The Toda system and multiple-end solutions of autonomous planar elliptic problems

Manuel del Pino [*] Michał Kowalczyk [†] Frank Pacard [‡] Juncheng Wei [§]

Abstract

We construct a new class of positive solutions for the classical elliptic problem

$$\Delta u - u + u^p = 0, p > 2, \quad \text{in } \mathbb{R}^2.$$

We establish a deep relation between them and the following Toda system

$$c^2 f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}} \quad \text{in } \mathbb{R}, \quad j = 1, \ldots, k.$$

We show that these solutions have the approximate form $u(x,z) \sim \sum_{j=1}^k w(x - f_j(z))$ where w is the unique even, positive, asymptotically vanishing solution of $w'' - w + w^p = 0$ in $\mathbb{R}$. Functions $f_j(z)$, representing the multiple ends of $u(x,z)$, solve the aforementioned Toda system, they are even, asymptotically linear, with

$$f_0 \equiv -\infty < f_1 \ll \cdots \ll f_k < f_{k+1} \equiv +\infty.$$

The solutions of the elliptic problem we construct have their counterpart in the theory of constant mean curvature surfaces. An analogy can also be made between their construction and the gluing of constant scalar curvature Fowler singular metrics in the sphere.

Contents

[*]Manuel del Pino - Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile. Email: delpino@dim.uchile.cl

[†]Michał Kowalczyk - Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile. Email kowalczy@dim.uchile.cl

[‡]Franck Pacard - Université Paris 12 and Institut Universitaire de France Email pacard@univ-paris12.fr

[§]Juncheng Wei - Department of Mathematics, Chinese University of Hong Kong, Shatin, Hong Kong Emailwei@math.cuhk.edu.hk

1 Introduction and statement of main results

This paper deals with the classical semilinear elliptic problem

$$\Delta u - u + u^p = 0, \quad u > 0, \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $p > 1$. Equation (1.1) arises for instance as the standing-wave problem for the standard nonlinear Schrödinger equation

$$i\psi_t = \Delta_y \psi + |\psi|^{p-1}\psi,$$

typically $p = 3$, corresponding to that of solutions of the form $\psi(y,t) = u(y)e^{-it}$. It also arises in nonlinear models in Turing's theory biological theory of pattern formation [39] such as the Gray-Scott or Gierer-Meinhardt systems, [16, 15]. The solutions of (1.1) which decay to zero at infinity are well understood. Problem (1.1) has a radially symmetric solution $w_N(y)$ which approaches 0 at infinity provided that

$$1 < p < \begin{cases} \frac{N+2}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}$$

see [38, 3]. This solution is unique [21], and actually any positive solution to (1.1) which vanishes at infinity must be radially symmetric around some point[14].

Problem (1.1) and its variations have been broadly treated in the PDE literature in the last two decades. These variations are mostly of one of the two types: (1.1) is changed to a non-autonomous problem with a potential depending on the space variable; or (1.1) is considered in a bounded domain under suitable boundary conditions. Typically, in both versions a small parameter is introduced rendering (1.1) a singular perturbation problem. We refer the reader to the works [1, 2, 5, 9, 10, 11, 13, 17, 18, 23, 25, 26, 34, 32, 33] and references therein. Many constructions in the literature refer to "multi-bump solutions", built by perturbation of a sum of copies of the basic radial bump w_N suitably scaled, with centers adjusted in equilibrium under appropriate constraints on the potential or the geometry of the underlying domain.

Much less is known about solutions to this equation in entire space which do not vanish at infinity (while they are all known to be bounded, see [36]). For example, the solution w_N of (1.1) in $\mathbb{R}^N$ induces a solution in $\mathbb{R}^{N+1}$ which only depends on N variables. This solution vanishes asymptotically in all but one variable. For simplicity, we restrict ourselves to the case $N = 2$, and consider positive solutions $u(x, z)$ to problem (1.1) which are even in z and vanish as $|x| \to +\infty$, namely

$$u(x, z) = u(x, -z) \quad \text{for all } (x, z) \in \mathbb{R}^2, \tag{1.2}$$

and

$$\lim_{|x|\to+\infty} u(x, z) = 0 \quad \text{for all } z \in \mathbb{R}. \tag{1.3}$$

A canonical example is thus built from the one-dimensional bump w_1, which we denote in the sequel just by w, namely the unique solution of the ODE

$$w'' - w + w^p = 0, \quad w > 0, \quad \text{in } \mathbb{R}, \tag{1.4}$$

$$w'(0) = 0, \quad w(x) \to 0 \quad \text{as } |x| \to +\infty, \tag{1.5}$$

corresponding in phase plane to a homoclinic orbit for the equilibrium 0. Using this function we can define a family of solutions u of equation (1.1) with the properties (1.2)-(1.3) setting $u(x, z) := w(x-a)$, $a \in \mathbb{R}$. By analogy with the above terminology, we may call these solutions "single bump-lines". A natural question is whether a solution that satisfies (1.2)-(1.3) and which is in addition even in x must equal $w(x)$. The solution w of (1.1) was found to be isolated by Busca and Felmer in [4] in a uniform topology which avoids oscillations at infinity. On the other hand, a second class of solutions which are even both in z and x was discovered by Dancer in [7] via local bifurcation arguments. They constitute a one-parameter family of solutions which are periodic in the z variable and originate from $w(x)$. Let us briefly review their construction: we consider problem (1.1) with T-periodic conditions in z,

$$u(x, z + T) = u(x, z) \quad \text{for all } (x, z) \in \mathbb{R}^2, \tag{1.6}$$

and regard $T > 0$ as a bifurcation parameter. The linearized operator around the single bump line is

$$L(\phi) = \phi_{zz} + \phi_{xx} + (pw^{p-1} - 1)\phi.$$

It is well known that the eigenvalue problem

$$\phi_{xx} + (pw^{p-1} - 1)\phi = \lambda\phi, \tag{1.7}$$

has a unique positive eigenvalue λ_1 with $Z(x)$ a positive eigenfunction. We observe that the operator L has a bounded element of its kernel given by

$$Z(x) \cos(\sqrt{\lambda_1} z),$$

which turns out to be the only one which is even, both in x and z variable, and in addition $T = \frac{2\pi}{\sqrt{\lambda_1}}$-periodic in z. Crandall-Rabinowitz bifurcation theorem can then be adapted to yield existence of a continuum of solutions bifurcating at this value of T, periodic in z with period $T_\delta = \frac{2\pi}{\sqrt{\lambda_1}} + O(\delta)$. They are uniformly close to $w(x)$ and their asymptotic formula is:

$$w_\delta(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2)e^{-|x|}.$$

We refer to the functions w_δ in what follows as *Dancer solutions.*

The purpose of this paper is to construct a new type of solutions of (1.1) in $\mathbb{R}^2$ that have multiple ends in the form of multiple bump-lines, and satisfy in addition properties (1.2)-(1.3). To explain let us consider the function

$$w_*(x) = \sum_{j=0}^{k} w(x - a_j).$$

While for numbers $a_1 \ll a_2 \ll \cdots \ll a_k$, w_* appears to be a very good approximation to a solution of the ODE (1.4), the only solutions of (1.4) which go to zero at infinity must be single translates of $w(x)$. Our main result in this paper is that, in the two dimensions, there exist solutions $u(x,z)$ with a profile of this type for each fixed z. Of course in this case the numbers a_j must be replaced by non-constant functions $f_j(z)$. In addition, it will be necessary to use as basic cells not just the standard bump-lines, but rather the *wiggling bump lines* found by Dancer.

Thus, what we actually look for is a solution $u(x,z)$ which is close, up to lower order terms, to a multi bump-line of the form

$$w_*(x,z) = \sum_{j=0}^{k} w_{\delta_j}(x - f_j(z), z), \tag{1.8}$$

for suitable small numbers δ_j and even functions

$$f_1(z) \ll f_2(z) \ll \cdots \ll f_k(z),$$

which have uniformly small derivatives. The functions f_j cannot be arbitrary and they turn out to satisfy (asymptotically) a second order system of differential equations, the *Toda system*, given by

$$c_p^2 f_j'' = e^{f_{j-1}-f_j} - e^{f_j - f_{j+1}} \quad \text{in } \mathbb{R}, \quad j = 1,\dots,k, \tag{1.9}$$

with the conventions $f_0 = -\infty$, $f_{k+1} = +\infty$, where c_p is an explicit positive constant that will be specified later. In agreement with the symmetry requirement (1.2) we consider even solutions of system (1.9), namely

$$f_j(z) = f_j(-z) \quad \text{for all } z \in \mathbb{R}. \tag{1.10}$$

We observe that for an even solution $\mathbf{f} = (f_1,\dots,f_k)$ of this system, function $\mathbf{f}_\alpha$ defined by

$$\mathbf{f}_\alpha = (f_{\alpha 1},\dots,f_{\alpha k}), \quad f_{\alpha j}(z) := f_{1j}(\alpha z) + (j - \frac{k+1}{2})\log\frac{1}{\alpha} \tag{1.11}$$

is also an even solution of the system. Our main result in this paper asserts that given such an $\mathbf{f}$ whose values at $z = 0$ are ordered, for all sufficiently small α there exists a multi bump-line solution of approximate form (1.8) with $\mathbf{f}$ replaced with $\mathbf{f}_\alpha$ and suitably chosen small numbers δ_j dependent on α. Thus, we fix numbers

$$a_1 < a_2 < \cdots < a_k, \qquad \sum_{i=1}^{k} a_i = 0, \tag{1.12}$$

(as we will see shortly the latter condition can be assumed without loss of generality) and consider the unique solution $\mathbf{f}$ of system (1.9) for which

$$f_j(0) = a_j, \quad f_j'(0) = 0, \quad j = 1, \dots, k, \tag{1.13}$$

and their associated scalings $\mathbf{f}_\alpha$. As an explicit example, we immediately check that for $k = 2$ we have

$$\begin{aligned} f_1(z) &= -\frac{1}{2}\Big(\log(2\lambda^{-2}c_p) - \log \frac{1}{2\cosh^2(\lambda z/2)} \Big), \\ f_2(z) &= \frac{1}{2}\Big(\log(2\lambda^{-2}c_p) - \log \frac{1}{2\cosh^2(\lambda z/2)} \Big), \end{aligned} \tag{1.14}$$

where

$$\lambda = \sqrt{\frac{2c_p}{e^{a_2 - a_1}}}.$$

As we will see later, in the case of a general k the functions f_j are asymptotically linear: the limits $\nu_j = f_{1j}'(+\infty)$, exist and

$$\nu_1 < \nu_2 < \cdots < \nu_k, \quad \sum_{j=0}^{k} \nu_j = 0. \tag{1.15}$$

For instance, for $k = 2$ we have $\nu_2 = -\nu_1 = \frac{1}{2}\sqrt{\frac{2c_p}{e^{a_2-a_1}}}$ according to formula (1.14). Besides, we have (globally) for α small

$$f_{\alpha 1}(z) \ll f_{\alpha 2}(z) \ll \cdots \ll f_{\alpha k}(z), \qquad f_{\alpha j}'(+\infty) = \nu_j \alpha,$$

and

$$f_{\alpha j}(z) \;=\; \nu_j \alpha\,|z| + b_j \;+\; (j - \frac{k+1}{2}) \log\frac{1}{\alpha} + O(e^{-\vartheta\alpha|z|}), \quad \text{as } |z| \to +\infty,$$

for certain scalars b_j and $\vartheta > 0$. These are standard facts about the Toda system that can be found for instance in [20]. Thus, each of the multiple ends of $u(x, z)$ is a bump-line that is nearly straight but bent, with an angle slightly distinct than the angles of the other ends. The Toda system is a classical model describing scattering of k particles distributed on a straight line, which interact only with their closest neighbors with a forces given by a potential depending on the exponentials of their mutual distances. Here the z variable is interpreted as time. In this context, $\mathbf{f}_\alpha$ corresponds to a setting in which the particles starting from the rest, scatter at slightly different, nearly constant small velocities whose average is zero. The latter fact follows the identity $\sum_{j=1}^{k} f_{\alpha j}''(z) = 0$, which also implies conservation of the center of mass used in (1.12).

Our main result is:

Theorem 1.1 *Assume that $N = 2$ and $p \geq 2$. Given $k \geq 2$ and numbers a_j as in* (1.12)*, for any sufficiently small number $\alpha > 0$, there exists a solution u_α of equation* (1.1) *which satisfies conditions* (1.2)-(1.3)*, and that has the form*

$$u_\alpha(x, z) = \sum_{j=1}^{k} w_{\delta_j}(x - f_{\alpha j}(z), z)\,(1 + o(1)).$$

Here $\mathbf{f}_\alpha$ is the scaling (1.11) *of $\mathbf{f}$, the unique solution of* (1.9) *satisfying* (1.13)*, and $o(1) \to 0$, $\delta_j \to 0$ as $\alpha \to 0$.*

Remark 1.1 By no means we have intended to state Theorem 1.1 in its most general form. For instance, the same result holds for the more general problem

$$\Delta u + g(u) = 0, \quad u > 0, \quad \text{in } \mathbb{R}^2. \tag{1.16}$$

where g is of class C^2, $g(0) = 0$, $g'(0) < 0$, $\int_0^c g(s)ds = 0$ for some $c > 0$. In this case a homoclinic solution w analogous to that of (1.4)-(1.5) exists. On the other hand, for problem (1.1) in $\mathbb{R}^N$, $N \geq 3$, we may take as the basis of the construction the unique radially symmetric decaying solution $w_{N-1}(x)$ of (1.1) in $\mathbb{R}^{N-1}$, provided that $p < \frac{N+1}{N-3}$, for which Dancer solutions $w_{N-1,\delta}(x)$ are equally available. In such a case we look for solutions close to a function of the form

$$\sum_{j=1}^{k} w_{N-1,\delta_j}(x_1, x_2, \ldots, x_{N-2}, x_{N-1} - f_{\alpha j}(z), z),$$

which are radial in the first $N - 2$ variables. In both extensions the necessary changes in the constructions are straightforward, so that for simplicity we only consider here the case $N = 2$ for (1.1).

Remark 1.2 The structure of equation (1.1) makes it natural to look for solutions u which are even both in x and z variables. It turns out that if the numbers a_j in (1.12) satisfy the symmetry requirement

$$a_j = -a_{k+1-j}, \quad j = 1, \ldots, k,$$

then the solutions $\mathbf{f}_\alpha$ satisfy similarly

$$f_{\alpha j} = -f_{\alpha k+1-j}, \quad \text{for all } j = 1, \ldots, k. \tag{1.17}$$

Indeed, in such a case the construction of the solutions u_α in Theorem 1.1 yields that they are even in both variables:

$$u_\alpha(-x, z) = u_\alpha(x, z) = u_\alpha(x, -z), \quad \text{for all } (x, z) \in \mathbb{R}^2, \tag{1.18}$$

In particular for k even this solution satisfies

$$\lim_{x \to \pm\infty} u_\alpha(x, z) = 0, \quad \text{for all } z \in \mathbb{R}, \qquad \lim_{z \to \pm\infty} u_\alpha(x, z) = 0, \quad \text{for all } x \in \mathbb{R}. \tag{1.19}$$

By the well known result by Gidas, Ni and Nirenberg [14] a positive solution of equation (1.1) that satisfies (1.18) and the limit conditions (1.19) *uniformly* must be radially symmetric around the origin. Theorem 1.1 shows that uniformity cannot be relaxed in this classical result.

One of the striking features of the existence result in Theorem 1.1, which is a purely PDE result, is that its counterparts can be found in geometric framework. Indeed, there are many examples where correspondence between solutions of (1.1) and those of some geometric problem can be drawn. To illustrate this, we will concentrate on what is perhaps the most spectacular one: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the study of entire solutions of (1.1). For simplicity we will restrict ourselves to constant mean curvature surfaces in $\mathbb{R}^3$ which have embedded coplanar ends. In the following we will draw parallels between these geometric objects and families of solutions of (1.1).

Embedded constant mean curvature surfaces of revolution were found by Delaunay in the mid 19th century [8]. They constitute a smooth one-parameter family of singly periodic surfaces D_τ, for $\tau \in (0, 1]$, which interpolate between the cylinder $D_1 = S^1(1) \times \mathbb{R}$ and the singular surface $D_0 := \lim_{\tau\to 0} D_\tau$, which is the union of an infinitely many spheres of radius $1/2$ centered at each of the points $(0, 0, n)$ as $n \in \mathbb{Z}$. The Delaunay surface D_τ can be parametrized by

$$X_\tau(x, z) = (\varphi(z)\,\cos x, \varphi(z)\,\sin x, \psi(z)) \in D_\tau \subset \mathbb{R}^3,$$

for $(x, z) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. Here the function φ is smooth solution of

$$(\varphi')^2 + \left(\frac{\varphi^2 + \tau}{2}\right)^2 = \varphi^2,$$

and the function ψ is defined by

$$\psi' = \frac{\varphi^2 + \tau}{2}.$$

As already mentioned, when $\tau = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as the cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way that the single bump-line solution of (1.1) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between

$$\boxed{\begin{array}{c}\text{The cylinder}\\ D_1 = S^1 \times \mathbb{R}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The single bump-line}\\ (x, z) \longmapsto w(x)\end{array}}.$$

Let us denote by w_2 the unique radially symmetric, decaying solution of (1.1). Inspection of the other end of the Delaunay family, namely when the parameter τ tends to 0, suggests the correspondence between

$$\boxed{\begin{array}{c}\text{The sphere}\\ S^1(1/2)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The radially symmetric solution}\\ (x, z) \longmapsto w_2(\sqrt{x^2 + z^2})\end{array}}.$$

To justify this correspondence, let us observe that on the one hand, as the parameter τ tends to 0, the surfaces D_τ resemble a sequence of spheres of radius $1/2$ arranged along the x_3-axis which are connected together by small catenoidal necks. On the other hand an analogous solution of (1.1) can be built as follows. Let $S_R = \mathbb{R} \times (0, R)$ and consider a least energy (mountain pass) solution in $H^1(S_R)$ for the the energy

$$\frac{1}{2}\int_{S_R} |\nabla u|^2 + \frac{1}{2}\int_{S_R} u^2 - \frac{1}{p+1}\int_{S_R} u^{p+1},$$

for large $R > 0$, which we may assume to be even in x and with maximum located at the origin. For R very large, this solution, which satisfies zero Neumann boundary conditions, resembles half of the unique radial, decaying solution w_2 of (1.1). Extension by successive even reflections in z variable yields a solution to (1.1) which resembles a periodic array of radially symmetric solutions of (1.1), with a very large period, along the z-axis. While this is not known, these solutions may be understood as a limit of the branch solutions constructed by Dancer.

More generally, there is a natural correspondence between

$$\boxed{\begin{array}{c}\text{Delaunay surfaces}\\ D_\tau\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Dancer solutions}\\ (x,z)\longmapsto w_\delta(x,z)\end{array}}.$$

To give further credit to this correspondence, let us recall that the Jacobi operator about the cylinder D_1 corresponds to the linearized mean curvature operator when nearby surfaces are considered as normal graphs over D_1. In the above parameterization, the Jacobi operator reads $J_1 = \frac{1}{\varphi^2}\left(\partial_x^2 + \partial_z^2 + 1\right)$. In this geometric context, it plays the role of the linear operator defined in section 2 which is the linearization of (1.1) about the single bump-line solution w. Hence we have the correspondence

$$\boxed{\begin{array}{c}\text{The Jacobi operator}\\ J_1 = \frac{1}{\varphi^2}\left(\partial_x^2 + \partial_z^2 + 1\right)\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The linearized operator}\\ L = \partial_x^2 + \partial_z^2 - 1 + p\, w^{p-1}\end{array}}.$$

In our construction, the polynomially bounded kernel of the linearized operator L plays a crucial role. Similarly, the polynomially bounded kernel of the Jacobi operator J_1 has some geometric interpretation. Let us recall that we only consider surfaces whose ends are coplanar, the Jacobi fields associated to the action of rigid motions are then given by

$$(x,z) \longmapsto \cos x \qquad \text{and} \qquad (x,z) \longmapsto z\, \cos x,$$

which correspond respectively to the action of translation and the action of the rotation of the axis of the Delaunay surface D_1. Clearly, these Jacobi fields are the counterpart of the elements of the kernel of L which are given by

$$(x,z) \longmapsto \partial_x w(x) \qquad \text{and} \qquad (x,z) \longmapsto z\, \partial_x w(x),$$

since the latter are also generated using the invariance of the problem with respect to the same kind of rigid motions.

Two additional Jacobi fields associated to J_1 are given by

$$(x,z) \longmapsto \cos z \qquad \text{and} \qquad (x,z) \longmapsto \sin z,$$

which are associated to the existence of the family D_τ as τ is close to 1, as can be easily seen using a bifurcation analysis, in a similar way that the functions

$$(x,z) \longmapsto Z(x)\, \cos(\sqrt{\lambda_1} z) \qquad \text{and} \qquad (x,z) \longmapsto Z(x)\, \sin(\sqrt{\lambda_1} z),$$

are associated to the existence of Dancer solutions when the parameter δ is close to 0. These two bifurcation results have their origin in the fact that we have the correspondence between

$$\boxed{\begin{array}{c}\text{The ground state 1 of}\\ \partial_x^2 + 1\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{The first eigenfunction } Z(x) \text{ of}\\ \partial_x^2 - 1 + p\, w^{p-1}\end{array}},$$

both of them associated to negative eigenvalues. The fact that the least eigenvalue of these operators is negative is precisely the reason why a bifurcation analysis can be performed and gives rise to the existence of Delaunay surfaces close to D_1 or Dancer's solutions close to the bump-line w.

With these analogies in mind, we can now *translate* our main result into the constant mean curvature surface framework. The result of Theorem 1.1 corresponds to the connected sum of finitely many copies of the cylinder $S^1(1) \times \mathbb{R}$ which have a common plane of symmetry. The connected sum construction is performed by inserting small catenoidal necks between two consecutive cylinders and this can be done in such a way that the ends of the resulting surface are coplanar. Such a result, in the context of constant mean curvature surfaces, follows at once from [29]. It is observed that, once the connected sum is performed the ends of the cylinder have to be slightly bent and moreover, the ends cannot be kept asymptotic to the ends of right cylinders but have to be asymptotic to Delaunay ends with parameters close to 1, in agreement with the result of Theorem 1.1.

However there is a major difference. The Toda system which governs the level sets has found no analogy in the constant mean curvature surfaces. This is mainly due to the strong interactions in the elliptic equations.

Another (older) construction of complete noncompact constant mean curvature surfaces was performed by N. Kapouleas [19] (see also [28]) starting with finitely many halves of Delaunay surfaces with parameter τ close to 0 which are connected to a central sphere. The corresponding solutions of (1.1) have recently been constructed by A. Malchiodi in [24]. It should be clear that many more examples of solutions of (1.1) can be found using the above correspondence and we shall return to this in a forthcoming paper.

It is well known that the story of complete constant mean curvature surfaces in $\mathbb{R}^3$ parallels that of complete locally conformally flat metrics with constant, positive scalar curvature. Therefore, it is not surprising that there should be a correspondence between these objects in conformal geometry and solutions of (1.1). For example, Delaunay surfaces and Dancer solutions should now be replaced by Fowler solutions which correspond to constant scalar curvature metrics on the cylinder $\mathbb{R} \times S^{n-1}$ which are conformal to the product metric $dz^2 + g_{S^{n-1}}$, when $n \geq 3$. These are given by

$$v^{\frac{4}{n-2}} (dz^2 + g_{S^{n-1}}),$$

where $z \longmapsto v(z)$ is a smooth positive solution of

$$(v')^2 - v^2 + \tfrac{n-2}{n} \, v^{\frac{2n}{n-2}} = -\tfrac{2}{n} \, \tau^2.$$

When $\tau = 1$ and $v \equiv 1$ the solution is a straight cylinder while as τ tends to 0 the metrics converge on compacts to the round metric on the unit sphere. The connected sum construction for such Fowler type metrics was performed by R. Mazzeo, D. Pollack and K. Uhlenberk [31] (where it is called the dipole construction). N. Kapouleas' construction mentioned above was initially performed by R. Schoen [37] (see also R. Mazzeo and F. Pacard [28]).

2 Linear theory

In this section we will consider the basic linearized operator. The developments presented here are crucial for our paper later on. By w we will denote the homoclinic solution to $u'' - u + u^p = 0$ such that $w'(0) = 0$. Let

$$L_0(\phi) = \phi_{xx} + f'(w)\phi, \quad f'(w) = pw^{p-1} - 1.$$

We recall some well known facts about L_0. First notice that $L_0(w_x) = 0$ i.e. has one dimensional kernel. Second we observe that

$$\lambda_1 = \frac{1}{4}(p-1)(p+3), \qquad Z = \frac{w^{(p+1)/2}}{\sqrt{\int_{\mathbb{R}} w^{p+1}}},$$

correspond, respectively, to principal eigenvalue and eigenfunction. Except for λ_1 and $\lambda_2 = 0$ the rest of the spectrum of L_0 is negative. As a consequence of these facts we observe that problem

$$L_0(\phi) - \xi^2\phi = h, \tag{2.1}$$

is uniquely solvable whenever $\xi \neq \pm\sqrt{\lambda_1}, 0$ for $h \in L^2(\mathbb{R})$. Actually, rather standard argument, using comparison principle and the fact that L_0 is of the form

$$L_0(\phi) = \phi_{xx} - \phi + q(x)\phi, \quad |q(x)| \le Ce^{-c|x|},$$

can be used to show that whenever h is for instance a compactly supported function then the solution of (2.1) is an exponentially decaying function.

Let us consider operator

$$L\phi = L_0(\phi) + \phi_{zz},$$

defined in the whole plane $(x, z) \in \mathbb{R}^2$. Equation $L(\phi) = 0$, has 3 obvious bounded solutions

$$w_x(x), \quad Z(x)\cos\sqrt{\lambda_1}z, \quad Z(x)\sin(\sqrt{\lambda_1}z).$$

Our first result shows that converse is also true.

Lemma 2.1 *Let ϕ be a bounded solution of the problem*

$$L(\phi) = 0 \quad \text{in } \mathbb{R}^2. \tag{2.2}$$

Then $\phi(x, z)$ is a linear combination of the functions $w_x(x)$, $Z(x)\cos(\sqrt{\lambda_1}z)$, and $Z(x)\sin(\sqrt{\lambda_1}z)$.

Proof. Let assume that ϕ is a bounded function that satisfies

$$\phi_{zz} + \phi_{xx} + (pw^{p-1} - 1)\phi = 0. \tag{2.3}$$

Let us consider the Fourier transform of $\phi(x, z)$ in the z variable, $\hat{\phi}(x, \xi)$ which is by definition the distribution defined as

$$\langle\hat{\phi}(x, \cdot), \mu\rangle_{\mathbb{R}} = \langle\phi(x, \cdot), \hat{\mu}\rangle_{\mathbb{R}} = \int_{\mathbb{R}} \phi(x, \xi)\hat{\mu}(\xi)d\xi,$$

where $\mu(\xi)$ is any smooth rapidly decreasing function. Let us consider a smooth rapidly decreasing function of the two variables $\psi(x,\xi)$. Then from equation (2.3) we find

$$\int_{\mathbb{R}} \langle \hat{\phi}(x,\cdot), \psi_{xx} - \xi^2\psi + (pw^{p-1}-1)\psi\rangle_{\mathbb{R}} dx = 0.$$

Let $\varphi(x)$ and $\mu(\xi)$ be smooth and compactly supported functions such that

$$\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\} \cap Supp\,(\mu) = \emptyset.$$

Then we can solve the equation

$$\psi_{xx} - \xi^2\psi + (pw^{p-1}-1)\psi = \mu(\xi)\varphi(x), \quad x \in \mathbb{R},$$

uniquely as a smooth, rapidly decreasing function $\psi(x,\xi)$ such that $\psi(x,\xi) = 0$ whenever $\xi \notin Supp\,(\mu)$. We conclude then that

$$\int_{\mathbb{R}} \langle \hat{\phi}(x,\cdot), \mu\rangle_{\mathbb{R}} \varphi(x)\, dx = 0,$$

so that for all $x \in \mathbb{R}$, $< \hat{\phi}(x,\cdot), \mu >= 0$, whenever $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\} \cap Supp\,(\mu) = \emptyset$, in other words

$$Supp\,(\hat{\phi}(x,\cdot)) \subset \{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\}.$$

By distribution theory we find then that $\hat{\phi}(x,\cdot)$ is a linear combination, with coefficients depending on x, of derivatives up to a finite order of Dirac masses supported in $\{\sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0\}$. Taking inverse Fourier transform, we get that

$$\phi(x,z) = p_0(z,x) + p_1(z,x)\cos(\sqrt{\lambda_1}z) + p_2(z,x)\sin(\sqrt{\lambda_1}z),$$

where p_j are polynomials in z with coefficients depending on x. Since ϕ is bounded these polynomials are of zero order, i.e. $p_j(z,x) \equiv p_j(x)$, and the bounded functions p_j must satisfy the equations

$$L_0(p_0) = 0, \quad L_0(p_1) - \lambda_1 p_1 = 0, \quad L_0(p_2) - \lambda_1 p_2 = 0,$$

from where the desired result thus follows. □

Let $B(\phi)$ be an operator of the form

$$B(\phi) = b_1\partial_{xx}\phi + b_2\partial_{xz}\phi + b_3\partial_x\phi + b_4\partial_z\phi + b_5\phi,$$

where the coefficients b_i are small functions. In the sequel we will denote $\mathbf{b} = (b_1,\ldots,b_5)$ and assume that

$$\|\mathbf{b}\| \equiv \sum_{j=1}^{5} \|b_j\|_\infty + \|\nabla b_1\|_\infty + \|\nabla b_2\|_\infty < \delta_0, \tag{2.4}$$

where the small number δ_0 will be subsequently fixed. The linear theory used in this paper is based on a priori estimates for the solutions of the following problems

$$B(\phi) + L(\phi) = h, \quad \text{in } \mathbb{R}^2. \tag{2.5}$$

The results of Lemma 2.1 imply that such estimates without imposing extra conditions on ϕ may not exist. The form of the bounded solutions of $L(\phi) = 0$ suggests the following orthogonality conditions:

$$\int_{\mathbb{R}} \phi(x,z) w_x(x) d\mu(x) = 0 = \int_{\mathbb{R}} \phi(x,z) Z(x) d\mu(x), \quad \text{for all } z \in \mathbb{R}, \tag{2.6}$$

where $d\mu(x)$ is a fixed measure in $\mathbb{R}$ absolutely continuous with respect to the Lebesque measure. In the sequel we will in particular consider $d\mu(x) = \rho(x)\, dx$ where ρ is a compactly supported cut-off function, however our next result applies for a general $d\mu(x)$ as well. With these restrictions imposed we have the following result concerning a priori estimates for this problem.

Lemma 2.2 *There exist constants δ_0 and C such that if the bound* (2.4) *holds and $h \in L^\infty(\mathbb{R}^2)$, then any bounded solution ϕ of problem* (2.5)-(2.6) *satisfies*

$$\|\phi\|_\infty \le C \|h\|_\infty.$$

Proof. We will argue by contradiction. Assuming the opposite means that there are sequences b_j^n, ϕ_n, h_n such that

$$\sum_{j=1}^{5} \|b_j^n\|_\infty + \|\nabla b_1^n\|_\infty + \|\nabla b_2^n\|_\infty \to 0,$$

$$\|\phi_n\|_\infty = 1, \quad \|h_n\|_\infty \to 0,$$

and

$$B_n(\phi_n) + L(\phi_n) = h_n, \quad \text{in } \mathbb{R}^2, \tag{2.7}$$

$$\int_{\mathbb{R}} \phi_n(x,z) w_x(x) d\mu(x) = 0 = \int_{\mathbb{R}} \phi_n(x,z) Z(x) d\mu(x), \quad \text{for all } z \in \mathbb{R}. \tag{2.8}$$

Here

$$B_n(\phi) = b_1^n \partial_{xx}\phi + b_2^n \partial_{xz}\phi + b_3^n \partial_x \phi + b_4^n \partial_z \phi + b_5^n \phi.$$

Let us assume that $(x_n, z_n) \in \mathbb{R}^2$ is such that

$$|\phi_n(x_n, z_n)| \to 1.$$

We claim that the sequence x_n is bounded. Indeed, if not, using the fact that $L\phi = \Delta\phi - \phi + O(e^{-c|x|})\phi$ and employing elliptic estimates we find that the sequence of functions

$$\tilde{\phi}_n(x,z) = \phi_n(x_n + x, z_n + z),$$

converges, up to a subsequence, locally uniformly to a solution $\tilde{\phi}$ of the equation

$$\Delta\tilde{\phi} - \tilde{\phi} = 0, \quad \text{in } \mathbb{R}^2,$$

whose absolute value attains its maximum at $(0,0)$, This implies $\tilde{\phi} \equiv 0$, so that x_n is indeed bounded. Let now

$$\tilde{\phi}_n(x,z) = \phi_n(x, z_n + z).$$

Then $\tilde{\phi}_n$ converges uniformly over compacts to a bounded, nontrivial solution $\tilde{\phi}$ of

$$L(\tilde{\phi}) = 0 \quad \text{in } \mathbb{R}^2,$$

$$\int_{\mathbb{R}} \tilde{\phi}(x,z) w_x(x) d\mu(x) = 0 = \int_{\mathbb{R}} \tilde{\phi}(x,z) Z(x) d\mu(x), \quad \text{for all } z \in \mathbb{R}.$$

Lemma 2.1 then implies $\tilde{\phi} \equiv 0$, a contradiction and the proof is concluded. □

Using Lemma 2.2 we can also find a priori estimates with norms involving exponential weights. Let us consider the norm

$$\|\phi\|_{\sigma,a} \equiv \|e^{\sigma|x|+a|z|}\phi\|_\infty .$$

where numbers $\sigma, a \geq 0$ are fixed and will be subsequently adjusted. In the case $a = 0$ we have the following a priori estimates.

Corollary 2.1 *There are numbers C and δ_0 as in Lemma 2.2 for which, if $\|h\|_{\sigma,0} < +\infty$, $\sigma \in [0,1)$, then a bounded solution ϕ of (2.5)–(2.6) satisfies*

$$\|\phi\|_{\sigma,0} + \|\nabla\phi\|_{\sigma,0} \leq C\|h\|_{\sigma,0}. \tag{2.9}$$

Proof. Again we concentrate on estimates for the problem (2.5)–(2.6). We already know that

$$\|\phi\|_\infty \leq C\|h\|_{\sigma,0}.$$

We set $\tilde{\phi} = \phi\|h\|_{\sigma,0}^{-1}$. Then we have

$$(L+B)(\tilde{\phi}) = \tilde{h}, \quad \text{where } \|\tilde{h}\|_{\sigma,0} \leq 1,$$

and also $\|\tilde{\phi}\|_\infty \leq C$. Let us fix a number $R_0 > 0$ such that for $x > R_0$ we have

$$pw^{p-1}(x) < \frac{1-\sigma^2}{2},$$

which is always possible since $w(x) = O(e^{-c|x|})$. For an arbitrary number $\rho > 0$ let us set

$$\bar{\phi}(x,z) = \rho[\cosh(z/2) + e^{\sigma x}] + Me^{-\sigma x},$$

where M is to be chosen. Then we find that, reducing δ_0 if necessary,

$$(L+B)(\bar{\phi}) \leq -\frac{M(1-\sigma^2)}{4} e^{-\sigma x}, \quad \text{for } x > R_0.$$

Thus

$$(L+B)(\bar{\phi}) \leq \tilde{h}, \quad \text{for } x > R_0,$$

if

$$\frac{M(1-\sigma^2)}{4} \geq \|\tilde{h}\|_{\sigma,0} = 1.$$

If we also also assume

$$Me^{-\sigma R_0} \geq \|\tilde{\phi}\|_\infty,$$

we conclude from maximum principle that $\tilde{\phi} \leq \bar{\phi}$. Letting $\rho \to 0$ we then get by fixing M,

$$\tilde{\phi} \leq Me^{-\sigma x}, \quad \text{for } x > 0,$$

hence

$$\phi \leq M\|h\|_{\sigma,0}e^{-\sigma x}, \quad \text{for } x > 0.$$

In a similar way we obtain the lower bound

$$\phi \geq -M\|h\|_{\sigma,0}e^{-\sigma x}, \quad \text{for } x > 0.$$

Finally, the same argument for $x < 0$ yields

$$\|\phi\|_{\sigma,0} \leq C\|h\|_{\sigma,0},$$

while from local elliptic estimates we find

$$\|\nabla\phi\|_{\sigma,0} \leq C\|h\|_{\sigma,0},$$

and the proof is concluded. □

When $a > 0$ in the definition of the norm $\|\cdot\|_{\sigma,a}$ then we have the following a priori estimates.

Corollary 2.2 *There are numbers C, δ_0 as in Lemma* 2.2, *and $\alpha_0 > 0$ for which, if $\|h\|_{\sigma,a} < +\infty$, $\sigma \in (0,1)$, $a \in [0,a_0)$, then a bounded solution ϕ to problem* (2.5)-(2.6) *satisfies*

$$\|\phi\|_{\sigma,a} + \|\nabla\phi\|_{\sigma,a} \leq C_\sigma\|h\|_{\sigma,a}.$$

Proof. We already know that

$$\|\phi\|_{\sigma,0} + \|\nabla\phi\|_{\sigma,0} \leq C\|h\|_{\sigma,a}.$$

Then we may write

$$\psi(z) = \int_{\mathbb{R}} \phi^2(x,z)\,dx,$$

and differentiate twice weakly to get

$$\psi''(z) = 2\int_{\mathbb{R}} \phi_z^2(x,z)\,dx + 2\int_{\mathbb{R}} \phi_{zz}\phi(x,z)\,dx.$$

We have

$$\int_{\mathbb{R}} \phi_{zz}\phi\,dx = \int_{\mathbb{R}} \phi_x^2\,dx + \int_{\mathbb{R}} (1-pw^{p-1})\phi^2\,dx - \int_{\mathbb{R}} B(\phi)\phi + \int_{\mathbb{R}} h\phi. \tag{2.10}$$

Integrating by parts once in x we find

$$\begin{aligned} \Big|\int_{\mathbb{R}} B(\phi)\phi\Big| &= \Big|\int_{\mathbb{R}} [-(b_1\phi)_x\phi_x - (b_2\phi)_x\phi_z + b_3\phi_x\phi + b_4\phi_z\phi + b_5\phi^2]\Big| \\ &\leq C\delta_0 \int_{\mathbb{R}} (\phi_z^2 + \phi_x^2 + \phi^2)\,dx. \end{aligned} \tag{2.11}$$

Because of the orthogonality conditions (2.6) we also have that for a certain $\gamma > 0$,

$$\int_{\mathbb{R}} \phi_x^2\,dx + \int_{\mathbb{R}} (1-pw^{p-1})\phi^2\,dx \geq \gamma \int_{\mathbb{R}} (\phi_x^2 + \phi^2)\,dx.$$

Hence, reducing δ_0 if necessary, we find that for a certain constant $C > 0$

$$\psi''(z) \geq \frac{\gamma}{4}\psi(z) - C\int_{\mathbb{R}} h^2(x,z)\,dx,$$

so that

$$-\psi''(z) + \frac{\gamma}{4}\psi(z) \leq \frac{C}{\sigma}e^{-2a|z|}\|h\|^2_{\sigma,a}.$$

Since we also know that ψ is bounded by:

$$|\psi(z)| \leq \frac{C}{\sigma}\|h\|^2_{\sigma,0},$$

we can use a barrier of the form $\psi^+(z) = M\|h\|^2_{\sigma,a}e^{-2az} + \rho e^{2az}$, with M sufficiently large and $\rho > 0$ arbitrary, to get the bound $0 \leq \psi \leq \psi^+$ for $z \geq 0$ and any $a < \frac{\sqrt{\gamma}}{4} \equiv a_0$. A similar argument can be used for $z < 0$. Letting $\rho \to 0$ we get then

$$\int_{\mathbb{R}} \phi^2(x,z)\,dx \leq C_\sigma e^{-2a|z|}\|h\|^2_{\sigma,a}, \quad a < a_0.$$

Elliptic estimates yield that for R_0 fixed and large

$$|\phi(x,z)| \leq C_\sigma e^{-a|z|}\|h\|_{\sigma,a} \quad \text{for } |x| < R_0.$$

The corresponding estimate in the complementary region can be found by barriers. For instance in the quadrant $\{x > R_0, z > 0\}$ we may consider a barrier of the form

$$\bar{\phi}(x,z) = M\|h\|_{\sigma,a}e^{-(\sigma x + az)} + \rho e^{\frac{x}{2}+\frac{z}{2}},$$

with $\rho > 0$ arbitrarily small. Fixing M depending on R_0 we find the desired estimate for $\|\phi\|_{\sigma,a}$ letting $\rho \to 0$. Arguing similarly in the remaining quadrants is similar. The corresponding bound for $\|\nabla\phi\|_{\sigma,a}$ is then deduced from local elliptic estimates. This concludes the proof. $\square$

Notice that for a general right hand side h equation of the form $L(\phi) + B(\phi) = h$ with the orthogonality conditions imposed as above does not have a solution. On the other hand the problem

$$L(\phi) + B(\phi) = h + c(z)w_x + d(z)Z, \quad \text{in } \mathbb{R}^2, \tag{2.12}$$

under orthogonality conditions

$$\int_{\mathbb{R}} \phi(x,z)w_x(x)d\mu(x) = 0 = \int_{\mathbb{R}} \phi(x,z)Z(x)d\mu(x), \quad \text{for all } z \in \mathbb{R}. \tag{2.13}$$

has a solution in the sense that for given h one can find (ϕ, c, d) satisfying (2.12)–(2.13).

Corollary 2.3 *There exist $C > 0$, $\eta_0 > 0$, $\delta_0 > 0$ as in Lemma 2.2, and $\alpha_0 > 0$ for which, if $\|h\|_{\sigma,\alpha} < +\infty$, $\sigma \in (0,1)$, $a \in [0, a_0)$, and $d\mu(x) = \rho(x)\,dx$ is such that*

$$\int_{\mathbb{R}} e^{-\sigma|x|}[(|w_x| + |Z|)|\rho_{xx}| + 2(|w_{xx}| + |Z_x|)|\rho_x|]\,dx < \eta_0, \tag{2.14}$$

then a bounded solution ϕ *to problem* (2.12)-(2.13) *satisfies*

$$\|\phi\|_{\sigma,a} + \|\nabla\phi\|_{\sigma,a} \leq C\|h\|_{\sigma,a}. \tag{2.15}$$

Moreover we have

$$|c(z)| + |d(z)| \leq C\|h\|_{\sigma,a} e^{-a|z|}. \tag{2.16}$$

Proof. To find a priori estimate (2.15) we have to find bounds for the coefficients $c(z)$ and $d(z)$. Testing equation (2.12) against w_x and integrating with respect to $d\mu(x)$ we get

$$\int_{\mathbb{R}} \phi_{zz} w_x \, d\mu(x) + \int_{\mathbb{R}} L_0(\phi) w_x \, d\mu(x) + \int_{\mathbb{R}} B(\phi) w_x \, d\mu(x) = \int_{\mathbb{R}} h w_x \, d\mu(x) + c(z) \int_{\mathbb{R}} w_x^2 \, d\mu(x).$$

Let us assume that $\|h\|_{\sigma,0} < +\infty$ and that ϕ is a bounded solution. Integrating by parts and using $L_0(w_x) = 0$ and the orthogonality condition (2.13) we get

$$c(z) \int_{\mathbb{R}} w_x^2 \rho \, dx = \int_{\mathbb{R}} B(\phi) w_x \rho \, dx + \int_{\mathbb{R}} \phi(2w_{xx}\rho_x + w_x \rho_{xx}) \, dx - \int_{\mathbb{R}} h w_x \, dx. \tag{2.17}$$

To estimate term $\int_{\mathbb{R}} B(\phi) w_x \rho, dx$ we use an argument similar to that of (2.11), and to estimate $\int_{\mathbb{R}} \phi(w_x\rho)_{xx} \, dx$ we use (2.14) to get

$$|c(z)| \leq C \int_{\mathbb{R}} |h w_x| + C(\delta_0 + \eta_0)(\|\nabla\phi\|_{\sigma,0} + \|\phi\|_{\sigma,0}).$$

From the a priori estimates (Corollary 2.1) applied to (2.12) we know that

$$\|\nabla\phi\|_{\sigma,0} + \|\phi\|_{\sigma,0} \leq C(\|h\|_{\sigma,0} + \|c\|_\infty + \|d\|_\infty),$$

since

$$\|w_x e^{\sigma|x|}\|_\infty, \|Z e^{\sigma|x|}\|_\infty < \infty,$$

for $\sigma \in [0,1)$. Thus, reducing δ_0, η_0 if necessary, we find

$$\|c\|_\infty \leq C\big(\|h\|_{\sigma,0} + (\delta_0 + \eta_0)\|d\|_\infty\big).$$

Testing the equation against Z and using exactly the same argument we find

$$\|d\|_\infty \leq C(\|h\|_{\sigma,0} + (\delta_0 + \eta_0)\|c\|_\infty).$$

Hence

$$\|dZ\|_{\sigma,0} + \|c w_x\|_{\sigma,0} \leq C\|h\|_{\sigma,0},$$

and the estimate

$$\|\nabla\phi\|_{\sigma,0} + \|\phi\|_{\sigma,0} \leq C\|h\|_{\sigma,0}.$$

follows.

Finally, if we additionally have $\|h\|_{\sigma,a} < +\infty$, we obtain that

$$\int_{\mathbb{R}} |hw_x| \le C\|h\|_{\sigma,a} e^{-a|z|}.$$

The same procedure above and the a priori estimates found in Corollary 2.2 then yield

$$|c(z)| + |d(z)| \le C\|h\|_{\sigma,a} e^{-a|z|},$$

from where the relation (2.15) immediately follows. □

Concerning the existence of bounded solutions of (2.12)–(2.13) we have:

Proposition 2.1 *There exists numbers $C > 0$, $\delta_0 > 0$, $\eta_0 > 0$ such that whenever bounds* (2.4), (2.14) *hold, then given h with $\|h\|_{\sigma,a} < +\infty$, $\sigma \in (0,1)$, $a \in [0, a_0)$, there exists a unique bounded solution $\phi = T(h)$ to problem* (2.12)-(2.13) *which defines a bounded linear operator of h in the sense that*

$$\|\nabla\phi\|_{\sigma,a} + \|\phi\|_{\sigma,a} \le C\|h\|_{\sigma,a}.$$

Proof. We will first consider solvability of the following problem

$$(L+B)(\phi) = h, \quad \text{in } \mathbb{R}^2, \tag{2.18}$$

in the space V, where $\psi \in V$ if $\|\psi\|_{\sigma,0} < \infty$, $\sigma \in (0,1]$ and

$$\int_{\mathbb{R}} \psi(x,z) w_x(x) \rho(x)\, dx = 0 = \int_{\mathbb{R}} \psi(x,z) Z(x) \rho(x)\, dx, \quad \text{for all } z \in \mathbb{R}, \tag{2.19}$$

where the density $\rho(x)$ satisfies the hypothesis of Corollary 2.3. We claim that given $h \in V$ there exists a unique solution ϕ of (2.18) in V. We will argue by approximations. Let us replace h by the function $h(x,z)\chi_{(-R,R)}(z)$ extended $2R$-periodically to the whole plane. With this right hand side we can give to the problem (2.18) a weak formulation in the subspace of $H^1_R \subset H^1(\mathbb{R}^2)$ of functions that are $2R$-periodic in z. To be more precise let

$$[\psi, \eta] = \int_{-\infty}^{\infty}\int_{-R}^{R} \nabla\psi \cdot \nabla\eta \, dzdx + \int_{-\infty}^{\infty}\int_{-R}^{R} \psi\eta\, dzdx.$$

By W we will denote the subspace of functions in H^1_R that satisfy (2.19). Then (2.18) can be written in the form

$$-[(A+K)(\phi), \psi] = \int_{-\infty}^{\infty}\int_{-R}^{R} h\psi, \quad \psi \in W, \tag{2.20}$$

where $A : W \to W$ is defined by

$$[A(\phi), \psi] = \int_{-\infty}^{\infty}\int_{-R}^{R} (\nabla\phi\nabla\psi + b_1\phi_x\psi_x + b_2\phi_x\psi_z)\, dzdx + \int_{-\infty}^{\infty}\int_{-R}^{R} (1-b_5)\phi\psi\, dzdx, \psi \in W,$$

and $K : W \to W$ is a linear operator defined by

$$[K(\phi), \psi] = \int_{-\infty}^{\infty}\int_{-R}^{R} [(b_{1x}+b_{2z}-b_3)\phi_x - b_4\phi_z]\psi\, dzdx - p\int_{-\infty}^{\infty}\int_{-R}^{R} \phi\psi w^{p-1}\, dzdx, \ \psi \in W.$$

Using (2.4), (2.14), and the fact that $\|w^{(p-1)}\|_{(p-1),0} < \infty$ one can show that the operator A is invertible and the operator K is compact.

From Fredholm alternative and Lemma 2.1 which in addition can be extended periodically to a unique solution $\phi \in V$, of (2.18) with h replaced by $h(x,z)\chi_{(-R,R)}(z)$. Letting $R \to +\infty$ and using the uniform a priori estimates valid for the approximations completes the proof of the claim.

The existence of a solution to (2.12)–(2.13) as well as the rest of the Proposition follows from this claim. Indeed, given h such that $\|h\|_{\sigma,0} < \infty$ by $\Pi_V(h)$ we will denote the orthogonal projection of h onto V (in the sense of $L^2(\rho\, dx)$ as indicated by (2.19)). Using the claim we can solve then the following problem

$$(L+B)(\phi) = \Pi_V(h).$$

Now we only need to chose functions $c(z)$, $d(z)$ such that

$$(I - \Pi_V)[(L+B)(\phi)] = (I - \Pi_V)(h) + c(z)w_x + d(z)Z.$$

This ends the proof.

□

Remark 2.1 Incidentally, Lemma 2.1 helps us to sketch a proof of existence of Dancer solutions (essentially that in [7]) as follows. Consider the bifurcation problem

$$\lambda \Delta v - v + v^p = 0, \quad \text{in } \mathbb{R}^2,$$

with $\lambda > 0$, which we write, for $w_\lambda(x) = w(\sqrt{\lambda}x)$, as

$$\lambda\Delta\phi + (pw_\lambda^{p-1} - 1)\phi + N_\lambda(\phi) = 0,$$

with $N(\phi) = (w_\lambda + \phi)^p - w_\lambda^p - pw_\lambda^{p-1}\phi$. We consider the space $\mathcal{X}$ of all functions ϕ with $\|\phi\|_{\sigma,0} < +\infty$ which are $T = 2\pi/\sqrt{\lambda_1}$-periodic and even in the z-variable. The operator $\lambda\Delta - 1$ has a bounded inverse in $\mathcal{X}$ and thus the problem gets rewritten as

$$\phi + (\lambda\Delta - 1)^{-1}(pw_\lambda^{p-1}\phi + N_\lambda(\phi)) = 0.$$

The derivative of this operator in ϕ at $\lambda = 1$ and $\phi = 0$ is just

$$I + (\Delta - 1)^{-1}(pw^{p-1}),$$

which has the form: $I + K$, where K is a compact operator. Crandall-Rabinowitz theorem thus yields that bifurcation takes place at $\lambda = 1$ since Lemma 2.1 implies that this linearization has a simple eigenvalue at $\lambda = 1$ with eigenfunction $Z(x)\cos(\sqrt{\lambda_1}z)$, and a branch of nontrivial solutions in $\mathcal{X}$ constituted by a smooth curve $\delta \mapsto (\lambda(\delta), \phi(\delta))$ with $\phi(0) = 0$ and $\lambda(0) = 1$, and $\partial_\delta\phi(0) = Z(x)\cos(\sqrt{\lambda_1}z)$. Scaling out $\lambda(\delta)$ we obtain then solutions w_δ to the problem for $\lambda = 1$ with period $2\pi/\sqrt{\lambda_\delta\lambda}$ and even in the z-variable. With the aid of barriers, we find that these solutions have an expansion of the form

$$w_\delta(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2)e^{-|x|},$$

for all small δ. By the standard theory we have that derivatives of this solutions satisfy the corresponding relations

$$\begin{aligned} w_{\delta,x}(x,z) &= w_x(x) + \delta Z_x(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2 e^{-|x|}), \\ w_{\delta,xx}(x,z) &= w_{xx}(x) + \delta Z_{xx}(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2 e^{-|x|}), \\ w_{\delta,z}(x,z) &= -\sqrt{\lambda_1}\delta Z(x)\sin(\sqrt{\lambda_1}z) + O(\delta^2 e^{-|x|}), \\ w_{\delta,zz}(x,z) &= -\lambda_1\delta Z(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2 e^{-|x|}), \\ w_{\delta,xz}(x,z) &= -\sqrt{\lambda_1}Z_x(x)\sin(\sqrt{\lambda_1}z) + O(\delta^2 e^{-|x|}). \end{aligned} \tag{2.21}$$

3 Linear theory for multiple bump-lines

In this section we will choose a collection of approximations $\mathtt{W}$ which will constitute the basis of our construction of a solution to Problem (1.1). Let α be a small positive number and k be a fixed positive integer. Let us consider functions $f_j \in C^2(\mathbb{R})$, $j = 1, \dots, k$, ordered in the sense that

$$f_1(z) < f_2(z) < \cdots < f_k(z), \quad \forall\, z \in \mathbb{R}. \tag{3.1}$$

These functions represent approximate locations of the bump lines. Asymptotically these lines are straight lines with slopes proportional to α and whose mutual distances are at least $O(d_*)$, where

$$d_* \equiv \log\frac{1}{\alpha}.$$

Thus, in addition to (3.1), we assume that for certain large, fixed constant $M > 0$ these functions satisfy

$$f_{j+1}(z) - f_j(z) \geq 2d_* - M, \quad j = 1, \dots, k-1, \tag{3.2}$$

Here and in what follows we use the following weighted norm for those functions that depend on z only

$$\|g\|_{\theta_0\alpha} := \|e^{\theta_0\alpha|z|}g\|_\infty,$$

where $g : \mathbb{R} \to \mathbb{R}^m$ and $\theta_0 > 0$ is a fixed number to be determined later. Notice that the factor θ_0 in front of the exponent $\alpha|z|$ in the definition of the norm will be taken small but fixed independent on α. For convenience we will denote

$$\mathbf{f} = (f_1, f_2, \dots, f_k).$$

We assume that

$$\|\mathbf{f}''\|_{\theta_0\alpha} \leq M\alpha^2. \tag{3.3}$$

We will further suppose that the locations of the bump lines are symmetric with respect to x axis, which in terms of $\mathbf{f}$ means,

$$\mathbf{f}(z) = \mathbf{f}(-z), \quad \forall\, z \in \mathbb{R}. \tag{3.4}$$

Let us observe that under these assumptions the numbers

$$\beta_j := f_j'(+\infty),$$

are well defined and

$$|\beta_j| \le M\alpha. \tag{3.5}$$

In fact, from (3.3)–(3.4) we see that there exist numbers B_j such that we have

$$f_j(z) = \beta_j|z| + B_j + O(e^{-\theta_0\alpha|z|}), \quad \text{as } |z| \to +\infty.$$

From (3.2)–(3.4) it follows also that:

$$\beta_{j+1} - \beta_j > 0, \quad B_{j+1} - B_j > 2d_* - M, \quad j = 1, \dots, k, \tag{3.6}$$

and

$$\||f_j'| - \beta_j\|_{\theta_0\alpha} \le C\alpha, \qquad j = 1, \dots, k. \tag{3.7}$$

In addition we will assume that with for some θ_1, $\theta_1 > \theta_0 > 0$ with θ_0 defined as above we have

$$\beta_{j+1} - \beta_j > \theta_1\alpha. \tag{3.8}$$

With functions f_j we will associate a change of variables defined as follows: for any even function $f(z)$ such that

$$\|f''\|_{\theta_0\alpha} \le M\alpha^2, \quad \beta = f'(\infty),$$

we set

$$\mathtt{X}(x,z) = \frac{x - f(z)}{\sqrt{1 + \big(\beta\eta(\alpha|z|)\big)^2}}, \quad \mathtt{Z}(x,z) = |z|\sqrt{1 + \big(\beta\eta(\alpha|z|)\big)^2} + \frac{\beta\eta(x - f(z))}{\sqrt{1 + \big(\beta\eta(\alpha|z|)\big)^2}}, \tag{3.9}$$

where η is a smooth cut-off function such that $\eta(t) = 0$ if $t < T_1$, $\eta(t) = 1$, $t > T_2$, where T_1, T_2 will be chosen later independent on α. With $f = f_j$ and $\beta = \beta_j$, satisfying (3.1)–(3.8), we will denote $\mathtt{X} = \mathtt{X}_j$, $\mathtt{Z} = \mathtt{Z}_j$.

Let us consider Dancer solutions $w_\delta(x,z)$, even in z, which can be expanded at main order as

$$w_\delta(x,z) = w(x) + \delta Z(x)\cos(\sqrt{\lambda_1}z) + O(\delta^2)e^{-|x|}, \quad |\delta| < \delta_0.$$

Given $\beta \in \mathbb{R}$ and δ, $|\delta| < \delta_0$ we define the function

$$w_{\delta,\beta}(x,z) = w_\delta(\mathtt{X},\mathtt{Z}), \quad \text{whith } (\mathtt{X},\mathtt{Z}) \text{ defined in (3.9)}. \tag{3.10}$$

With this definition $w_{\delta,\beta}$ is an exact solution of the problem in the region where $\eta(\alpha|z|) = 0$. If we set $f(z) = \beta|z| + B$ then the function $w_{\delta,\beta}(x,z)$ is also an exact solution of the problem where $\eta(\alpha|z|) = 1$. Indeed, restricted to the half-plane $\mathbb{R}^2_+ = \{z > 0\}$, the latter situation corresponds simply to rotating and translating the axis of the solution w_δ from the z axis to the line $x = \beta z + B$.

We will also consider functions $e_j \in C^2(\mathbb{R})$, $j = 1, \dots, k$, such that

$$\|\mathbf{e}\|_{\theta_0\alpha} + \|\mathbf{e}'\|_{\theta_0\alpha} + \|\mathbf{e}''\|_{\theta_0\alpha} \le M\alpha^2, \tag{3.11}$$

where

$$\mathbf{e} = (e_1, e_2, \dots, e_k).$$

As in (3.4) we shall assume that functions e_j are even,

$$\mathbf{e}(z) = \mathbf{e}(-z), \quad \forall\, z \in \mathbb{R}. \tag{3.12}$$

We are now ready to set up our first approximation. Given functions f_j, e_j satisfying (3.1)–(3.4), (3.11)–(3.12), and small numbers δ_j, $j = 1, \ldots, k$ we let

$$\mathtt{W}(x,z) \,=\, \sum_{j=1}^{k} w_{\delta_j,\beta_j}(x,z) + \sum_{j=1}^{k} e_j(z)\, Z_j(x,z)\,, \tag{3.13}$$

where $\beta_j = f_j'(+\infty)$, $j = 1, \ldots, k$, and

$$Z_j(x,z) = Z(\mathtt{X}_j).$$

In the sequel we assume

$$\|\vec{\delta}\| \le M\alpha, \quad \vec{\delta} = (\delta_1, \ldots, \delta_k). \tag{3.14}$$

We want to develop a theory similar to that in the previous section now for the operator

$$\mathcal{L}(\phi) = \Delta\phi + (p\mathtt{W}^{p-1} - 1)\phi.$$

More precisely we define the weighted norm

$$\|\phi\|_{\sigma,\theta_0\alpha,*} := \Big\| \Big(\sum_{j=1}^{k} e^{-\sigma|x - f_j(z)| - \theta_0\alpha|z|} \Big)^{-1} \phi \Big\|_{\infty},$$

and search for a bounded left inverse for a projected problem for the operator $\mathcal{L}$ in the space of functions whose $\|\cdot\|_{\sigma,\theta_0\alpha,*}$ norm is finite. Thus we consider the problem

$$\mathcal{L}(\phi) = h + \sum_{j=1}^{k} c_j(z)\eta_j w_{j,x} + d_j(z)\eta_j Z_j, \quad \text{in } \mathbb{R}^2, \tag{3.15}$$

where now we *do not assume* necessarily orthogonality conditions on ϕ. Here and in what follows we denote

$$w_j(x,z) = w(x - f_j), \tag{3.16}$$
$$w_{j,x}(x,z) = w'(x - f_j), \tag{3.17}$$
$$Z_j(x,z) = Z(x - f_j). \tag{3.18}$$

We will now define several cut-off functions that will be important in the sequel. To this end let $\eta_a^b(s)$ be a smooth function with $\eta_a^b(s) = 1$ for $|s| < a$ and $= 0$ for $|s| > b$, where $0 < a < b < 1$. Then, with $d_* = \log\frac{1}{\alpha}$, we set

$$\begin{aligned}
\rho_j(x,z) &= \eta_a^b\Big(\frac{|\mathtt{X}_j|}{d_*}\Big), \quad a = \frac{2^5-1}{2^5}, b = \frac{2^6-1}{2^6},\\
\eta_j^-(x,z) &= \eta_a^b\Big(\frac{|\mathtt{X}_j|}{d_*}\Big), \quad a = \frac{2^6-1}{2^6}, b = \frac{2^7-1}{2^7},\\
\eta_j(x,z) &= \eta_a^b\Big(\frac{|\mathtt{X}_j|}{d_*}\Big), \quad a = \frac{2^7-1}{2^7}, b = \frac{2^8-1}{2^8},\\
\eta_j^+(x,z) &= \eta_a^b\Big(\frac{|\mathtt{X}_j|}{d_*}\Big), \quad a = \frac{2^8-1}{2^8}, b = \frac{2^9-1}{2^9}.
\end{aligned} \tag{3.19}$$

We will prove the following result:

Proposition 3.1 *There exist positive constants α_0, $\delta_0 > 0$ such that if* (2.4) *is satisfied, and if $\alpha \in [0, \alpha_0)$, $\sigma \in (0,1)$, and constraints* (3.1)-(3.6), (3.14) *hold, then problem* (3.15) *has a solution $\phi = \mathcal{T}(h)$ which defines a linear operator of h with $\|h\|_{\sigma,\theta_0\alpha,*} < +\infty$ and satisfies the estimate*

$$\|\phi\|_{\sigma,\theta_0\alpha,*} + \|\nabla\phi\|_{\sigma,\theta_0\alpha,*} \le C_\sigma \|h\|_{\sigma,\theta_0\alpha,*}.$$

In addition function ϕ satisfies the following orthogonality conditions

$$\int_{\mathbb{R}} \phi(x,z) w_{j,x}(x,z)\rho_j(x,z)\, dx = 0 = \int_{\mathbb{R}} \phi(x,z) Z_j(x,z)\rho_j(x,z)\, dx, \quad \text{for all } z \in \mathbb{R}. \tag{3.20}$$

Besides, the coefficients $c_j(z)$ and $d_j(z)$ in (3.15) *can be estimated as*

$$\sum_{j=1}^{k} (|c_j(z)| + |d_j(z)|) \le C\|h\|_{\sigma,\theta_0\alpha,*} e^{-\theta_0\alpha|z|}. \tag{3.21}$$

Proof. The main idea in the proof of this proposition is to decompose problem (3.15) into *interior* problems that can be handled with the help of the theory developed in the previous section and an *exterior* problem and then *glue* the solutions of the subproblems.

Form the definition of the functions η_j, $\eta_j^{\pm}$ we have

$$\eta_j \eta_j^- = \eta_j^-, \quad \eta_j^+ \eta_j = \eta_j. \tag{3.22}$$

We search for a solution of (3.15) of the form

$$\phi = \sum_{j=1}^{k} \eta_j \phi_j + \psi.$$

Substituting this expression into equation (3.15) and arranging terms we find

$$\sum_{j=1}^{k} \eta_j [\Delta\phi_j + (1 - p\mathtt{W}^{p-1})\phi_j - c_j(z) w_{j,x} - d_j(z) Z_j - h]$$
$$+ \Big[\Delta\psi + (1 - p\,\mathtt{W}^{p-1})\psi - \Big(1 - \sum_{j=1}^{k}\eta_j\Big)h - \sum_{j=1}^{k}(2\nabla\eta_j\nabla\phi_j + \Delta\eta_j\phi_j)\Big] = 0.$$

We will denote

$$h_j = \eta_j^+ h, \quad r_j = p\eta_j^+ (w_j^{p-1} - \mathtt{W}^{p-1}).$$

Let us observe that in the support of η_j we have, using (3.22),

$$\eta_j h_j = \eta_j h, \quad \eta_j r_j = p\eta_j (w_j^{p-1} - \mathtt{W}^{p-1}),$$

Then we find a solution to problem (3.15) if we solve the following linear system of equations

$$\Delta\phi_j - (1 - p w_j^{p-1})\phi_j + r_j\phi_j = h_j - p\mathtt{W}^{p-1}\eta_j^- \psi + c_j(z) w_{j,x} + d_j(z) Z_j, \tag{3.23}$$

in $\mathbb{R}^2$, for $j = 1, \ldots, k$, and

$$\Delta\psi - \Big[1 - p\Big(1 - \sum_{j=1}^{k} \eta_j^-\Big) \mathtt{W}^{p-1}\Big]\psi = \Big(1 - \sum_{j=1}^{k} \eta_j\Big) h - \sum_{j=1}^{k} (2\nabla\eta_j \nabla\phi_j + \Delta\eta_j \phi_j), \quad (3.24)$$

in $\mathbb{R}^2$. To solve equations (3.23) we denote $\tilde\phi_j = \phi_j + \eta_i^- \psi$ and use (3.23)–(3.24) to write the equation for $\tilde\phi_j$

$$\Delta\tilde\phi_j - (1 - pw_j^{p-1})\tilde\phi_j + r_j\tilde\phi_j = h_j + \eta_j^- \psi\Big[p(w^{p-1} - \mathtt{W}^{p-1}) + r_j + p\Big(1 - \sum \eta_m^-\Big)\mathtt{W}^{p-1}\Big] + c_j w_{j,x} + d_j Z_j. \quad (3.25)$$

We observe that equation (3.24) written in terms of $\tilde\phi_j$ has form

$$\Delta\psi - \Big[1 - p\Big(1 - \sum_{j=1}^{k} \eta_j^-\Big) \mathtt{W}^{p-1}\Big]\psi = \Big(1 - \sum_{j=1}^{k} \eta_j\Big) h - \sum_{j=1}^{k} (2\nabla\eta_j \nabla\tilde\phi_j + \Delta\eta_j \tilde\phi_j), \quad (3.26)$$

since for instance $\nabla\eta_j\nabla(\eta_j^-\psi) \equiv 0$.

Let us denote

$$L_j(\phi) = \Delta\phi - (1 - pw_j^{p-1})\phi,$$

and consider first the auxiliary problem

$$L_j(\phi) = \tilde h + c(z) w_{j,x} + d(z) Z_j, \quad \text{in } \mathbb{R}^2, \quad (3.27)$$

under orthogonality conditions

$$\int_{\mathbb{R}} \phi(x,z) w_{j,x}(x,z)\rho_j(x,z)\, dx = 0 = \int_{\mathbb{R}} \phi(x,z) Z_j(x,z)\rho_j(x,z)\, dx, \quad \text{for all } z \in \mathbb{R}. \quad (3.28)$$

For future references observe that if β_j is sufficiently small then

$$\rho_j \eta_j^- = \rho_j, \quad \rho_j(1 - \eta_j^-) \equiv 0. \quad (3.29)$$

We want to solve (3.27)–(3.28) using Proposition 2.1. To this end we consider the natural change of coordinates

$$x \mapsto \mathtt{x} \equiv x - f_j, \quad z \mapsto \mathtt{z}.$$

and set

$$\phi(x,z) = \tilde\phi(\mathtt{x},\mathtt{z}).$$

Direct computation then shows that problem (3.27)-(3.28) is equivalent to

$$L(\tilde\phi) + B_j(\tilde\phi) = \tilde h + c(\mathtt{z}) w_{\mathtt{x}}(\mathtt{x}) + d(\mathtt{z}) Z(\mathtt{x}), \quad \text{in } \mathbb{R}^2. \quad (3.30)$$

under orthogonality conditions

$$\int_{\mathbb{R}} \tilde\phi(\mathtt{x},\mathtt{z}) w_{\mathtt{x}}(\mathtt{x})\rho(\mathtt{x})\, d\mathtt{x} = 0 = \int_{\mathbb{R}} \phi(\mathtt{x},\mathtt{z}) Z(\mathtt{x})\rho(\mathtt{x})\, d\mathtt{x}, \quad \text{for all } \mathtt{z} \in \mathbb{R}, \quad (3.31)$$

where

$$\rho(\mathtt{x}) = \eta_a^b\Big(\frac{\mathtt{x}}{d^*}\Big), \quad a = \frac{2^5-1}{2^5}, b = \frac{2^6-1}{2^6}.$$

Here

$$L(\tilde\phi) = \tilde\phi_{\mathtt{zz}} + \tilde\phi_{\mathtt{xx}} + (pw^{p-1}-1)\tilde\phi,$$

and

$$B_j(\tilde\phi) = \Big(\frac{\partial \mathtt{x}}{\partial z}\Big)^2 \tilde\phi_{\mathtt{xx}} + 2\Big(\frac{\partial \mathtt{x}}{\partial z}\Big)\tilde\phi_{\mathtt{xz}} + \Big(\frac{\partial^2 \mathtt{x}}{\partial z^2}\Big)\tilde\phi_{\mathtt{x}}, \quad \frac{\partial \mathtt{x}}{\partial z} = -f_j'(\mathtt{z}), \quad \frac{\partial^2 \mathtt{x}}{\partial z^2} = -f_j''(\mathtt{z}). \tag{3.32}$$

The operator B_j satisfies the assumptions of Proposition 2.1, since from (3.3) we have using the notation of (2.4) and denoting the vector of the coefficients of B_j by $\mathbf{b}_j$:

$$\|\mathbf{b}_j\| \le C(\|f_j'\|_\infty + \|f_j''\|_\infty) \le C\alpha. \tag{3.33}$$

Problem (3.30)–(3.31) has a unique bounded solution $\tilde\phi = \tilde T_j(\tilde\phi)$ where $\tilde T_j$ is the linear operator T predicted by the proposition for $B = B_j$. Besides, if $\|\tilde h\|_{\sigma,\alpha} < +\infty$ then we have the estimate

$$\|\nabla_{\mathtt{x},\mathtt{z}}\tilde\phi\|_{\sigma,\theta_0\alpha} + \|\tilde\phi\|_{\sigma,\theta_0\alpha} \le C\|\tilde h\|_{\sigma,\theta_0\alpha}.$$

going back to the original variables we see then that there is a unique solution to (3.27)-(3.28), $\phi = T_j(h)$ where T_j is a linear operator. In addition we have the estimate

$$\|\nabla\phi\|_{\sigma,\theta_0\alpha,j} + \|\phi\|_{\sigma,\theta_0\alpha,j} \le C\|h\|_{\sigma,\theta_0\alpha,j},$$

where

$$\|\phi\|_{\sigma,\theta_0\alpha,j} \;=\; \|\, e^{\sigma|x-f_j(z)|+\theta_0\alpha|z|}\,\phi\,\|_\infty.$$

Moreover, the coefficients c and d are estimated using (2.16) by

$$|c(z)| + |d(z)| \le C\|h\|_{\sigma,\theta_0\alpha,j}e^{-\theta_0\alpha|z|}. \tag{3.34}$$

Let us observe that given ψ we can recast the equations (3.23) for $\tilde\phi_j$ as a system of the form

$$\tilde\phi_j + T_j(r_j\tilde\phi_j) = T_j\Big(h_j - p\mathtt{W}^{p-1}\eta_j^-\psi\Big[p(w^{p-1}-\mathtt{W}^{p-1}) + r_j + p\Big(1-\sum\eta_m^-\Big)\mathtt{W}^{p-1}\Big]\Big),$$
$$j = 1,\dots,k, \tag{3.35}$$

We will solve next equation (3.26) for ψ as a linear operator

$$\psi = \Psi(\Phi, h),$$

where Φ denotes the k-tuple $\Phi = (\tilde\phi_1,\dots,\tilde\phi_k)$. To this end let us consider first the problem

$$\Delta\psi - (1-\theta)\psi = g \quad \text{in } \mathbb{R}^2. \tag{3.36}$$

where

$$\theta = p\Big(1 - \sum_{j=1}^k \eta_j^-\Big)\mathtt{W}^{p-1}.$$

Observe that if the number d_* is large enough then θ is uniformly small, indeed $\theta = o(1)$ as $\alpha \to 0$. Let us assume that g satisfies

$$|g(x,z)| \le A \sum_{j=1}^{k} e^{-\mu|x-f_j(z)|-\theta_0\alpha|z|},$$

for some $0 \le \mu < 1$. Then, given that if d_* is sufficiently large then number θ is small, and also that (3.3) holds, the use of barriers and elliptic estimates proves that this problem has a unique bounded solution with

$$|\nabla\psi(x,z)| + |\psi(x,z)| \le C \sum_{j=1}^{k} e^{-\mu|x-f_j(z)|-\theta_0\alpha|z|}.$$

Thus if we take

$$g = (1 - \sum_{j=1}^{k} \eta_j)h - \sum_{j=1}^{k}(2\nabla\eta_j\nabla\tilde\phi_j + \Delta\eta_j\tilde\phi_j),$$

we clearly have that

$$\begin{aligned}|g(x,z)| \le & C\Big[\|h\|_{\sigma,\theta_0\alpha,*} + o(1)\sum_{j=1}^{k}\big(\|\tilde\phi_j\|_{\sigma,\theta_0\alpha,j} + \|\nabla\tilde\phi_j\|_{\sigma,\theta_0\alpha,j}\big)\Big] \\ & \times \sum_{j=1}^{k} e^{-\mu|x-f_j(z)|-\theta_0\alpha|z|},\end{aligned}$$

and hence equation (3.26) has a unique bounded solution

$$\psi = \Psi(\Phi, h),$$

which defines a linear operator in its argument and satisfies the estimate

$$\begin{aligned}|\Psi(\Phi,h)| \le & C\Big[\|h\|_{\sigma,\theta_0\alpha,*} + o(1)\sum_{j=1}^{k}\big(\|\tilde\phi_j\|_{\sigma,\theta_0\alpha,j} + \|\nabla\tilde\phi_j\|_{\sigma,\theta_0\alpha,j}\big)\Big] \\ & \times \sum_{j=1}^{k} e^{-\mu|x-f_j(z)|-\theta_0\alpha|z|}.\end{aligned} \tag{3.37}$$

In addition, we find that

$$\|\Psi(\Phi,h)\|_{\sigma,\theta_0\alpha,*} \le C\Big[\|h\|_{\sigma,\theta_0\alpha,*} + o(1)\sum_{j=1}^{k}\big(\|\tilde\phi_j\|_{\sigma,\theta_0\alpha,j} + \|\nabla\tilde\phi_j\|_{\sigma,\theta_0\alpha,j}\big)\Big]. \tag{3.38}$$

Now we have the ingredients to solve the full system (3.25)-(3.26). Accordingly to (3.35) we obtain a solution if we solve the system in Φ

$$\tilde\phi_j + T_j\big(\eta_j^-\Psi(\Phi,0)\chi_j + r_j\tilde\phi_j\big) = T_j\big(h_j - \eta_j^-\Psi(0,h)\chi_j\big), \quad j = 1,\dots,k, \tag{3.39}$$

where

$$\chi_j = \Big[p(w^{p-1} - \mathtt{W}^{p-1}) + r_j + p\Big(1 - \sum \eta_m^-\Big)\mathtt{W}^{p-1}\Big], \quad j = 1, \ldots, k.$$

We consider this system defined in the space X of all C^1 functions Φ such that the norm

$$\|\Phi\|_X := \sum_{j=1}^{k} \|\nabla\tilde{\phi}_j\|_{\sigma,\theta_0\alpha,j} + \|\tilde{\phi}_j\|_{\sigma,\theta_0\alpha,j},$$

is finite. System (3.39) can be written as

$$\Phi + \mathcal{A}(\Phi) = \mathcal{B}(h),$$

where $\mathcal{A}$ and $\mathcal{B}$ are linear operators. Thanks to the estimates for the operators T_j and the bound (3.37) we see that

$$\|\mathcal{B}(h)\|_X \le C\|h\|_{\sigma,\theta_0\alpha,*}.$$

On the other hand we have that

$$\|\mathcal{A}(\Phi)\|_X \le C\Big[\sum_{j=1}^{k} \|\eta_j^- \Psi(\Phi, 0)\chi_j\|_{\sigma,\theta_0\alpha,j} + \sum_{j=1} \|r_j\tilde{\phi}_j\|_{\sigma,\theta_0\alpha,j}\Big].$$

Using estimates (3.37) and (3.38) we find

$$\sum_{j=1}^{k} \|\eta_j^- \Psi(\Phi, 0)\chi_j\|_{\sigma,\theta_0\alpha,j} \le o(1)\|\Phi\|_X. \tag{3.40}$$

From the definition of r_j and (3.14) we get $\|r_j\|_\infty = o(1)$ which implies

$$\sum_{j=1}^{k} \|r_j\tilde{\phi}_j\|_{\sigma,\theta_0\alpha,j} \le o(1)\|\Phi\|_X.$$

Summarizing the last estimates we obtain

$$\|\mathcal{A}(\Phi)\|_X \le o(1)\|\Phi\|_X,$$

hence the operator $\mathcal{A}$ is a uniformly small operator in the norm $\|\cdot\|_X$ provided that α is sufficiently small. We conclude that system (3.39) has a unique solution $\Phi = \Phi(h)$, which in addition is a linear operator of h such that

$$\|\Phi(h)\|_X \le C\|h\|_{\sigma,\theta_0\alpha,*}.$$

Thus we get a solution to problem (3.15) by setting

$$\phi = \sum_{j=1}^{k} \eta_j\tilde{\phi}_j(h) + \Big(1 - \sum \eta_j^-\Big)\Psi(\Phi(h), h). \tag{3.41}$$

Using (3.29) we get $\rho_j\phi = \rho_j\tilde{\phi}_j$ hence from and (3.28) we obtain (3.20). Estimate (3.21) follows directly from (3.34). The proof of the proposition is complete. □

4 The nonlinear projected problem

Let us recall that our goal is to find a solution of the problem

$$S[u] := \Delta u + u^p - u = 0, \quad p \geq 2, \quad \text{in } \mathbb{R}^2,$$

which is close to the function $\mathtt{W}$ defined in (3.13). We will denote by E the error of approximation at $\mathtt{W}$:

$$E := S[\mathtt{W}] = \Delta \mathtt{W} + \mathtt{W}^p - \mathtt{W}.$$

A main observation we make is that, under the assumptions (3.1)-(3.6), (3.14), $\mathtt{W}$ is such that

$$E_* := \|S[\mathtt{W}]\|_{\sigma,\theta_0\alpha,*} \leq C\alpha^{2-2\sigma}, \tag{4.1}$$

where θ_0 is defined in (3.8). We will postpone the proof of (4.1) for now and for the purpose of the present section we will simply accept it as a fact. We look for a solution to our problem in the form

$$u = \mathtt{W} + \phi,$$

where ϕ is a small perturbation of $\mathtt{W}$. Thus the equation for u is equivalent to

$$\mathcal{L}(\phi) = E - N(\phi), \quad \text{in } \mathbb{R}^2, \tag{4.2}$$

where

$$\mathcal{L}(\phi) := \Delta\phi + (p\,\mathtt{W}^{p-1} - 1)\phi, \tag{4.3}$$

and

$$N(\phi) = (\mathtt{W} + \phi)^p - \mathtt{W}^p - p\,\mathtt{W}^{p-1}\phi. \tag{4.4}$$

Rather than solving problem (4.2) directly we consider an intermediate projected version of it,

$$\mathcal{L}(\phi) = E - N(\phi) + \sum_{j=1}^{k} c_j(z)\eta_j w_{j,x} + d_j(z)\eta_j Z_j, \quad \text{in } \mathbb{R}^2, \tag{4.5}$$

where η_j are the cut-off functions defined by (3.19) for problem (3.15).

We will establish next that this nonlinear problem is solvable with similar estimates to those obtained in Proposition in 3.1 for problem (3.15) with $h = E$, namely we show the following result.

Proposition 4.1 *There exist positive numbers α_0, δ_0, σ_0, such that for any number $\sigma \in (0, \sigma_0)$, $\alpha \in [0, \alpha_0)$ and any $\mathbf{f}$, $\mathbf{e}$ and δ satisfying constraints (3.1)-(3.6), (3.14), Problem (4.5) has a solution ϕ with $\|\phi\|_{\sigma,\theta_0\alpha,*} \leq C_{\sigma'}\alpha^{2-2\sigma'}$, where $\sigma' \in (\sigma, 3\sigma/2)$ such that*

$$\int_{\mathbb{R}} \phi(x,z) w_{j,x}(x,z)\rho_j(x,z)\,dx = 0 = \int_{\mathbb{R}} \phi(x,z) Z_j(x,z)\rho_j(x,z)\,dx, \quad \text{for all } z \in \mathbb{R}. \tag{4.6}$$

The coefficients c_j and d_j can be estimated as follows:

$$\sum_{j=1}^{k} (|c_j(z)| + |d_j(z)|) \leq C_{\sigma'}\alpha^{2-2\sigma'} e^{-\theta_0\alpha|z|}. \tag{4.7}$$

Proof. Let us observe that we obtain a solution of the problem (4.5) if we solve the fixed point problem

$$\phi = \mathcal{T}\big(E - N(\phi)\big) := \mathcal{M}(\phi), \tag{4.8}$$

where $\mathcal{T}$ is the operator found in Proposition 3.1. Assume that $\|\phi_j\|_{\sigma,\theta_0\alpha,*} < 1$, $j = 1, 2$. We have that

$$|\, N(\phi_2) - N(\phi_1)\,| \le C(|\phi_1| + |\phi_2|)\,|\phi_2 - \phi_1|,$$

and hence

$$\|\, N(\phi_2) - N(\phi_1)\,\|_{\sigma,\theta_0\alpha,*} \;\le\; C(\,\|\phi_1\|_{\sigma,\theta_0\alpha,*} + \|\phi_2\|_{\sigma,\theta_0\alpha,*})\,\|\phi_2 - \phi_1\|_{\sigma,\theta_0\alpha,*}, \tag{4.9}$$

in particular

$$\|\, N(\phi)\,\|_{\sigma,\theta_0\alpha,*} \;\le\; \|\phi\|^2_{\sigma,\theta_0\alpha,*}.$$

Then, from (4.1), the following holds: for each fixed $\sigma' \in (\sigma, 3\sigma/2)$ there exists a number $\nu > 0$ such that for all small α the operator $\mathcal{M}$ is a contraction mapping in a region of the form

$$\mathcal{B} = \{\phi \;/\; \|\phi\|_{\sigma,\theta_0\alpha,*} \le \nu\alpha^{2-2\sigma'}\},$$

and hence a solution of the fixed point problem (4.8) in $\mathcal{B}$ exists. Furthermore ϕ solves (4.5), and by (3.20) we find that ϕ satisfies (4.6). The proof of the proposition is complete. □

5 Estimates of the error of the initial approximation

In this section we will show estimate (4.1) announced above for the error

$$E = S[\mathtt{W}] = \Delta\mathtt{W} - \mathtt{W} + \mathtt{W}^p.$$

We will denote

$$E_1 = \Delta\mathtt{W}, \quad E_2 = -\mathtt{W} + \mathtt{W}^p, \tag{5.1}$$

and let for $j = 1, \ldots, k$,

$$\mathtt{X}_j(x,z) = \frac{x - f_j(z)}{\sqrt{1 + \big(\beta_j\eta(\alpha|z|)\big)^2}}, \quad \mathtt{Z}_j(x,z) = |z|\sqrt{1 + \big(\beta_j\eta(\alpha|z|)\big)^2} + \frac{\beta_j\eta(x - f_j(z))}{\sqrt{1 + \big(\beta_j\eta(\alpha|z|)\big)^2}}.$$

With a j fixed we will set for brevity of the notation $f_j = f$, $\delta_j = \delta$, $\beta_j = \beta$, $\mathtt{X}_j = \mathtt{X}$, $\mathtt{Z}_j = \mathtt{Z}$ and let

$$w_{\delta,\beta}(x - f(z), z) := w_\delta(\mathtt{X}(x,z), \mathtt{Z}(x,z)). \tag{5.2}$$

We observe that $\mathtt{X}(x,z) = \mathtt{X}(x,-z)$ and $\mathtt{Z}(x,z) = \mathtt{Z}(x,-z)$ and therefore we only need to consider the error of the approximation assuming $z \ge 0$. For brevity we will denote

$$a_0 = \sqrt{1 + (\beta\eta)^2}, \qquad a_1 = \frac{1}{a_0} = \frac{1}{\sqrt{1 + (\beta\eta)^2}},$$

$$a_2 = a_1' = \frac{-\alpha\beta^2\eta\eta'}{a_0^3}, \qquad a_3 = a_2' = \frac{-\alpha^2\beta^2[(\eta')^2 + \eta\eta'']}{a_0^3} + \frac{3\alpha^2\beta^4\eta\eta'}{a_0^5}.$$

Using this notation we can write, whenever $z \geq 0$

$$\mathtt{X} = (x - f)a_1, \quad \mathtt{Z} = za_0 + \beta\eta\mathtt{X}.$$

We have that

$$\begin{aligned} \mathtt{X}_x &= a_1, & \mathtt{X}_z &= -f'a_1 + \mathtt{X}a_2, \\ \mathtt{Z}_x &= \beta\eta a_1, & \mathtt{Z}_z &= a_0 + \alpha\beta^2 z\eta'\eta a_1 + \alpha\beta\eta'\mathtt{X} + \beta\eta\mathtt{X}_z \\ & & &= a_0 + \alpha\beta^2 z\eta'\eta a_1 - f'\beta\eta a_1 + \alpha\beta\eta'\mathtt{X} + \beta\eta a_2\mathtt{X}^2, \end{aligned}$$

and therefore

$$\begin{aligned} A_{11} := \mathtt{X}_x^2 + \mathtt{X}_z^2 &= a_1^2\big(1 + (f')^2\big) - 2f'a_1a_2\mathtt{X} + a_2^2\mathtt{X}^2 \\ &= 1 + a_1^2\big((f')^2 - \beta^2\eta^2\big) + O(\alpha^3)(1 + \mathtt{X}^2)e^{-\theta_0\alpha|z|}, \\ A_{12} = A_{21} := \mathtt{X}_x\mathtt{Z}_x + \mathtt{X}_z\mathtt{Z}_z &= \beta\eta a_1^2 - f'a_1(a_0 + \alpha\beta\eta'\mathtt{X} + \beta\eta\mathtt{X}_z) + a_2\mathtt{X}\mathtt{Z}_z \\ &= \beta\eta - f' + a_1^2\big((f')^2 - \beta^2\eta^2\big) + O(\alpha^3)(1 + \mathtt{X}^4)e^{-\theta_0\alpha|z|}, \\ A_{22} := \mathtt{X}_z^2 + \mathtt{Z}_z^2 &= 1 + 2\beta\eta(\beta\eta - f') + \big((f')^2 - \beta^2\eta^2\big) + 2\alpha\beta^2 z\eta'\eta + 2a_0\alpha\beta\eta'\mathtt{X} \\ &\quad + O(\alpha^3)(1 + \mathtt{X}^4)e^{-\theta_0\alpha|z|}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} l_1 &:= \Delta\mathtt{X} = -f''a_1 + a_2(-f' + \mathtt{X}_z) + \mathtt{X}a_3, \\ l_2 &:= \Delta\mathtt{Z} = 2\beta^2\alpha\eta\eta'a_1 + \beta^2\alpha z(\eta'\eta a_1)' + \beta\eta\Delta\mathtt{X} + 2\alpha\beta\mathtt{X}_z\eta' + \alpha^2\beta\eta''\mathtt{X}. \end{aligned}$$

Denoting $\mathbf{A} = (A_{ij} - \delta_i^j)$, $\mathbf{l} = (l_1, l_2)$ we have for any function $u = u(\mathtt{X}(x,z), \mathtt{Z}(x,y))$,

$$\Delta_{x,z}u = \Delta_{\mathtt{X},\mathtt{Z}}u + Tr\big(\mathbf{A}\cdot Hess_{\mathtt{X},\mathtt{Z}}(u)\big) + \mathbf{l}\cdot\nabla_{\mathtt{X},\mathtt{Z}}u,$$

From (3.3)–(3.7) we also have the relations

$$\mathbf{A} = \begin{bmatrix} O(\alpha^2) + O(\alpha^3)(1 + \mathtt{X}^2) & O(\alpha) + O(\alpha^3)(1 + \mathtt{X}^4) \\ O(\alpha) + O(\alpha^3)(1 + \mathtt{X}^4) & O(\alpha^2)(1 + \mathtt{X}^4) \end{bmatrix} e^{-\theta_0\alpha|z|}.$$

and

$$\mathbf{l} = \big(O(\alpha^2) + O(\alpha^4)\mathtt{X}, O(\alpha^3) + O(\alpha^3)\mathtt{X}\big)e^{-\theta_0\alpha|z|}.$$

Using now the fact that

$$\mathtt{W}(x,z) = \sum_{j=1}^k w_{\delta_j}(\mathtt{X}_j, \mathtt{Z}_j) + \sum_{j=1}^k e_j(z)Z(\mathtt{X}_j),$$

for each $j = 1, \ldots, k$ and (2.21) and (3.11) we obtain

$$\Delta\mathtt{W} = \sum_{j=1}^k \Delta_{\mathtt{X}_j,\mathtt{Z}_j} w_{\delta_j}(\mathtt{X}_j, \mathtt{Z}_j) + E_{12}, \tag{5.3}$$

where

$$\|E_{12}\|_{\sigma,\alpha,*} \leq C\alpha^2.$$

We now turn to computing E_2. For brevity we will set: $f(u) = -u + u^p$, $p \geq 2$. We fix a j, and if $1 < j < k$ we define sets

$$A_j = \left\{ (x,z) \;\middle|\; \frac{f_{j-1}(\varepsilon z) + f_j(\varepsilon z)}{2} \leq x < \frac{f_{j+1}(\varepsilon z) + f_j(\varepsilon z)}{2} \right\},$$

while when $j = 1$ or $j = k$ we set

$$A_1 = \left\{ (x,z) \;\middle|\; -\infty < x < \frac{f_2(\varepsilon z) + f_1(\varepsilon z)}{2} \right\},$$
$$A_k = \left\{ (x,z) \;\middle|\; \frac{f_{k-1}(\varepsilon z) + f_k(\varepsilon z)}{2} \leq x < \infty \right\}.$$

For $x \in A_j$, with $1 \leq j \leq k$, fixed we write

$$\begin{aligned} E_2 = f(\mathtt{W}) &= f(w_{\delta_j,\beta_j}) + [f(\mathtt{W}) - f(w_{\delta_j,\beta_j})] \\ &= f(w_{\delta_j,\beta_j}) + f'(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j}) \\ &\quad + \frac{1}{2} f''(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j})^2 + E_{2j} \\ &= \sum_{i=1}^k f(w_{\delta_i,\beta_i}) + \left[f'(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j}) - \sum_{i\neq j} f(w_{\delta_i,\beta_i}) \right] \\ &\quad + \frac{1}{2} f''(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j})^2 + E_{2j}. \end{aligned}$$

We first assume that $p > 2$. Since $p > 2$ we have

$$\begin{aligned} |E_{2j}| &\leq C\Big(\max_{i\neq j} e^{-\frac{1}{2}(p-3)|f_i - f_j|} \Big) \Big[\sum_{i\neq j} |w_{\delta_i,\beta_i}|^3 + \sum_{i=1}^k |e_i|^3 |Z(\mathtt{X}_i)|^3 \Big] \\ &\leq C\alpha^{(p-3)} e^{-\frac{1}{2}(p-3)\theta_0\alpha|z|} \Big[\sum_{i\neq j} e^{-\sigma|x-f_i|} e^{-\frac{1}{2}(3-\sigma)|f_j - f_i|} + \alpha^3 \sum_{i=1}^k e^{-3\theta_0\alpha|z|} e^{-3|x-f_i|} \Big] \\ &\leq C\Big(\alpha^{p-\sigma} e^{-\frac{1}{2}(p-\sigma)\theta_0\alpha|z|} \sum_{i=1}^k e^{-\sigma|x-f_i|} \Big). \end{aligned}$$

Thus when $\sigma < p - 2$ then

$$\|E_{2j}\|_{\sigma,\theta_0\alpha,*} \leq C\alpha^{p-\sigma}.$$

Going back to estimation of E_2, we have

$$\begin{aligned} &f'(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j}) - \sum_{i\neq j} f(w_{\delta_i,\beta_i}) \\ &\quad = \sum_{i\neq j} [p w_{\delta_j,\beta_j}^{p-1} w_{\delta_i,\beta_i} - w_{\delta_i,\beta_i}^p] + f'(w_{\delta_j,\beta_j}) \sum_{i=1}^k e_i Z(\mathtt{X}_i) \\ &\quad = \max_{i\neq j} O(e^{-(p-1)|x-f_j| - |x-f_i|}) + \max_{i\neq j} O(e^{-p|x-f_i|}) \\ &\qquad + O(\alpha^2) e^{-\theta_0\alpha|z|} e^{-|x-f_j|} \sum_{i=1}^k e^{-(1+\mu)|x-f_i|}. \end{aligned}$$

When $0 < \sigma < p-2$ we have for $i \neq j$

$$\begin{aligned}(p-1)|x-f_j| + |x-f_i| &= \sigma|x-f_j| + (p-1-\sigma)|x-f_j| + |x-f_i| \\ &\geq \sigma|x-f_j| + \min\{1,(p-1-\sigma)\}(|x-f_j|+|x-f_i|) \\ &\geq \sigma|x-f_j| + |f_j - f_i| \\ &\geq \sigma|x-f_j| + (|\beta_i-\beta_j||z| + 2d_* - M),\end{aligned}$$

and

$$p|x-f_i| \geq \sigma|x-f_i| + \frac{p-\sigma}{2}(|\beta_j-\beta_i||z| + 2d_* - M).$$

Using $|\beta_i - \beta_j| \geq \theta_0\alpha$ we get for $x \in A_j$ and $p > 2$

$$\Delta \mathtt{W} + f(\mathtt{W}) = O(\alpha^{2-2\sigma})e^{-\sigma|x-f_i(z)|-\theta_0\alpha|z|} + E_{2j}, \tag{5.4}$$

hence in A_j we have

$$\|E\|_{\sigma,\theta_0\alpha,*} \leq C\alpha^{2-2\sigma}. \tag{5.5}$$

When $p = 2$, we trivially have

$$0 = \|E_{2j}\|_{\sigma,\theta_0\alpha,*} \leq C\alpha^{p-\sigma},$$

and

$$\begin{aligned}|x-f_j| + |x-f_i| &= \sigma|x-f_j| + (1-\sigma)|x-f_j| + |x-f_i| \\ &\geq \sigma|x-f_j| + (1-\sigma)|f_j - f_i| \\ &\geq \sigma|x-f_j| + (1-\sigma)(\theta_1\alpha|z| + 2d_* - M).\end{aligned}$$

Since $\theta_1 > \theta_0$, we may choose σ small such that $(1-\sigma)\theta_1 > \theta_0$ and hence (5.5) also holds. The ends the proof of the estimate. □

6 Dependence of the solution on the parameters

Now we will study the dependence of the function ϕ found in the proposition above on the parameters $\mathbf{f}$, $\mathbf{e}$ and δ. More specifically we are interested in establishing the Lipschitz character of ϕ as a function of variables $(\mathbf{f}'', \mathbf{e}'')$. We will begin with the observation that the error term $S[\mathtt{W}]$ can be written as follows:

$$S[\mathtt{W}] = E_1(\mathbf{f}'', \mathbf{e}''; \mathbf{f}; \vec{\delta}) + E_2(\mathbf{f}', \mathbf{e}', \mathbf{f}, \mathbf{e}; \vec{\delta}),$$

where for functions $\mathbf{g}$, $\mathbf{h}$ which are even and a vector parameter $\vec{\delta} = (\delta_1, \dots, \delta_k)$ such that

$$\|\mathbf{g}''\|_\alpha, \|\mathbf{h}''\|_\alpha \leq C\alpha^2, \qquad \|\vec{\delta}\| \leq C\alpha, \tag{6.1}$$

we have defined:

$$\begin{aligned}E_1(\mathbf{g}'', \mathbf{h}''; \mathbf{f}; \vec{\delta}) &= -\sum_{i=1}^k g_i'' a_{i1}(1,\beta_i)\cdot\nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} + \sum_{i=1}^k h'' Z(\mathtt{X}_i), \\ E_2(\mathbf{f}', \mathbf{e}'; \mathbf{f}, \mathbf{e}; \vec{\delta}) &= S[\mathtt{W}] + \sum_{i=1}^k f_i'' a_{i1}(1,\beta_i)\cdot\nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} - \sum_{i=1}^k e'' Z(\mathtt{X}_i).\end{aligned} \tag{6.2}$$

Notice that E_2 depends only on $\mathbf{f},\mathbf{f}',\mathbf{e},\mathbf{e}'$, $\vec\delta$ and it has exactly the same expression as in the previous section. Given $\mathbf{g}$, $\mathbf{h}$, $\tilde{\mathbf{f}}$, $\tilde{\mathbf{e}}$, $\mathbf{f}$, $\mathbf{e}$, $\vec\delta$ let us now consider the following problem

$$\mathcal{L}(\phi)=E_1(\mathbf{g}'',\mathbf{h}'';\mathbf{f};\vec\delta)+E_2(\tilde{\mathbf{f}}',\tilde{\mathbf{e}}';\mathbf{f},\mathbf{e};\vec\delta)-N(\phi)+\sum_{j=1}^{k}c_j(z)\eta_j w_{j,x}+d_j(z)\eta_j Z_j,\quad \text{in } \mathbb{R}^2. \tag{6.3}$$

The solution of this problem can be obtained by the argument of Section 4. In fact the analog of Proposition 4.1 yields $\|\phi\|_{\alpha,\sigma,*}\le C\alpha^{2-\sigma'}$, where $\sigma'\in(\sigma,3\sigma/2)$.

We will consider functions $\mathbf{g}_i$, $\mathbf{h}_i$, $\tilde{\mathbf{f}}_i$, $\tilde{\mathbf{e}}_i$, $\mathbf{f}_i$, $\mathbf{e}_i$, and vectors $\vec\delta_i$, $i=1,2$ such that

$$\begin{aligned}\|\mathbf{g}_1''-\mathbf{g}_2''\|_{\theta_0\alpha}+\alpha\|\tilde{\mathbf{f}}_1'-\tilde{\mathbf{f}}_2'\|_{\theta_0\alpha}+\alpha^2\|\mathbf{f}_1-\mathbf{f}_2\|_\infty&\le C\alpha^2,\\ \|\mathbf{h}_1''-\mathbf{h}_2''\|_{\theta_0\alpha}+\|\tilde{\mathbf{e}}_1'-\tilde{\mathbf{e}}_2'\|_{\theta_0\alpha}+\|\mathbf{e}_1-\mathbf{e}_2\|_{\theta_0\alpha}&\le C\alpha^2,\\ \|\vec\delta_1-\vec\delta_2\|&\le C\alpha.\end{aligned} \tag{6.4}$$

First we want to show that functions E_1 and E_2 are Lipschitz functions of its variables. To make it more precise we will distinguish the norms taken with respect to different functions $\mathbf{f}$. Thus we will denote in this section:

$$\|\phi\|_{\sigma,\theta_0\alpha,\mathbf{f}}:=\Big\|\Big(\sum_{j=1}^{k}e^{-\sigma|x-f_j(z)|-\theta_0\alpha|z|}\Big)^{-1}\phi\Big\|_\infty,\quad \mathbf{f}=(f_1,\dots,f_k).$$

Lemma 6.1 *Under the assumptions (6.4) we have the following estimates:*

$$\begin{aligned}&\|E_1(\mathbf{g}_1'',\mathbf{h}_1;\mathbf{f}_1;\vec\delta_1)-E_1(\mathbf{g}_2'',\mathbf{h}_2;\mathbf{f}_2;\vec\delta_2)\|_{\sigma,\theta_0\alpha,\mathbf{f}_1}\\ &\quad\le C(\|\mathbf{g}_1''-\mathbf{g}_2''\|_\alpha+\|\mathbf{h}_1''-\mathbf{h}_2''\|_{\theta_0\alpha})+C(\max_{i=1,2}\|\mathbf{g}_i''\|_{\theta_0\alpha}+\max_{i=1,2}\|\mathbf{h}_i''\|_{\theta_0\alpha})\|\mathbf{f}_1-\mathbf{f}_2\|_\infty\\ &\qquad+C\max_{i=1,2}\|\mathbf{g}_i''\|_{\theta_0\alpha}\|\vec\delta_1-\vec\delta_2\|,\end{aligned} \tag{6.5}$$

$$\begin{aligned}&\|E_2(\tilde{\mathbf{f}}_1',\tilde{\mathbf{e}}_1';\mathbf{f}_1,\mathbf{e}_1;\vec\delta_1)-E_2(\tilde{\mathbf{f}}_2',\tilde{\mathbf{e}}_2';\mathbf{f}_2,\mathbf{e}_2;\vec\delta_2)\|_{\sigma,\theta_0\alpha,\mathbf{f}_1}\\ &\quad\le C(\alpha^{1-2\sigma}\|\tilde{\mathbf{f}}_1'-\tilde{\mathbf{f}}_2'\|_{\theta_0\alpha}+\|\tilde{\mathbf{e}}_1'-\tilde{\mathbf{e}}_2'\|_{\theta_0\alpha})+C(\alpha^{2-2\sigma}\|\mathbf{f}_1-\mathbf{f}_2\|_\infty+\|\mathbf{e}_1-\mathbf{e}_2\|_{\theta_0\alpha})\\ &\qquad+C\alpha^{2-2\sigma}\|\vec\delta_1-\vec\delta_2\|.\end{aligned} \tag{6.6}$$

Proof. First notice that to prove Lipschitz estimates (6.5)–(6.6) it suffices to show corresponding inequalities varying one component at a time (for instance considering $\mathbf{g}_i$, $i=1,2$ and fixing the rest of the parameters). When $\tilde{\mathbf{f}}$ and $\mathbf{f}$ are fixed then the estimates are rather standard. We will therefore concentrate on the case when all parameters except $\tilde{\mathbf{f}}$ and $\mathbf{f}$ are fixed.

We will indicate the dependence of the change of variables $\mathtt{X}_j,\mathtt{Z}_j$ on $\mathbf{f}$ by writing $\mathtt{X}_j(\mathbf{f}),\mathtt{Z}_j(\mathbf{f})$. Similarly, we will write $\mathbf{A}_j(\mathbf{f},\tilde{\mathbf{f}}')$ for the components of the matrix $\mathbf{A}_j$. Finally we will set

$$\tilde{\mathbf{l}}_j(\mathbf{f},\tilde{\mathbf{f}}')=\mathbf{l}_j(\mathbf{f},\tilde{\mathbf{f}}',\mathbf{g}'')+g_i''a_{i1}(1,\beta_i).$$

We will gather first some estimates that are important in the proof. We have

$$\begin{aligned}|w(\mathtt{X}_j(\mathbf{f}_1)-w'(\mathtt{X}_j(\mathbf{f}_2)|+|w(\mathtt{X}_j(\mathbf{f}_1)-w'(\mathtt{X}_j(\mathbf{f}_2)|&\le C|w(\mathtt{X}_j(\mathbf{f}_1)||\mathbf{f}_1-\mathbf{f}_2|,\\ |\tilde{\mathbf{l}}_j(\mathbf{f}_1,\tilde{\mathbf{f}}')-\tilde{\mathbf{l}}_j(\mathbf{f}_2,\tilde{\mathbf{f}}')|&\le C\alpha^2|\mathbf{f}_1-\mathbf{f}_2|,\\ |\tilde{\mathbf{l}}_j(\mathbf{f},\tilde{\mathbf{f}}_1')-\tilde{\mathbf{l}}_j(\mathbf{f},\tilde{\mathbf{f}}_2')|&\le C\alpha|\tilde{\mathbf{f}}_1'-\tilde{\mathbf{f}}_2'|.\end{aligned} \tag{6.7}$$

From the definition of E_1 and (6.7) one gets (6.5). Estimate (6.6) is somewhat tedious but rather standard. For instance let us consider a typical term coming from $\Delta \mathtt{W}$:

$$\begin{aligned}
&|Tr\big(\mathbf{A}.Hess_{\mathtt{X}_j,\mathtt{Z}_j}(w_{\delta_j,\beta_j})\big)(\mathbf{f}_1,\tilde{\mathbf{f}}_1') - Tr\big(\mathbf{A}_j\cdot Hess_{\mathtt{X}_j,\mathtt{Z}_j}(w_{\delta_j,\beta_j})\big)(\mathbf{f}_2,\tilde{\mathbf{f}}_2')| \\
&\le |Tr\big(\mathbf{A}_j\cdot Hess_{\mathtt{X}_j,\mathtt{Z}_j}(w_{\delta_j,\beta_j})\big)(\mathbf{f}_1,\tilde{\mathbf{f}}_1') - Tr\big(\mathbf{A}_j\cdot Hess_{\mathtt{X}_j,\mathtt{Z}_j}(w_{\delta_j,\beta_j})\big)(\mathbf{f}_2,\tilde{\mathbf{f}}_1')| \\
&\quad + |Tr\big(\mathbf{A}_j\cdot Hess_{\mathtt{X}_j,\mathtt{Z}_j}(w_{\delta_j,\beta_j})\big)(\mathbf{f}_2,\tilde{\mathbf{f}}_2') - Tr\big(\mathbf{A}_j\cdot Hess_{\mathtt{X}_j,\mathtt{Z}_j}(w_{\delta_j,\beta_j})\big)(\mathbf{f}_2,\tilde{\mathbf{f}}_1')| \\
&\quad = B_1 + B_2.
\end{aligned}$$

Using the (6.7) we can estimate the first term above by

$$\begin{aligned}
B_1 \le\ & C|\mathbf{A}_{j,11}(\mathbf{f}_1,\tilde{\mathbf{f}}_1') - \mathbf{A}_{j,11}(\mathbf{f}_2,\tilde{\mathbf{f}}_1')||w(\mathtt{X}_j)(\mathbf{f}_1)| \\
& + C|\mathbf{A}_{j,11}(\mathbf{f}_2,\tilde{\mathbf{f}}_1')||w''(\mathtt{X}_j)(\mathbf{f}_1) - w''(\mathtt{X}_j)(\mathbf{f}_2)| \\
& + C\delta_j|\mathbf{A}_j(\mathbf{f}_1,\tilde{\mathbf{f}}_1') - \mathbf{A}_j(\mathbf{f}_2,\tilde{\mathbf{f}}_1')||Z(\mathtt{X}_j)(\mathbf{f}_1)| \\
& + C\delta_j|\mathbf{A}_j(\mathbf{f}_2,\tilde{\mathbf{f}}_1')||Z(\mathtt{X}_j)(\mathbf{f}_1) - Z(\mathtt{X}_j)(\mathbf{f}_2)| \\
\le\ & C\alpha^2|\mathbf{f}_1 - \mathbf{f}_2|e^{-\theta_0\alpha|z|}\big|w\big(\mathtt{X}_j(\mathbf{f}_1)\big)\big|^\sigma.
\end{aligned}$$

In an analogous way we can estimate B_2. Another typical term we have to deal with is (considered in the set A_j defined in the previous section) the following

$$\begin{aligned}
&|f'(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j})(\mathbf{f}_1) - f'(w_{\delta_j,\beta_j})(\mathtt{W} - w_{\delta_j,\beta_j})(\mathbf{f}_2)| \\
&\le |f'(w_{\delta_j,\beta_j})(\mathbf{f}_1) - f'(w_{\delta_j,\beta_j})(\mathbf{f}_2)||(\mathtt{W} - w_{\delta_j,\beta_j})(\mathbf{f}_1)| \\
&\quad + |f'(w_{\delta_j,\beta_j})(\mathbf{f}_2)||(\mathtt{W} - w_{\delta_j,\beta_j})(\mathbf{f}_1) - (\mathtt{W} - w_{\delta_j,\beta_j})(\mathbf{f}_2)| \\
&\le C\alpha^{2-\sigma}e^{-\theta_0\alpha|z|}|\mathbf{f}_1 - \mathbf{f}_2|\big|w\big(\mathtt{X}_j(\mathbf{f}_1)\big)\big|^\sigma + C\max_{i\neq j} e^{-|f_{1j}-f_{1i}|}|\mathbf{f}_1 - \mathbf{f}_2|\big|w\big(\mathtt{X}_j(\mathbf{f}_1)\big)\big|^\sigma \\
&\le C\alpha^{2-\sigma}e^{-\theta_0\alpha|z|}|\mathbf{f}_1 - \mathbf{f}_2|\big|w\big(\mathtt{X}_j(\mathbf{f}_1)\big)\big|^\sigma.
\end{aligned}$$

Other terms in the definition of E_2 are handled similarly and from this we get (6.6). We leave the details to the reader. □

In what follows we will emphasize the dependence of ϕ and c_j, d_j on parameters by writing

$$\begin{aligned}
\phi^{(i)} &= \phi(\mathbf{g}_i'', \mathbf{h}_i''; \tilde{\mathbf{f}}_i', \tilde{\mathbf{e}}_i'; \mathbf{f}_i, \mathbf{e}_i; \vec{\delta}_i), \\
c_j^{(i)} &= c_j^{(i)}(\mathbf{g}_i'', \mathbf{h}_i''; \tilde{\mathbf{f}}_i', \tilde{\mathbf{e}}_i'; \mathbf{f}_i, \mathbf{e}_i; \vec{\delta}_i), \\
d_j^{(i)} &= d_j^{(i)}(\mathbf{g}_i'', \mathbf{h}_i''; \tilde{\mathbf{f}}_i', \tilde{\mathbf{e}}_i'; \mathbf{f}_i, \mathbf{e}_i; \vec{\delta}_i).
\end{aligned} \tag{6.8}$$

Proposition 6.1 *Let ϕ, be the solution of (6.3). Then for $j = 1, \ldots, k$ functions ϕ, c_j, d_j are continuous with respect to the parameters $\mathbf{g}''$, $\mathbf{h}''$,$\tilde{\mathbf{f}}'$, $\tilde{\mathbf{e}}'$, $\mathbf{f}$, $\mathbf{e}$. Moreover assuming (6.4), in the notation of (6.8) we have the following estimates*

$$\begin{aligned}
&\|\phi^{(1)} - \phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_1} + \|\nabla\phi^{(1)} - \nabla\phi\|_{\sigma,\theta_0\alpha\mathbf{f}_1} \\
&\quad \le C\alpha^{-2\sigma'}(\|\mathbf{g}_1'' - \mathbf{g}_2''\|_{\theta_0\alpha} + \|\mathbf{h}_1'' - \mathbf{h}_2''\|_{\theta_0\alpha}) \\
&\qquad + C\alpha^{-2\sigma'}(\alpha\|\tilde{\mathbf{f}}_1' - \tilde{\mathbf{f}}_2'\|_{\theta_0\alpha} + \|\tilde{\mathbf{e}}_1' - \tilde{\mathbf{e}}_2'\|_{\theta_0\alpha}) \\
&\qquad + C\alpha^{-2\sigma'}(\alpha^2\|\mathbf{f}_1 - \mathbf{f}_2\|_\infty + \|\mathbf{e}_1 - \mathbf{e}_2\|_{\theta_0\alpha}) \\
&\qquad + C\alpha^{2-2\sigma'}\|\vec{\delta}_1 - \vec{\delta}_2\|,
\end{aligned} \tag{6.9}$$

where $\sigma' \in (\sigma, 3\sigma/2)$.

Proof. First observe that it is sufficient to prove Lipschitz dependence considering dependence of ϕ on different components taken separately. For instance let us fix $\tilde{\mathbf{f}}$, $\tilde{\mathbf{e}}$, $\mathbf{f}$ and $\mathbf{e}, \vec{\delta}$ and let vary $\mathbf{g}_i$, $\mathbf{h}_i$. Then the proof of the result follows directly from Proposition (3.1) and Proposition (4.1) applied to the equation for the difference $\phi(\mathbf{g}_1'', \mathbf{h}_1''; \tilde{\mathbf{f}}', \tilde{\mathbf{e}}'; \mathbf{f}, \mathbf{e}; \vec{\delta}) - \phi(\mathbf{g}_2'', \mathbf{h}_2''; \tilde{\mathbf{f}}', \tilde{\mathbf{e}}'; \mathbf{f}, \mathbf{e}; \vec{\delta})$. We leave the details to the reader.

We will now consider Lipschitz dependence of ϕ on $\tilde{\mathbf{f}}', \tilde{\mathbf{e}}'$, $\mathbf{e}$. Let us fix $\mathbf{g}$ and $\mathbf{h}$ as above as well $\mathbf{f}$ and $\vec{\delta}$. We will denote by $\phi^{(i)}$ the solution of the following problem in $\mathbb{R}^2$:

$$\mathcal{L}(\phi^{(i)}) = E_1(\mathbf{g}'', \mathbf{h}''; \mathbf{f}; \vec{\delta}) + E_2(\tilde{\mathbf{f}}_i', \tilde{\mathbf{e}}_i'; \mathbf{f}, \mathbf{e}_i; \vec{\delta}) - N(\phi^{(i)}) + \sum_{j=1}^{k} c_j^{(i)}(z)\eta_j w_{j,x} + d_j^{(i)}(z)\eta_j Z_j. \tag{6.10}$$

It follows from Lemma 6.1 that

$$\begin{aligned} &\|E_2(\tilde{\mathbf{f}}_1', \tilde{\mathbf{e}}_1'; \mathbf{f}, \mathbf{e}_1; \vec{\delta}) - E_2(\tilde{\mathbf{f}}_2', \tilde{\mathbf{e}}_2'; \mathbf{f}, \mathbf{e}_2; \vec{\delta})\|_{\sigma,\theta_0\alpha,\mathbf{f}_1} \\ &\leq C\alpha^{-2\sigma'}(\alpha\|\tilde{\mathbf{f}}_1' - \tilde{\mathbf{f}}_2'\|_{\theta_0\alpha} + \|\mathbf{e}_1 - \mathbf{e}_2\|_{\theta_0\alpha}) + \|\tilde{\mathbf{e}}_1' - \tilde{\mathbf{e}}_2'\|_{\theta_0\alpha}). \end{aligned} \tag{6.11}$$

Applying now the theory developed for the linear problem for the difference $\phi^{(1)} - \phi^{(2)}$ we get

$$\begin{aligned} &\|\phi^{(1)} - \phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_1} + \|\nabla\phi^{(1)} - \nabla\phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_1} \\ &\leq C\alpha^{-2\sigma'}(\alpha\|\tilde{\mathbf{f}}_1' - \tilde{\mathbf{f}}_2'\|_{\theta_0\alpha} + \|\mathbf{e}_1 - \mathbf{e}_2\|_{\theta_0\alpha}) + \|\tilde{\mathbf{e}}_1' - \tilde{\mathbf{e}}_2'\|_{\theta_0\alpha}). \end{aligned} \tag{6.12}$$

Finally, it remains to consider the Lipschitz dependence of solutions on $\mathbf{e}$ and $\mathbf{f}$. This case is somewhat more complicated since the norms involved depend on the variables $\mathtt{X}_j$, which in turn depend on the functions f_j.

Now, by $\phi^{(i)}$ we will denote solutions of

$$\begin{aligned} \mathcal{L}_{\mathbf{f}_i}(\phi^{(i)}) = {} & E_1(\mathbf{g}'', \mathbf{h}''; \mathbf{f}_i; \vec{\delta}) + E_2(\tilde{\mathbf{f}}', \tilde{\mathbf{e}}'; \mathbf{f}_i, \mathbf{e}; \vec{\delta}) - N(\phi^{(i)}) \\ & + \sum_{j=1}^{k} c_j^{(i)}(z)\eta_j^{(i)} w_{j,x}^{(i)} + d_j^{(i)}(z)\eta_j^{(i)} Z_j^{(i)}, \quad \text{in } \mathbb{R}^2, \end{aligned} \tag{6.13}$$

with fixed functions $\mathbf{g}$, $\mathbf{h}$, $\tilde{\mathbf{f}}$, $\tilde{\mathbf{e}}$, and function $\mathbf{f}_i$, $i = 1, 2$ such that

$$\|\mathbf{f}_1 - \mathbf{f}_2\|_\infty \leq C. \tag{6.14}$$

The equation for the difference $\phi^{(1)} - \phi^{(2)} \equiv \tilde{\phi}$ can be written in the form:

$$\begin{aligned} \mathcal{L}_{\mathbf{f}_1}\tilde{\phi} = {} & E_1(\mathbf{g}'', \mathbf{h}''; \mathbf{f}_1; \vec{\delta}) - E_1(\mathbf{g}'', \mathbf{h}''; \mathbf{f}_2; \vec{\delta}) \\ & + E_2(\tilde{\mathbf{f}}', \tilde{\mathbf{e}}'; \mathbf{f}_1, \tilde{\mathbf{e}}; \vec{\delta}) - E_2(\tilde{\mathbf{f}}', \tilde{\mathbf{e}}'; \mathbf{f}_2, \tilde{\mathbf{e}}; \vec{\delta}) \\ & + p[(\mathtt{W}^{(1)})^{p-1} - (\mathtt{W}^{(2)})^{p-1}]\phi^{(2)} + [N(\phi^{(1)}) - N(\phi^{(2)})] \\ & + \sum_{j=1}^{k} c_j^{(2)}(\eta_j^{(1)} w_{j,x}^{(1)} - \eta_j^{(2)} w_{j,x}^{(2)}) + \sum_{j=1}^{k} d_j^{(2)}(\eta_j^{(1)} Z_j^{(1)} - \eta_j^{(2)} Z_j^{(2)}) \\ & + \sum_{j=1}^{k}(c_j^{(1)} - c_j^{(2)})\eta_j^{(1)} w_{j,x}^{(1)} + \sum_{j=1}^{k}(d_j^{(1)} - d_j^{(2)})\eta_j^{(1)} Z_j^{(1)}. \end{aligned} \tag{6.15}$$

Observe that if we denote $\tilde{\mathbf{r}} = \mathbf{f}_1 - \mathbf{f}_2$ then we have $\|\tilde{\mathbf{r}}\|_{\theta_0\alpha} \le C$ and

$$|\mathtt{W}^{(1)} - \mathtt{W}^{(2)}| \le C|\tilde{\mathbf{r}}| \sum_{j=1}^{k} e^{-|x-f_{1j}|}.$$

It follows

$$\|\mathtt{W}^{(1)} - \mathtt{W}^{(2)}\|_{\sigma,\infty,\mathbf{f}_1} \le C\|\tilde{\mathbf{r}}\|_{\infty}. \tag{6.16}$$

Similar estimates hold for other terms involved in the right hand side of (6.15). Now, we will further decompose

$$\tilde{\phi} = \tilde{\phi}_0 + \sum_{j=1}^{k} \tilde{\phi}_j \rho_j^{(1)} w_{j,x}^{(1)},$$

where

$$\tilde{\phi}_j = \frac{\int_{\mathbb{R}} \phi^{(2)} (\rho_j^{(1)} w_{j,x}^{(1)} - \rho_j^{(2)} w_{j,x}^{(2)})\, dx}{\int_{\mathbb{R}} (\rho_j^{(1)} w_{j,x}^{(1)})^2\, dx}.$$

The Lipschitz character of functions $\tilde{\phi}_j$, $j = 1, \dots, k$ can be established directlly from the definition. On the other hand function $\tilde{\phi}_0$ satisfies an equation similar to (6.15) together with the orthogonality conditions. This, estimates in Proposition 4.1 and Lemma 6.1 yield finally

$$\|\phi^{(1)} - \phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_1} + \|\nabla\phi^{(1)} - \nabla\phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_1} \le C\alpha^{2-2\sigma'} \|\tilde{\mathbf{r}}\|_{\infty} e^{-\theta_0\alpha|z|}. \tag{6.17}$$

7 Projection of the error on the elements of the kernel

Let us now compute the projections of the error on the elements of the approximate kernel. Thus for each $j = 1, \dots, k$ we will compute

$$\Pi_{f_j} := \int_{\mathbb{R}} S[\mathtt{W}] w_x(\mathtt{X}_j)\, dx,$$
$$\Pi_{e_j} := \int_{\mathbb{R}} S[\mathtt{W}] Z(\mathtt{X}_j)\, dx,$$

For a fixed j we begin with Π_{f_j}. We will use decomposition of $S[\mathtt{W}]$ similar to the one in section 5:

$$\begin{aligned} S[\mathtt{W}] = & \sum_{i=1}^{k} \left[Tr(\mathbf{A}_i \cdot Hess_{\mathtt{X}_i,\mathtt{Z}_i}(w_{\delta_i,\beta_i}) + \mathbf{l}_i \cdot \nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} \right] \\ & + \sum_{i=1}^{k} \Delta_{x,z}[e_i(z) Z(\mathtt{X}_i)] \\ & + [f(\mathtt{W}) - \sum_{i=1}^{k} f(w_{\delta_i,\beta_i})] \\ := & S_{1j}[\mathtt{W}] + S_{2j}[\mathbf{e}, Z(\mathtt{X}_1), \dots, Z(\mathtt{X}_k)] + S_{3j}[\mathtt{W}]. \end{aligned} \tag{7.1}$$

Using (2.21) we obtain:

$$\begin{aligned} Hess_{\mathtt{X}_i,\mathtt{Z}_i}(w_{\delta_i,\beta_i}) &= \begin{bmatrix} w''(\mathtt{X}_i) & 0 \\ 0 & 0 \end{bmatrix} \\ &+ \delta_i \begin{bmatrix} Z''(\mathtt{X}_i)\cos(\sqrt{\lambda_1}\mathtt{Z}_i) & -\sqrt{\lambda_1}Z'(\mathtt{X}_i)\sin(\sqrt{\lambda_1}\mathtt{Z}_i) \\ -\sqrt{\lambda_1}Z'(\mathtt{X}_i)\sin(\sqrt{\lambda_1}\mathtt{Z}_i) & -\lambda_1 Z(\mathtt{X}_i)\cos(\sqrt{\lambda_1}\mathtt{Z}_i) \end{bmatrix} \\ &+ O(\delta_i^2)e^{-|\mathtt{X}_i|}. \end{aligned}$$

From the formulas for $\mathtt{X}_i, \mathtt{Z}_i$ and their derivatives we also get

$$\begin{aligned} \mathbf{A}_i = &\begin{bmatrix} a_{i1}^2\big((f_i')^2 - \beta_i^2\eta^2\big) & \beta_i\eta - f_i' + a_{i1}^2\big((f_i')^2 - \beta_i^2\eta^2\big) \\ \beta_i\eta - f_i' + a_{i1}^2\big((f_i')^2 - \beta_i^2\eta^2\big) & 2\beta_i\eta(\beta_i\eta - f_i') + \big((f_i')^2 - \beta_i^2\eta^2\big) + 2\alpha\beta_i^2 z\eta'\eta \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 2a_0\alpha\beta_i\eta'\mathtt{X}_i \end{bmatrix} + \begin{bmatrix} O(\alpha^3) & O(\alpha^3) \\ O(\alpha^3) & O(\alpha^3) \end{bmatrix}(1 + \mathtt{X}_i^2 + \mathtt{X}_i^4)e^{-\theta_0\alpha|z|} \\ \equiv\ & \mathbf{A}_{i0} + \mathbf{A}_{i1} + \mathbf{A}_{i2}. \end{aligned}$$

Similarly we have

$$\nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} = (w'(\mathtt{X}_i), 0) + \delta_i\big(Z'(\mathtt{X}_i)\cos(\sqrt{\lambda_1}\mathtt{Z}_i), -\sqrt{\lambda_1}Z(\mathtt{X}_i)\sin(\sqrt{\lambda_1}\mathtt{Z}_i)\big) + O(\delta_i^2)e^{-|\mathtt{X}_i|},$$

and

$$\mathbf{l}_i = -f_i'' a_{i1}(1, \beta_i\eta) + O(\alpha^3)(1 + |f_i' - \beta_i\eta| + |\mathtt{X}_i|)e^{-\theta_0\alpha|z|}.$$

We observe that the entries of $\mathbf{A}_i$ and $\mathbf{l}_i$ are polynomials with respect to $\mathtt{X}_i$ and they depend on f_i only through $\mathtt{X}_i$. In the sequel we will use also expansion

$$\cos(\sqrt{\lambda_1}\mathtt{Z}_i) = \cos(\sqrt{\lambda_1}a_{i0}z) + O(\alpha)|\mathtt{X}_i|, \quad \sin(\sqrt{\lambda_1}\mathtt{Z}_i) = \sin(\sqrt{\lambda_1}a_{i0}z) + O(\alpha)|\mathtt{X}_i|.$$

Since functions w'', Z and Z'' are even functions of its arguments and w' is an odd function therefore, changing variables $x \mapsto \mathtt{X}_j$, we get

$$\begin{aligned} \int_{\mathbb{R}} Tr\big(\mathbf{A}_j \cdot Hess(w_{\delta_j,\beta_j})\big)w'(\mathtt{X}_j)\,dx = &-2\sqrt{\lambda_1}a_{j0}\delta_j\left[\beta_j\eta - f_j' + a_{j1}^2\big((f_j')^2 - \beta_j^2\eta^2\big)\right] \\ &\times \sin(\sqrt{\lambda_1}a_{i0}z)\int_R w'(s)Z'(s)\,ds \\ &+ \delta_j O(\delta_j + \beta_j)(f_j' - \beta_j\eta) \\ &+ \alpha^2 O(\delta_j + \alpha)e^{-\theta_0\alpha|z|}. \end{aligned}$$

When $i \neq j$ we have

$$\int_{\mathbb{R}} w'(\mathtt{X}_j)w(\mathtt{X}_i)\,dx = O(e^{-|f_j - f_i|}), \qquad \int_{\mathbb{R}} Z(\mathtt{X}_j)w(\mathtt{X}_i)\,dx = O(e^{-(1+\mu)|f_j - f_i|}),$$

with similar formulas for the integrals involving $Z'(\mathtt{X}_j)$ and $Z''(\mathtt{X}_j)$. It follows then

$$\begin{aligned} \int_{\mathbb{R}} Tr\big(\mathbf{A}_i \cdot Hess(w_{\delta_i,\beta_i})\big)w'(\mathtt{X}_j)\,dx = &\big((f_i')^2 - \beta_i^2\eta^2\big)O(e^{-|f_j - f_i|}) \\ &+ (f_i' - \beta_i\eta)(\alpha + \delta_i)\delta_i O(e^{-|f_j - f_i|}) \\ &+ O(\alpha^3 + \delta_i\alpha^2)e^{-\theta_0\alpha|z|}. \end{aligned}$$

We have also

$$\int_{\mathbb{R}} \mathbf{l}_j \cdot \nabla_{\mathtt{X}_j,\mathtt{Z}_j} w_{\delta_j,\beta_j} w'(\mathtt{X}_j)\,dx = -f_j''\Big[\int_{\mathbb{R}} (w')^2 + \delta_j \cos(\sqrt{\lambda_1}a_{i0}z)\int_{\mathbb{R}} Z'w' + \delta_j O(\delta_j+\beta_j)\Big] + O(\alpha^3)(f_j'+1)e^{-\theta_0\alpha|z|},$$

and similarly for $i \neq j$ we get

$$\int_{\mathbb{R}} \mathbf{l}_j \cdot \nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} w'(\mathtt{X}_j)\,dx = -f_i'' O(e^{-|f_j-f_i|}) + O(\alpha^3)(f_i'+1)e^{-\theta_0\alpha|z|}.$$

To compute the corresponding projections of S_{2j} we notice that

$$\begin{aligned}\Delta_{x,z}\big(e_i(z)Z(\mathtt{X}_i)\big) &= e_i''(z)Z(\mathtt{X}_i) + 2e_i'(z)Z'(\mathtt{X}_i)\mathtt{X}_{iz} + e_i(z)\Delta_{x,z}Z(\mathtt{X}_i)\\ &= e_i''Z(\mathtt{X}_i) + \big[2e_i'(-f_i'a_{i1}+\mathtt{X}_i a_{i2}) - e_i f_i'' a_{i1}\\ &\quad + e_i(f_i'+\mathtt{X}_i)e^{-\theta_0\alpha|z|}O(\alpha^3)\big]Z'(\mathtt{X}_i)\\ &\quad + e_i\big[1 + a_{i1}^2\big((f_i')^2 - \beta_i^2\eta^2\big) + O(\alpha^3)(1+\mathtt{X}_i^2+\mathtt{X}_i^4)e^{-\theta_0\alpha|z|}\big]Z''(\mathtt{X}_i).\end{aligned} \tag{7.2}$$

Then we get

$$\begin{aligned}\int_{\mathbb{R}} S_{2j}w'(\mathtt{X}_j)\,dx &= -(2e_j'f_j' + e_j f_j'')\int_{\mathbb{R}} Z'(s)w'(s)\,ds\\ &\quad + \sum_{i\neq j}\big\{e_i'' - (2e_i f_i' a_{i1} + e_i f_i'' a_{i1})\\ &\qquad + e_i\big[1 + a_{i1}^2\big((f_i')^2 - \beta_i^2\eta^2\big)\big]\big\}O(e^{-|f_i-f_j|})\\ &\quad + \sum_{i\neq j}(e_i' + e_i f_i')O(\alpha^3)e^{-\theta_0\alpha|z|}.\end{aligned}$$

Using the sets A_l, $l = 1,\dots,k$ introduced above the projection of $S_{3j}[\mathtt{W}]$ on $w'(\mathtt{X}_j)$ can be written as follows

$$\begin{aligned}\int_{\mathbb{R}} S_{3j}[\mathtt{W}]w'(\mathtt{X}_j)\,dx &= \int_{A_j}\Big[f'(w_{\delta_j,\beta_j})\sum_{i\neq j} w_{\delta_i,\beta_i} - \sum_{i\neq j} f(w_{\delta_i,\beta_i})\Big]w'(\mathtt{X}_j)\,dx\\ &\quad + \sum_{l\neq j}\int_{A_l}\Big[f'(w_{\delta_l,\beta_l})\sum_{i\neq l} w_{\delta_i,\beta_i} - \sum_{i\neq l} f(w_{\delta_i,\beta_i})\Big]w'(\mathtt{X}_j)\,dx\\ &\quad + \sum_{l=1}^{k}\int_{A_l} f'(w_{\delta_l,\beta_l})\sum_{i=1}^{k} e_i Z(\mathtt{X}_i)w'(\mathtt{X}_j)\,dx\\ &\quad + \sum_{l=1}^{k}\int_{A_l} O\big((\mathtt{W} - w_{\delta_l,\beta_l})^2\big)w'(\mathtt{X}_j)\,dx\\ &:= S_{3j1} + S_{3j2} + S_{3j3} + S_{3j4}.\end{aligned}$$

For S_{3j1} we get

$$S_{3j1} = \sum_{i\neq j}\int_{A_j}[p w_{\delta_j,\beta_j}^{p-1} w_{\delta_i,\beta_i} - w_{\delta_i,\beta_i}^{p}]w'(\mathtt{X}_j)\,dx.$$

For a fixed $i \neq j$, using formulas (2.21), we have

$$\begin{aligned}
&\int_{A_j} [p w_{\delta_j,\beta_j}^{p-1} w_{\delta_i,\beta_i} - w_{\delta_i,\beta_i}^{p}] w'(\mathtt{X}_j)\, dx \\
&= \int_{A_j} [p w(\mathtt{X}_j)^{p-1} w(\mathtt{X}_i) - w^p(\mathtt{X}_i)] w'(\mathtt{X}_j)\, dx \\
&\qquad + \delta_j \int_{A_j} [O(e^{-(1+\mu)|\mathtt{X}_j|}) w(\mathtt{X}_i) + O(e^{-(1+\mu)|\mathtt{X}_i|}) w(\mathtt{X}_j)]\, dx \\
&= I_{1i} + I_{2i}.
\end{aligned}$$

To compute I_{1i} let us observe that for any $\lambda \in \mathbb{R}$ and $a < b$ we have:

$$\begin{aligned}
\int_a^b p w^{p-1}(s) w'(s) w(s-\lambda) - \int_{\mathbb{R}} w^p(s-\lambda) w'(s) &= p w^p(s) w(s-\lambda)\,\Big|_a^b \\
&\quad - \int_a^b [w^p(s) w'(s-\lambda) + w^p(s-\lambda) w'(s)] \\
&= \big[p w^p(s) w(s-\lambda) + w'(s) w'(s-\lambda) - w(s) w(s-\lambda)\big]\,\Big|_a^b\ ,
\end{aligned}$$

hence, changing variables, we get, with $s_j = \frac{f_{j-1}-f_j}{2}$, $t_j = \frac{f_{j+1}-f_j}{2}$, $\lambda_{ij} = f_i - f_j$,

$$\begin{aligned}
I_{1i} &= \int_{s_j}^{t_j} p w^{p-1}(s) w'(s) w\big(s a_{j0} a_{i1} + (f_j - f_i) a_{i1}\big)\, ds \\
&\quad - \int_{\mathbb{R}} w^p\big(s a_{j0} a_{i1} + (f_j - f_i) a_{i1}\big) w'(s)\, ds \\
&= \big[p w^p(s) w(s-\lambda_{ij}) + w'(s) w'(s-\lambda_{ij}) - w(s) w(s-\lambda_{ij})\big]\,\Big|_{s_j}^{t_j} \\
&\quad + O(\beta_i^2 + \beta_j^2) e^{-|f_j - f_i|}.
\end{aligned}$$

Similarly,

$$I_{2i} = O(\delta_j) e^{-|f_j - f_i|}.$$

Using this formula with $i = 1, \dots, k$ and noting that for a fixed j the dominating terms in S_{3j1} come from $i = j-1$ and $i = j+1$ we get

$$\begin{aligned}
S_{3j1} &= w\big(\frac{f_j - f_{j-1}}{2}\big)^2 + w'\big(\frac{f_j - f_{j-1}}{2}\big)^2 - w\big(\frac{f_j - f_{j+1}}{2}\big)^2 - w'\big(\frac{f_j - f_{j+1}}{2}\big)^2 \\
&\quad + O(\alpha) \sum_{i \neq j} e^{-|f_j - f_i|} \\
&= C_p\big(e^{-|f_j - f_{j-1}|} - e^{-|f_{j+1} - f_j|}\big) + O(\alpha) \sum_{i \neq j} e^{-|f_j - f_i|},
\end{aligned}$$

where we have denoted

$$C_p = \lim_{s \to \infty} \{e^{s} [w(s/2)^2 + w'(s/2)^2]\}. \tag{7.3}$$

The remaining term S_{3j2} can be estimated in a similar way. Observing that when $l \neq j$ and $x \in A_l$ we have $w'(\mathtt{X}_j) = w'(\mathtt{X}_l)O(e^{-|f_j - f_l|})$, we get that

$$S_{3j2} = O(\alpha) \sum_{i \neq j} e^{-|f_j - f_i|}.$$

To compute S_{3j3} we observe that

$$\int_{\mathbb{R}} f'(s)Z(s)w'(s)\, ds = 0,$$

hence

$$\begin{aligned} S_{3j3} &= \int_{A_j} f'(w(\mathtt{X}_j)) \sum_{i \neq j} e_i Z(\mathtt{X}_i) w'(\mathtt{X}_j)\, dx + e_j O(\delta_j) \\ &\quad + \sum_{l \neq j}^{k} \int_{A_l} f'(w_{\delta_l,\beta_l}) \sum_{i=1}^{k} e_i Z(\mathtt{X}_i) w'(\mathtt{X}_j)\, dx \\ &= O(e_j \delta_j) + \sum_{i \neq j} e_i O(e^{-|f_i - f_j|}). \end{aligned}$$

Finally we have

$$S_{4j1} = O(e_j^2) + \sum_{i \neq j} e_i^2 O(e^{-|f_i - f_j|}).$$

Summarizing we get the following expression

$$\begin{aligned} \Pi_{f_j} &= -f_j''[c_0 + c_1 e_j + O(\delta_j)] + 2\sqrt{\lambda_1} a_{j0} \delta_j (f_j' - \beta_j \eta) \sin(\sqrt{\lambda_1} a_{i0} z) \\ &\quad + C_p (e^{-|f_j - f_{j-1}|} - e^{-|f_j - f_{j+1}|}) \\ &\quad - 2c_1 f_j' e_j' + \gamma_j (f_j' - \beta_j \eta) + O(e_j^2) + e_j O(\delta_j) \\ &\quad + \sum_{i \neq j} [f_i'' p_{1ij} + (f_i' - \beta_i \eta) p_{2ij}] + \sum_{i \neq j} [f_i' e_i q_{1ij} + (e_i'' + e_i' + e_i + e_i^2) q_{2ij}] \\ &\quad + \sum_{i \neq j} \big(e_i + O(\alpha)\big) e^{-|f_j - f_i|}, \end{aligned} \tag{7.4}$$

where functions γ_j, p_{lij} and q_{lij} are smooth functions of their arguments satisfying estimates

$$\begin{aligned} \gamma_j &= \gamma_j(f_j', \beta_j, \delta_j, \alpha, z) = O(\delta_j^2) + O(\alpha^2), \\ p_{lij} &= p_{lij}(f_j, f_i, e_i, \alpha, \delta, z) = O\big(e^{-|f_j - f_i|}(1 + e_i)\big), \\ q_{lij} &= q_{lij}(f_j, f_i, e_i, \alpha, \delta, z) = O(\delta_j) + O(e^{-|f_j - f_i|}) + O(\alpha^3 e^{-\theta_0 \alpha |z|}), \end{aligned}$$

and C_p, c_0, c_1 are constants defined by

$$C_p = \lim_{t \to \infty} e^{t}\big(w(t/2)^2 + w'(t/2)^2\big), \quad c_0 = \int_{\mathbb{R}} w'(s)^2\, ds, \quad c_1 = \int_{\mathbb{R}} w'(s) Z'(s)\, ds.$$

Now we compute projection of the error on $Z(\mathtt{X}_j)$ denoted by Π_{e_j} above. We will use the expression for $S[\mathtt{W}]$, (7.1). We will denote the entries of the matrix $\mathbf{A}_{0i}$ by $A_{0i,lm}$. First we observe that

$$
\begin{aligned}
\int_{\mathbb{R}} Tr\big(\mathbf{A}_i \cdot Hess_{\mathtt{X}_i,\mathtt{Z}_i}(w_{\delta_i,\beta_i})\big) Z(\mathtt{X}_i) = {} & A_{0i,11} a_{i0} \int_{\mathbb{R}} w''(s) Z(s)\, ds \\
& + \delta_i \Big[A_{0i,11} \cos(\sqrt{\lambda_1} a_{i0} z) \int_{\mathbb{R}} Z''(s) Z(s)\, ds \\
& - 2A_{0i,12} a_{i0} \sqrt{\lambda_1} \sin(\sqrt{\lambda_1} a_{i0} z) \int_{\mathbb{R}} \big(Z'(s)\big)^2\, ds \\
& - \lambda_1 A_{0i,22} a_{i0} \cos(\sqrt{\lambda_1} a_{i0} z) \int_{\mathbb{R}} Z^2(s)\, ds \\
& \quad + \delta_i O\big(\|f_i' - \beta_i \eta\|_{\theta_0 \alpha} + \alpha^2\big) e^{-\theta_0 \alpha |z|} \Big] \\
& \equiv h_{0i}.
\end{aligned} \tag{7.5}
$$

It is convenient to write

$$
\mathbf{l}_i = -f_i'' a_{i1}(1, \beta_i \eta) + \tilde{\mathbf{l}}_i.
$$

Notice that

$$
\|\tilde{\mathbf{l}}_i\|_\alpha + \alpha^{-1} \|\tilde{\mathbf{l}}_i'\|_\alpha \le C\alpha^3 (1 + \|f_i' - \beta_i \eta\|_\alpha + |\mathtt{X}_i|).
$$

Then we have

$$
\begin{aligned}
\int_{\mathbb{R}} \mathbf{l}_i \cdot \nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} Z(\mathtt{X}_i) = {} & \delta_i f_i'' a_{i1} \beta_i \eta \sin(\sqrt{\lambda_1} a_{i0} z) \Big[\int_{\mathbb{R}} Z^2 + O(\alpha) \Big] \\
& + O(\alpha^3)(1 + \|f_i' - \beta_i \eta\|_\alpha + \delta_i) e^{-\theta_0 \alpha |z|}.
\end{aligned}
$$

When $i \neq j$ we get

$$
\begin{aligned}
& \int_{\mathbb{R}} Tr\big(\mathbf{A}_i \cdot Hess_{\mathtt{X}_i,\mathtt{Z}_i}(w_{\delta_i,\beta_i})\big) Z(\mathtt{X}_j) \\
& = O\big(\|f_i' - \beta_i \eta\|_\alpha + \alpha^2\big) e^{-\theta_0 \alpha |z|} (1 + \delta_i) e^{-|f_i - f_j|},
\end{aligned}
$$

and

$$
\int_{\mathbb{R}} \mathbf{l}_i \cdot \nabla_{\mathtt{X}_i,\mathtt{Z}_i} w_{\delta_i,\beta_i} Z(\mathtt{X}_j) = O\big(\|f_i''\|_\alpha + \|f_i' - \beta_i \eta\|_\alpha + \alpha^2\big) e^{-\theta_0 \alpha |z|} (1 + \delta_i) e^{-|f_i - f_j|}.
$$

Now we compute the projection of $S_{2j}[\mathtt{W}]$. We notice first that

$$
\begin{aligned}
\int_{\mathbb{R}} \Delta_{x,z}\big(e_j Z(\mathtt{X}_j)\big) Z(\mathtt{X}_j)\, dx = {} & e_j'' d_0 + \big[2e_j' a_{j2} + e_j e^{-\theta_0 \alpha |z|} O(\alpha^3)\big] d_1 \\
& + e_j \big[1 + a_{j1}\big((f_j')^2 - \beta_j^2 \eta^2\big)\big] d_2,
\end{aligned}
$$

where

$$
d_0 = \int_{\mathbb{R}} Z^2(s)\, ds, \quad d_1 = \int_{\mathbb{R}} s Z'(s) Z(s)\, ds, \quad d_2 = \int_{\mathbb{R}} Z''(s) Z(s)\, ds.
$$

On the other hand

$$
\begin{aligned}
&\sum_{i\neq j}\int_{\mathbb{R}} \Delta_{x,z}\big(e_i Z(\mathtt{X}_i)\big) Z(\mathtt{X}_j)\,dx \\
&= \sum_{i\neq j} e_i'' O(e^{\,(1+\mu)|f_i-f_j|}) + \sum_{i\neq j} e_i'\big[f_i' + \alpha^3 O(|f_i-f_j|)\big] O(e^{\,(1+\mu)|f_i-f_j|}) \\
&\quad + \sum_{i\neq j} e_i\big[1 + f_i'' a_{i1} + \big((f_i')^2 - \beta_i^2\eta^2\big)\big] O(e^{\,(1+\mu)|f_i-f_j|}) \\
&\quad + \sum_{i\neq j} e_i\alpha^3\big(O(|f_i-f_j|) + f_i'\big) O(e^{\,-\theta_0\alpha|z|}) O(e^{\,(1+\mu)|f_i-f_j|}).
\end{aligned}
$$

To calculate the projection of $S_{3j}[\mathtt{W}]$ we observe that

$$
\int_{\mathbb{R}} S_{3j}[\mathtt{W}] Z(\mathtt{X}_j)\,dx = \sum_{l=1}^{k}\int_{A_l} f'(w_{\delta_l,\beta_l}) \sum_{i=1}^{k} e_i Z(\mathtt{X}_i) Z(\mathtt{X}_j)\,dx + \sum_{i\neq j} O(e^{\,-|f_i-f_j|}).
$$

We have

$$
\int_{A_j} f'(w_{\delta_j,\beta_j}) e_j Z^2(\mathtt{X}_j)\,dx = e_j\big(d_3 + O(\delta_j+\alpha^3)\big), \quad d_3 = \int_{\mathbb{R}} f'\big(w(s)\big) Z^2(s)\,ds.
$$

On the other hand when $l\neq j$ then

$$
\int_{A_l} e_i f'(w_{\delta_l,\beta_l}) Z(\mathtt{X}_i) Z(\mathtt{X}_j)\,dx = \begin{cases} e_j e^{\,-(1+\mu)|f_j-f_l|}, & i=j,\\ e_i e^{\,-(1+\mu)|f_j-f_i|}, & i\neq j.\end{cases}
$$

Summarizing we get

$$
\Pi_{e_j} = (e_j'' + \lambda e_j) d_0 + e_j g_{0j} + e_j' g_{1j} + \sum_{i\neq j}(e_i r_{0ij} + e_i' r_{1ij} + e_i'' r_{2ij}) + h_j, \tag{7.6}
$$

where functions g_{lj}, r_{lij} are smooth functions of their arguments that satisfy

$$
\begin{aligned}
g_{0j} &= g_{0j}(f_j',\beta_j,\delta_j,z) = O(\delta_j+\alpha^3) + (f_j'-\beta_j\eta) O(|f_j'|+\beta_j),\\
g_{1j} &= g_{1j}(z) = O(\alpha^3) e^{\,-\theta_0\alpha|z|},\\
r_{lij} &= r_{lij}(f_j,f_i,f_i',f_i'',\beta_i,z) = [\alpha^3 O(|f_i-f_j|) + O(|f_i'|+|f_i''|+1)] e^{\,-(1+\mu)|f_i-f_j|},
\end{aligned}
$$

Function h_j satisfies

$$
\begin{aligned}
h_j &= h_{0j} + \delta_j f_j'' a_{j1}\beta_i\eta \sin(\sqrt{\lambda_1} a_{j0} z)\Big[\int_{\mathbb{R}} Z^2 + O(\alpha)\Big]\\
&\quad + O\big((\alpha^3(1+\|f_i'-\beta_i\eta\|_\alpha)\big)(1+\delta_j)\\
&\quad + \sum_{i\neq j} O\big(\|f_i''\|_\alpha + \|f_i'-\beta_i\eta\|_\alpha + \alpha^2\big) e^{\,-\theta_0|z|}(1+\delta_i) O(e^{\,-|f_i-f_j|}),
\end{aligned}
$$

with h_{0j} defined in (7.5).

8 Derivation of the reduced problem

In this section we will derive conditions for $\mathbf{f}$, $\mathbf{e}$ and δ which imply that

$$c_j(z) \equiv 0, \qquad d_j(z) \equiv 0, \qquad j = 1, \dots, k, \tag{8.1}$$

and consequently lead to a solution of our original problem. We recall that with

$$\rho_j(x,z) = \eta_a^b\Big(\frac{|\mathtt{X}_j|}{d_*}\Big), \quad a = \frac{2^5-1}{2^5}, b = \frac{2^6-1}{2^6},$$

we have

$$\int_{\mathbb{R}} \phi(x,z) w_{j,x}(x,z)\rho_j(x,z)\, dx = 0 = \int_{\mathbb{R}} \phi(x,z) Z_j(x,z)\rho_j(x,z)\, dx, \quad \text{for all } z \in \mathbb{R},$$

where ϕ is the solution of

$$\mathcal{L}(\phi) = E - N(\phi) + \sum_{j=1}^{k} c_j(z)\eta_j w_{j,x} + d_j(z)\eta_j Z_j \quad \text{in } \mathbb{R}^2, \tag{8.2}$$

(see (4.5).

To derive the reduced problem we will multiply (8.2) by $\rho_j w_{j,x}$ and $\rho_j Z_j$, $j = 1, \dots, k$ and integrate over $\mathbb{R}$ with respect to x. To begin with let us observe that

$$\int_{\mathbb{R}} w_{j,x}^2 \eta_j \rho_j\, dx = \delta_i^j \int_{\mathbb{R}} (w')^2 + O(\alpha), \qquad \int_{\mathbb{R}} Z_j^2 \eta_j \rho_j\, dx = \delta_i^j \int_{\mathbb{R}} Z^2 + O(\alpha),$$

hence for (8.2) to be satisfied we need for $j = 1, \dots, k$:

$$\int_{\mathbb{R}} [E - N(\phi) - \mathcal{L}(\phi)] w_{j,x}\rho_j\, dx \equiv 0, \tag{8.3}$$

$$\int_{\mathbb{R}} [E - N(\phi) - \mathcal{L}(\phi)] Z_j\rho_j\, dx \equiv 0, \tag{8.4}$$

where ϕ is the solution found in Proposition 4.1. These equations can be written also in the form

$$\Pi_{f_j} = \int_{\mathbb{R}} [N(\phi) + \mathcal{L}(\phi)] w_{j,x}\rho_j\, dx + \int_{\mathbb{R}} E[w'(\mathtt{X}_j) - w_{j,x}\rho_j]\, dx \equiv Q_j(\mathbf{f}, \mathbf{e}), \tag{8.5}$$

$$\Pi_{e_j} = \int_{\mathbb{R}} [N(\phi) + \mathcal{L}(\phi)] Z_j\rho_j\, dx + \int_{\mathbb{R}} E[Z(\mathtt{X}_j) - Z_j\rho_j]\, dx \equiv P_j(\mathbf{f}, \mathbf{e}), \tag{8.6}$$

hence, by the results of the previous section, it only remains to compute the right hand sides of the above expressions.

We observe first that $|N(\phi)| \le C|\phi|^2$ hence we have

$$\Big|\int_{\mathbb{R}} N(\phi) w_{j,x}\rho_j\, dx\Big| + \Big|\int_{\mathbb{R}} N(\phi) Z_j\rho_j\, dx\Big| \le C\|\phi\|^2_{\sigma,\alpha,*} e^{-\alpha|z|} \le C\alpha^{4-4\sigma'} e^{-2\theta_0\alpha|z|}, \tag{8.7}$$

$\sigma' \in (\sigma, 3/2\sigma)$. To estimate the term involving $\mathcal{L}(\phi)$ we use integration by parts and the orthogonality condition (4.6) to get

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{L}(\phi) w_{j,x}\rho_j\,dx &= \int_{\mathbb{R}} \phi[2w_{j,xx}\rho_{j,x} + w_{j,x}\rho_{j,xx}]\,dx \\ &\quad - \int_{\mathbb{R}} [\phi_z (w_{j,x}\rho_j)_z + \phi(w_{j,x}\rho_j)_{zz}]\,dx \\ &\quad + \int_{\mathbb{R}} [f'(\mathtt{W}) - f'(w_j)]\phi w_{j,x}\rho_j\,dx \\ &= I + II + III.\end{aligned}$$

To estimate the first term above we notice that in the support of $\rho_{j,x}$ and $\rho_{j,xx}$ we have

$$\frac{(2^6-1)d^*}{2^6} > |x - f_j| > \frac{(2^5-1)d^*}{2^5}, \tag{8.8}$$

hence

$$|I| \le C\|\phi\|_{\sigma,\alpha,*} e^{-\theta_0\alpha|z|}\alpha^{15/16} \le C\alpha^{2-2\sigma'+15/16} e^{-\theta_0\alpha|z|}.$$

The second term is estimated similarly, using additionally $|f_j'| = O(\alpha)$, so that

$$|II| \le C\alpha^{2-2\sigma'+15/16} e^{-\theta_0\alpha|z|}.$$

To estimate the last term we notice that

$$|[f'(\mathtt{W}) - f'(w_j)]\rho_j| \le C(\alpha + \|\vec{\delta}\| + \|e\|_{\theta_0\alpha}),$$

hence

$$|III| \le C\alpha^{2-2\sigma'}(\alpha + \|\vec{\delta}\| + \|e\|_{\theta_0\alpha}).$$

To estimate the last term on the right hand side of the equation (8.5) we observe that

$$\begin{aligned}|w'(\mathtt{X}_j) - w_{j,x}\rho_j| &\le |w'(\mathtt{X}_j) - w'(x-f_j)|\rho_j + (1-\rho_j)w'(\mathtt{X}_j) \\ &\le C\alpha^2|w'(x-f_j)|\rho_j + C(1-\rho_j)\alpha^{31/32},\end{aligned}$$

hence

$$\left|\int_{\mathbb{R}} E[w'(\mathtt{X}_j) - w_{j,x}\rho_j]\,dx\right| \le C\|E\|_{\sigma,\alpha,*}\alpha^{31/32}e^{-\theta_0\alpha|z|} \le C\alpha^{2-2\sigma'+15/16}e^{-\theta_0\alpha|z|}.$$

Analogous argument applies for the components of the right hand side of (8.6). Taking now $\sigma < 2^{-10}$ we get that there exists a $\mu > 7/8$ such that

$$|Q_j(\mathbf{f}, \mathbf{e}, \vec{\delta})| \le C\alpha^{2+\mu}, \qquad |P_j(\mathbf{f}, \mathbf{e}, \vec{\delta})| \le C\alpha^{2+\mu}. \tag{8.9}$$

Now we will consider Lipschitz dependence of Q_j and P_j on their parameters. To do this we can argue in a way similar to the way we proved Lemma 6.1 and Proposition 6.1. This means we need to introduce parameters $\mathbf{g}_i$, $\mathbf{h}_i$, $\tilde{\mathbf{f}}_i$, $\tilde{\mathbf{e}}_i$, $\mathbf{e}_i$, $\mathbf{f}_i$ and $\vec{\delta}_i$ and consider Lipschitz dependence with respect to them making use of Lemma 6.1 and Proposition

6.1. In fact, when $\mathbf{f}$ is fixed then by a similar argument as above we get (using the familiar notation)

$$\begin{aligned}\|Q_j^{(1)}-Q_j^{(2)}\|_{\theta_0\alpha}+\|P_j^{(1)}-P_j^{(2)}\|_{\theta_0\alpha} \le C\alpha^\mu(\|\mathbf{g}_1''-\mathbf{g}_2''\|_{\theta_0\alpha}+\|\mathbf{h}_1''-\mathbf{h}_2''\|_{\theta_0\alpha})\\ +C\alpha^\mu(\alpha\|\tilde{\mathbf{f}}_1'-\tilde{\mathbf{f}}_2'\|_{\theta_0\alpha}+\|\tilde{\mathbf{e}}_1'-\tilde{\mathbf{e}}_2'\|_{\theta_0\alpha})\\ +C\alpha^\mu(\|\mathbf{e}_1-\mathbf{e}_2\|_{\theta_0\alpha}+\alpha^2\|\vec{\delta}_1-\vec{\delta}_2\|).\end{aligned} \tag{8.10}$$

When all parameters except $\mathbf{f}$ are fixed then the argument is similar. Let us for example consider one typical term in the projection Q_j:

$$\begin{aligned}\left|\int_R[\phi^{(1)}w_{j,xx}^{(1)}\rho_{j,x}^{(1)}-\phi^{(2)}w_{j,xx}^{(2)}\rho_{j,x}^{(2)}]\,dx\right| &\le \int_{\mathbb{R}}|\phi^{(1)}-\phi^{(2)}||w_{j,xx}^{(1)}\rho_{j,x}^{(1)}|\,dx\\ &\quad+\int_{\mathbb{R}}|\phi^{(2)}||\phi^{(1)}w_{j,xx}^{(1)}\rho_{j,x}^{(2)}-w_{j,xx}^{(2)}\rho_{j,x}^{(2)}|\,dx\\ &\le C\alpha^{31/32}e^{-\theta_0\alpha|z|}(\|\phi^{(1)}-\phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_1}\\ &\quad+\|\mathbf{f}_1-\mathbf{f}_2\|_\infty\|\phi^{(2)}\|_{\sigma,\theta_0\alpha,\mathbf{f}_2})\\ &\le C\alpha^{2+\mu}e^{-\theta_0\alpha|z|}\|\mathbf{f}_1-\mathbf{f}_2\|_\infty.\end{aligned}$$

Other terms can be handled in a similar way. In summary we will get then, taking into account the Lipschitz dependence on $\mathbf{f}$ only:

$$\|Q_j^{(1)}-Q_j^{(2)}\|_{\theta_0\alpha}+\|P_j^{(1)}-P_j^{(2)}\|_{\theta_0\alpha}\le C\alpha^{2+\mu}e^{-\theta_0\alpha|z|}\|\mathbf{f}_1-\mathbf{f}_2\|_\infty. \tag{8.11}$$

We will now consider the right hand sides of (8.5) and (8.6). Let us go back to formulas (7.4) and (7.6). Using the same notation we will write now (8.5) and (8.6) in the form:

$$\begin{aligned}&-c_0f_j''+2\sqrt{\lambda_1}a_{j0}\delta_j(f_j'-\beta_j\eta)\sin(\sqrt{\lambda_1}a_{i0}z)+C_p(e^{-|f_j-f_{j-1}|}-e^{-|f_j-f_{j+1}|})\\ &\quad=\widetilde{\Pi}_{f_j}+Q_j,\end{aligned} \tag{8.12}$$

$$(e_j''+\lambda_1e_j)d_0+h_{0j}=\widetilde{\Pi}_{e_j}+P_j, \tag{8.13}$$

where we have

$$\widetilde{\Pi}_{f_j}=\widetilde{\Pi}_{f_j}(\mathbf{f}'',\mathbf{e}'',\mathbf{f}',\mathbf{e}',\mathbf{f},\vec{\delta}),\qquad \widetilde{\Pi}_{e_j}=\widetilde{\Pi}_{e_j}(\mathbf{f}'',\mathbf{e}'',\mathbf{f}',\mathbf{e}',\mathbf{f},\vec{\delta}).$$

Functions $\widetilde{\Pi}_{f_j}$ and $\widetilde{\Pi}_{e_j}$ are smooth functions of their parameters and under the assumptions (3.1)–(3.8) and (3.11)–(3.14) we have:

$$\|\widetilde{\Pi}_{f_j}\|_{\theta_0\alpha}+\|\widetilde{\Pi}_{e_j}\|_{\theta_0\alpha}\le C\alpha^3. \tag{8.14}$$

The rest of this paper will be devoted to solving system (8.12)–(8.13).

9 The Toda system

In this section we will determine the main order behavior of function $\mathbf{f}$. Intuitively it should be given by solving the following system

$$-c_0f_j''+C_p(e^{-|f_j-f_{j-1}|}-e^{-|f_j-f_{j+1}|})=0,\quad j=1,\dots,k. \tag{9.1}$$

Let us consider first the case $k = 2$. Remembering that in this case $f_3 = \infty$ and $f_0 = -\infty$ we can reduce (9.1) to a single equation for $u = f_1 - f_2$:

$$u'' + 2c_p e^{-u} = 0, \quad u(0) = a_1 - a_2 \quad c_p = \frac{C_p}{\int_{\mathbb{R}} w_x^2}, \quad \text{where } C_p \text{ is defined in (7.3),} \tag{9.2}$$

and $f_j(0) = a_j$, see (1.14). For $\alpha > 0$ a family (parametrized by α) of explicit even solutions of (9.2) is given by:

$$u_\alpha(z) := 2\log\frac{1}{\alpha} + \log 2\lambda^{-2}c_p - \log\Big(\frac{-1+\tanh^2(\lambda\alpha z/2)}{2}\Big), \quad \alpha > 0, \tag{9.3}$$

where $\lambda = \sqrt{\frac{2c_p}{e^{a_2-a_1}}}$. Starting with this family it is natural to define

$$f_1(z) = -\frac{1}{2}u_\alpha(z), \quad f_2(z) = \frac{1}{2}u_\alpha(z). \tag{9.4}$$

Let us observe that in particular $f_1(z) + f_2(z) \equiv 0$ and

$$\|f_i''\|_\alpha = O(\alpha^2), \quad \beta_1 = -\frac{\lambda}{2}\alpha, \quad \beta_2 = \frac{\lambda}{2}\alpha, \quad f_2 - f_1 \geq 2\log\frac{1}{\alpha} + \log 2\lambda^2 c_p, \tag{9.5}$$

as needed. In this case we take $\theta_0 = \frac{\lambda}{4}$ and $\theta_1 = \frac{\lambda}{2}$. In the sequel we will denote $\mathbf{f}_0(z) = (f_1(z), f_2(z))$.

We will assume now $k > 2$ since the case $k = 2$ has just been treated above. It is convenient to consider our problem in a slightly more general framework then that of the system (9.1). Thus for given functions $q_j(t), p_j(t)$, $j = 1, \ldots, k$ such that

$$\sum_{j=1}^k q_j = \sum_{j=1}^k p_j = 0,$$

we define the Hamiltonian

$$H = \sum_{j=1}^k \frac{p_j^2}{2} + V, \quad V = \sum_{j=1}^{k-1} e^{(q_j - q_{j+1})}.$$

We consider the following Toda system

$$\begin{aligned} \frac{dq_j}{dt} &= p_j, \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial q_j} \\ q_j(0) &= q_{0j}, \quad p_j(0) = 0 \qquad j = 1, \ldots, k. \end{aligned} \tag{9.6}$$

Solutions to (9.6) are of course even. Observe that also that the location of their center of mass remains fixed. Thus to mode out translations we will assume that

$$\sum_{j=1}^k q_{0j} = 0. \tag{9.7}$$

We will now give a more precise of these solutions and in particular their asymptotic behavior as $z \to \pm\infty$. To this end we will often make use of classical results of Konstant [20] and in particular we will use the explicit formula for the solutions of (9.6) (see formula (7.7.10) in [20]).

We will first introduce some notation. Given numbers $w_1, \dots, w_k \in \mathbb{R}$ such that

$$\sum_{j=1}^{k} w_j = 0, \quad \text{and } w_j > w_{j+1}, \quad j = 1, \dots, k \tag{9.8}$$

we define the matrix

$$\mathbf{w}_0 = \operatorname{diag}(w_1, \dots, w_k).$$

Furthermore, given numbers $g_1, \dots, g_k \in \mathbb{R}$ such that

$$\prod_{j=1}^{k} g_j = 1, \quad \text{and } g_j > 0, \quad j = 1, \dots, k, \tag{9.9}$$

we define the matrix

$$\mathbf{g}_0 = \operatorname{diag}(g_1, \dots, g_N).$$

The matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ can be parameterized by introducing the following two sets of parameters

$$c_j = w_j - w_{j+1}, \quad d_j = \log g_{j+1} - \log g_j, \quad j = 1, \dots, k. \tag{9.10}$$

Furthermore, we define functions $\Phi_j(\mathbf{g}_0, \mathbf{w}_0; t)$, $t \in \mathbb{R}$, $j = 0, \dots, k$, by

$$\begin{aligned} &\Phi_0 = \Phi_k \equiv 1 \\ &\Phi_j(\mathbf{g}_0, \mathbf{w}_0; t) = \\ &(-1)^{j(k-j)} \sum_{1 \le i_i < \dots < i_j \le k} r_{i_1 \dots i_j}(\mathbf{w}_0) g_{i_1} \dots g_{i_j} \exp[-t(w_{i_1} + \dots + w_{i_j})], \end{aligned} \tag{9.11}$$

where $r_{i_1 \dots i_j}(\mathbf{w}_0)$ are rational functions of the entries of the matrix $\mathbf{w}_0$. It is proven in [20] that all solutions of (9.6) are of the form

$$q_j(t) = \log \Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; t) - \log \Phi_j(\mathbf{g}_0, \mathbf{w}_0; t), \quad j = 1, \dots, k, \tag{9.12}$$

Namely, given initial conditions in (9.6) there exist matrices $\mathbf{w}_0$ and $\mathbf{g}_0$ satisfying (9.8)–(9.9). According to Theorem 7.7.2 of [20], it holds

$$q_j^{'}(+\infty) = w_{k+1-j}, \quad q_j^{'}(-\infty) = w_j, \quad j = 1, ..., k. \tag{9.13}$$

We introduce variables

$$u_j = q_j - q_{j+1}. \tag{9.14}$$

In terms of $\mathbf{u} = (u_1, \dots, u_{k-1})$ system (9.6) becomes

$$\begin{aligned} &\mathbf{u}'' - M e^{\mathbf{u}} = 0, \\ &u_j(0) = q_{0j} - q_{0j+1}, \quad u_j'(0) = 0, \quad j = 1, \dots, k-1, \end{aligned} \tag{9.15}$$

where

$$M = \begin{pmatrix} 2 & -1 & 0\cdots & 0 \\ -1 & 2 & -1\cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 2 & -1 \\ 0 & \cdots & -1 & 2 \end{pmatrix}, \quad e^{-\mathbf{u}} = \begin{pmatrix} e^{u_1} \\ \vdots \\ e^{u_{k-1}} \end{pmatrix}.$$

As a consequence of (9.11) all solutions to (9.15) are given by

$$u_j(t) = q_j(t) - q_{j+1}(t) \quad = \quad -2\log\Phi_j(\mathbf{g}_0, \mathbf{w}_0; t) + \log\Phi_{j-1}(\mathbf{g}_0, \mathbf{w}_0; t) + \log\Phi_{j+1}(\mathbf{g}_0, \mathbf{w}_0; t). \tag{9.16}$$

Our first goal is to prove the following:

Lemma 9.1 *Let* $\mathbf{w}_0$ *be such that*

$$\min_{j=1,\dots,k-1}(w_j - w_{j+1}) = \vartheta > 0. \tag{9.17}$$

Then there holds

$$u_j(t) = \begin{cases} -c_{k-j}t - d_{k-j} + \tau_j^+(\mathbf{c}) + O(e^{-\vartheta|t|}), \text{ as } t \to +\infty, & j = 1, \dots, k-1, \\ c_j t + d_j + \tau_j^-(\mathbf{c}) + O(e^{-\vartheta|t|}), \text{ as } t \to -\infty, & j = 1, \dots, k-1, \end{cases} \tag{9.18}$$

where $\tau_j^\pm(\mathbf{c})$ *are smooth functions of the vector* $\mathbf{c} = (c_1, \dots, c_{k-1})$.

Proof. This Lemma has been proved in [12]. We include a proof here for completeness. Let q_j, $j = 1, \dots, k$ be a solution of the system (9.6) depending on the (matrix valued) parameters $\mathbf{w}_0$, $\mathbf{g}_0$ and defined in (9.12). We need to study the asymptotic behavior of $\Phi_j(\mathbf{w}_0, \mathbf{g}_0; t)$ as $t \to \pm\infty$ with the entries of $\mathbf{w}_0$ satisfying (9.17) and still undetermined $\mathbf{g}_0$.

By (9.11) and (9.8), we get that as $t \to -\infty$

$$\Phi_j = (-1)^{j(k-j)} r_{1\dots j}(\mathbf{w}_0) g_1 \dots g_j e^{-(w_1+\dots+w_j)t}(1 + O(e^{-(w_j - w_{j-1})t}),$$

hence

$$\frac{\Phi_{j+1}\Phi_{j-1}}{\Phi_j^2} = \frac{g_{j+1} r_{1\dots(j-1)}(\mathbf{w}_0) r_{1\dots(j+1)}(\mathbf{w}_0) e^{-c_j t}}{g_j r_{1\dots j}^2(\mathbf{w}_0)}(1 + O(e^{-\vartheta|t|})). \tag{9.19}$$

It follows that as $t \to -\infty$

$$\begin{aligned} u_j(t) &= \log\left(\frac{\Phi_{j+1}\Phi_{j-1}}{\Phi_j^2}\right) \\ &= -c_j t + \log\left(\frac{g_{j+1} r_{1\dots(j-1)}(\mathbf{w}_0) r_{1\dots(j+1)}(\mathbf{w}_0)}{g_j r_{1\dots j}^2(\mathbf{w}_0)}\right) + O(e^{-\vartheta|t|}) &(9.20) \\ &= c_j t + d_j + \tau_k^-(\mathbf{c}) + O(e^{-\vartheta|t|}), &(9.21) \end{aligned}$$

where

$$\tau_j^-(\mathbf{c}) = \log\left(\frac{r_{1\dots(j-1)}(\mathbf{w}_0) r_{1\dots(j+1)}(\mathbf{w}_0)}{r_{1\dots j}^2(\mathbf{w}_0)}\right).$$

Similarly, as $t \to +\infty$ we get

$$\begin{aligned} u_j(t) &= \log\left(\frac{\Phi_{j+1}\Phi_{j-1}}{\Phi_j^2}\right) \\ &= -c_{k-j}t - d_{k-j} + \tau_j^+ + O(e^{-\vartheta|t|}), \end{aligned} \tag{9.22}$$

where

$$\tau_j^+(\mathbf{c}) = \log\left(\frac{r_{k+2-j\dots k}(\mathbf{w}_0)r_{k-j\dots k}(\mathbf{w}_0)}{r^2_{k+1-j\dots k}(\mathbf{w}_0)}\right).$$

This ends the proof. □

To find a family of solutions parameterized by α starting from a solution of (9.6) we use functions u_j and set

$$u_{j\alpha}(z) = u_j(\alpha z) - 2\log\frac{1}{\alpha} - \log c_p. \tag{9.23}$$

Then functions $f_{j\alpha}(z)$ are obtained from the relations

$$\begin{aligned} u_{j\alpha}(z) &= f_{j\alpha}(z) - f_{j+1\alpha}(z), \\ \sum_{j=1}^k f_{j\alpha}(z) &= 0. \end{aligned} \tag{9.24}$$

Observe that as a consequence of Lemma 9.1 we get that there exist w_j, g_j, $j = 1, \dots, k$ such that (9.8)–(9.9) holds, that

$$\min_{j=1,\dots,k}(w_j - w_{j+1}) = \vartheta > 0,$$

and functions f_j satisfy

$$\begin{aligned} &\|f_j\|_{\vartheta\alpha} = \|f_j'' e^{\vartheta\alpha|z|}\|_\infty \le C\alpha^2, \\ &f_j'(\infty) = \beta_j = f_j'(-\infty), \quad \text{where } \beta_{j+1} - \beta_j = (w_j - w_{j+1})\alpha > \vartheta\alpha, \\ &f_j(z) - f_{j+1}(z) \ge 2\log\frac{1}{\alpha} + \log c_p. \end{aligned} \tag{9.25}$$

In this case we take $\theta_0 = \frac{1}{4}\vartheta$ and $\theta_1 = \frac{1}{2}\vartheta$.

10 Bounded solvability of some equations in the line

In this section will continue with preliminaries needed to solve (8.12)–(8.13). First we will study the linearization of the system (8.12) around the solution $\mathbf{f}_0$ of the Toda system defined in (9.4) and (9.25). We will always assume that $\alpha > 0$ is small and $\theta_0 > 0$ has value defined in the previous section.

Again, we will consider the case $k = 2$ first. Let $p_j(z)$, $j = 1, 2$ be given even and continuous functions. The linearized Toda system takes form

$$\begin{aligned} &\varphi_1'' + c_p e^{-u_\alpha}(\varphi_1 - \varphi_2) = p_1(z), \\ &\varphi_2'' - c_p e^{-u_\alpha}(\varphi_1 - \varphi_2) = p_2(z), \\ &\quad \varphi_j(0) = x_j, \quad \varphi'(0) = 0, \quad j = 1, 2, \end{aligned} \tag{10.1}$$

which can be reduced to a single ODE for $h = \varphi_1 - \varphi_2$ with $p(z) = p_1(z) - p_2(z)$:

$$\begin{aligned} \mathcal{L}_\alpha(h) := h'' + 2c_p e^{-u_\alpha} h = p(z), \quad z \in \mathbb{R}, \\ h(0) = h_0, \quad h'(0) = 0. \end{aligned} \tag{10.2}$$

Notice that we do not impose the initial condition in (10.1) and (10.2). In fact we will require that the solution to (10.1) and (10.2) is *bounded and even* and this determines uniquely the initial data.

We will assume that function $p(z)$ is even and satisfies

$$\|p\|_{\theta_0\alpha} \le C\alpha^{2+\mu}, \quad \text{some } \mu > 0, \tag{10.3}$$

and look for a solution to (10.2) in the space of C^2 even functions such that

$$\|h''\|_{\theta_0\alpha} + \alpha\|h'\|_{\theta_0\alpha} + \alpha^2\|h\|_\infty < \infty. \tag{10.4}$$

We will denote the space of such functions by $\mathcal{X}$. Observe that evenness of solutions to (10.2) is guaranteed by the initial conditions and the evenness of the right hand side.

To solve problem (10.2) we will construct a suitably bounded left inverse for the linearized operator $\mathcal{L}_\alpha$. By $\psi_{j\alpha}$ we will denote the elements of the fundamental set of $\mathcal{L}_\alpha$. They are known explicitly:

$$\psi_{1\alpha}(z) = u'_\alpha(z), \quad \psi_{2\alpha}(z) = zu'_\alpha(z) + 2,$$

and their Wronskian is $W(\psi_{1\alpha}, \psi_{2\alpha}) = \alpha^2$. Then there exists a unique even and bounded solution solution to (10.2). It is given by:

$$\begin{aligned} h(z) = &-\frac{1}{\alpha^2}\psi_{1\alpha}(z)\int_0^z \psi_{2\alpha}(s)p(s)\,ds + \frac{1}{\alpha^2}\psi_{2\alpha}(z)\int_0^z \psi_{1\alpha}(s)p(s)\,ds \\ &- \frac{1}{\alpha^2}\psi_{2\alpha}(z)\int_0^\infty u'_\alpha(s)p(s)\,ds \\ = &\frac{1}{\alpha^2}\int_0^z u'_\alpha(z)u'_\alpha(s)(z-s)p(s)\,ds + \frac{1}{\alpha^2}\int_0^z \big(u'_\alpha(s) - u'_\alpha(z)\big)p(s)\,ds \\ &- \frac{1}{\alpha^2}\psi_{2\alpha}(z)\int_0^\infty u'_\alpha(s)p(s)\,ds. \end{aligned} \tag{10.5}$$

Directly examining this formula we get that

$$\|h''\|_{\theta_0\alpha} \le C\|p\|_{\theta_0\alpha}, \qquad |h(\infty)| \le C\alpha^{-2}\|p\|_{\theta_0\alpha} \le C\alpha^\mu,$$

and using $h'(0) = 0$ we obtain the following estimate for the inverse of $\mathcal{L}_\alpha$:

$$\|h''\|_{\theta_0\alpha} + \alpha\|h'\|_{\theta_0\alpha} + \alpha^2\|h\|_\infty \le C\|p\|_{\theta_0\alpha}. \tag{10.6}$$

These considerations provide us with a framework needed to solve (10.2). We have the following result.

Lemma 10.1 *There exists a constants $\mu > 0, C > 0$ independent of α such that if p is an even function with*

$$\|p\|_{\theta_0\alpha} \le C\alpha^{2+\mu}, \tag{10.7}$$

then Problem 10.2 possesses an even solution h of the form (10.5 such that the estimate (10.6) holds. Denoting this solution for a given p by $\mathcal{R}_\alpha[p]$ we have additionally the following estimate

$$\|\mathcal{R}_\alpha[p_1]'' - \mathcal{R}_\alpha[p_2]''\|_{\theta_0\alpha} + \alpha\|\mathcal{R}_\alpha[p_1]' - \mathcal{R}_\alpha[p_2]'\|_{\theta_0\alpha} + \alpha^2\|\mathcal{R}_\alpha[p_1] - \mathcal{R}_\alpha[p_2]\|_\infty \\ \leq C\|p_1 - p_2\|_{\theta_0\alpha},$$

for all p_1, p_2 satisfying (10.7).

The proof of this result is left to the reader.

Now we will consider the general case $k > 2$. We are lead to the following linear system

$$\vec{\phi}'' - \begin{pmatrix} 2e^{u_{1\alpha}} & -e^{u_{2\alpha}} & 0\cdots & 0 \\ -e^{-u_{1\alpha}} & 2e^{u_{2\alpha}} & -e^{u_{3\alpha}}\cdots & 0 \\ & & \ddots & \\ 0 & \cdots & 2e^{u_{k-2\alpha}} & -e^{u_{k-1\alpha}} \\ 0 & \cdots & -e^{u_{k-2\alpha}} & 2e^{u_{k-1\alpha}} \end{pmatrix} \vec{\phi}^T = \vec{p}, \tag{10.8}$$

$$\vec{\phi} = (\phi_1, \dots, \phi_{k-1}), \quad \vec{p} = (p_1, \dots, p_{k-1}),$$

where $\vec{p}$ is an even function such that

$$\|\vec{p}\|_{\theta_0\alpha} \leq C\alpha^{2+\mu}. \tag{10.9}$$

We will analyze the solvability of this problem in the space of even C^2 functions $\vec{\phi}$ such that

$$\|\vec{\phi}''\|_{\theta_0\alpha} + \alpha\|\vec{\phi}'\|_{\theta_0\alpha} + \alpha^2\|\vec{\phi}\|_\infty < \infty. \tag{10.10}$$

Thus in addition to (10.8) we will require that

$$\vec{\phi}(0) = \vec{x}, \quad \vec{\phi}'(0) = 0. \tag{10.11}$$

We first observe that

$$g_j\frac{\partial u_{m\alpha}}{\partial g_j} = \alpha \begin{cases} 1, & j = m+1, \quad z \to \infty, \\ -1, & j = m, \quad z \to \infty \\ 1, & j = k+2-m, \quad z \to -\infty, \\ -1, & j = k+1-m, \quad z \to -\infty, \\ 0, & \text{otherwise.} \end{cases}$$

Hence by a transformation we can find a set of linearly independent solutions to the homogenous version of (10.8)

$$\psi_{1j\alpha}(z) = \begin{cases} \alpha\vec{e}_j, & z \to \infty, \\ -\alpha\vec{e}_j, & z \to -\infty. \end{cases}$$

Similarly, considering derivatives of u_k with respect to w_j we can find solutions of (10.8), $\psi_{2j}(z)$, $j = 1, \dots, k-1$ such that

$$\psi_{2j\alpha}(t) = \alpha\vec{e}_j|z| + O(1).$$

The functions $\{(\psi_{1j}(z), \psi'_{1j}(z)), (\psi_{2j}(z), \psi'_{2j}(z))\}$ form a fundamental set for the system (10.8), whose behavior as $z \to \pm\infty$ is analogous to that of the functions $(\psi_1(z)\psi'_1(z))$, $(\psi_2(z), \psi'_2(z))$, respectively. Let us denote the fundamental matrix of the system (10.8) described above by $\Psi_\alpha(z)$ and the right hand side of the transformed system by $\vec{q}$. We observe that as $z \to \infty$, matrices $\Psi_\alpha, \Psi_\alpha^{-1}$ are block matrices of the form

$$\Psi_\alpha(z) = \begin{pmatrix} \alpha I + o(1) & \alpha z I + O(1) \\ \mathbf{0} & \alpha I + o(1) \end{pmatrix}, \quad \Psi_\alpha^{-1} = \begin{pmatrix} \alpha I + o(1) & -\alpha z I + O(1) \\ \mathbf{0} & \alpha I + o(1) \end{pmatrix}. \tag{10.12}$$

Let us denote these blocks by $\Psi_{mn\alpha}$, $\Psi^{-1}_{mn\alpha}$, $m, n = 1, 2$, respectively. Then, from variation of parameters formula we get that the solution of our problem has form

$$(\vec{\phi}(z), \vec{\phi}'(z)) = \Psi_\alpha(z) \cdot \int_0^z \Psi_\alpha^{-1}(s) \cdot (\mathbf{0}, \vec{q}(s))^T \, ds - \Psi_{12\alpha}(z) \int_0^\infty \Psi^{-1}_{22\alpha}(s)\vec{q}(s)\, ds. \tag{10.13}$$

Using this we can directly estimate

$$\|\vec{\phi}''\|_{\theta_0\alpha} \le C\|\vec{q}\|_{\theta_0\alpha},$$

form which it follows that (using the notation of Lemma 10.1 and going back to the original variables)

$$\|\mathcal{R}[\vec{p}]''\|_{\theta_0\alpha} + \alpha\|\mathcal{R}[\vec{p}]'\|_{\theta_0\alpha} + \alpha^2\|\mathcal{R}[\vec{p}]\|_\infty \le C\|\vec{p}\|_{\theta_0\alpha}, \tag{10.14}$$

$$\|\mathcal{R}[\vec{p}_1]'' - \mathcal{R}[\vec{p}_2]''\|_{\theta_0\alpha} + \alpha\|\mathcal{R}[\vec{p}_1]' - \mathcal{R}[\vec{p}_2]'\|_{\theta_0\alpha} + \alpha^2\|\mathcal{R}[\vec{p}_1] - \mathcal{R}[\vec{p}_2]\|_\infty \le C\|\vec{p}_1 - \vec{p}_2\|_{\theta_0\alpha}. \tag{10.15}$$

A second problem, important for our purposes is that of finding an even solution of the equation

$$e'' + \lambda_1 e = q(z), \quad z \in \mathbb{R}, \tag{10.16}$$

where, again, q is an even function with $\|q\|_\alpha < +\infty$. This time we want e to satisfy

$$\|e''\|_{\theta_0\alpha} + \|e'\|_{\theta_0\alpha} + \|e\|_{\theta_0\alpha} < +\infty.$$

We need to assume the following solvability condition on q for this to be the case:

$$\int_0^\infty q(z)\cos(\sqrt{\lambda_1}z)\, dz \; = \; 0. \tag{10.17}$$

Under this assumption the solution turns out to be unique. Denoting the solution of (10.16) by $\mathcal{S}[q]$ we get explicitly

$$\mathcal{S}[q] = \frac{1}{\sqrt{\lambda_1}}\sin(\sqrt{\lambda_1}z)\int_z^\infty q(t)\cos(\sqrt{\lambda_1}t)\, dt - \frac{1}{\sqrt{\lambda_1}}\cos(\sqrt{\lambda_1}z)\int_z^\infty q(t)\sin(\sqrt{\lambda_1}t)\, dt. \tag{10.18}$$

Clearly this operator is bounded in the sense that $e = S[q]$ satisfies

$$\|e\|_{\theta_0\alpha} + \|e'\|_{\theta_0\alpha} + \|e''\|_{\theta_0\alpha} \le C\alpha^{-1}\|q\,\|_{\theta_0\alpha}. \tag{10.19}$$

On the other hand, a better estimate is available in case that we know, in addition that $\|q'\|_{\theta_0\alpha} < +\infty$. Indeed, in this case integrating by parts in the formula (10.18) we get

$$\|e\|_{\theta_0\alpha} + \|e'\|_{\theta_0\alpha} + \|e''\|_{\theta_0\alpha} \le C(\,\|q\|_{\theta_0\alpha} + \alpha^{-1}\|q'\|_{\theta_0\alpha}\,). \tag{10.20}$$

Finally it is clear that the operator $\mathcal{S}[q]$ is Lipschitz in the norms used above.

11 Solving the reduced system for $(\mathbf{f}, \mathbf{e}, \vec{\delta})$

We will now go back to (8.12)–(8.13). We will solve this system using a fixed point argument around and approximate solution $\mathbf{f} = \mathbf{f}_0$, $\mathbf{e} = 0$ and $\vec{\delta} = 0$, where $\mathbf{f}_0$ is a solution to Toda system will be determined shortly. We let $\mathbf{f}_0$ to be an even solution to (9.25) satisfying the initial data

$$f_{0j}(0) = x_j, \quad \sum_{j=1}^{k} x_j = 0, \qquad f'_{0j}(0) = 0, \quad j = 1, \dots, k.$$

We will set

$$\beta_j = f_{0j}(\pm\infty). \tag{11.1}$$

In the sequel we will often use the fact that $\mathbf{f}_0$ is a smooth function which in addition satisfies

$$\alpha^{-1}\|\mathbf{f}_0'''\|_{\theta_0\alpha} + \|\mathbf{f}_0''\|_{\theta_0\alpha} + \alpha\|\mathbf{f}_0'\|_{\theta_0\alpha} \le C_0\alpha^2. \tag{11.2}$$

With $\mathbf{f}_0$ fixed now we will will assume that the solution of our problem is of the form

$$\mathbf{f} = \mathbf{f}_0 + \mathbf{f}_1,$$

where

$$\|\mathbf{f}_1''\|_{\theta_0\alpha} + \alpha\|\mathbf{f}_1'\|_{\theta_0\alpha} + \alpha^2\|\mathbf{f}_1\|_\infty \le C\alpha^{2+\nu}, \tag{11.3}$$

and for $\mathbf{e}$ and $\vec{\delta}$ we will assume that

$$\begin{aligned} \|\mathbf{e}''\|_{\theta_0\alpha} + \|\mathbf{e}'\|_{\theta_0\alpha} + \|\mathbf{e}\|_{\theta_0\alpha} &\le C\alpha^2, \\ |\vec{\delta}| &\le C\alpha^{1+\nu}, \qquad \text{some } \nu > 0. \end{aligned} \tag{11.4}$$

Let ν be a fixed small number to be specified later and let

$$\mathcal{Y} = \left\{ (\mathbf{f}_1, \mathbf{e}, \vec{\delta}) \;\middle|\; \begin{aligned} \|(\mathbf{f}_1, \mathbf{e}, \vec{\delta})\|_{\mathcal{Y}} = \alpha^{-2}(\|\mathbf{f}_1''\|_{\theta_0\alpha} + \alpha\|\mathbf{f}_1'\|_{\theta_0\alpha} + \alpha^2\|\mathbf{f}_1\|_\infty) \\ + \alpha^{-2+\nu}(\|\mathbf{e}''\|_{\theta_0\alpha} + \|\mathbf{e}'\|_{\theta_0\alpha} + \|\mathbf{e}\|_{\theta_0\alpha}) + \alpha^{-1}|\vec{\delta}| < \infty \end{aligned} \right\}.$$

Let also

$$B_R = \{(\mathbf{f}_1, \mathbf{e}, \vec{\delta}) \mid \|(\mathbf{f}_1, \mathbf{e}, \vec{\delta})\|_{\mathcal{Y}} < R\}.$$

We want to use Banach fixed point theorem to solve (8.12)–(8.13) in $B_{R\alpha^\nu}$, some $\nu > 0$. It is convenient to linearize this system around the approximate solution first. We will denote $f_j = f_{0j} + f_{1j}$ and:

$$\begin{aligned}
\mathtt{L}_j(\mathbf{f}_1) &= -c_1 f''_{j1} + C_p\big(e^{-(f_{0j-1}-f_{0j})}(f_{1j} - f_{1j-1}) - e^{-(f_{0j}-f_{0j+1})}(f_{1j+1} - f_{1j}),\\
\mathcal{M}_j(\mathbf{f}, \vec{d}) &= -C_p\big[(e^{-|f_j - f_{j-1}|} - e^{-|f_j - f_{j+1}|}) - (e^{-(f_{0j-1}-f_{0j})}(f_{1j} - f_{1j-1})\\
&\qquad + e^{-(f_{0j}-f_{0j+1})}(f_{1j+1} - f_{1j})\big]\\
&\quad - 2\sqrt{\lambda_1} a_{j0} d_j (f'_j - \beta_j \eta)\sin(\sqrt{\lambda_1} a_{j0} z),\\
\mathtt{K}_j(\mathbf{e}) &= (e''_j + \lambda_1 e_j) d_0,\\
B_{0j} &= [A_{0j,11}(\mathbf{f}_0)\int_{\mathbb{R}} Z''(s)Z(s)\,ds - \lambda_1 A_{0j,22}(\mathbf{f}_0)\int_{\mathbb{R}} Z^2(s)\,ds]\cos(\sqrt{\lambda_1} a_{i0} z),\\
\tilde{h}_{0j}(\mathbf{f}_0 + \tilde{f}, \vec{d}) &= -h_{0j}(\mathbf{f}_0 + \tilde{\mathbf{f}}, \vec{d}) + d_j B_{0j}.
\end{aligned}$$

Let $\tilde{\mathbf{f}}$, $\tilde{\mathbf{e}}$ and $\vec{d} \in \mathbb{R}^k$ satisfying (11.3)–(11.4) be fixed. We will consider the linearized system (8.12)–(8.13) written in the form:

$$\mathtt{L}_j(\mathbf{f}_1) = \mathcal{M}_j(\tilde{\mathbf{f}}, \vec{d}) + \widetilde{\Pi}_{f_j}(\mathbf{f}_0 + \tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}) + Q_j(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}), \tag{11.5}$$

$$\mathtt{K}_j(\mathbf{e}) + \delta_j B_{0j} = \tilde{h}_{0j}(\mathbf{f}_0 + \tilde{\mathbf{f}}, \vec{d}) + \widetilde{\Pi}_{e_j}(\mathbf{f}_0 + \tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}) + P_j(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}). \tag{11.6}$$

Our first goal is to show that given $\tilde{\mathbf{f}}$, $\tilde{\mathbf{e}}$ and $\vec{d}$ one can chose $\vec{\delta}$ so that the solvability condition for the equation (11.6) is satisfied. Thus we need to compute projections of (11.6) onto $\cos(\sqrt{\lambda_1} z)$ and show that adjusting $\vec{\delta}$ we can accomplish that for $j = 1, \dots, k$:

$$\int_0^\infty B_{0j}\cos(\sqrt{\lambda_1} z)\,dz \neq 0. \tag{11.7}$$

We first notice that given function $q(z)$ such that

$$\alpha^{-1}\|q'\|_{\theta_0\alpha} + \|q\|_{\theta_0\alpha} < \infty,$$

we have

$$\begin{aligned}
\int_0^\infty \cos(\sqrt{\lambda_1} z)\cos(\sqrt{\lambda_1} a_{j0} z) q(z)\,dz &= \frac{1}{2}\int_0^\infty q(z)\,dz + \frac{1}{2}\int_0^\infty \cos(2\sqrt{\lambda_1} z) q(z)\,dz\\
&\quad + \int_0^\infty z\cos(\sqrt{\lambda_1} z)\sin(\sqrt{\lambda_1} z) O(\beta_j^2\eta^2) q(z)\,dz\\
&= \frac{1}{2}\int_0^\infty q(z)\,dz + O(\alpha^{-1}\|q'\|_{\theta_0\alpha}) + O(\|q\|_{\theta_0\alpha}).
\end{aligned} \tag{11.8}$$

The first term above which is of order $O(\alpha^{-1}\|q\|_{\theta_0\alpha})$ is expected to be the leading order term. Thus to find asymptotic values of the integrals involved in (11.7) we need to consider

$$\begin{aligned}
I_0 &= \int_0^\infty A_{0j,11}(\mathbf{f}_0)\,dz \int_{\mathbb{R}} Z''(s)Z(s)\,ds,\\
II_0 &= -\lambda_1 \int_0^\infty A_{0j,22}(\mathbf{f}_0)\,dz \int_{\mathbb{R}} Z^2(s)\,ds.
\end{aligned}$$

Let us recall that

$$A_{0j,11}(\mathbf{f}_0) = a_{j1}^2\big((f_{0j}')^2 - \beta_j^2\eta^2\big),$$
$$A_{0j,22}(\mathbf{f}_0) = 2\beta_j\eta(\beta_j\eta - f_{0j}') + \big((f_{0j}')^2 - \beta_j^2\eta^2\big) + 2\alpha\beta_j^2 z\eta'\eta.$$

We will first prove the following

Lemma 11.1 *Let $\mathbf{f}_0$ be the solution of the Toda system described above. For any $B > 0$ there exists a cut-off function $\eta(t) = 0$, $t < T_1$, $\eta(t) = 0$, $t > T_2$ such that*

$$|I_0 + II_0| \geq B\alpha. \tag{11.9}$$

Proof. Let us denote $b_1 = -\int_0^\infty (Z')^2$, $b_2 = -\lambda_1\int_0^\infty Z^2$. Since $a_{j1}^2 = 1 + O(\beta_j^2\eta^2)$, therefore

$$\begin{aligned} I_0 &= b_1\int_0^\infty \big((f_{0j}')^2 - \beta_j^2\eta^2\big)\,dz + O(\alpha^2) \\ &= b_1\int_0^\infty \big((f_{0j}')^2 - \beta_j^2\big)\,dz + b_1\beta_j^2\alpha^{-1}\int_0^\infty \big(1-\eta^2(t)\big)\,dt + O(\alpha^2) \\ &= I_{01} + I_{02} + O(\alpha^2). \end{aligned}$$

Likewise we get

$$\begin{aligned} II_0 &= b_2\int_0^\infty \big[\big((f_{0j}')^2 - \beta_j^2\big) + 2\beta_j\eta(\beta_j - f_j')\big]\,dz \\ &\quad + b_2\beta_j\int_0^\infty \big[(1-\eta^2(t))(\beta_j\alpha^{-1} - 1)\big]\,dt + 2\beta_j\alpha^{-1}\int_0^\infty \eta(t)(\eta(t)-1)\,dt \\ &= II_{01} + II_{02} + II_{03}. \end{aligned}$$

We notice that there exists a constant $C_0 > 0$, independent of the cut off function η such that

$$|I_{01}| + |II_{01}| \leq C_0\alpha. \tag{11.10}$$

To estimate the rest of the integrals we will consider

$$b_3 = b_1\beta_j^2\alpha^2 + \beta_2\beta_j\alpha^{-1}.$$

Notice that $b_3 = O(1)$ as $\alpha \to 0$. Let us assume first that $b_3 \neq 0$. Then we can take $T_2 = T_1 + R$ (see the definition of η) with R fixed. Increasing T_1 if necessary we can make then $|I_{02} + II_{02}|$ as large as we wish while $|II_{03}|$ will remain bounded so that

$$|I_0 + II_0| \geq |I_{02} + II_{02}| - |II_{03}| - |I_{01}| - |II_{01}| + O(\alpha^2) \geq B\alpha.$$

If on the other hand $b_3 = 0$ then by taking $T_2 - T_1 = R$ large we can make II_{03} as large as we want and then

$$|I_0 + II_0| \geq |II_{03}| - |I_{01}| - |II_{01}| + O(\alpha^2) \geq B\alpha.$$

The proof of the Lemma is complete. □

We are ready now to set up the fixed point argument. We will begin by outlining the steps that are needed to complete the proof.

1. Given $(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}) \in \mathcal{Y}$ we will chose δ so that the solvability condition for (11.6) is satisfied.
2. Using Step 1 we define an operator $\mathcal{F} : B_{R\alpha^\nu} \to \mathcal{Y}$ by

$$\mathcal{F}(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}) = (\mathbf{f}_1, \mathbf{e}, \vec{\delta}), \quad \text{where } (\mathbf{f}_1, \mathbf{e}, \vec{\delta}) \text{ is a solution of (11.5)–(11.6).}$$

3. We will show that there exist $\nu > 0$ and $R > 0$ such that

$$\|\mathcal{F}(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d})\|_{\mathcal{Y}} \le R\alpha^\nu, \tag{11.11}$$

given that

$$\|(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d})\|_{\mathcal{Y}} \le R\alpha^\nu. \tag{11.12}$$

4. We will show that the operator $\mathcal{F}$ restricted to $B_{R\alpha^\nu}$ is a contraction.

Once steps (1)–(4) are executed will conclude by applying Banach fixed point theorem.
Step 1 is an easy consequence of Lemma 11.1 and give the following formula

$$\delta_j \int_0^\infty B_{0j} \cos(\sqrt{\lambda_1} z)\, dz = \int_0^\infty \left[\tilde{h}_{0j}(\mathbf{f}_0 + \tilde{\mathbf{f}}, \vec{d}) + \widetilde{\Pi}_{e_j}(\mathbf{f}_0 + \tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d}) + P_j(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d})\right] \times \cos(\sqrt{\lambda_1} z)\, dz. \tag{11.13}$$

We will need a precise estimate for $\vec{\delta}$ and this is the computation we will carry on now. We will make use of the following:

Lemma 11.2 *Let $q : \mathbb{R}_+ \to \mathbb{R}$ be a function such that $\|q'\|_{\theta_0 \alpha} < \infty$. Then for $s \in \mathbb{N}$ we have*

$$\begin{aligned} \left| \int_0^\infty q(z) z^s \cos(\sqrt{\lambda_1} z)\, dz \right| &\le \frac{C}{\alpha^{s+1}} \|q'\|_{\theta_0\alpha}, \qquad s \ge 0, \\ \left| \int_0^\infty q(z) z^s \sin(\sqrt{\lambda_1} z)\, dz \right| &\le \frac{C}{\alpha^{s+1}} \|q'\|_{\theta_0\alpha}, \qquad s \ge 1. \end{aligned} \tag{11.14}$$

Similar estimates hold when $\|q''\|_{\theta_0\alpha} < \infty$, and and $q'(0) = 0$, with $\|q'\|_{\theta_0\alpha}$ above replaced by $\|q''\|_{\theta_0\alpha}$.

Proof. We have

$$\begin{aligned} \int_0^\infty q(z) \cos(\sqrt{\lambda_1} z)\, dz &= -\frac{1}{\sqrt{\lambda_1}} \int_0^\infty q'(z) \sin(\sqrt{\lambda_1} z)\, dz \\ &= -\frac{1}{\lambda_1} \int_0^\infty q''(z) \cos(\sqrt{\lambda_1} z)\, dz, \qquad \text{using } q'(0) = 0, \end{aligned}$$

hence we get

$$\left| \int_0^\infty q(z) \cos(\sqrt{\lambda_1} z)\, dz \right| \le \frac{C}{\alpha} \|q'\|_{\theta_0\alpha}, \qquad \left| \int_0^\infty q(z) \cos(\sqrt{\lambda_1} z)\, dz \right| \le \frac{C}{\alpha} \|q''\|_{\theta_0\alpha}.$$

if $\|q''\|_{\theta_0\alpha} < \infty$. The first inequality in (11.14) follows now by induction. Similarly we get

$$\begin{aligned}\int_0^\infty zq(z)\sin(\sqrt{\lambda_1}z)\,dz &= \frac{1}{\sqrt{\lambda_1}}\int_0^\infty \big(zq(z)\big)'\cos(\sqrt{\lambda_1}z)\,dz - \frac{1}{\lambda_1}\int_0^\infty q'(z)\sin(\sqrt{\lambda_1}z)\,dz\\ &= \Big(\frac{2}{\lambda_1^{3/2}} - \frac{1}{\lambda_1}\Big)\int_0^\infty zq''(z)\cos(\sqrt{\lambda_1}z)\,dz\\ &\quad - \frac{2}{\lambda_1^2}\int_0^\infty q''(z)\cos(\sqrt{\lambda_1}z)\,dz,\\ &\quad \text{using } q'(0)=0.\end{aligned}$$

From these identities, using the above and induction, one can show the second estimate in (11.14). □

With some abuse of notation we will write

$$\|(\tilde{\mathbf{f}},\mathbf{0},\mathbf{0})\|_{\mathcal{Y}} = \|\tilde{\mathbf{f}}\|_{\mathcal{Y}}, \qquad \|(\mathbf{0},\tilde{\mathbf{e}},\mathbf{0})\|_{\mathcal{Y}} = \|\tilde{\mathbf{e}}\|_{\mathcal{Y}}, \qquad \text{etc.}$$

Will will now compute the integrals on the right hand side of (11.13). We begin with

$$\begin{aligned}&\int_0^\infty \big[\tilde{h}_{0j}(\mathbf{f}_0+\tilde{\mathbf{f}})\big]\cos(\sqrt{\lambda_1}z)\,dz\\ &= \int_{\mathbb{R}} w''(s)Z(s)\,ds\int_0^\infty A_{0j,11}a_{j0}\cos(\sqrt{\lambda_1}z)\,dz\\ &\quad - 2d_j\int_{\mathbb{R}}\big(Z'(s)\big)^2\,ds\int_0^\infty A_{0j,12}a_{j0}\sqrt{\lambda_1}\sin(\sqrt{\lambda_1}a_{j0}z)\cos(\sqrt{\lambda_1}z)\,dz\\ &\quad + d_j^2\int_0^\infty O\big(\|f_{0j}'+\tilde{f}_j'-\beta_j\eta\|_{\theta_0\alpha}+\alpha^2\big)e^{-\theta_0\alpha|z|}\cos(\sqrt{\lambda_1}z)\,dz\\ &= \Big(\int_{\mathbb{R}} w''(s)Z(s)\,ds\Big)I_j - 2d_j\Big(\int_{\mathbb{R}}\big(Z'(s)\big)^2\,ds\Big)II_j + d_j^2 III_j.\end{aligned}$$

Since $\mathbf{f}_0$ is a smooth function, therefore from (11.2) we get using $f_{j0}'(0)=0$ and Lemma 11.2:

$$\begin{aligned}I_j &= \int_0^\infty a_{j0}\big((f_{0j}'+\tilde{f}_j')^2-\beta_j^2\eta^2\big)\cos(\sqrt{\lambda_1}z)\,dz\\ &= \int_0^\infty a_{j0}\big((f_{0j}')^2-\beta_j^2\eta^2)\big)\cos(\sqrt{\lambda_1}z)\,dz\\ &\quad + \int_0^\infty a_{j0}\big(2f_{j0}'\tilde{f}_j'+(\tilde{f}_j')^2\big)\cos(\sqrt{\lambda_1})\,dz,\end{aligned}$$

hence

$$|I_j| \le C(\alpha^3+\alpha^2\|\tilde{\mathbf{f}}\|_{\mathcal{Y}}).$$

Similarly we get

$$\begin{aligned}|II_j| &\le C\alpha\|d\|_{\mathcal{Y}}(\alpha^2+\alpha\|\tilde{\mathbf{f}}\|_{\mathcal{Y}}),\\ |III_j| &\le C\alpha^2\|d\|_{\mathcal{Y}}^2(1+\alpha\|\tilde{\mathbf{f}}\|_{\mathcal{Y}}).\end{aligned}$$

Some tedious but standard calculations yield:

$$\left|\int_0^\infty \widetilde{\Pi}_{e_j}(\mathbf{f}_0+\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})\cos(\sqrt{\lambda_1}z)\,dz\right| \le C\big[\alpha^3+\alpha\|d\|_{\mathcal{Y}}(\alpha^2+\alpha\|\tilde{\mathbf{f}}\|_{\mathcal{Y}}+\alpha^{1-\nu}\|\tilde{\mathbf{e}}\|_{\mathcal{Y}}) + \alpha^{2+\frac12-\nu}\|\tilde{\mathbf{e}}\|_{\mathcal{Y}}\big].$$

Using (8.8) and $Z(s)\approx e^{-\frac{p+1}{2}|s|}$, $p\ge 2$ we can estimate

$$\left|\int_0^\infty P_j(\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})\cos(\sqrt{\lambda_1}z)\,dz\right| \le C\alpha^{2+\frac14-2\sigma'}.$$

Since $\sigma' < 2^{-10}$ the above estimates and Lemma 11.1 yield whenever $\nu < \frac14$:

$$\|\vec{\delta}\|_{\mathcal{Y}} \le \frac{C}{B}\big[\alpha^{\frac18}+R^2\alpha^{\nu}\big], \tag{11.15}$$

as long as (11.11) is satisfied.

Step 2 follows directly from the results of Section 10. We will now show that the claim of Step 3. From (10.18) we have

$$e_j = -\delta_j S[B_{0j}] + S[\tilde{h}_{0j}(\mathbf{f}_0+\tilde{\mathbf{f}},\vec{d})] + S[\widetilde{\Pi}_{e_j}(\mathbf{f}_0+\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})] + S[P_j(\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})], \tag{11.16}$$

where S is the operator defined in (10.18). We have, using estimates (10.19)–(10.20):

$$\|\delta_j S[B_{0j}]\|_{\theta_0\alpha} \le C\alpha^2\|\vec{\delta}\|_{\mathcal{Y}}.$$

Likewise

$$\|S[\tilde{h}_{0j}(\mathbf{f}_0+\tilde{\mathbf{f}},\vec{d})]\|_{\theta_0\alpha} \le R_0\alpha^2 + C\alpha^2\|\tilde{\mathbf{f}}\|_{\mathcal{Y}} + C\alpha^2\|\vec{d}\|_{\mathcal{Y}}\|\tilde{\mathbf{f}}\|_{\mathcal{Y}}.$$

where $R_0>0$ is a constant depending on $\mathbf{f}_0$ only. Using the formulas for $\widetilde{\Pi}_{e_j}$ and P_j we get

$$\|S[\widetilde{\Pi}_{e_j}(\mathbf{f}_0+\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})]\|_{\theta_0\alpha} \le C\alpha^{-1}\big[\alpha^{2-\nu}\|\tilde{\mathbf{e}}\|_{\mathcal{Y}}(\alpha\|\vec{d}\|_{\mathcal{Y}}+\alpha^2)\big],$$
$$\|S[P_j(\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})]\|_{\theta_0\alpha} \le C\alpha^{2+\frac18},$$

These estimates can be summarized as follows:

$$\|\mathbf{e}\|_{\mathcal{Y}} \le R_0\alpha^{\nu} + C\alpha^{\nu}(\alpha^{\frac18}+\alpha^{\nu}). \tag{11.17}$$

We will now estimate the right hand side of (11.5).

$$\|\mathcal{M}_j(\tilde{\mathbf{f}},\vec{d})\|_{\theta_0\alpha} \le R_0\alpha^2\|\vec{d}\|_{\mathcal{Y}} + C\alpha^2(\|\tilde{\mathbf{f}}\|^2_{\mathcal{Y}} + \|\vec{d}\|_{\mathcal{Y}}\|\tilde{\mathbf{f}}\|_{\mathcal{Y}}).$$

We also get in $B_{R\alpha^{\nu}}$:

$$\|\widetilde{\Pi}_{f_j}(\mathbf{f}_0+\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})\|_{\theta_0\alpha} + \|Q_j(\tilde{\mathbf{f}},\tilde{\mathbf{e}},\vec{d})\|_{\theta_0\alpha} \le C\alpha^2(\alpha^{\frac18}+\alpha^{2\nu}).$$

The last two estimates yield

$$\|\mathbf{f}_1\|_{\mathcal{Y}} \le R_0\|\vec{d}\|_{\mathcal{Y}} + C(\alpha^{\frac18}+\alpha^{2\nu}). \tag{11.18}$$

Combining now (11.15)–(11.18) we get

$$\|\mathcal{F}(\tilde{\mathbf{f}}, \tilde{\mathbf{e}}, \vec{d})\|_{\mathcal{Y}} \le R_0\alpha^\nu + \frac{C(1+R_0)}{B}(\alpha^{\frac{1}{8}} + \alpha^\nu) + C(\alpha^{\frac{1}{8}} + \alpha^{2\nu}) \tag{11.19}$$

$$\le \frac{R}{2}\alpha^\nu. \tag{11.20}$$

provided that $\frac{1}{8} < \nu < \frac{1}{4}$, R in the definition of $B_{R\alpha^\nu}$ is taken sufficiently large, α is taken small and B in the Lemma 11.1 is taken large. This shows the claim of Step 3.

It now remain to show the claim of Step 4. This follows from a rather straightforward application of the theory of Lipschitz dependence of various terms involved in (11.5)–(11.6). We have for instance

$$\|\vec{\delta}^{(1)} - \vec{\delta}^{(2)}\|_{\mathcal{Y}} \le \frac{C}{B}\|(\tilde{\mathbf{f}}^{(1)}, \tilde{\mathbf{e}}^{(1)}, \vec{d}^{(1)}) - (\tilde{\mathbf{f}}^{(2)}, \tilde{\mathbf{e}}^{(2)}, \vec{d}^{(2)})\|_{\mathcal{Y}}, \tag{11.21}$$

and

$$\|\mathbf{f}_1^{(1)} - \mathbf{f}_1^{(2)}\|_{\mathcal{Y}} + \|\mathbf{e}^{(1)} - \mathbf{e}^{(2)}\|_{\mathcal{Y}} \le C\alpha^\nu\|(\tilde{\mathbf{f}}^{(1)}, \tilde{\mathbf{e}}^{(1)}, \vec{d}^{(1)}) - (\tilde{\mathbf{f}}^{(2)}, \tilde{\mathbf{e}}^{(2)}, \vec{d}^{(2)})\|_{\mathcal{Y}}, \tag{11.22}$$

We leave the details to the reader. From these estimates, taking B larger if necessary we get that $\mathcal{F}$ is a Lipschitz contraction as claimed. This shows that $\mathcal{F}$ has a fixed point in $B_{R\alpha^\nu}$. The proof of the theorem is complete. □

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References

[1] Ambrosetti, A., Badiale, M., Cingolani, S., Semiclassical states of nonlinear Schrödinger equations, *Arch. Rational Mech. Anal.* 140 (1997), 285–300.

[2] Byeon, J., Wang, Z.-Q. Standing waves with a critical frequency for nonlinear Schrödinger equations. *Arch. Rat. Mech. Anal.* 65(2002), 295–316.

[3] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state.*Arch. Rational Mech. Anal.* 82 (1983), no. 4, 313–345.

[4] J. Busca, P. Felmer, Qualitative properties of some bounded positive solutions to scalar field equations.*Calc. Var. Partial Differential Equations* 13 (2001), no. 2, 191–211.

[5] V. Coti Zelati, P.H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on R^n.* Comm. Pure Appl. Math. 45 (1992), no. 10, 1217–1269.

[6] M. Crandall, P. Rabinowitz, Bifurcation from simple eigenvalues, *J. Funct. Anal.* 8 (1971), 321–340.

[7] E.N. Dancer, New solutions of equations on $\mathbf{R}^n$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 30 no. 3-4, 535–563 (2002).

[8] C. Delaunay, Sur la surface de revolution dont la courbure moyenne est constante, *Jour. de Mathématiques* 6 (1841), 309-320.

[9] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, em Calc. Var. Partial Differential Equations 4 (1996), 121–137.

[10] M. del Pino, P. Felmer, Semi-classcal states for nonlinear Schrödinger equations, *J. Funct. Anal.* 149(1997), 245–265.

[11] M. del Pino, M. Kowalczyk, J. Wei, Concentration on curves for nonlinear Schrödinger equations, em Comm. Pure Appl. Math. 70(2007), 113-146.

[12] M. del Pino, M. Kowalczyk, J. Wei, The Toda system and clustering interfaces in the Allen-Cahn equation, *Arch. Rat. Mech. Anal.*, to appear.

[13] Floer, A., Weinstein, A., Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* 69 (1986), 397–408.

[14] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^N, *Adv. Math. Suppl. Stud.* 7A (1981) 369–402.

[15] A. Gierer and H. Meinhardt, A theory of biological pattern formation, *Kybernetik (Berlin)* 12 (1972) 30–39.

[16] P. Gray and S.K. Scott, Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability. *Chem. Eng. Sci.* 38 (1983), 29–43.

[17] C. Gui, J. Wei, Multiple interior peak solutions for some singular perturbation problems, *J. Differential Equations* 158 (1999) 1–27.

[18] Kang, X., Wei, J. On interacting bumps of semi-classical states of nonlinear Schrödinger equations., *Adv. Diff. Eqn.* 5(7-9), 899–928 (2000).

[19] N. Kapouleas, Complete constant mean curvature surfaces in Euclidean three-space, *Ann. of Math.* (2) 131 (1990) 239–330.

[20] B. Konstant, The solution to a generalized Toda lattice and Representation Theory, *Adv. Math.* 34 (1979), 195–338.

[21] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n. *Arch. Rational Mech. Anal.* 105 (1989), no. 3, 243–266.

[22] Li, Y.Y. On a singularly perturbed elliptic equation, *Adv. Diff. Eqn.* 2(6), 955–980 (1997).

[23] F.H. Lin, W.-M. Ni, J. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem. *Comm. Pure Appl. Math.* 60 (2007), no. 2, 252–281.

[24] A. Malchiodi, Some new entire solutions of semilinear elliptic equations on R^n. *Preprint 2007.*

[25] A. Malchiodi, M. Montenegro, Boundary concentration phenomena for a singularly perturbed elliptic problem. *Comm. Pure Appl. Math.* 55 (2002), no. 12, 1507–1568.

[26] A. Malchiodi, M. Montenegro, Multidimensional boundary layers for a singularly perturbed Neumann problem.*Duke Math. J.* 124 (2004), no. 1, 105–143.

[27] R. Mazzeo and F. Pacard, Constant scalar curvature metrics with isolated singularities. *Duke Math. J.* 99 (1999), no. 3, 353–418.

[28] R. Mazzeo and F. Pacard, Constant mean curvature surfaces with Delaunay ends. *Comm. Anal. Geom.* 9 (2001), no. 1, 169–237.

[29] R. Mazzeo, F. Pacard and D. Pollack, Connected sums of constant mean curvature surfaces in Euclidean 3 space. *J. Reine Angew. Math.* 536 (2001), 115–165.

[30] R. Mazzeo, and D. Pollack, *Gluing and moduli for noncompact geometric problems.* 17–51, Sympos. Math., XXXVIII, Cambridge Univ. Press, Cambridge, 1998.

[31] R. Mazzeo, D. Pollack, and K. Uhlenbeck, Connected sum constructions for constant scalar curvature metrics, *Topol. Methods Nonlinear Anal.* 6 (1995), 207–233.

[32] W.-M. Ni, I. Takagi, On the shape of least-energy solutions to a semilinear Neumann problem. *Comm. Pure Appl. Math.* 44 (1991), no. 7, 819–851.

[33] W.-M. Ni, I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem. *Duke Math. J.* 70 (1993), no. 2, 247–281.

[34] W.-M. Ni, Diffusion, cross-diffusion, and their spike-layer steady states. *Notices Amer. Math. Soc.* 45 (1998), no. 1, 9–18.

[35] Oh, Y.-G. Stability of semiclassical bound states of nonlinear Schroedinger equations with potentials, *Commun. Math. Phys.* 121 (1989), 11–33.

[36] P. Quittner, P. Polacik and Ph. Souplet, Singularity and decay estimates in superlinear problems via Liouville-type theorems, I: Elliptic equations and systems, *Duke Math. J.* 139 (2007), 555-579

[37] R. Schoen, The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation, *Comm. Pure Appl. Math.* 41 (1988) 317–392.

[38] Strauss, Walter A. Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* 55 (1977), no. 2, 149–162

[39] A. M. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. Lond. B* 237 (1952) 37–72.