_{\mathcal{H}^2(\mathcal{O}\setminus B)} + \|\mathbf{V}\|_{\mathcal{H}^2(B)} + \|P\|_{H^1(\mathcal{O})}\right\}. \qquad ∎$$

5. Definition and properties of the change of variables. In order to prove Theorem 3.2, we should be able to compare the exact solution, which is rigid in the ball $B(\boldsymbol{\zeta}(t_k))$, with the approximate solution, which is rigid in the ball $B(\boldsymbol{\zeta}_h^k)$. This will be achieved by the use of a change of variables that maps the exact ball onto the approximate one. This section is devoted to the description and main properties of this transformation.

5.1. Change of variables. In this section, we suppose that $\mathcal{O}$ is convex. In what follows, we need a change of variables, transforming a function in $\widehat{\mathcal{K}}(\boldsymbol{\zeta}_1)$ into a function in $\widehat{\mathcal{K}}(\boldsymbol{\zeta}_2)$, where $\boldsymbol{\zeta}_i \in \mathcal{O}$ are such that

$$\operatorname{dist}(\boldsymbol{\zeta}_i, \partial\mathcal{O}) > 1 + 2\eta, \quad i \in \{1,2\}, \quad \text{with } \eta > 0. \tag{5.1}$$

In this case, $B(\boldsymbol{\zeta}_i)$ is contained in $\mathcal{O}$ and the distance between $B(\boldsymbol{\zeta}_i)$ and $\partial\mathcal{O}$ is greater than 2η. Let $\xi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ be a compactly supported function such that

- $\xi = 1$ if $\mathbf{x} \in \mathcal{O}$ and $\operatorname{dist}(\mathbf{x}, \partial\mathcal{O}) > 2\eta$,
- $\xi = 0$ if $\mathbf{x} \notin \mathcal{O}$ or $\operatorname{dist}(\mathbf{x}, \partial\mathcal{O}) \leqslant \eta$.

Let $\boldsymbol{\Lambda}$ be the mapping defined by

$$\boldsymbol{\Lambda}(\mathbf{x}) = \left[(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) \cdot \mathbf{x}^\perp\right] (\mathbf{rot}\ \xi) + \xi(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2) \quad \forall \mathbf{x} \in \mathbb{R}^2. \tag{5.2}$$

We need several properties of the field $\boldsymbol{\Lambda}$ and of the associated flow. Since these properties are similar to those proved in [27] we state them here without proof.

LEMMA 5.1. *Let $\boldsymbol{\Lambda}$ be the mapping defined by* (5.2). *Then we have*

(i) $\boldsymbol{\Lambda} = 0$ *outside* $\mathcal{O}$,
(ii) $\operatorname{div} \boldsymbol{\Lambda} = 0$ *in* $\mathbb{R}^2$,
(iii) $\boldsymbol{\Lambda}(\mathbf{x}) = \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2$ *if* $\mathbf{x} \in \mathcal{O}$ *and if* $\operatorname{dist}(\mathbf{x}, \partial\mathcal{O}) > 2\eta$.

In other words, the restriction of $\boldsymbol{\Lambda}$ to a neighborhood of $\partial\mathcal{O}$ is zero and $\boldsymbol{\Lambda}$ is a translation when restricted to points of $\mathcal{O}$ at distance to $\partial\mathcal{O}$ larger than 2η.

We consider next the initial value problem

$$\begin{cases} \dfrac{d}{d\lambda}\boldsymbol{\psi}(\lambda) = \boldsymbol{\Lambda}(\boldsymbol{\psi}(\lambda)), \quad \lambda > 0, \\ \boldsymbol{\psi}(0) = \mathbf{y}, \end{cases} \tag{5.3}$$

with $\boldsymbol{\Lambda}$ given by (5.2).

LEMMA 5.2. *For all $\mathbf{y} \in \mathbb{R}^2$, the initial value problem* (5.3) *admits a unique solution $\boldsymbol{\psi}(\lambda, \mathbf{y})$ on* $[0,1]$. *Denote*

$$\mathbf{X}_{\boldsymbol{\zeta}_2, \boldsymbol{\zeta}_1}(\mathbf{y}) = \mathbf{X}(\mathbf{y}) = \boldsymbol{\psi}(1, \mathbf{y}). \tag{5.4}$$

Then $\mathbf{X}$ is a C^∞-diffeomorphism from $\mathcal{O}$ onto itself, and $\mathbf{X}(B(\boldsymbol{\zeta}_2)) = B(\boldsymbol{\zeta}_1)$. If we denote by

$$\mathbf{J}_{\mathbf{X}} = \left(\frac{\partial X_i}{\partial y_j}\right)_{i,j}$$

the jacobian matrix of the transformation $\mathbf{y} \mapsto \mathbf{X}(\mathbf{y})$, then the above change of variables satisfies

$$\det \mathbf{J}_{\mathbf{X}}(\mathbf{y}) = 1 \quad \forall \mathbf{y} \in \mathbb{R}^2. \tag{5.5}$$

We denote by

$$\mathbf{Y}_{\boldsymbol{\zeta}_2, \boldsymbol{\zeta}_1} = \mathbf{Y} = \mathbf{X}^{-1} \tag{5.6}$$

the inverse of $\mathbf{X}$ on $\mathcal{O}$.

5.2. Properties of the change of variables. In this subsection, we use the change of variables defined by the mapping $\mathbf{X}$ in Lemma 5.2 to transform functions in $\widehat{\mathcal{K}}(\boldsymbol{\zeta}_1)$ (resp., $\mathcal{K}(\boldsymbol{\zeta}_1)$, $M(\boldsymbol{\zeta}_1)$) into functions in $\widehat{\mathcal{K}}(\boldsymbol{\zeta}_2)$ (resp., $\mathcal{K}(\boldsymbol{\zeta}_2)$, $M(\boldsymbol{\zeta}_2)$). We also give the expressions of $\Delta\mathbf{u}$ and ∇p after the transformation.

Consider $(\mathbf{u},\ p) \in \mathcal{H}^1(\mathcal{O}) \times L^2(\mathcal{O})$ and define as in [18] the functions $(\mathbf{U},\ P) \in \mathcal{H}^1(\mathcal{O}) \times L^2(\mathcal{O})$ by

$$\mathbf{U}(\mathbf{y}) = \mathbf{J}_{\mathbf{Y}}(\mathbf{X}(\mathbf{y}))\mathbf{u}(\mathbf{X}(\mathbf{y})) \quad \forall \mathbf{y} \in \mathcal{O}, \tag{5.7}$$

$$P(\mathbf{y}) = p(\mathbf{X}(\mathbf{y})) \quad \forall \mathbf{y} \in \mathcal{O}. \tag{5.8}$$

We can easily check, by using the definition of $\boldsymbol{\Lambda}$, that

$$\mathbf{X}(\mathbf{y}) = \mathbf{y} + \boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2 \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}_2), \tag{5.9}$$

$$\mathbf{Y}(\mathbf{x}) = \mathbf{x} - \boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2 \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}_1), \tag{5.10}$$

Consequently, if $\mathbf{u} \in \mathcal{K}(\boldsymbol{\zeta}_1)$, then $\mathbf{U} \in \mathcal{K}(\boldsymbol{\zeta}_2)$ and if $p \in M(\boldsymbol{\zeta}_1)$, then $P \in M(\boldsymbol{\zeta}_2)$.

By using (5.5), we obtain the following result (see, for instance, [18, Proposition 2.4]).

LEMMA 5.3. *If* $\mathbf{X}$ *is defined by* (5.4)*, then for all* $\mathbf{u} \in \mathcal{H}^1(\mathcal{O})$*, the function* $\mathbf{U}$ *defined as above satisfies the relation*

$$\operatorname{div}[\mathbf{U}(\mathbf{y})] = \operatorname{div}[\mathbf{u}(\mathbf{X}(\mathbf{y}))] \quad \forall \mathbf{y} \in \mathcal{O}.$$

This lemma implies in particular that if $\mathbf{u} \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}_1)$, then $\mathbf{U} \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}_2)$.

In order to write down the expressions of $\Delta\mathbf{u}$ and ∇p after the change of variables, we define (see [18])

$$[\mathbf{L}\mathbf{U}]_i = \sum_{j,k} \frac{\partial}{\partial y_j}\left(g^{jk}\frac{\partial U_i}{\partial y_k}\right) + 2\sum_{j,k,l} g^{kl}\Gamma^i_{jk}\frac{\partial U_j}{\partial y_l} + \sum_{j,k,l}\left\{\frac{\partial}{\partial y_k}(g^{kl}\Gamma^i_{jl}) + \sum_m g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right\}U_j, \tag{5.11}$$

$$[\mathbf{G}P]_i = \sum_{j=1}^{2} g^{ij}\frac{\partial P}{\partial y_j}, \tag{5.12}$$

where we denote (see, for instance, [7])

$$g^{ij} = \sum_k \frac{\partial Y_i}{\partial x_k}\frac{\partial Y_j}{\partial x_k} \quad \text{(metric contravariant tensor)}, \tag{5.13}$$

$$g_{ij} = \sum_k \frac{\partial X_k}{\partial y_i}\frac{\partial X_k}{\partial y_j} \quad \text{(metric covariant tensor)}, \tag{5.14}$$

and

$$\Gamma^k_{ij} = \frac{1}{2}\sum_l g^{kl}\left\{\frac{\partial g_{il}}{\partial y_j} + \frac{\partial g_{jl}}{\partial y_i} - \frac{\partial g_{ij}}{\partial y_l}\right\} \quad \text{(Christoffel symbol)}. \tag{5.15}$$

We are now in position to write down the expressions of $\Delta\mathbf{u}$ and ∇p after the change of variables (see again [18] for details).

PROPOSITION 5.4. *Suppose that*

$$(\mathbf{u}, p) \in \mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}_1)) \times H^1(\mathcal{O} \setminus B(\boldsymbol{\zeta}_1)).$$

Then, we have that

$$(\mathbf{U}, P) \in \mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}_2)) \times H^1(\mathcal{O} \setminus B(\boldsymbol{\zeta}_2)).$$

Moreover, for all $\mathbf{y} \in \mathcal{O} \setminus B(\boldsymbol{\zeta}_2)$*, we have that*

$$[\mathbf{LU}](\mathbf{y}) = \mathbf{J}_{\mathbf{Y}}(\mathbf{X}(\mathbf{y}))\ [(\Delta \mathbf{u}) \circ \mathbf{X}]\,(\mathbf{y}), \quad [\mathbf{G}P](\mathbf{y}) = \mathbf{J}_{\mathbf{Y}}(\mathbf{X}(\mathbf{y}))\ [(\nabla p) \circ \mathbf{X}]\,(\mathbf{y}).$$

In the remaining part of this section, we denote by C a positive constant which may depend only on ξ and $\mathcal{O}$. We give below (without proofs) several estimates of the dependence of the change of variables defined in (5.4) on the points $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$. For the proofs of these estimates, we refer to [27] and [28].

LEMMA 5.5. *Let* $\boldsymbol{\Lambda}$ *be the function defined by* (5.2)*. Then, for all* $\boldsymbol{\zeta}_1,\ \boldsymbol{\zeta}_2 \in \mathcal{O}$ *satisfying* (5.1) *we have*

$$\|\boldsymbol{\Lambda}\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,, \quad \|\nabla \boldsymbol{\Lambda}\|_{[L^\infty(\mathcal{O})]^4} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,,$$

$$\left\|\frac{\partial^2 \boldsymbol{\Lambda}}{\partial x_i \partial x_j}\right\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,, \quad \left\|\frac{\partial^3 \boldsymbol{\Lambda}}{\partial x_i \partial x_j \partial x_k}\right\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,.$$

LEMMA 5.6. *Let* $\boldsymbol{\Lambda}$*,* $\boldsymbol{\zeta}_1$*,* $\boldsymbol{\zeta}_2$ *be as in Lemma* 5.5*. Then the functions* $\mathbf{X}$ *and* $\mathbf{Y}$ *defined by* (5.4) *and* (5.6) *satisfy the following inequalities:*

$$\|\mathbf{X}\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C, \quad \|\mathbf{Y}\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C,$$

$$\|\mathbf{J}_{\mathbf{X}} - \mathbf{Id}\|_{[L^\infty(\mathcal{O})]^4} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,, \quad \|\mathbf{J}_{\mathbf{Y}} - \mathbf{Id}\|_{[L^\infty(\mathcal{O})]^4} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,,$$

$$\left\|\frac{\partial^2 Y_i}{\partial x_j \partial x_k}\right\|_{L^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,, \quad \left\|\frac{\partial^2 X_i}{\partial y_j \partial y_k}\right\|_{L^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,,$$

$$\left\|\frac{\partial^3 Y_i}{\partial x_j \partial x_l \partial x_k}\right\|_{L^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,, \quad \left\|\frac{\partial^3 X_i}{\partial y_j \partial y_l \partial y_k}\right\|_{L^\infty(\mathcal{O})} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\,.$$

LEMMA 5.7. *Let* $\boldsymbol{\Lambda}$*,* $\boldsymbol{\zeta}_1$*,* $\boldsymbol{\zeta}_2$ *be as in Lemma* 5.5*. Moreover, suppose that*

$$(\mathbf{U}, P) \in \mathcal{H}^2(\mathcal{O} \setminus B(\boldsymbol{\zeta}_2)) \times H^1(\mathcal{O} \setminus B(\boldsymbol{\zeta}_2))$$

and that $\mathbf{L}$ *and* $\mathbf{G}$ *are given by* (5.11) *and* (5.12)*. Then we have*

(i) $\|\nu[(\mathbf{L} - \Delta)\mathbf{U}]\|_{\mathcal{L}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}_2))} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\ \|\mathbf{U}\|_{\mathcal{H}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}_2))}$,

(ii) $\|[(\nabla - \mathbf{G})P]\|_{\mathcal{L}^2(\mathcal{O}\setminus B(\boldsymbol{\zeta}_2))} \leqslant C\,|\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2|\ \|P\|_{H^1(\mathcal{O}\setminus B(\boldsymbol{\zeta}_2))}$.

6. Consistency of the fully discretized scheme. This section is devoted to the consistency of our fully discretized scheme. The main result in this section asserts that the solution $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ of (1.1)–(1.8) satisfies the scheme (3.2)–(3.6) with consistency errors that will be estimated. Since $(\mathbf{u}(t_k), p(t_k))$ belongs to $\mathcal{K}(\boldsymbol{\zeta}(t_k)) \times M(\boldsymbol{\zeta}(t_k))$ and not to $\mathcal{K}(\boldsymbol{\zeta}_h^k) \times M(\boldsymbol{\zeta}_h^k)$, we need the change of variables introduced in the previous section.

6.1. Consistency in time. In this subsection we show that the exact values at instants $t = t_k$ of a strong solution of (1.1)–(1.8) satisfy a perturbed version of the semidiscretized problem introduced in subsection 2.2 and we estimate these perturbations with respect to the time step. The precise statement is given in Lemma 6.1 below.

Consider the solution $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ of (1.1)–(1.8) and assume that (3.8) and (3.10) hold. In what follows, we will use the notation

$$\widetilde{\boldsymbol{X}}(\mathbf{x}) = \widetilde{\boldsymbol{\psi}}(t_k; t_{k+1}, \mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{O}, \tag{6.1}$$

where $\widetilde{\boldsymbol{\psi}}$ is defined by relation (2.6). Note that $\widetilde{\boldsymbol{X}}(\mathcal{O}) = \mathcal{O}$.

Let $\boldsymbol{\varepsilon}_k$, $\boldsymbol{\delta}_k$, $\boldsymbol{\alpha}_k$, $\boldsymbol{\beta}_k$, γ_k be quantities defined by

$$\boldsymbol{\varepsilon}_k = \boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}'(t_k)\Delta t, \tag{6.2}$$

$$\boldsymbol{\delta}_k(t, \mathbf{x}) = \mathbf{u}(\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x}), t) - \mathbf{u}(\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x}), t_k), \tag{6.3}$$

$$\boldsymbol{\alpha}_k = \frac{\mathbf{u}(t_{k+1}) - \mathbf{u}(t_k) \circ \widetilde{\boldsymbol{X}}}{\Delta t} - \frac{d}{dt}\left[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}\right](t_{k+1}), \tag{6.4}$$

$$\boldsymbol{\beta}_k = \frac{\boldsymbol{\zeta}'(t_{k+1}) - \boldsymbol{\zeta}'(t_k)}{\Delta t} - \boldsymbol{\zeta}''(t_{k+1}), \tag{6.5}$$

$$\gamma_k = \frac{\omega'(t_{k+1}) - \omega'(t_k)}{\Delta t} - \omega''(t_{k+1}). \tag{6.6}$$

By using the fact that $\mathbf{u}\left(\boldsymbol{\zeta}(t_k), t_k\right) = \boldsymbol{\zeta}'(t_k)$ and relations (2.6), (1.1), (1.5), and (1.6) together with the above definitions, we infer that the exact solution $(\mathbf{u}, p, \boldsymbol{\zeta}, \omega)$ satisfies

$$\boldsymbol{\zeta}(t_{k+1}) = \boldsymbol{\zeta}(t_k) + \mathbf{u}\left(\boldsymbol{\zeta}(t_k), t_k\right)\Delta t + \boldsymbol{\varepsilon}_k, \tag{6.7}$$

$$\begin{cases} \dfrac{d}{dt}\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x}) = \mathbf{u}\left(\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{x}), t_k\right) + \boldsymbol{\delta}_k(t, \mathbf{x}), \\ \widetilde{\boldsymbol{\psi}}(t_{k+1}; t_{k+1}, \mathbf{x}) = \mathbf{x} \end{cases} \tag{6.8}$$

for all $\mathbf{x} \in \mathcal{O}$ and for all $t \in [t_k, t_{k+1}]$, together with

$$\frac{\mathbf{u}(t_{k+1}) - \mathbf{u}(t_k) \circ \widetilde{\boldsymbol{X}}}{\Delta t} - \nu\Delta\mathbf{u}(t_{k+1}) + \nabla p(t_{k+1}) = \mathbf{f}^{k+1} + \boldsymbol{\alpha}_k \quad \text{in } \mathcal{O} \setminus B(\boldsymbol{\zeta}(t_{k+1})), \tag{6.9}$$

$$M\frac{\boldsymbol{\zeta}'(t_{k+1}) - \boldsymbol{\zeta}'(t_k)}{\Delta t} = -\int_{\partial B(\boldsymbol{\zeta}(t_{k+1}))} \boldsymbol{\sigma}(t_{k+1})\mathbf{n}\, \mathrm{d}\Gamma + \int_{B(\boldsymbol{\zeta}(t_{k+1}))} \mathbf{f}^{k+1}\, \mathrm{d}\mathbf{x} + \boldsymbol{\beta}_k, \tag{6.10}$$

$$J\frac{\omega(t_{k+1}) - \omega(t_k)}{\Delta t} = -\int_{\partial B(\boldsymbol{\zeta}(t_{k+1}))} (\mathbf{y} - \boldsymbol{\zeta}(t_{k+1}))^{\perp} \cdot \boldsymbol{\sigma}(t_{k+1})\mathbf{n}\, \mathrm{d}\Gamma + \int_{B(\boldsymbol{\zeta}(t_{k+1}))} (\mathbf{y} - \boldsymbol{\zeta}(t_{k+1}))^{\perp} \cdot \mathbf{f}^{k+1}\, \mathrm{d}\mathbf{x} + \gamma_k. \tag{6.11}$$

Moreover, if we denote

$$\theta(t) = \int_0^t \omega(s)\ \mathrm{d}s$$

and by $\mathbf{R}_\theta$ the rotation matrix of angle θ, then we also define the matrix $\mathbf{E}_k$ by

$$\mathbf{R}_{\theta(t_{k+1})-\theta(t_k)} = \mathbf{Id} - \Delta t\,\omega(t_{k+1})\mathbf{R}_{-\pi/2} + \mathbf{E}_k. \tag{6.12}$$

By using the Taylor–Lagrange inequality, we easily obtain the following consistency error estimates.

LEMMA 6.1. *The elements $\boldsymbol{\alpha}_k$, $\boldsymbol{\beta}_k$, γ_k, $\boldsymbol{\delta}_k$, $\boldsymbol{\varepsilon}_k$, and $\mathbf{E}_k$ defined by* (6.2)–(6.6) *satisfy the following inequalities:*

$$\begin{gathered}
|\boldsymbol{\varepsilon}_k| \leqslant C\left(\Delta t\right)^2, \quad \|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} \leqslant C\Delta t \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}, \\
\|\boldsymbol{\alpha}_k\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\sqrt{\Delta t} \left\| \frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}] \right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}, \\
|\boldsymbol{\beta}_k| \leqslant C\Delta t, \quad |\gamma_k| \leqslant C\Delta t, \quad |\mathbf{E}_k| \leqslant C\left(\Delta t\right)^2.
\end{gathered} \tag{6.13}$$

6.2. Transformed system. We need to compare $\mathbf{u}(t_k) \in \mathcal{K}(\boldsymbol{\zeta}(t_k))$, which is a rigid velocity field in $B(\boldsymbol{\zeta}(t_k))$, with $\mathbf{u}_h^k \in \mathcal{K}(\boldsymbol{\zeta}_h^k)$, which is a rigid velocity field in $B(\boldsymbol{\zeta}_h^k)$. This will be done by using the change of variables introduced in section 5.1. To this end, we suppose that $|\boldsymbol{\zeta}_h^k - \boldsymbol{\zeta}(t_k)| < \eta$. This hypothesis and (3.10) imply that

$$\operatorname{dist}\left(B(\boldsymbol{\zeta}(t_k)), \partial\mathcal{O}\right) > 2\eta. \tag{6.14}$$

With this assumption, we can transform $\mathbf{u}(t_k)$ by using the change of variables introduced in section 5.1: we denote (see (5.4), (5.6))

$$\mathbf{X}^k = \mathbf{X}_{\boldsymbol{\zeta}_h^k,\boldsymbol{\zeta}(t_k)}, \quad \mathbf{Y}^k = \mathbf{Y}_{\boldsymbol{\zeta}_h^k,\boldsymbol{\zeta}(t_k)}. \tag{6.15}$$

We also define (see (5.7) and (5.8))

$$\begin{gathered}
\mathbf{U}^k(\mathbf{y}) = \mathbf{J}_{\mathbf{Y}^k}(\mathbf{X}^k(\mathbf{y}))\mathbf{u}\left(\mathbf{X}^k(\mathbf{y}), t_k\right), \quad P^k(\mathbf{y}) = p^k(\mathbf{X}^k(\mathbf{y})), \\
\boldsymbol{S}^k = -P^k\,\mathbf{Id} + 2\nu\mathbf{D}(\mathbf{U}^k), \quad \mathbf{F}^k(\mathbf{y}) = \mathbf{J}_{\mathbf{Y}^k}(\mathbf{X}^k(\mathbf{y}))\mathbf{f}(\mathbf{X}^k(\mathbf{y}), t_k).
\end{gathered} \tag{6.16}$$

We recall that, according to Lemma 5.3, $\mathbf{U}^k \in \widehat{\mathcal{K}}(\boldsymbol{\zeta}_h^k)$ and $P^k \in M(\boldsymbol{\zeta}_h^k)$. We introduce the following notation that will be useful in what follows:

$$\widehat{\boldsymbol{X}} = \mathbf{Y}^k \circ \widetilde{\boldsymbol{X}} \circ \mathbf{X}^{k+1} \tag{6.17}$$

and

$$\widehat{\mathbf{J}} = \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\boldsymbol{X}}\right). \tag{6.18}$$

Before stating the main result of this section, we give some properties on the characteristics. First note that, according to Lemma 5.2, we have

$$\mathbf{X}^{k+1}(B(\boldsymbol{\zeta}_h^{k+1})) = B(\boldsymbol{\zeta}(t_{k+1})), \tag{6.19}$$

$$\mathbf{Y}^k(B(\boldsymbol{\zeta}(t_k))) = B(\boldsymbol{\zeta}_h^k).$$

Moreover, we can easily check that the function $\widetilde{\boldsymbol{X}}$ defined by (6.1) satisfies

$$\widetilde{\boldsymbol{X}}(\mathbf{x}) = \boldsymbol{\zeta}(t_k) + \mathbf{R}_{\theta(t_{k+1})-\theta(t_k)}(\mathbf{x} - \boldsymbol{\zeta}(t_{k+1})) \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}(t_{k+1})). \tag{6.20}$$

Consequently, we have

$$\widetilde{\boldsymbol{X}}(B(\boldsymbol{\zeta}(t_{k+1}))) = B(\boldsymbol{\zeta}(t_k)),$$

and therefore, we obtain

$$\widehat{\boldsymbol{X}}(B(\boldsymbol{\zeta}_h^{k+1})) = B(\boldsymbol{\zeta}_h^k). \tag{6.21}$$

We summarize some of the above properties in the following diagram:

$$\begin{array}{ccc} B(\boldsymbol{\zeta}_h^{k+1}) & \xrightarrow{\mathbf{X}^{k+1}} & B(\boldsymbol{\zeta}(t_{k+1})) \\ \widehat{\boldsymbol{x}}\big\downarrow & & \big\downarrow\widetilde{\boldsymbol{x}} \\ B(\boldsymbol{\zeta}_h^k) & \xleftarrow[\mathbf{Y}^k]{} & B(\boldsymbol{\zeta}(t_k)) \end{array}$$

Next, we turn to the main result of this subsection: we show that $\mathbf{U}^{k+1}$ and P^{k+1} satisfy a mixed weak formulation with test functions in $\mathcal{K}(\boldsymbol{\zeta}_h^{k+1})$ and $M(\boldsymbol{\zeta}_h^{k+1})$.

PROPOSITION 6.2. *The functions* $(\mathbf{U}^{k+1}, P^{k+1})$ *defined by* (6.16) *satisfy*

$$\left(\frac{1}{\Delta t}\left[\mathbf{U}^{k+1} - \widehat{\mathbf{J}}\left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}}\right)\right], \boldsymbol{\varphi}\right) + a(\mathbf{U}^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P^{k+1}) = (\mathbf{f}_h^{k+1}, \boldsymbol{\varphi}) + (\mathbf{A}_k, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1}), \tag{6.22}$$

$$b(\mathbf{U}^{k+1}, q) = 0 \quad \forall q \in M(\boldsymbol{\zeta}_h^{k+1}), \tag{6.23}$$

with

$$\|\mathbf{A}_k\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\left(|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| + h + \Delta t + C\sqrt{\Delta t}\left\|\frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}]\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\right). \tag{6.24}$$

Proof.

First step. We transform (6.9).

By using Proposition 5.4, we have that $\mathbf{U}^{k+1}$ and P^{k+1} satisfy

$$\left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right) \frac{\mathbf{u}(t_{k+1}) - \mathbf{u}(t_k) \circ \widetilde{\boldsymbol{X}}}{\Delta t} \circ \mathbf{X}^{k+1} - \nu[\mathbf{L}^{k+1}\mathbf{U}^{k+1}] + [\mathbf{G}^{k+1}P^{k+1}] = \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)(\mathbf{f}(\mathbf{X}^{k+1}, t_{k+1})) + \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)(\boldsymbol{\alpha}_{k+1} \circ \mathbf{X}^{k+1}), \quad \text{in } \mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1}).$$

The above relation and (6.16) imply

$$\frac{1}{\Delta t}\left[\mathbf{U}^{k+1} - \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\boldsymbol{X}}\right)\left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}}\right)\right] - \nu\Delta\mathbf{U}^{k+1} + \nabla P^{k+1} = \nu[(\mathbf{L}^{k+1} - \Delta)\mathbf{U}^{k+1}] + [(\nabla - \mathbf{G}^{k+1})P^{k+1}] + \mathbf{F}^{k+1} + \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)(\boldsymbol{\alpha}_{k+1} \circ \mathbf{X}^{k+1}), \quad \text{in} \quad \mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1}), \tag{6.25}$$

where $\widehat{\boldsymbol{X}}$ is defined by (6.17).

By taking the inner product of the previous equation with $\boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1})$ and by using (6.18), we obtain

$$\int_{\mathcal{O}\setminus B(\boldsymbol{\zeta}_h^{k+1})} \left(\frac{1}{\Delta t} \left[\mathbf{U}^{k+1} - \widehat{\mathbf{J}} \left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}} \right) \right] \cdot \boldsymbol{\varphi} \right) \, \mathrm{d}\mathbf{y} - \int_{\mathcal{O}\setminus B(\boldsymbol{\zeta}_h^{k+1})} \left(\operatorname{div} \boldsymbol{S}^{k+1} \cdot \boldsymbol{\varphi} \right) \, \mathrm{d}\mathbf{y} = \int_{\mathcal{O}\setminus B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y} + A_1 \tag{6.26}$$

with

$$A_1 = \int_{\mathcal{O}\setminus B(\boldsymbol{\zeta}_h^{k+1})} \left(\nu[(\mathbf{L}^{k+1} - \Delta)\mathbf{U}^{k+1}] + [(\nabla - \mathbf{G}^{k+1})P^{k+1}] \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y} + \int_{\mathcal{O}\setminus B(\boldsymbol{\zeta}_h^{k+1})} \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1} \right) \left(\boldsymbol{\alpha}_{k+1} \circ \mathbf{X}^{k+1} \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y}. \tag{6.27}$$

Second step. We transform the integral

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \frac{\mathbf{U}^{k+1} - \widehat{\mathbf{J}} \left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}} \right)}{\Delta t} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y}$$

by using (6.10)–(6.11). From (5.3) (with $\mathbf{Y}^{k+1}$ as in (6.15)), combined with (5.9) and with (5.10), we obtain that

$$\mathbf{J}_{\mathbf{Y}^{k+1}}(\mathbf{x}) = \mathbf{Id} \quad \forall \mathbf{x} \in B(\boldsymbol{\zeta}(t_{k+1})). \tag{6.28}$$

The above relation, (6.16), and (5.9) imply that for all $\mathbf{y} \in B(\boldsymbol{\zeta}_h^{k+1})$,

$$\mathbf{U}^{k+1}(\mathbf{y}) = \mathbf{u} \left(\mathbf{y} + \boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}, t_{k+1} \right). \tag{6.29}$$

In particular, we have that

$$\mathbf{U}^{k+1}(\mathbf{y}) = \boldsymbol{\zeta}'(t_{k+1}) + \omega(t_{k+1})(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}_h^{k+1}). \tag{6.30}$$

Similarly, we have

$$\mathbf{U}^{k}(\mathbf{y}) = \boldsymbol{\zeta}'(t_{k}) + \omega(t_{k})(\mathbf{y} - \boldsymbol{\zeta}_h^{k})^\perp \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}_h^{k}). \tag{6.31}$$

Relations (6.19) and (6.21) yield

$$\left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1} \right) \left(\mathbf{J}_{\mathbf{X}^{k}} \circ \widehat{\boldsymbol{X}} \right) = \mathbf{Id} \quad \text{in} \quad B(\boldsymbol{\zeta}_h^{k+1}). \tag{6.32}$$

Simple calculations combined with relations (5.9) and (6.20) yield

$$\widehat{\boldsymbol{X}}(\mathbf{y}) = \mathbf{R}_{\theta(t_{k+1})-\theta(t_k)}(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1}) + \boldsymbol{\zeta}_h^{k} \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}_h^{k+1}).$$

The above relation, (6.32), and (6.31) imply that for all $\mathbf{y} \in B(\boldsymbol{\zeta}_h^{k+1})$, we have that

$$\widehat{\mathbf{J}}(\mathbf{U}^k \circ \widehat{\boldsymbol{X}})(\mathbf{y}) = \boldsymbol{\zeta}'(t_k) + \omega(t_k)\mathbf{R}_{\theta(t_{k+1})-\theta(t_k)}(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp.$$

By using (6.12), the previous equality can be written as

$$\widehat{\mathbf{J}}(\mathbf{U}^k \circ \widehat{\boldsymbol{X}})(\mathbf{y}) = \boldsymbol{\zeta}'(t_k) + \omega(t_k)(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp + \Delta t\ \omega(t_k)\omega(t_{k+1})(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1}) + \omega(t_k)\mathbf{E}_k(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp \quad \forall \mathbf{y} \in B(\boldsymbol{\zeta}_h^{k+1}).$$

By taking the inner product of the above relation with $\boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1})$ and by integrating on $B(\boldsymbol{\zeta}_h^{k+1})$, we obtain that

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \widehat{\mathbf{J}}(\mathbf{U}^k \circ \widehat{\boldsymbol{X}})(\mathbf{y}) \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y} = M\mathbf{l}_{\boldsymbol{\varphi}} \cdot \boldsymbol{\zeta}'(t_k) + J\omega(t_k)\omega_{\boldsymbol{\varphi}} + \omega(t_k) \int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{E}_k(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y}. \tag{6.33}$$

Relation (6.30) implies that, for all $\boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1})$, we have

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{U}^{k+1} \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y} = M\mathbf{l}_{\boldsymbol{\varphi}} \cdot \boldsymbol{\zeta}'(t_{k+1}) + J\omega(t_{k+1})\omega_{\boldsymbol{\varphi}}.$$

The above equality and (6.33) yield that, for all $\boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1})$, we have

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \frac{\mathbf{U}^{k+1} - \widehat{\mathbf{J}}(\mathbf{U}^k \circ \widehat{\boldsymbol{X}})}{\Delta t} \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y} = M\mathbf{l}_{\boldsymbol{\varphi}} \cdot \frac{\boldsymbol{\zeta}'(t_{k+1}) - \boldsymbol{\zeta}'(t_k)}{\Delta t} + J\frac{\omega(t_{k+1}) - \omega(t_k)}{\Delta t}\omega_{\boldsymbol{\varphi}} - \frac{\omega(t_k)}{\Delta t} \int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{E}_k(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y}.$$

The above relation and (6.10)–(6.11) imply that

$$\begin{aligned} \int_{B(\boldsymbol{\zeta}_h^{k+1})} \frac{\mathbf{U}^{k+1} - \widehat{\mathbf{J}}(\mathbf{U}^k \circ \widehat{\boldsymbol{X}})}{\Delta t} \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y} &= -\mathbf{l}_{\boldsymbol{\varphi}} \cdot \int_{\partial B(\boldsymbol{\zeta}(t_{k+1}))} \boldsymbol{\sigma}^{k+1}\mathbf{n}\ \mathrm{d}\Gamma \\ - \omega_{\boldsymbol{\varphi}} \int_{\partial B(\boldsymbol{\zeta}(t_{k+1}))} (\mathbf{y} - \boldsymbol{\zeta}(t_{k+1}))^\perp \cdot \boldsymbol{\sigma}^{k+1}\mathbf{n}\ \mathrm{d}\Gamma + \mathbf{l}_{\boldsymbol{\varphi}} \cdot \int_{B(\boldsymbol{\zeta}(t_{k+1}))} \mathbf{f}^{k+1}\ \mathrm{d}\mathbf{x} \\ + \omega_{\boldsymbol{\varphi}} \int_{B(\boldsymbol{\zeta}(t_{k+1}))} (\mathbf{x} - \boldsymbol{\zeta}(t_{k+1}))^\perp \cdot \mathbf{f}^{k+1}(x)\ \mathrm{d}\mathbf{x} \\ + \mathbf{l}_{\boldsymbol{\varphi}} \cdot \boldsymbol{\beta}_k + \omega_{\boldsymbol{\varphi}}\gamma_k - \frac{\omega(t_k)}{\Delta t} \int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{E}_k(\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp \cdot \boldsymbol{\varphi}\ \mathrm{d}\mathbf{y}. \end{aligned} \tag{6.34}$$

On the other hand, by using relations (5.9), (5.10), and (6.28), we easily obtain that

$$\int_{\partial B(\boldsymbol{\zeta}_h^{k+1})} \boldsymbol{S}^{k+1}\mathbf{n}\ \mathrm{d}\Gamma = \int_{\partial B(\boldsymbol{\zeta}(t_{k+1}))} \boldsymbol{\sigma}^{k+1}\mathbf{n}\ \mathrm{d}\Gamma$$

and that

$$\int_{\partial B(\boldsymbol{\zeta}_h^{k+1})} (\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^\perp \cdot \boldsymbol{S}^{k+1}\mathbf{n}\ \mathrm{d}\Gamma = \int_{\partial B(\boldsymbol{\zeta}(t_{k+1}))} (\mathbf{y} - \boldsymbol{\zeta}(t_{k+1}))^\perp \cdot \boldsymbol{\sigma}^{k+1}\mathbf{n}\ \mathrm{d}\Gamma.$$

The above relations and (6.34) yield that

$$\int_{B(\boldsymbol{\zeta}_h^{k+1})} \frac{\mathbf{U}^{k+1} - \widehat{\mathbf{J}}(\mathbf{U}^k \circ \widehat{\boldsymbol{X}})}{\Delta t} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y} = - \int_{\partial B(\boldsymbol{\zeta}_h^{k+1})} \left(\boldsymbol{S}^{k+1} \mathbf{n} \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\Gamma + \int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{F}^{k+1} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y} + \mathbf{l}_{\boldsymbol{\varphi}} \cdot \boldsymbol{\beta}_k + \omega_{\boldsymbol{\varphi}} \gamma_k - \frac{\omega(t_k)}{\Delta t} \int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{E}_k (\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^{\perp} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y}. \tag{6.35}$$

Third step. By integrating by parts, we have that

$$2\nu \int_{\mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{D}(\mathbf{U}^{k+1}) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\mathbf{y} - \int_{\mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1})} P^{k+1} \operatorname{div}(\boldsymbol{\varphi}) \, \mathrm{d}\mathbf{y} = \int_{\partial B(\boldsymbol{\zeta}_h^{k+1})} \left(\boldsymbol{S}^{k+1} \mathbf{n} \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\Gamma - \int_{\mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1})} \operatorname{div}(\boldsymbol{S}^{k+1}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y}. \tag{6.36}$$

Summing (6.36), (6.35), and (6.26) yields (6.22) with

$$\begin{aligned}(\mathbf{A}_k, \boldsymbol{\varphi}) = (\mathbf{F}^{k+1} - \mathbf{f}_h^{k+1}, \boldsymbol{\varphi}) + \mathbf{l}_{\boldsymbol{\varphi}} \cdot \boldsymbol{\beta}_k + \omega_{\boldsymbol{\varphi}} \gamma_k - \frac{\omega(t_k)}{\Delta t} \int_{B(\boldsymbol{\zeta}_h^{k+1})} \mathbf{E}_k (\mathbf{y} - \boldsymbol{\zeta}_h^{k+1})^{\perp} \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y} \\ + \int_{\mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1})} \left(\nu [(\mathbf{L}^{k+1} - \Delta) \mathbf{U}^{k+1}] + [(\nabla - \mathbf{G}^{k+1}) P^{k+1}] \right) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y} \\ + \int_{\mathcal{O} \setminus B(\boldsymbol{\zeta}_h^{k+1})} \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1} \right) (\boldsymbol{\alpha}_{k+1} \circ \mathbf{X}^{k+1}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\mathbf{y}.\end{aligned}$$

The above relation, combined with relation (3.7) and Lemmas 5.6, 5.7, and 6.1, implies the proposition. ∎

6.3. Some results on characteristics. In this subsection, we give some results on the functions $\mathbf{X}^k$, $\widehat{\boldsymbol{X}}$, and $\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}$ that will be used in the proof of the main result.

LEMMA 6.3. *There exists a positive constant C independent of h and k such that*

$$\|\mathbf{X}^{k+1} - \mathbf{X}^k\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C \left(\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \Delta t + |\boldsymbol{\varepsilon}_k| \right).$$

Proof. We denote by $\boldsymbol{\Lambda}^k$ (resp., $\boldsymbol{\Lambda}^{k+1}$) the mapping defined by (5.2) with $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}(t_k)$ and $\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_h^k$ (resp., $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}(t_{k+1})$ and $\boldsymbol{\zeta}_2 = \boldsymbol{\zeta}_h^{k+1}$). Let $\boldsymbol{\psi}^k$ and $\boldsymbol{\psi}^{k+1}$ be the solution of (5.3) corresponding to the velocity fields $\boldsymbol{\Lambda}^k$ and $\boldsymbol{\Lambda}^{k+1}$, respectively.

By using (5.3), we have that

$$(\boldsymbol{\psi}^{k+1} - \boldsymbol{\psi}^k)(\lambda) = \int_0^\lambda \boldsymbol{\Lambda}^{k+1}(\boldsymbol{\psi}^{k+1}(\mu)) - \boldsymbol{\Lambda}^k(\boldsymbol{\psi}^k(\mu)) \, \mathrm{d}\mu.$$

Therefore, by Lemma 5.5, there exists a positive constant C such that for all $\lambda \in [0,1]$, we have that

$$\left| (\boldsymbol{\psi}^{k+1} - \boldsymbol{\psi}^k)(\lambda) \right| \leqslant \|\boldsymbol{\Lambda}^{k+1} - \boldsymbol{\Lambda}^k\|_{\mathcal{L}^\infty(\mathcal{O})} + C \int_0^\lambda \left| (\boldsymbol{\psi}^{k+1}(\mu) - \boldsymbol{\psi}^k(\mu)) \right| \, \mathrm{d}\mu.$$

The above inequality and Gronwall's lemma yield

$$\left| (\boldsymbol{\psi}^{k+1} - \boldsymbol{\psi}^k)(\lambda) \right| \leqslant C \|\boldsymbol{\Lambda}^{k+1} - \boldsymbol{\Lambda}^k\|_{\mathcal{L}^\infty(\mathcal{O})}$$

for all $\lambda \in [0,1]$. In particular, for $\lambda = 1$, we have that

$$\|\mathbf{X}^{k+1} - \mathbf{X}^k\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C\|\mathbf{\Lambda}^{k+1} - \mathbf{\Lambda}^k\|_{\mathcal{L}^\infty(\mathcal{O})}. \tag{6.37}$$

By using relation (5.2), there exists a positive constant C such that

$$\|\mathbf{\Lambda}^{k+1} - \mathbf{\Lambda}^k\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1} - \boldsymbol{\zeta}(t_k) + \boldsymbol{\zeta}_h^k|.$$

The above relation, combined with (3.2) and (6.7), yields

$$\|\mathbf{\Lambda}^{k+1} - \mathbf{\Lambda}^k\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k) - \mathbf{u}\left(\boldsymbol{\zeta}(t_k), t_k\right)|\Delta t + C|\boldsymbol{\varepsilon}_k|. \tag{6.38}$$

On the other hand, by (6.29), we have $\mathbf{u}\left(\boldsymbol{\zeta}(t_k), t_k\right) = \mathbf{U}^k(\boldsymbol{\zeta}_h^k)$ and, moreover, $\mathbf{u}_h^k - \mathbf{U}^k \in \mathcal{K}(\boldsymbol{\zeta}_h^k)$. Then, owing to (2.5), we readily check that

$$|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k) - \mathbf{U}^k(\boldsymbol{\zeta}_h^k)| \leqslant \frac{1}{\sqrt{M}}\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}. \tag{6.39}$$

Therefore, the above relation and (6.38) imply that

$$\|\mathbf{\Lambda}^{k+1} - \mathbf{\Lambda}^k\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}\Delta t + C|\boldsymbol{\varepsilon}_k|. \tag{6.40}$$

Relations (6.37) and (6.40) yield the conclusion of the lemma. ∎

A similar estimate holds for the jacobian matrices $\mathbf{J}_{\mathbf{X}^{k+1}}$ and $\mathbf{J}_{\mathbf{X}^k}$. Since the proof of this estimate is completely similar to the proof of Lemma 6.3, we give below only its statement and skip the proof.

LEMMA 6.4. *There exists a positive constant C independent of k and h such that*

$$\|\mathbf{J}_{\mathbf{X}^{k+1}} - \mathbf{J}_{\mathbf{X}^k}\|_{\mathcal{L}^\infty(\mathcal{O})} \leqslant C\left(\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}\Delta t + |\boldsymbol{\varepsilon}_k|\right).$$

The functions $\widehat{\boldsymbol{X}}$ and $\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}$ are close to the identity in the sense made precise below.

LEMMA 6.5. *The functions $\widehat{\boldsymbol{X}}$ and $\overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}$ defined by* (6.17) *and* (3.4) *satisfy the following estimates:*

$$\|\widehat{\boldsymbol{X}} - \mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\left(|\boldsymbol{\varepsilon}_k| + \Delta t\|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} + \sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + \Delta t\right), \tag{6.41}$$

$$\|\widehat{\boldsymbol{X}} - \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\left(|\boldsymbol{\varepsilon}_k| + \Delta t\|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} + \sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + h\Delta t\right). \tag{6.42}$$

Proof. Let us define

$$\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y}) = \mathbf{Y}^k(\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{X}^{k+1}(\mathbf{y}))), \tag{6.43}$$

where $\widetilde{\boldsymbol{\psi}}$ is defined by (2.6). Note that $\widehat{\boldsymbol{\psi}}(t_k; t_{k+1}, \mathbf{y}) = \widehat{\boldsymbol{X}}(\mathbf{y})$ for all $\mathbf{y} \in \mathcal{O}$.

We have that

$$\frac{d}{dt}\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y}) = \mathbf{J}_{\mathbf{Y}^k}(\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{X}^{k+1}(\mathbf{y})))\frac{d}{dt}\widetilde{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{X}^{k+1}(\mathbf{y})).$$

By using (6.8) we obtain that

$$\begin{aligned}\frac{d}{dt}\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y}) = &\left[\mathbf{J}_{\mathbf{Y}^k} \circ \mathbf{X}^k\right](\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y}))\left[\mathbf{u}\left(\mathbf{X}^k\left(\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y})\right), t_k\right)\right] \\ &+ \left[\mathbf{J}_{\mathbf{Y}^k} \circ \mathbf{X}^k\right](\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y}))\left[\boldsymbol{\delta}_k\left(t, \mathbf{X}^k\left(\widehat{\boldsymbol{\psi}}(t; t_{k+1}, \mathbf{y})\right)\right)\right].\end{aligned}$$

The above relation and (6.16) yield

$$\frac{d}{dt}\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y}) = \mathbf{U}^k(\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y})) + \left[\mathbf{J}_{\mathbf{Y}^k}\circ\mathbf{X}^k\right](\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y}))\left[\boldsymbol{\delta}_k\left(t,\mathbf{X}^k\left(\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y})\right)\right)\right]. \tag{6.44}$$

On the other hand, we have that

$$\widehat{\boldsymbol{\psi}}(t_{k+1};t_{k+1},\mathbf{y}) = \mathbf{Y}^k\circ\mathbf{X}^{k+1}(\mathbf{y}). \tag{6.45}$$

Therefore, by using (6.44) and (6.45), we get

$$\widehat{\boldsymbol{X}}(\mathbf{y})-\mathbf{y} = \mathbf{Y}^k\circ\mathbf{X}^{k+1}(\mathbf{y})-\mathbf{y}-\int_{t_k}^{t_{k+1}}\mathbf{U}^k(\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y}))\,\mathrm{d}t - \int_{t_k}^{t_{k+1}}\left[\mathbf{J}_{\mathbf{Y}^k}\circ\mathbf{X}^k\right](\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y}))\left[\boldsymbol{\delta}_k\left(t,\mathbf{X}^k\left(\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y})\right)\right)\right]\,\mathrm{d}t,$$

which yields

$$\|\widehat{\boldsymbol{X}}-\mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant \|\mathbf{Y}^k\circ\mathbf{X}^{k+1}-\mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} + \int_{t_k}^{t_{k+1}}\left\|\mathbf{U}^k(\widehat{\boldsymbol{\psi}}(s))\right\|_{\mathcal{L}^2(\mathcal{O})}\,\mathrm{d}s + C\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}. \tag{6.46}$$

By Lemma 5.6, there exists a positive constant C such that

$$\|\mathbf{Y}^k\circ\mathbf{X}^{k+1}-\mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\|\mathbf{X}^{k+1}-\mathbf{X}^k\|_{\mathcal{L}^\infty(\mathcal{O})}.$$

The above relation and Lemma 6.3 yield

$$\|\mathbf{Y}^k\circ\mathbf{X}^{k+1}-\mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\left(\Delta t\|\mathbf{u}_h^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\varepsilon}_k|\right). \tag{6.47}$$

Relations (6.46) and (6.47), together with (3.8) and (6.16), imply

$$\|\widehat{\boldsymbol{X}}-\mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\left(\Delta t\|\mathbf{u}_h^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\varepsilon}_k|\right)+C\Delta t+\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}.$$

Therefore, we deduce (6.41).

Now we turn to the proof of (6.42): by using (3.3), (6.44), and (6.45), we obtain

$$\widehat{\boldsymbol{\psi}}(t;t_{k+1},\mathbf{y})-\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(t;t_{k+1},\mathbf{y}) = \mathbf{Y}^k\circ\mathbf{X}^{k+1}(\mathbf{y})-\mathbf{y} - \int_t^{t_{k+1}}\left(\mathbf{U}^k(\widehat{\boldsymbol{\psi}}(s;t_{k+1},\mathbf{y}))-\mathbf{P}\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(s;t_{k+1},\mathbf{y}))\right)\,\mathrm{d}s - \int_t^{t_{k+1}}(\mathbf{J}_{\mathbf{Y}^k}\circ\mathbf{X}^k)(\widehat{\boldsymbol{\psi}}(s;t_{k+1},\mathbf{y}))\left[\boldsymbol{\delta}_k\left(s,\mathbf{X}^k\left(\widehat{\boldsymbol{\psi}}(s;t_{k+1},\mathbf{y})\right)\right)\right]\,\mathrm{d}s,$$

which yields

$$\|\widehat{\boldsymbol{\psi}}(t)-\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(t)\|_{\mathcal{L}^2(\mathcal{O})} \leqslant \|\mathbf{Y}^k\circ\mathbf{X}^{k+1}-\mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})} + \int_t^{t_{k+1}}\left\|\mathbf{U}^k(\widehat{\boldsymbol{\psi}}(s))-\mathbf{P}\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}_{\boldsymbol{h}}^{\boldsymbol{k}}(s))\right\|_{\mathcal{L}^2(\mathcal{O})}\,\mathrm{d}s + C\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}. \tag{6.48}$$

Relations (6.48) and (6.47) imply

$$\|\widehat{\boldsymbol{\psi}}(t)-\overline{\boldsymbol{\psi}}^{\boldsymbol{k}}_{\boldsymbol{h}}(t)\|_{\mathcal{L}^2(\mathcal{O})}\leqslant C\left(\Delta t\|\mathbf{u}_h^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\varepsilon}_k|\right)+\int_t^{t_{k+1}}\left\|\mathbf{U}^k(\widehat{\boldsymbol{\psi}}(s))-\mathbf{P}\mathbf{u}_h^k(\overline{\boldsymbol{\psi}}^{\boldsymbol{k}}_{\boldsymbol{h}}(s))\right\|_{\mathcal{L}^2(\mathcal{O})}\,\mathrm{d}s+C\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}.$$

By using (3.8) and Remark 3.1, we have that

$$\|\widehat{\boldsymbol{\psi}}(t)-\overline{\boldsymbol{\psi}}^{\boldsymbol{k}}_{\boldsymbol{h}}(t)\|_{\mathcal{L}^2(\mathcal{O})}\leqslant C\left(\Delta t\|\mathbf{u}_h^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\varepsilon}_k|+\Delta t\|\mathbf{U}^k-\mathbf{P}\mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})}\right)+C\int_t^{t_{k+1}}\left\|\widehat{\boldsymbol{\psi}}(s)-\overline{\boldsymbol{\psi}}^{\boldsymbol{k}}_{\boldsymbol{h}}(s)\right\|_{\mathcal{L}^2(\mathcal{O})}\,\mathrm{d}s+C\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}.$$

Therefore, by Gronwall's lemma, we get that

$$\|\widehat{\boldsymbol{\psi}}(t)-\overline{\boldsymbol{\psi}}^{\boldsymbol{k}}_{\boldsymbol{h}}(t)\|_{\mathcal{L}^2(\mathcal{O})}\leqslant C\Big(\Delta t\|\mathbf{u}_h^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\varepsilon}_k|+\Delta t\|\mathbf{U}^k-\mathbf{P}\mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})}+\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\Big).$$

In particular for $t=t_k$, we obtain that

$$\|\widehat{\boldsymbol{X}}-\overline{\boldsymbol{X}}^{\boldsymbol{k}}_{\boldsymbol{h}}\|_{\mathcal{L}^2(\mathcal{O})}\leqslant C\Big(\Delta t\|\mathbf{u}_h^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\varepsilon}_k|+\Delta t\|\mathbf{U}^k-\mathbf{P}\mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})}+\sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\Big).\tag{6.49}$$

Since $\mathbf{P}$ is an orthogonal projection in $\mathcal{L}^2(\mathcal{O})$, we have that

$$\|\mathbf{U}^k-\mathbf{P}\mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})}\leqslant\|\mathbf{U}^k-\mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})}+\|\mathbf{P}\mathbf{U}^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}.\tag{6.50}$$

Now, since $\mathbf{U}^k\in\mathcal{H}_0^1(\mathcal{O})$ and $\operatorname{div}(\mathbf{U}^k)=0$, there exists a stream function $\psi\in H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$ of $\mathbf{U}^k$, i.e., $\mathbf{U}^k=\mathbf{rot}\ \psi$. Let ψ_h be the Lagrange interpolated function of ψ on the triangulation $\mathcal{T}_h$. We denote $\widetilde{\mathbf{U}_h^k}=\mathbf{rot}\ \psi_h$. Since $\widetilde{\mathbf{U}_h^k}\in\mathcal{R}_h$, we have that

$$\begin{aligned}\|\mathbf{P}\mathbf{U}^k-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}&\leqslant\|\widetilde{\mathbf{U}_h^k}-\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}=\|\mathbf{rot}\ (\psi-\psi_h)\|_{\mathcal{L}^2(\mathcal{O})}\\&\leqslant Ch\|\psi\|_{H^2(\mathcal{O})}\leqslant Ch\|\mathbf{U}^k\|_{\mathcal{H}^1(\mathcal{O})}.\end{aligned}$$

The above equation, (6.49), and (6.50) imply the result. ∎

7. Proof of the main result. We can now prove Theorem 3.2.

First step. Assume that $h\leq C(\Delta t)^2$. We first show that if (3.10) holds and if

$$\operatorname{dist}(B(\boldsymbol{\zeta}_h^k),\partial\mathcal{O})>2\eta,\quad \operatorname{dist}(B(\boldsymbol{\zeta}_h^{k+1}),\partial\mathcal{O})>2\eta,\tag{7.1}$$

then there exist two positive constants C_0 and C_1 independent of Δt and h such that the error $e_h^k=\|\mathbf{U}^k-\mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})}+|\boldsymbol{\zeta}(t_k)-\boldsymbol{\zeta}_h^k|$ satisfies the following inequality:

$$e_h^{k+1}\leqslant e_h^k(1+C_0\Delta t)+C_0\Delta t\beta_h^k,\tag{7.2}$$

where

$$\sum_{k=0}^{N}\beta_h^k\leqslant C_1.$$

Let us remark that assumption (7.1) together with (3.10) allows us to perform the change of variables defined in section 5 and to define $\mathbf{U}^k$, $\mathbf{U}^{k+1}$, and P^{k+1} (see (6.16)).

By using (4.50), there exists $(\mathbf{U}_h^{k+1}, P_h^{k+1}) \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}) \times M_h(\boldsymbol{\zeta}_h^{k+1})$ such that

$$\left\{ \begin{array}{rcl} a\left(\mathbf{U}^{k+1} - \mathbf{U}_h^{k+1}, \boldsymbol{\varphi}\right) + b\left(\boldsymbol{\varphi}, P^{k+1} - P_h^{k+1}\right) &=& 0 \quad \forall \boldsymbol{\varphi} \in \mathcal{K}_h(\boldsymbol{\zeta}_h^{k+1}) \\ b\left(\mathbf{U}^{k+1} - \mathbf{U}_h^{k+1}, q\right) &=& 0 \quad \forall q \in M_h(\boldsymbol{\zeta}_h^{k+1}). \end{array} \right. \tag{7.3}$$

Subtracting (7.3) and (3.5) from (6.22) yields

$$\begin{aligned} &\frac{1}{\Delta t}\left(\mathbf{U}^{k+1} - \mathbf{u}_h^{k+1}, \boldsymbol{\varphi}\right) + a(\mathbf{U}_h^{k+1} - \mathbf{u}_h^{k+1}, \boldsymbol{\varphi}) + b(\boldsymbol{\varphi}, P_h^{k+1} - p_h^{k+1}) \\ &\qquad = \frac{1}{\Delta t}\left(\widehat{\mathbf{J}}\left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}}\right) - \mathbf{u}_h^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}, \boldsymbol{\varphi}\right) + (\mathbf{A}_k, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \mathcal{K}(\boldsymbol{\zeta}_h^{k+1}), \\ &\qquad\quad b(\mathbf{U}_h^{k+1} - \mathbf{u}_h^{k+1}, q) = 0 \quad \forall q \in M_h(\boldsymbol{\zeta}_h^{k+1}). \end{aligned}$$

In particular, for $\boldsymbol{\varphi} = \mathbf{U}_h^{k+1} - \mathbf{u}_h^{k+1}$ and $q = P_h^{k+1} - p_h^{k+1}$, we easily obtain that

$$\begin{aligned} \left\|\mathbf{U}_h^{k+1} - \mathbf{u}_h^{k+1}\right\|_{\mathcal{L}^2(\mathcal{O})} \leqslant& \left\|\widehat{\mathbf{J}}\left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}}\right) - \mathbf{u}_h^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\right\|_{\mathcal{L}^2(\mathcal{O})} \\ &+ \Delta t \|\mathbf{A}_k\|_{\mathcal{L}^2(\mathcal{O})} + \left\|\mathbf{U}^{k+1} - \mathbf{U}_h^{k+1}\right\|_{\mathcal{L}^2(\mathcal{O})}. \end{aligned} \tag{7.4}$$

On the other hand, since

$$\widehat{\mathbf{J}} = \left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\boldsymbol{X}}\right),$$

we have that

$$\begin{aligned} \left\|\widehat{\mathbf{J}}\left(\mathbf{U}^k \circ \widehat{\boldsymbol{X}}\right) - \mathbf{u}_h^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\right\|_{\mathcal{L}^2(\mathcal{O})} \leqslant& C\left\|\left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\boldsymbol{X}}\right) - \mathbf{Id}\right\|_{\mathcal{L}^2(\mathcal{O})} \\ &+ \left\|\mathbf{U}^k \circ \widehat{\boldsymbol{X}} - \mathbf{U}^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\right\|_{\mathcal{L}^2(\mathcal{O})} \\ &+ \left\|\mathbf{U}^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}} - \mathbf{u}_h^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\right\|_{\mathcal{L}^2(\mathcal{O})}. \end{aligned} \tag{7.5}$$

Since $\left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\mathbf{J}_{\mathbf{X}^{k+1}} = \mathbf{Id}$, we infer from Lemma 5.6 that

$$\begin{aligned} \left\|\left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\boldsymbol{X}}\right) - \mathbf{Id}\right\|_{\mathcal{L}^2(\mathcal{O})} \leqslant& C\|\widehat{\boldsymbol{X}} - \mathbf{Id}\|_{\mathcal{L}^2(\mathcal{O})}\,|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| \\ &+ C\|\mathbf{J}_{\mathbf{X}^k} - \mathbf{J}_{\mathbf{X}^{k+1}}\|_{\mathcal{L}^2(\mathcal{O})}. \end{aligned}$$

By using Lemmas 6.4 and 6.5 and the above inequality, we obtain that

$$\begin{aligned} \left\|\left(\mathbf{J}_{\mathbf{Y}^{k+1}} \circ \mathbf{X}^{k+1}\right)\left(\mathbf{J}_{\mathbf{X}^k} \circ \widehat{\boldsymbol{X}}\right) - \mathbf{Id}\right\|_{\mathcal{L}^2(\mathcal{O})} \leqslant& C\Big(\Delta t|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| \\ &+ \Delta t\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} + \sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + |\boldsymbol{\varepsilon}_k|\Big). \end{aligned} \tag{7.6}$$

By using (3.8) and Lemma 5.6, we easily check that

$$\|\mathbf{U}^k \circ \widehat{\boldsymbol{X}} - \mathbf{U}^k \circ \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\|\widehat{\boldsymbol{X}} - \overline{\boldsymbol{X}}_{\boldsymbol{h}}^{\boldsymbol{k}}\|_{\mathcal{L}^2(\mathcal{O})}.$$

The above inequality, relations (7.4), (7.5), and (7.6), Lemma 6.5, and the fact that $\det \mathbf{J}_{\overline{\boldsymbol{X}}_h^k} = 1$ imply that

$$\begin{aligned}(7.7)\quad \|\mathbf{U}_h^{k+1} - \mathbf{u}_h^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} &\leqslant C\Big(\Delta t|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \Delta t\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \\ &\quad + \sqrt{\Delta t}\|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + |\boldsymbol{\varepsilon}_k| + h\Delta t\Big) \\ &\quad + \|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} + \Delta t\|\mathbf{A}_k\|_{\mathcal{L}^2(\mathcal{O})} + \|\mathbf{U}^{k+1} - \mathbf{U}_h^{k+1}\|_{\mathcal{L}^2(\mathcal{O})}.\end{aligned}$$

By using Lemma 4.4, Proposition 6.2, and Lemma 6.1, we have the following inequalities:

$$\begin{aligned}\|\mathbf{U}^{k+1} - \mathbf{U}_h^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} &\leqslant Ch, \\ \|\mathbf{A}_k\|_{\mathcal{L}^2(\mathcal{O})} &\leqslant C\left(|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| + h + \Delta t + C\sqrt{\Delta t}\left\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\right), \\ \|\boldsymbol{\delta}_k\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} &\leqslant C\Delta t\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}, \\ |\boldsymbol{\varepsilon}_k| &\leqslant C\left(\Delta t\right)^2.\end{aligned}$$

The above inequalities and (7.7) yield that

$$\begin{aligned}(7.8)\quad \|\mathbf{U}^{k+1} - \mathbf{u}_h^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} &\leqslant \|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} + C\Bigg(\left(\Delta t\right)^2 + h\Delta t + h \\ &\quad + \Delta t|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| + \Delta t\|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} \\ &\quad + (\Delta t)^{3/2}\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + (\Delta t)^{3/2}\left\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\Bigg).\end{aligned}$$

On the other hand, (3.2), (6.7), (6.31), and (6.39) imply that

$$\begin{aligned}|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| &\leqslant |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \Delta t|\mathbf{u}_h^k(\boldsymbol{\zeta}_h^k) - \mathbf{u}\left(\boldsymbol{\zeta}(t_k), t_k\right)| + |\boldsymbol{\varepsilon}_k| \\ &\leqslant |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + C\Delta t\|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} + |\boldsymbol{\varepsilon}_k|. \qquad (7.9)\end{aligned}$$

Combining (7.8) and (7.9), we obtain that

$$\begin{aligned}&\|\mathbf{U}^{k+1} - \mathbf{u}_h^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} + |\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \\ &\qquad \leqslant (1 + C\Delta t)\left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}\right) \\ &\qquad\quad + C\Bigg(h + \left(\Delta t\right)^2 + h\Delta t + (\Delta t)^{3/2}\left\|\frac{\partial \mathbf{u}}{\partial t}\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} \\ &\qquad\qquad + (\Delta t)^{3/2}\left\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\right\|_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\Bigg).\end{aligned}$$

The above inequality and the hypothesis $h \leqslant C\left(\Delta t\right)^2$ imply the existence of a positive constant C_0 such that

$$\begin{aligned}&\|\mathbf{U}^{k+1} - \mathbf{u}_h^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} + |\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| \\ &\qquad \leqslant (1 + C_0\Delta t)\left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})}\right) \\ &\qquad\quad + C_0\Delta t\left(\Delta t + \left\|\frac{\partial \mathbf{u}}{\partial t}\right\|^2_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + \left\|\frac{d^2}{dt^2}[\mathbf{u}\circ\widetilde{\boldsymbol{\psi}}]\right\|^2_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))}\right),\end{aligned}$$

which is exactly (7.2).

Second step. We show that if Δt is small enough, then the error $e_h^k = \|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} + |\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k|$ satisfies $e_h^k \leqslant C_1 \Delta t$ with a constant C_1 independent of k, Δt, and h. This fact implies, in particular, that (7.1) holds.

Define

$$C_1 = C_0 \exp(C_0 T) \left(\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,T))} + \left\| \frac{d^2}{dt^2} [\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,T))} \right) + \exp(C_0 T).$$

It can be easily checked that

$$(1 + C_0 \Delta t)^n C_0 \left(\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,T))} + \left\| \frac{d^2}{dt^2} [\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,T))} \right) + (1 + C_0 \Delta t)^n - 1 \leqslant C_1 \quad \forall n \in \{0, \dots, N\}.$$

Moreover, there exists a positive constant C_2 such that

$$\|\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C_2.$$

Let $N_0 \in \mathbb{N}$ be such that $(2C_1 + C_2)\Delta t < \eta$ for all $N \geqslant N_0$. Next we prove by induction over k that for $N \geqslant N_0$ and for $k \in \{0, \dots, N\}$ we have

$$|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \leqslant \Bigg[(1 + C_0 \Delta t)^k - 1 + C_0 (1 + C_0 \Delta t)^k \left(\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,t_k))} + \left\| \frac{d^2}{dt^2} [\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,t_k))} \right) \Bigg] \Delta t. \tag{7.10}$$

The relation (7.10) is true for $k = 0$. Suppose that we have shown (7.10) for a given $k \geqslant 0$. Then, we deduce that

$$|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| \leqslant C_1 \Delta t < \eta \tag{7.11}$$

and therefore, by using (3.10), we have that $\operatorname{dist}(B(\boldsymbol{\zeta}_h^k), \partial\mathcal{O}) > 2\eta$.

By using (3.2) and (3.10), we also have that

$$\begin{aligned} |\boldsymbol{\zeta}_h^{k+1} - \boldsymbol{\zeta}_h^k| &\leqslant \frac{1}{\sqrt{\pi}} \left(\|\mathbf{U}^k - \mathbf{u}_h^k\|_{\mathcal{L}^2(\mathcal{O})} + \|\mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \right) \Delta t \\ &\leqslant \frac{C_1 + C_2}{\sqrt{\pi}} \Delta t. \end{aligned}$$

The above relation, the fact that $(2C_1 + C_2)\Delta t < \eta$, and (7.11) imply that

$$\operatorname{dist}(B(\boldsymbol{\zeta}_h^{k+1}), \partial\mathcal{O}) > 2\eta.$$

Thus, we can apply the first step of the proof to obtain that

$$\begin{aligned} &|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| + \|\mathbf{u}_h^{k+1} - \mathbf{U}^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} \\ &\quad \leqslant (1 + C_0 \Delta t) \left(|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \right) \\ &\qquad + C_0 \Delta t \left(\Delta t + \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + \left\| \frac{d^2}{dt^2} [\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} \right). \end{aligned}$$

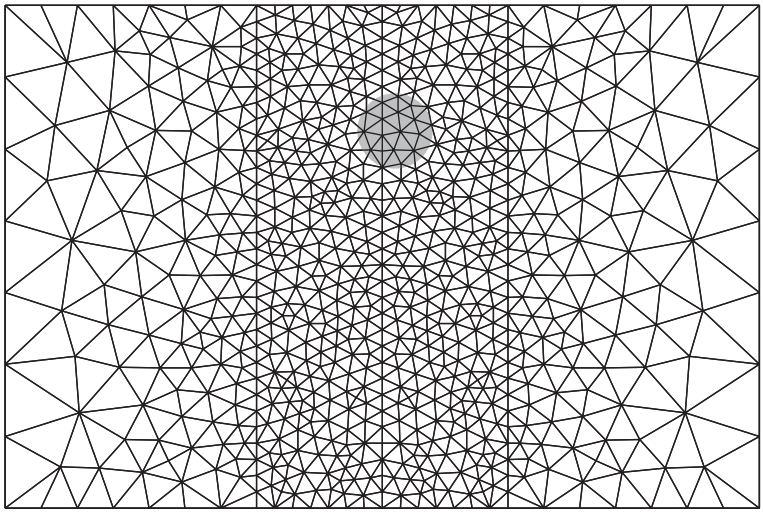

FIG. 2. *Initial position and mesh.*

The above relation and (7.10) imply that

$$
\begin{aligned}
|\boldsymbol{\zeta}(t_{k+1}) - \boldsymbol{\zeta}_h^{k+1}| + \|\mathbf{u}_h^{k+1} - \mathbf{U}^{k+1}\|_{\mathcal{L}^2(\mathcal{O})} &\leqslant (1 + C_0\Delta t)[(1 + C_0\Delta t)^k - 1]\Delta t \\
&+ C_0(1 + C_0\Delta t)^{k+1} \left(\left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,t_k))} + \left\| \frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(0,t_k))} \right) \Delta t \\
&+ C_0\Delta t \left(\Delta t + \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} + \left\| \frac{d^2}{dt^2}[\mathbf{u} \circ \widetilde{\boldsymbol{\psi}}] \right\|^2_{\mathcal{L}^2(\mathcal{O}\times(t_k,t_{k+1}))} \right),
\end{aligned}
$$

which implies (7.10) for $k + 1$.

Third step. From the previous steps we conclude that if Δt is small enough and if $h \leqslant C(\Delta t)^2$, then

$$
|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}_h^k - \mathbf{U}^k\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C_1\Delta t \qquad \forall\, k \in \{0, \ldots, N\}.
$$

The above relation, Lemma 5.6, (3.8), and Lemma 4.4 imply that if Δt is small enough and if $h \leqslant C(\Delta t)^2$, then

$$
|\boldsymbol{\zeta}(t_k) - \boldsymbol{\zeta}_h^k| + \|\mathbf{u}_h^k - \mathbf{u}(t_k)\|_{\mathcal{L}^2(\mathcal{O})} \leqslant C\Delta t \qquad \forall\, k \in \{0, \ldots, N\},
$$

which is the conclusion of the theorem. □

8. Concluding remarks. We implemented the numerical method we proposed, and several numerical tests have been performed. Let us briefly describe the results obtained in the case of a rigid ball falling vertically under the action of a vertical force oriented downward. At instant $t = 0$ the velocity field in the fluids and in the solid is supposed to vanish.

We use a mesh with 1432 triangles and 752 vertices (see Figure 2).

Far from the ball the space discretization parameter is $h_1 \approx 0.57$, whereas in the neighborhood of the ball it is given by $h_2 \approx 0.12$. For the time discretization, we choose the time step $\Delta t = 0.1$. Moreover, we choose the radius of the ball equal

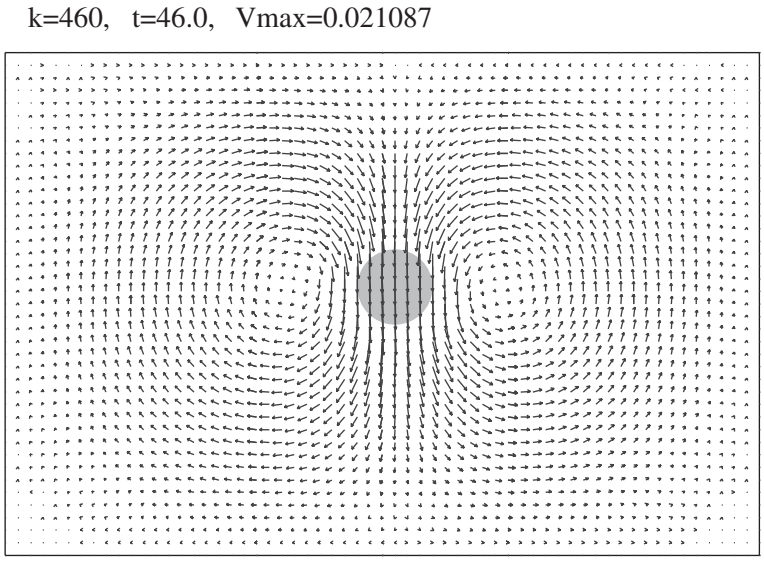

FIG. 3. *Position and velocity field at time* $t = 46.0$.

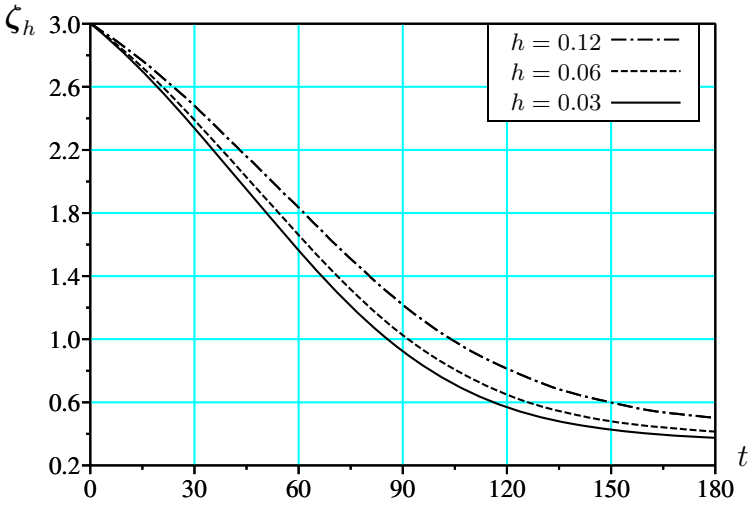

FIG. 4. *Position of the ball.*

to 0.3, the viscosity $\mu = 1$, and the downward force of intensity equal to one (all quantities are given in International System (IS) units). In Figure 3 we represent the configuration of the system for $k = 460$ (corresponding to $t = 46.0$).

We repeated the calculation twice by dividing each mesh size by two (this means that each triangle was each time divided into four smaller triangles). More precisely, we used the meshes described in the table below.

	h	Triangles	Vertices	CPU time
Mesh 1	0.12	1432	752	3 hours
Mesh 2	0.06	5728	2935	11 hours
Mesh 3	0.03	22912	11597	8 days

The last column represents the time used by a Pentium IV computer with a 2.4 GHz CPU clock to achieve the calculation.

In Figure 4 we represented the height of the center of the ball versus the time t for the different meshes.

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