Stochastics and Statistics

Optimal maintenance service contract negotiation with aging equipment

Canek Jackson *, Rodrigo Pascual

Department of Mechanical Engineering, Universidad de Chile, Casilla 2777, Santiago, Chile

Abstract

In recent years, there has been a growing trend to out-source service operations in which the equipment maintenance is carried out by an external agent rather than in-house. Often, the agent (service provider) offers more than one option and the owners of equipment (customers) are faced to the problem of selecting the optimal option, under the terms of a contract. In the current work, we develop a model and report results to determine the agent's optimal strategy for a given type of contract. The model derives in a non-cooperative game formulation in which the decisions are taken by maximizing expected profits. This work extends previous models by considering the realistic case of equipments having an increasing failure intensity due to imperfect maintenance, instead of the standard assumption that considers failure times are exponentially distributed (constant failure intensity). We develop a model using a linear function of time to characterize the failure intensity. The main goal, for the agent, is to determine the pricing structure in the contract and the number of customers to service. On the other hand, for the clients, the main goal is to define the period between planned actions for preventive maintenance and the time to replace equipments. In order to give a complete characterization of the results, we also carry out a sensitivity analysis over some of the factors that would influence over the failure intensity.

Keywords: Game theory; Maintenance; Optimization; Reliability; Stochastic processes

1. Introduction

Reliability of industrial equipments is poor at long-run in the sense that they deteriorate with age and at some point replacement is economically convenient. Maintenance actions have significant impact on the business performance because they are used to control the failure intensity that introduces downtime costs for the owner of failed unit. However, it is usually uneconomical for owners of such equipment to have the specialist tools and personnel in-house, so that it could be needed to out-source the maintenance operations. Here we list some advantages of out-sourcing maintenance: (i) access to high level specialists and latest maintenance technology, (ii) better maintenance due to expertise of service provider, (iii) fixed cost service contracts remove the

* Corresponding author. Tel.: +56 2 978 46 90.
E-mail addresses: cjackson@ing.uchile.cl (C. Jackson), rpascual@ing.uchile.cl (R. Pascual).

risk of high costs, (iv) less capital investment for the owner of equipment, and (v) managers can devote more time to other facets of the business since maintenance management involves less of their time and effort. However there could exist some disadvantages as well: (i) cost of out-sourcing, (ii) dependency on the service provider, and (iii) loss of maintenance knowledge and personnel.

An example situation occurs when the knowledge to carry out the maintenance and the spares for replacement need to be obtained from the original equipment manufacturer (OEM). In this case, the owner of equipment is forced into having a maintenance service contract with the OEM, which may tend to act as a virtual monopolist. Then, maintenance operations are necessarily carried out by an external agent and the owner of the equipment can be viewed as a client of the agent for service providing.

There is a vast literature on maintenance service contracts using qualitative approaches. However, the number of papers dealing with mathematical models is small. As mentioned by Asgharizadeh [1], there is no study which deals with all the issues relevant to maintenance service contracts in a integrated manner.

Murthy and Padmanabhan [2] developed a model for service contracts as extended warranties. In their model, a warranty is an agreement by the manufacturer/seller prior to the sale of a product which allows the customer to seek redress if the product does not perform satisfactorily over a certain period. Then, service contracts are a kind of warranties that extend the coverage period of original warranty. Warranties are an important part of servicing since they determine the post-sales service. A comprehensive literature on extended warranties can be found in [3–5].

Murthy and Yeung [6], using game theory, formulated two models where maintenance actions are carried out by an independent service agent under a service contract. They assume that the failure intensity of equipment being maintained increases with age and consider service contracts involving planned actions policies for preventive maintenance. In the first model, the service provider decides on the price structure and the owner of equipment decides on the time between preventive maintenance actions. In the second model, the situation is extended to the case in which the agent orders spare parts for delivery (inventory for replacement) at optimal periods of time. They derived optimal strategies for the customer and the agent using a Bertrand-Stackelberg game formulation. More information on service contracts concerning about preventive maintenance policies can be found in [7,8].

Murthy and Asgharizadeh [9] developed a Stackelberg game theoretic model formulation to obtain the optimal pricing structure with the service provider as the leader and the customer as the follower. The model assumes exponential failure times so that there is no need for preventive maintenance. By doing so, no consideration is made to the natural increasing failure intensity of real equipments.

Later, Asgharizadeh and Murthy [10] extended their earlier model to include multiple customers but with only a single service channel. This implies that when unit fails, its repair cannot commence immediately if there are one or more failed equipments needing repair. In this case, the number of customers to service is an extra decision variable (in addition to the pricing structure) which the agent must select optimally.

With multiple customers and a single service channel, the mean waiting time for failed unit increases with the numbers of customers that the agent services. One way of reducing this is to have more than one service channel so that more than one failed equipment can be repaired at any given time. However, this results in additional (set up) costs to the agent. Here, the number of service channels is an extra decision variable, so the personnel sizing of the business is determined. Murthy and Asgharizadeh [11] give a complete characterization for this extended case.

The comparative advantage of models described in [6] over those proposed in [9–11] is that they take into consideration equipment having an increasing failure intensity like real mechanical equipments (so carrying out preventive maintenance actions do have sense). The disadvantage lies in the fact that they consider only a single customer to interact with the service provider.

In this paper, we develop a model to take into account the relevant issues of each reference models in an integrated way. The parties are faced to the problem of selecting the price value to be paid for maintenance service providing. Based on that decision the agent determines the number of clients to service in order to maximize its own profit. The pricing structure in the contract is negotiated by solving a non-cooperative game where the Nash equilibrium is encountered. Such interaction is shown in Fig. 1. Once the equilibrium is reached, the surplus generated due to negotiation is maximized by both parties. We use this approach to extend the situation described in previous models concerning monopolist service providers. Then, we study

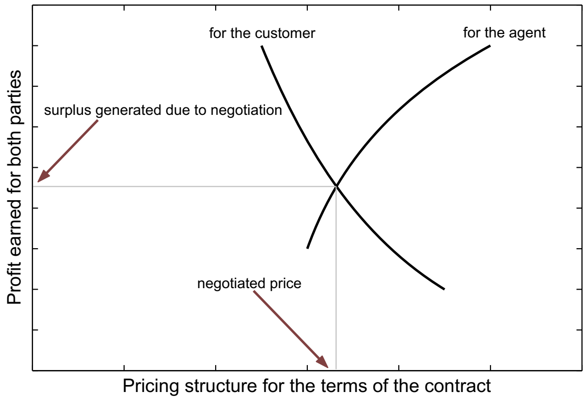

Fig. 1. Characterization of the Nash equilibrium involved in the negotiation.

the case in which neither agent nor customer is leader or follower, such as it happens in a Stackelberg game formulation. A numerical example is presented to illustrate the applicability and other good properties of the proposed formulation.

At the end of the paper, we state some comments concerning the limitations of model developed and propose interest extensions for future models to research about.

2. Methodology

The description of the contract considered in the current work is as follows. Statement (constant fee based contract): For a fixed price of P, the agent offers to carry out maintenance operations (corrective and preventive) over the unit life-cycle. If the failed equipment is not returned to working state within a period τ subsequent to the failure, the agent incurs a penalty. The penalty is proportional to the time delayed after τ, at a cost rate of α for the service provider. This penalty clause is associated to downtime costs that should be partially paid by the agent.

In order to formulate the model, we have first built a mathematical characterization of equipment failures and repairs. Then objective functions for both parties, agent and customer, have been explicitly defined under the terms of the contract described above. The formulation is expressed in terms of the expected value of the random variables involved.

We use a game theoretic formulation to derive the optimal decision for the pricing structure P, the period T between preventive maintenance actions, the period NT between equipment replacements, and the number of customers to service M. In a first stage, the study is carried out by obtaining results for the case in which there is only a single customer to service ($M = 1$). The game solution corresponds to a bargaining value for the pricing structure, $P^{\star}$, as a function of N and T. The remaining variables, $N^{\star}$ and $T^{\star}$, are decided by simple maximization on the goal function of the agent, which is defined as the expected profit per unit time over the life-cycle of equipment. This optimization criterion (definition of the objective function) is appropriate when the life-cycle duration is a variable to decide, and it is mathematically supported by the elementary renewal reward processes theorem [12]. The theorem states that if a cycle is completed every time a renewal process occurs, then the long-run average return is just the expected return earned during a cycle, divided by the expected time of a cycle.

For the analysis to be completed, we extend the results for the case in which there are multiple customers to service ($M > 1$).

2.1. Model formulation

2.1.1. Equipment failures

Each customer owns a single unit which is used to generate revenue for itself. The revenue generated is R per unit time when the equipment is in working state and no income when in failed state. The purchase price of a unit is C_r. In general, such price may include costs associated to investment and labor. Since we use a long-run approach, we will assume that at the end of the life-cycle the equipment is absolutely devalued so that its market recovery value is negligible.

All units are statistically similar in terms of reliability. Such reliability is characterized by an increasing failure intensity due to mechanical wear out. Typically, this kind of aging behavior is well-described by a failure hazard, λ, given by

$$\lambda(t) = \lambda_0 + rt, \tag{1}$$

where λ_0 is the initial failure intensity when the equipment is new, at $t = 0$, and r is the aging rate of the equipment. Note that t is the age of the unit, and not the calendar time since the equipment is new. The aging rate is the parameter that characterizes the reliability of the equipment in terms of its deterioration suffered due to usage.

2.1.2. Equipment repairs

The equipment is subjected to corrective and preventive maintenance. The time to repair is exponentially distributed with repair rate μ. Since preventive maintenance actions are planned, we assume the time taken to carry them out is short compared to the mean time to repair, $1/\mu$, associated to corrective (emergency) actions.

During corrective actions, the failed unit is returned back to working state by minimal repair (the equipment reliability stays "as bad as old"). This approach is appropriate for large complex systems where the failure occurs due to one (or a few) component(s) failing. As a result, the age of the system after repair is nearly the same as that before (since the repaired components have a negligible impact on the system as a whole).

For preventive actions, the unit receives $N - 1$ imperfect overhauls during its life ($N \geqslant 1$, integer). An overhaul improves the equipment in term of its failure intensity λ. Let $\lambda_n(t)$ be the failure intensity after the nth overhaul ($n = 1, \ldots, N - 1$). Zhang and Jardine [13] expressed such failure intensity as

$$\lambda_n(t) = p\lambda_{n-1}(t - T) + (1 - p)\lambda_{n-1}(t), \tag{2}$$

where p is an improvement factor that characterizes the quality of the overhauls ($0 < p < 1$). Such situation is shown in Fig. 2. If $p = 0$, the system behaves as minimally repaired (reliability stays "as bad as old" after repair). If $p = 1$, it means overhaul actions are perfect in term of improving unit reliability (reliability becomes "as good as new " after repair). In order to seek for more detail, Pham and Wang [14] present a review for models concerning imperfect maintenance.

The interval between overhauls T is constant. The equipment is periodically replaced once its life-cycle has been reached. Then, the life-cycle of the unit, $L = NT$, is also a variable to negotiate since N and T are variables too.

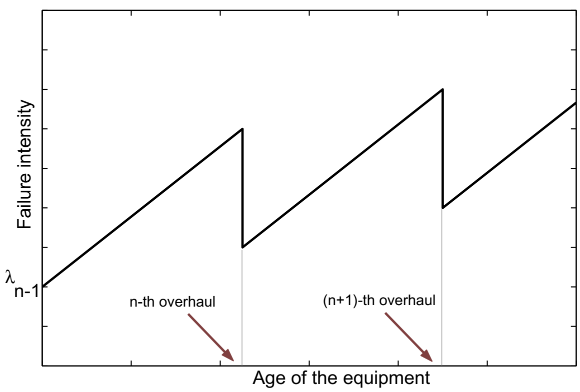

Fig. 2. Characterization of the failure intensity modeled by imperfect overhauls.

2.1.3. Customer's decision problem

For customer j ($j = 1, \ldots, M$), let the number of failures over $[0, NT)$ be F_j. Let Y_{ji} denote the time taken to make the equipment operational after ith failure. This time includes the waiting time and the time taken to repair. Let ω denote the profit to jth customer. Then, it is easily seen that

$$\omega = R\left(NT - \sum_{i=1}^{F_j} Y_{ji}\right) + \alpha\left(\sum_{i=1}^{F_j} \max\{0, Y_{ji} - \tau\}\right) - C_r - P. \tag{3}$$

Note that ω is a random variable since failures occur in an uncertain manner. In addition, it can be easily seen that F_j and Y_{ji} describe stochastic processes. Then, these would be characterized in terms of their expected values.

2.1.4. Agent's decision problem

Let C_m and C_o be the costs of each repair (corrective actions) and overhaul (preventive ones) for the agent, respectively. These costs include material and workforce. Let π denote the service provider's profit. It is easily seen that

$$\pi = \sum_{j=1}^{M}\left[P - C_m F_j - C_o(N-1) - \alpha\left(\sum_{i=1}^{F_j} \max\{0, Y_{ji} - \tau\}\right)\right]. \tag{4}$$

Both parties, service provider and unit owner, negotiate the pricing value P by solving the involved game formulation via a Nash bargaining solution. This solution corresponds to the well-known Nash equilibrium for non-cooperative games (see, for example, [15,16]).

2.2. Model analysis

2.2.1. Simplifying assumptions

For the analysis to be tractable, we state the following assumptions:

1. All clients are identical in their attitudes to risk. If not, the negotiation of each service contract may result into different equilibria.
2. Failed equipments are repaired on a first-come, first-repair basis. This implies that the system behaves like a stochastic process queue model.
3. The life-cycle $L = NT$ of all units is assumed to be sufficiently large. This implies that we can use a steady state distribution for Y_{ji} in our analysis.
4. We assume that $M\lambda_0 < \mu$. If not, the queue builds up with time and as a result the total time of each failed unit that is in the system waiting to be repaired increases.

2.2.2. Expected value for time to wait and repair

The model formulation is identical to a Markovian queue with finite population (M) with a single server and first-come, first-served rule. When there are k failed units, then the arrival rate of failed units is $\lambda_k = (M-k)\lambda$ for $0 \leqslant k \leqslant M$. The service (repair) rate is μ since there is only one server. Let $f(y)$ denote the steady state density function for Y_{ji}. Then, using results from queueing theory [17], we have that

$$f(y) = \sum_{k=0}^{M-1} P_k \mu e^{-\mu y} \frac{(\mu y)^k}{k!}, \tag{5}$$

where P_k ($k = 0, \ldots, M-1$) is given by

$$P_k = \frac{(M-k)(\lambda/\mu)^k \{M!/(M-k)!\}}{\sum_{k=0}^{M-1}(M-k)(\lambda/\mu)^k \{M!/(M-k)!\}}. \tag{6}$$

It can be shown that the expected value of Y_{ji} is given by

$$E[Y_{ji}] = \int_0^\infty y f(y)\,\mathrm{d}y = \sum_{k=0}^{M-1} \frac{P_k(k+1)}{\mu}. \tag{7}$$

Also, we can obtain the following expression for the expected value of $\max\{0, Y_{ji} - \tau\}$

$$E[\max\{0, Y_{ji} - \tau\}] = \int_\tau^\infty (y-\tau) f(y)\,\mathrm{d}y = \sum_{k=0}^{M-1} P_k \mu^k \mathrm{e}^{-\mu\tau} \left[(k+1-\mu\tau) \sum_{l=0}^{k} \frac{\tau^{k-l}}{\mu^{l+1}(k-l)!} + \frac{\tau^{k+1}}{k!} \right]. \tag{8}$$

The expression given by (7) is easy to derive. However, the one defined by the integral in (8) could be a little more difficult to obtain. Here we present the formula that allows derive this last expression. The interested reader could demonstrate it by mathematical induction on k

$$\int_\tau^\infty y^k \mathrm{e}^{-\mu y}\,\mathrm{d}y = k!\,\mathrm{e}^{-\mu\tau} \sum_{l=0}^{k} \frac{\tau^{k-l}}{\mu^{l+1}(k-l)!}. \tag{9}$$

Note that the expressions obtained depend on P_k, so on λ since $P_k = P_k(\lambda)$. Then, since λ is defined by time dependency (1) and overhaul dependency (2), we need an analytical expression for the mean value of failure intensity, $\bar{\lambda}$.

2.2.3. Expected value for number of failure times

Let $H(t)$ be the expected value of the number of failure times in the period $[0, t)$ when the unit is not subjected to any overhaul. On the other hand, let $\widehat{H}(t)$ denote the expected value of the number of failure times when there are overhauls that were carried out over the period $[0, t)$. Then, it can be demonstrated [13] that

$$\widehat{H}(NT) = \sum_{n=0}^{N} \binom{N}{n} p^{N-n} (1-p)^{n-1} H(nT), \tag{10}$$

where

$$H(nT) = \int_0^{nT} \lambda(t)\,\mathrm{d}t = \lambda_0 nT + r\frac{(nT)^2}{2}. \tag{11}$$

Then, using (10) we finally have that the expected number of failure times over the equipment life-cycle, $\widehat{H}(NT) = E[F_j]$, is given by

$$\widehat{H}(NT) = \lambda_0 NT + rT^2 \frac{N^2(1-p) + Np}{2}. \tag{12}$$

Here we present the formulas that allow calculate (12) from (10) and (11). The interested reader could derive them by mathematical induction on N

$$\sum_{n=0}^{N} \binom{N}{n} p^{N-n} (1-p)^{n-1} n = N, \tag{13}$$

$$\sum_{n=0}^{N} \binom{N}{n} p^{N-n} (1-p)^{n-1} n^2 = N^2(1-p) + Np. \tag{14}$$

Also, it is easily seen that we can obtain the mean value of failure intensity as

$$\bar{\lambda} = \frac{\widehat{H}}{NT} = \lambda_0 + rT\frac{N(1-p) + p}{2}, \tag{15}$$

where for notation simplicity, we have suppressed the argument (NT) from the expected number of failure times $\widehat{H}(NT)$. In addition, note that the probabilities P_k that appear should be referred to the mean value of the failure intensity; i.e., $P_k = P_k(\bar{\lambda}) = P_k(NT)$.

2.2.4. Customer's decision problem

Then, the profit function (3) is given by the following expression, in terms of expected values for the random variables involved, in the following way:

$$E[\omega] = R\left(NT - \widehat{H}\sum_{k=0}^{M-1}\frac{P_k(k+1)}{\mu}\right) + \alpha\widehat{H}\sum_{k=0}^{M-1}P_k\mu^k\,\mathrm{e}^{-\mu\tau}\left(\sum_{l=0}^{k}\frac{\tau^{k-l}(k+1-\mu\tau)}{\mu^{l+1}(k-l)!} + \frac{\tau^{k+1}}{k!}\right) - C_r - P. \tag{16}$$

2.2.5. Agent's decision problem

Likewise, the profit function (4) is given by the next expression, in terms of expected values for the random variables involved, in the following way:

$$E[\pi] = M\left[P - C_m\widehat{H} - C_o(N-1) - \alpha\widehat{H}\sum_{k=0}^{M-1}P_k\mu^k\,\mathrm{e}^{-\mu\tau}\left(\sum_{l=0}^{k}\frac{\tau^{k-l}(k+1-\mu\tau)}{\mu^{l+1}(k-l)!} + \frac{\tau^{k+1}}{k!}\right)\right]. \tag{17}$$

3. Results

As mentioned earlier, for simplicity, we first consider the case in which there is a single customer negotiation, $M = 1$. Then, the respective expected profits for client and service provider are simplified as follows:

$$\omega = R\left(NT - \frac{\widehat{H}}{\mu}\right) + \alpha\widehat{H}\frac{\mathrm{e}^{-\mu\tau}}{\mu} - C_r - P, \tag{18}$$

$$\pi = P - C_m\widehat{H} - C_o(N-1) - \alpha\widehat{H}\frac{\mathrm{e}^{-\mu\tau}}{\mu}. \tag{19}$$

The bargaining solution for P is obtained from the condition of a Nash equilibrium in which, as a result from negotiation, the total surplus generated (monetary income earned with respect to the situation in which no business is done) is divided into equal parts (a half for the customer and a half for the agent). Note that the applicability of the bargaining solution, in which surplus is divided into equal parts, could be subjected to the cases where client is not more powerful than the contractor and viceversa. These are common situations in which clients are customers for industrial and commercial products. Often, these products are either manufactured by a monopolist or by a small number of manufacturers, so the terms and conditions of the contract are determined jointly by the customer and the service provider.

As a result, the bargaining condition gives $\omega = \pi$. Then, the price to be bargained satisfies the following condition

$$R\left(NT - \frac{\widehat{H}}{\mu}\right) + \alpha\widehat{H}\frac{\mathrm{e}^{-\mu\tau}}{\mu} - C_r - P^\star = P^\star - C_m\widehat{H} - C_o(N-1) - \alpha\widehat{H}\frac{\mathrm{e}^{-\mu\tau}}{\mu}. \tag{20}$$

Then, it is easily seen that

$$P^\star = \frac{R}{2}\left(NT - \frac{\widehat{H}}{\mu}\right) + \alpha\widehat{H}\frac{\mathrm{e}^{-\mu\tau}}{\mu} + \frac{C_m}{2}\widehat{H} + \frac{C_o}{2}(N-1) - \frac{C_r}{2}. \tag{21}$$

Evaluating on the expected profit given by (19), we have

$$\pi(N,T) = \frac{R}{2}\left(NT - \frac{\widehat{H}}{\mu}\right) - \frac{C_m}{2}\widehat{H} - \frac{C_o}{2}(N-1) - \frac{C_r}{2}. \tag{22}$$

Finally, we can define explicitly the objective function $f(N,T)$ to optimize, as the amount to be returned (by the agent) at long-run due to negotiation, in the following way

$$f(N,T) = \frac{\pi(N,T)}{NT} = \frac{R}{2}\left(1 - \frac{\bar{\lambda}}{\mu}\right) - \frac{C_m}{2}\bar{\lambda} - \frac{C_o}{2}\left(\frac{1}{T} - \frac{1}{NT}\right) - \frac{C_r}{2NT}. \tag{23}$$

3.1. Numerical example

We consider the following data from reference [10], taking the extra parameters needed for our model from [13]: $\lambda_0 = 0.08$ (10^{-2}/hours), $\alpha = 0.06$(10^3 \$/hours), $r = 0.01$ (10^{-5}/hours^2), $C_r = 200$ (10^3 \$), $C_m = 1$ (10^3 \$), $C_o = 8$ (10^3 \$), $\mu = 0.02$ (1/hours), $p = 0.7$, $R = 0.015$ (10^3 \$/hours), $\tau = 70$ (hours).

The example was solved using the optimization solver from Microsoft Excel©. The following results are obtained: $N^\star = 7$, $T^\star = 12025$ (hours), so that the life-cycle to replace the unit is around 41.50 years assuming the equipment operates 45 hours a week, and 45 weeks a year. As a result, the value negotiated for the pricing structure is given by $P^\star = 736.11$ (10^3 \$).

In Fig. 3, we observe the topology of the objective function around the optimal solution $(N^\star, T^\star)$. It is easily distinguished from it a set of level curves showing the existence of a global maximum.

Table 1 gives optimal solutions for each N. It is clearly shown that the maximum expected profit per unit time is reached at $N = 7$, where an optimal value of $f^\star = 7.80$ (10^3 \$/year) is obtained for the objective function.

Table 2 gives optimal solutions for various improvement factors, at fixed nominal value $N = 7$. The sensitivity analysis is carried out by varying this factor from $p = 0.7$ to $p = 0.3$. Note that for notation simplicity the symbol (★) has been suppressed from the decision variables.

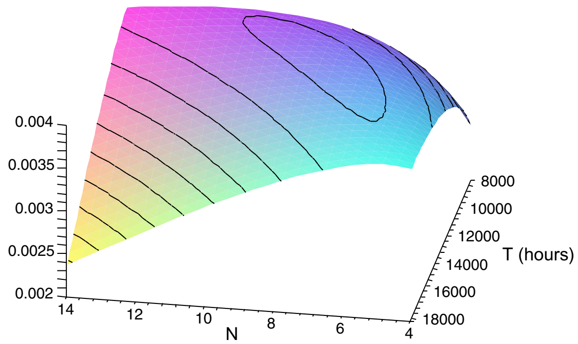

Fig. 3. Expected profit per unit time (10^3 \$/hour).

Table 1
Optimal solutions for the nominal values of a single customer to service

N	T (hours)	P (10^3 \$)	L (years)	f (10^3 \$/year)
2	30,237	502.18	29.75	6.81
3	22,678	572.06	33.50	7.34
4	18,353	624.08	36.25	7.59
5	15,525	666.56	38.25	7.72
6	13,522	703.24	40.00	7.78
7	12,025	736.11	41.50	7.81
8	10,862	766.31	43.00	7.80
9	9930	794.54	44.25	7.79

Table 2
Optimal solutions for different improvement factors

p	T (hours)	P (10^3 \$)	L (years)	f (10^3 \$/year)
0.7	12,025	736.11	41.50	7.80
0.6	10,913	672.32	37.75	7.20
0.5	10,061	623.48	34.75	6.64
0.4	9382	584.54	32.50	6.12
0.3	8824	522.55	30.50	5.64

Table 3
Optimal solutions for different aging rates

r (10^{-5}/hours^2)	T (hours)	P (10^3 \$)	L (years)	f (10^3 \$/year)
0.01	12025	736.11	41.50	7.80
0.02	8503	534.15	29.50	5.33
0.03	6943	444.67	24.00	3.44
0.04	6013	391.33	20.75	1.84
0.05	5378	354.93	18.50	0.43

Table 3 gives the optimal solutions for various aging rates, at fixed nominal value $N = 7$. The sensitivity analysis is carried out by increasing this rate to a maximum value five times larger than the reference value. Note that the reference values used for the current sensitivity analysis are $p = 0.7$ and $r = 0.01$ (10^{-5}/hours^2).

It is seen that larger values of the improvement factor result in longer periods of time between preventive maintenance actions, which means that the reliability of the equipment has been improved. As a result, the life-cycle of the equipment is larger and a larger pricing should be negotiated as well. The reduction over the failure intensity, due to a better overhaul performance, increases the expected profit since downtime costs are diminished via the reduction over the expected number of failure times. We also observe the same behavior as the aging rate diminishes, since the reliability of equipments gets worse as long as the unit deteriorates rapidly with time.

3.2. Multiple customers to service

For the case in which there is more than one client, the condition over the Nash solution for the price value of the contract should be applied on each customer, so the negotiation on the pricing structure, P, is given by

$$\omega(P^\star, M, N, T) = \frac{\pi(P^\star, M, N, T)}{M}. \tag{24}$$

This bargaining condition relies on the following negotiated price value, $P^\star$, obtained from the expressions (16) and (17)

$$P^\star = \frac{R}{2}\left(NT - \widehat{H}\sum_{k=0}^{M-1}\frac{P_k(k+1)}{\mu}\right) + \alpha\widehat{H}\sum_{k=0}^{M-1}P_k\mu^k e^{-\mu\tau}\left(\sum_{l=0}^{k}\frac{\tau^{k-l}(k+1-\mu\tau)}{\mu^{l+1}(k-l)!} + \frac{\tau^{k+1}}{k!}\right) - \frac{C_r}{2} + \frac{C_m}{2}\widehat{H} + \frac{C_o}{2}(N-1). \tag{25}$$

By replacing $P = P^\star$ on the expression (17), we obtain the expected profit for the agent as a result from negotiation

$$\pi(M, N, T) = M\left[\frac{R}{2}\left(NT - \widehat{H}\sum_{k=0}^{M-1}\frac{P_k(k+1)}{\mu}\right) - \frac{C_m}{2}\widehat{H} - \frac{C_o}{2}(N-1) - \frac{C_r}{2}\right]. \tag{26}$$

Then, using the bargaining value $P^\star$, we can define explicitly the objective function $f(M, N, T)$ in the following way:

$$f(M, N, T) = \frac{\pi(M, N, T)}{NT} = M\left[\frac{R}{2}\left(1 - \bar{\lambda}\sum_{k=0}^{M-1}\frac{P_k(k+1)}{\mu}\right) - \frac{C_m}{2}\bar{\lambda} - \frac{C_o}{2}\left(\frac{1}{T} - \frac{1}{NT}\right) - \frac{C_r}{2NT}\right]. \tag{27}$$

The optimal solution was encountered by exhaustive finding on M; i.e., solving the optimization problem for $f(2, N, T), f(3, N, T), \ldots, f(M, N, T)$, where the largest value used for M is given by the restriction $M < \mu/\lambda_0 = 0.02/0.0008 = 25$. In Fig. 4, we can observe that there's a point in which the income earned by the negotiation with multiple customers is less than the outcome due to excessive downtime costs. The results are compared to the reference model proposed by Asgharizadeh and Murthy [10], shown in red color. Note that the restriction associated to the number of customers becomes active for a shorter value than

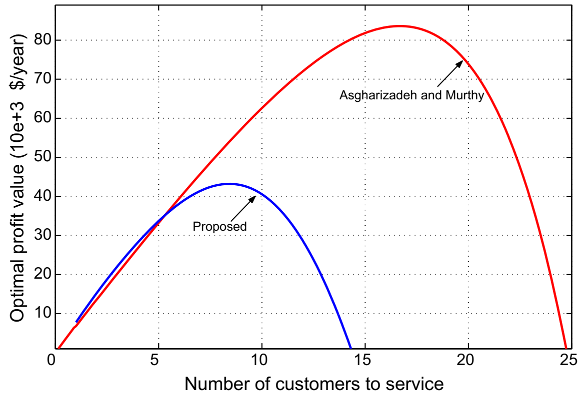

Fig. 4. Model from Asgharizadeh and Murthy versus proposed.

$M = 25$ (it is reached for $M = 14$). The explanation relies in the fact that the failure intensity, defining the maximum value for M, is larger than λ_0 since the units age with time. Then, as a result, the agent selects a smaller number of clients to optimize its objective function ($M^\star = 8$, for the current model, versus $M^\star = 17$, for the reference model). Because of this, the expected profit per unit time also relies on a shorter value compared to the reference model.

Using the same data defined on the previous example, we obtain the following results: $M^\star = 8$, $N^\star = 7$, $T^\star = 7554$ (hours), so that the life-cycle for equipment replacement is around 26 years. As a result, the bargained pricing is given by $P^\star = 708.86$ (10^3 \$).

Note that, as a result from interaction with multiple clients, the equilibrium price increases. The reason for this is as follows. Larger M values result in longer waiting time for repair, and hence, greater expected penalty cost. This implies that the service provider's profit (as a function of P) moves downward and the customer's profit moves upward. Then, the Nash equilibrium due to bargaining is reached for a greater value of the pricing structure and a greater generated surplus as well.

Table 4 shows optimal solutions for each M. It is clearly observed that the maximum expected profit per unit time is reached at $M = 8$, where an optimal value of $f^\star = 43.05$ (10^3 \$/year) is obtained for the objective function.

Table 5 shows the effect of improvement factor variations over the optimal number of customers to service. The sensitivity analysis is carried out by varying p from 0.7 to 0.3.

Table 6 shows the effect of aging rate variations over the optimal number of customers to service. The sensitivity analysis is carried out by varying r from 0.01 (10^{-5}/hours^2) to 0.05 (10^{-5}/hours^2).

As observed, the optimal number of clients to service, $M^\star$, decreases as the improvement factor decreases and/or the aging rate increases. This implies that as the unit becomes more unreliable, the optimal strategy for the agent is to reduce the number of customers in order to maximize the expected profit at the long-run. As a

Table 4
Optimal solutions for the nominal values of multiple customers to service

M	P (10^3 \$)	f (10^3 \$/year)
2	755.25	15.25
3	775.93	22.09
4	790.33	28.28
5	759.10	33.70
6	752.78	38.15
7	747.38	41.37
8	708.86	43.05
9	703.95	42.88

Table 5
Effect of improvement factor variations

p	M	f (10^3 \$/year)
0.7	8	42.61
0.6	7	34.68
0.5	7	29.41
0.4	6	25.24
0.3	6	22.02

Table 6
Effect of aging rate variations

r (10^{-5}/hours^2)	M	f (10^3 \$/year)
0.01	8	42.61
0.02	5	17.76
0.03	3	7.38
0.04	2	2.57
0.05	1	0.43

result, the blue curve shown in Fig. 4 moves downward and its maximum value is reached for a smaller number of clients.

4. Discussion

This work has presented a Nash game based formulation to negotiate pricing in service contracts such that both parties share expected profits in a bargaining way. The current model considers a failure intensity which is linear with time, in order to characterize the mechanical wear out due to the aging of the unit. In a first approach, both parties determine the optimal strategy to decide on the number of preventive maintenance actions to be carried out and also decide on the life-cycle of the unit. Furthermore, the situation is extended to the case in which the agent has to select the optimal number of clients to be served.

In the model studied, we have used a long-run criterion for optimization. If the life-cycle is not very large (compared to any characteristic time such as the total mean time to repair), this criterion could be useless. In addition, the assumption of a steady state distribution for the total time (waiting + repair) associated to the Markovian queue is no longer appropriate. In this case, one would need to use simulation approaches to determine the solutions with a non-negligible market recovery value for the purchased unit.

Another limitation on the model is that the time taken to overhaul equipments was assumed short. If it is not, larger values of downtime costs are obtained due to overhauling. Then, as a result, the cost structure that is evaluated in the current paper corresponds to an underestimated value.

Finally, the customer could have a strong input into the post-sale support (including maintenance operations) of a product. If so, the Nash equilibrium (solution for the bargaining game) is no longer valid. This case corresponds to a single customer for the product, in which a leader–follower solution takes place. A typical example of this is the federal government buying specialist products such as rockets, tanks, ships, among others. Here the government acts like the leader and the service provider like the follower. On the other hand, if the customer is not well informed about the product and lacks the knowledge to maintain it, the contractor may act as the leader.

The model can be extended in several ways. Here we list a few of those:

1. There is more than one type of service contract and these differ in their price structure and penalty clauses.
2. The model assumes that the failure intensity is linear with time. One would use another type of behavior, such as, Weibull's failure time distributions and exponential increasing failure rates.
3. The case of two or more service providers such that the competition between them becomes an important variable.

4. The model assumes that the repair times are exponentially distributed. Mahon and Bailey [18] suggest that distributions with decreasing repair rate are more appropriate for modelling repair times.
5. The agent can repair more than one failed unit at any given time; i.e., it employs more than one service channel.
6. The case in which one of the parties is more powerful than the other, so that a Stackelberg game formulation may be more appropriate to model the interaction between client and contractor.

Acknowledgments

The authors acknowledge the partial funding for this work by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grant 1030943. We also thank comments from José Miguel Benavente, Department of Economy, Universidad de Chile. Finally, the authors acknowledge comments from Dragan Banjevic and Andrey Pak, Condition Based Maintenance Laboratory, University of Toronto.

References

[1] E. Asgharizadeh, Modelling and analysis of maintenance service contracts, Thesis for the degree: Doctor of Philosophy, The University of Queensland, 1997.
[2] D.N.P. Murthy, V. Padmanabhan, A Continuous Time Model of Warranty, Graduate School of Business, Stanford University, Stanford, 1993.
[3] D.N.P. Murthy, W.R. Blischke, Strategic warranty management: A life-cycle approach, IEEE Transactions on Engineering Management 47 (1) (2000) 40–50.
[4] I. Djamaludin, D.N.P. Murthy, Warranty and preventive maintenance, International Journal of Reliability, Quality and Safety Engineering 8 (2) (2001) 89–107.
[5] D.N.P. Murthy, O. Solem, T. Roren, Product warranty logistics: Issues and challenges, European Journal of Operational Research 156 (2004) 110–126.
[6] D.N.P. Murthy, V. Yeung, Modelling and analysis of maintenance service contracts, Mathematical and Computer Modelling 22 (1995) 219–225.
[7] W.P. Pierskella, J.A. Voelker, The control and surveillance of deteriorating system, Naval Research Logistics Quarterly 23 (1976) 353–388.
[8] L.C. Thomas, A survey of maintenance and replacement models for maintainability and reliability of multi-item systems, Reliability Engineering 16 (1986) 297–309.
[9] D.N.P. Murthy, E. Asgharizadeh, A stochastic model for service contracts, International Journal of Reliability Quality and Safety (1996).
[10] E. Asgharizadeh, D.N.P. Murthy, Service contracts: A stochastic model, Mathematical and Computer Modelling 31 (2000) 11–20.
[11] D.N.P. Murthy, E. Asgharizadeh, Optimal decision making in a maintenance service operation, European Journal of Operational Research 116 (1999) 259–273.
[12] S.M. Ross, Stochastic Processes, Wiley, New York, 1983.
[13] F. Zhang, A.K.S. Jardine, Optimal maintenance models with minimal repair, periodic overhaul and complete renewal, IIE Transactions 30 (1998) 1109–1119.
[14] H. Pham, H. Wang, Imperfect maintenance, European Journal of Operational Research 94 (1996) 425–438.
[15] A. Rubinstein, Perfect equilibrium in a bargaining model, Econometrica 50 (1) (1982) 97–109.
[16] K. Binmore, A. Rubinstein, A. Wolinsky, The Nash bargaining solution in economic modelling, Rand Journal of Economics 17 (2) (1986) 176–188.
[17] D. Gross, C.M. Harris, Fundamentals of Queueing Theory, Wiley, New York, 1985.
[18] B.H. Mahon, R.J.M. Bailey, A proposed improvement replacement policy for army vehicles, Operational Research Quarterly 26 (1975) 477–494.