

SUPER-LINEAR ELLIPTIC EQUATION FOR FULLY NONLINEAR OPERATORS WITHOUT GROWTH RESTRICTIONS FOR THE DATA

MARIA J. ESTEBAN, PATRICIO L. FELMER, AND ALEXANDER QUAAS

ABSTRACT. In this paper we deal with existence and uniqueness of solution to the fully nonlinear equation

$$-F(D^2u) + |u|^{s-1}u = f(x) \quad \text{in } \mathbb{R}^n,$$

where $s > 1$ and f satisfies only local integrability conditions. This result is well known when, instead of the fully nonlinear elliptic operator F , the Laplacian or a divergence form operator is considered. Our existence results use the Alexandroff-Bakelman-Pucci inequality since we cannot use any variational formulation. For radially symmetric f , and in the particular case where F is a maximal Pucci operator, we can prove our results under less integrability assumptions, taking advantage of an appropriate variational formulation. We also obtain an existence result with boundary explosion in smooth domains.

1. INTRODUCTION

The problem we study in this article is the solvability of the differential equation

$$-F(D^2u) + |u|^{s-1}u = f(x) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

when F is a fully nonlinear, uniformly elliptic operator, $s > 1$ and for f having only local integrability properties, but without assuming any growth condition at infinity.

When F is replaced by the Laplace operator, Brezis showed in [1] that whenever $s > 1$, one can find a (unique) solution to the above problem assuming only local integrability of f . This very weak assumption is enough when the nonlinearity is increasing and super-linear, as in the case of $|u|^{s-1}u$ with $s > 1$. This result was extended to the case of a general quasilinear operator, including the p -Laplace operator, and to parabolic equations by Boccardo, Gallouet and Vázquez in [2] and [3], respectively. See also the work by Leoni in [17] where more general

Key words and phrases. Pucci operator, super-linear elliptic problem, boundary explosion, local data.

nonlinearities are considered. In all these works, the existence of the solution is obtained using in a crucial way the variational structure of the equation by choosing appropriate test functions to obtain *a priori* estimates.

On the operator F we assume uniform ellipticity, that is:

$$M_{\lambda,\Lambda}^-(M - N) \leq F(M) - F(N) \leq \mathcal{M}_{\lambda,\Lambda}^+(M - N) \text{ for all } N, M \in S_N,$$

and $F(0) = 0$. Here $0 < \lambda \leq \Lambda$, $\mathcal{M}_{\lambda,\Lambda}^+$ and $\mathcal{M}_{\lambda,\Lambda}^-$ are the extremal Pucci operators as defined in [5] and S_N is the set of $N \times N$ symmetric matrices. We assume throughout the paper that F satisfies this condition. Whenever no confusion arises we will simply write \mathcal{M}^+ and \mathcal{M}^- , omitting the parameters. In order to find a solution to (1.1), we have to work in the viscosity solution framework and we cannot use test functions and integration by parts to derive *a priori* estimates. The use of the viscosity theory forces us to work in the $L^N(\mathbb{R}^N)$ framework and indeed, the presence of the $|u|^{s-1}u$ term in the equation allows us also to prove the existence of a unique L^N -viscosity for (1.1) whenever $f \in L_{loc}^N(\mathbb{R}^N)$. Since there is no available theory for viscosity solution when $f \in L_{loc}^1(\mathbb{R}^N)$, at this point we cannot expect to obtain results under this weaker condition as in [1]. However, in view of our results in Section §3 for the radially symmetric case, one may expect to find solutions when f has less than L^N -integrability, but at this point we are not able to do it. Our first theorem is the following

Theorem 1.1. *Assume that $s > 1$. For every function $f \in L_{loc}^N(\mathbb{R}^N)$, the equation (1.1) possesses a unique solution in the L^N -viscosity sense and if $f \geq 0$ a.e. then $u(x) \geq 0$ for all $x \in \mathbb{R}^N$.*

The formal definition of solution is given in Section §2.

It is well known that in the case of super-linear problems one can find solutions which explode at the boundary of a bounded domain. This has been shown for various cases of linear and nonlinear second order elliptic operators in divergence form. See for instance the work by Keller [14], Loewner and Nirenberg [18], Kondrat'ev and V. Nikishkin [15], Diaz and Letelier [11], Diaz and Diaz [10], Del Pino and Letelier [9] and Marcus and Veron [19].

In the case of fully nonlinear operators, the techniques used to prove Theorem 1.1 can also be used to prove the following theorem on the existence of solutions in a bounded set, with explosion on the boundary. The simplest situation is the following

Theorem 1.2. *Let $s > 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain in \mathbb{R}^N . Assume that $f \in L_{loc}^N(\Omega)$ and for some $g \in L^N(\Omega)$ we have $f \geq g$, then*

the equation

$$-F(D^2u) + |u|^{s-1}u = f \quad \text{in } \Omega, \quad (1.2)$$

$$\lim_{x \rightarrow \partial\Omega} u(x) = \infty \quad (1.3)$$

possesses at least one solution in the L^N -viscosity sense.

Here we only address this simple situation, but the same kind of results should also hold true under more general assumptions. Moreover, the asymptotic study of the blow-up rate, both when $f \in L^N(\Omega)$ or as when f itself explodes at the boundary, is an interesting problem due to the nonlinearity of the differential operator.

At this point we would like to mention the work of Labutin [16], where the author studies the local behavior of solutions to the same type of equations as ours, without the right hand side, establishing removability of singularities.

In the second part of this paper we analyze the case of radially symmetric data f . Here we can prove existence and uniqueness of solutions under weaker integrability assumptions on f but only in the particular case when F is the maximal Pucci operator. The reason for this is that in the radial case we can re-write equation (1.1) as a divergence form quasilinear ordinary differential equations, for which one can define a notion of weak solution. In this case we are back to integration by parts techniques.

Comparison between radial solutions and positivity results however, are not obtained in a direct way. This is because the coefficient of the second order derivative in the equation depends on the solution and its first derivative in a nonlinear way. Thus, when comparing two solutions we do not have an obvious common factor for the second derivative of the difference or, if we have it we do not control its integrability at the origin. An *ad hoc* argument has to be found to do comparison in this case, see Lemma 3.3.

Theorem 1.3. *Assume $s > 1$ and f is a radially symmetric function satisfying*

$$\int_0^R r^{N_+-1} |f(r)| dr < \infty, \quad (1.4)$$

for all $R > 0$. Here $N_+ := \frac{\lambda}{\Lambda}(N-1) + 1$, with λ and Λ being the parameters defining the Pucci operator $\mathcal{M}_{\lambda,\Lambda}^+$. Then equation (1.1) with $F = \mathcal{M}_{\lambda,\Lambda}^+$ has a unique weak radially symmetric solution and if f is nonnegative then u is also nonnegative.

The formal definition of radially symmetric weak solution and the proof of Theorem 1.3 are given in Section §3. See also Remark 3.1 where we discuss the assumptions on f in this case.

Remark 1.1. *In all our results, the power function $|u|^{s-1}u$ could be replaced with nonlinear functions which are super-linear at infinity, however for simplicity all throughout the paper we will only deal with the pure power case. In this direction see [1], [2] and [17]. Let us also stress that the assumption $s > 1$ is essential for our results to hold, as we can see from the discussion in [2].*

Acknowledgments. The authors wish to thank the referee for suggesting to extend the original result to the setting of general fully nonlinear elliptic operators.

2. THE GENERAL CASE WITH $f \in L^N_{loc}(\mathbb{R}^N)$.

We devote this section to prove Theorem 1.1 by an approximation procedure together with a local estimate based on a truncation argument and the application of the Alexandroff-Bakelman-Pucci inequality.

We start recalling the notion of solution suitable when the right hand side in (1.1) is only in $L^p_{loc}(\mathbb{R}^N)$. Following the work by Caffarelli, Crandall, Kocan and Swiech [4], we first notice that the framework requires $p > N - \varepsilon_0$, where $\varepsilon_0 > 0$ depends on the ellipticity constants λ and Λ . Thus the case $p = N$, which is our framework, is covered by the theory. Even though the context of the definitions in [4] is much more general, for the purposes of this article we only consider a 'semilinear' case (1.1)

$$-F(D^2u) + G(u) = f(x) \quad \text{in } \mathbb{R}^N, \quad (2.1)$$

where G is an increasing continuous odd function. According to [4] we have the following definition:

Definition 2.1. *Assume that $f \in L^p_{loc}(\mathbb{R}^N)$, then we say that a continuous function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is an L^p -viscosity subsolution (supersolution) of the equation (2.1) in \mathbb{R}^N if for all $\varphi \in W^{2,p}_{loc}(\mathbb{R}^N)$ and a point $\hat{x} \in \mathbb{R}^N$ at which $u - \varphi$ has a local maximum (respectively, minimum) one has*

$$\text{ess lim inf}_{x \rightarrow \hat{x}} (-F(D^2\varphi(x)) + G(u(x)) - f(x)) \leq 0 \quad (2.2)$$

$$(\text{ess lim sup}_{x \rightarrow \hat{x}} (-F(D^2\varphi(x)) + G(u(x)) - f(x)) \geq 0). \quad (2.3)$$

Moreover, u is an L^p -viscosity solution of (2.1) if it is both an L^p -viscosity subsolution and an L^p -viscosity supersolution.

In what follows we say that u is a C -viscosity (sub or super) solution of (2.1) when in the definition above we replace the test function space $\varphi \in W_{loc}^{2,p}(\mathbb{R}^N)$ by $C^2(\mathbb{R}^N)$. In this case the limits (2.2) and (2.3) become simple evaluation at \hat{x} , as given in [7].

As we mentioned above, the idea is to consider a sequence of approximate problems and then take the limit at the end. So, given $f \in L_{loc}^N(\mathbb{R}^N)$ we assume $\{f_n\}$ is a sequence of $C^\infty(\mathbb{R}^N)$ functions so that for every bounded set Ω

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f|^N dx = 0. \quad (2.4)$$

The sequence $\{f_n\}$ is easily constructed by mollification and a diagonal argument.

The following is a basic existence and regularity result we need in our construction of a solution to (1.1).

Lemma 2.1. *For every $n \in \mathbb{N}$ there is a solution $u_n \in C^1(B_n)$ of the equation*

$$-F(D^2 u_n) + \frac{1}{n} u_n + |u_n|^{s-1} u_n = f_n(x) \quad \text{in } B_n \quad (2.5)$$

where $B_n = B(0, n)$ is the ball centered at 0 and with radius n .

Proof. We observe that there is a constant M_n so that

$$-M_n^s \leq f_n(x) \leq M_n^s \quad \text{for all } x \in B_n$$

and then $v_- = -M_n$ and $v_+ = M_n$ are subsolution and supersolution of (2.5), respectively. Then we can use the existence Theorem 4.1 in [7] for viscosity solutions of (2.5) to find u_n a C -viscosity solution. We observe that the hypothesis of Theorem 4.1 are fully satisfied by our operator, which is proper and satisfies the other hypothesis with $\gamma = 1/n$, see [7]. Noticing that u_n solves the equation

$$F(D^2 u_n) = g_n \quad (2.6)$$

for the continuous function $g_n(x) = u_n(x)/n + |u_n(x)|^s u_n(x) - f_n(x)$, then $u_n \in C^{1,\beta}(B_n)$, for certain $\beta > 0$, by applying the regularity theory of Caffarelli [6].

Our next lemma is a version of Kato's inequality for C -viscosity solutions of equation (2.1) with continuous right hand side.

Lemma 2.2. *Assume $\Omega \subset \mathbb{R}^N$ and $u, v, f : \Omega \rightarrow \mathbb{R}$ are continuous functions and let $H(x) = G(u(x)) - G(v(x))$. If $u - v$ is a C -viscosity solution of equation*

$$-F(D^2(u - v)) + H(x) \leq f \quad \text{in } \Omega \quad (2.7)$$

then $(u - v)^+$ is a C -viscosity solution of

$$-F(D^2(u - v)^+) + H^+ \leq f^+ \quad \text{in } \Omega. \quad (2.8)$$

Proof. If $x \in \Omega$ satisfies $u(x) - v(x) > 0$ or $u(x) - v(x) < 0$ then obviously $u - v$ satisfies (2.8) at x . If $u(x) - v(x) = 0$ then we choose a test function φ so that $(u - v)^+ - \varphi$ has a local maximum at x , but then $(u - v) - \varphi$ has a local maximum at x and then we may use (2.7) to obtain

$$-F(D^2\varphi(x)) \leq f^+$$

so that (2.8) is satisfied in x , since $H(x) = 0$. \square

Now we give a generalization of Kato's inequality (see [13]) for C -viscosity solutions of equation (2.1).

Lemma 2.3. *If we assume that $u, f : \Omega \rightarrow \mathbb{R}$ are continuous functions and u is a C -viscosity solution of equation*

$$-F(D^2u) + G(u) = f \quad \text{in } \Omega, \quad (2.9)$$

then $|u|$ satisfies

$$-\mathcal{M}^+(D^2|u|) + G(|u|) \leq |f| \quad \text{in } \Omega \quad (2.10)$$

in the C -viscosity sense.

Proof. We first use $v = 0$ in Lemma 2.2 to get that u^+ is a subsolution with f^+ as right hand side, and then observe that

$$-\mathcal{M}^+(D^2(-u)) + G(-u) \leq f^-,$$

since $F \leq \mathcal{M}^+$ and $\mathcal{M}^- \leq \mathcal{M}^+$, that gives that u^- is a subsolution with f^- as right hand side. We conclude that $|u| = \max\{u^+, u^-\}$ satisfies (2.10) \square

The following lemma contains the crucial local estimate for solutions of (2.5). This result was proved by Brezis [1] in the context of the Laplacian and tells that solutions have local estimates independent of the global behavior of f . The approach in [1], see also [2], is to use suitable test functions and integration by parts. This cannot be done here since the differential operator does not have divergence form. For this result the fact that $s > 1$ is essential.

Lemma 2.4. *Let $s > 1$ and g continuous in $\Omega \subset \mathbb{R}^N$, an open set. Suppose that $g \geq 0$ in Ω and u is a $C^1(\Omega)$ nonnegative C -viscosity solution of*

$$-\mathcal{M}^+(D^2u) + \frac{1}{n}u + |u|^{s-1}u \leq g \quad \text{in } \Omega,$$

then for all $R > 0$ and $R' > R$ such that $B_{R'} \subset \Omega$

$$\sup_{B_R} u \leq C(1 + \|g\|_{L^N(B_{R'})}), \quad (2.11)$$

where $C = C(s, R, R', N, \lambda, \Lambda)$ does not depend on g nor n .

Proof. Let $\xi(x) = (R')^2 - |x|^2$ and $\beta = 2/(s-1)$ and consider $v = \xi^\beta u$. Now we want to find the equation satisfied by v . Suppose that $v - \varphi$ has a local maximum, $v(\hat{x}) = \varphi(\hat{x})$, $Dv(\hat{x}) = D\varphi(\hat{x})$ and $\varphi \in C^2$. Then $u - \xi^{-\beta}\varphi$ has a local maximum at \hat{x} . Therefore $\xi^{-\beta}\varphi$ is a test function for u and so

$$-\mathcal{M}^+(D^2\varphi) + \frac{1}{n}\varphi + \xi^{-2}|\varphi|^{s-1}\varphi \leq \xi^\beta g + I + II + III, \quad (2.12)$$

$$I := -\beta\xi^{-1}v\mathcal{M}^-(D^2\xi) \quad (2.13)$$

$$II := \beta(\beta+1)\xi^{-2}v\mathcal{M}^+(D\xi \otimes D\xi) \quad (2.14)$$

$$III := -\beta\xi^{-1}\mathcal{M}^-(D\xi \otimes D\varphi + D\varphi \otimes D\xi) \quad (2.15)$$

So v satisfies the equation

$$-\mathcal{M}^+(D^2v) + \frac{1}{n}\xi^{-2}v + v|v|^{s-1} \leq \xi^\beta g + I + II + III \quad (2.16)$$

in $B(R')$ in the C -viscosity sense. Here in I, II and III we replace $D\varphi$ by Dv .

In what follows we write $\Omega^+ = \{x \in \Omega / v(x) > 0\}$. Consider the contact set for the function v , which is defined as

$$\Gamma_v^+ = \{x \in B_{R'} / \exists p \in \mathbb{R}^N \text{ with } v(y) \leq v(x) + \langle p, y - x \rangle, \forall y \in B_{R'}\}.$$

We observe that $\Gamma_v^+ \subset \Omega^+ \cap B_{R'}$ and that if \bar{v} is the concave envelope of v in $\bar{B}_{R'}$ then for $x \in B_{R'}$ we have $v(x) = \bar{v}(x)$ if and only if $x \in \Gamma_v^+$. The function \bar{v} , being concave, satisfies

$$\bar{v}(y) \leq v(x) + \langle Dv(x), y - x \rangle,$$

for all $x \in \Gamma_v^+$ and $y \in \bar{B}_{R'}$. Choosing adequately $y \in \partial B_{R'}$ we obtain

$$|Dv(x)| \leq \frac{v(x)}{R' - |x|}, \quad \text{for all } x \in \Gamma_v^+. \quad (2.17)$$

Now we claim that the function v satisfies

$$-\mathcal{M}^+(D^2v) + \xi^{-2}v(|v|^{s-1} - C) \leq \xi^\beta g \quad \text{for all } x \in \Gamma_v^+.$$

Notice that in Γ_v^+ we have $I, II \leq C\xi^{-2}v$, and

$$III \leq c\xi^{-1}|Dv| \leq c(R' + |x|)\xi^{-2}v \leq C\xi^{-2}v,$$

where we used (2.17). Here c and C are constants depending on R' and s . Therefore the claim follows.

Now we define $w = \max\{v - C^{1/(p-1)}, 0\}$ in $B_{R'}$ and we observe that $\Gamma_w^+ \subset \Gamma_v^+$ and $\Gamma_w^+ \subset \{x \in B_{R'} / w > 0\}$. Consequently

$$-\mathcal{M}^+(D^2w) \leq \xi^\beta g, \quad \text{a.e in } \Gamma_w^+.$$

Thus, from Alexandroff-Bakelman-Pucci inequality (see for example [5]),

$$\sup_{B_{R'}} w \leq C\|\xi^\beta g\|_{L^N(B_{R'})},$$

but then

$$c \sup_{B_R} u \leq \sup_{B_{R'}} v \leq \sup_{B_{R'}} w + C^{1/(p-1)} \leq C(1 + \|g\|_{L^N(B_{R'})}),$$

where c and C represent generic constants depending only on s, R, R', N, λ and Λ but not on n , as desired. \square

Remark 2.1. *Observe that in this estimate the constant C does not even depend on the possibly arbitrary values of u on $\partial\Omega$. This fact is very important in the study solutions of this equation having explosion on the boundary of Ω , as we see in Theorem 1.2.*

Proof of Theorem 1.1 (Existence) We start with a sequence of smooth functions $\{f_n\}$ such that for every bounded set Ω (2.4) holds. Then we use Lemma 2.1 to construct a sequence of solutions $\{u_n\}$ of equation (2.5). According to Lemma 2.3 and 2.4, for every $0 < R < R' < n$ we have

$$\sup_{B_R} |u_n| \leq C(1 + \|f\|_{L^N(B_{R'})}),$$

where C does not depend on f nor in n . With this inequality in hand we look at equation (2.5) and use Proposition 4.10 in [5] to obtain, for every bounded open set Ω ,

$$\|u_n\|_{C^\alpha(\Omega)} \leq C,$$

where C does not depend on n , but only on f, Ω and the other parameters. By a diagonal procedure, we then obtain a subsequence of solutions of equation

$$-F(D^2u_n) + c_n u_n + |u_n|^{s-1} u_n = f_n,$$

that we keep calling $\{u_n\}$, such that u_n converges uniformly over every bounded subset of \mathbb{R}^N . Here the equation holds in B_{1/c_n} , with $c_n \rightarrow 0$ as $n \rightarrow \infty$, and f_n has been renamed. Then using Theorem 3.8 in [4] we conclude that u is an L^N -viscosity solution of (1.1), completing the proof of the existence part of the Theorem 1.1. \square

The next lemma gives the positivity part of Theorem 1.1.

Lemma 2.5. *Assume $s > 1$. If $f \leq 0$ a.e. and u solves equation (1.1) in the L^N -viscosity sense then $u \leq 0$ in \mathbb{R}^N and if $f \geq 0$ a.e. then $u \geq 0$ in \mathbb{R}^N .*

Proof. We proceed as in [1], considering the function defined by Oserman in [20]:

$$U(x) = \frac{CR^\beta}{(R^2 - |x|^2)^\beta} \quad \text{in } B_R \quad R > 0,$$

where $\beta = 2/(s-1)$ and $C^{s-1} = 2\beta\Lambda \max\{N, \beta + 1\}$. Since U' and U'' are positive, we see that $\mathcal{M}^+(D^2U) = \Lambda\Delta U$ and then a direct computation gives that

$$-\mathcal{M}^+(D^2U) + U^s \geq 0 \quad \text{in } B_R. \quad (2.18)$$

From here, the equation for u and the non-positivity of f we obtain

$$-\mathcal{M}^+(D^2(u - U)) + |u|^{s-1}u - U^s \leq 0.$$

We observe that this inequality is in the L^N -viscosity sense, however since f was dropped, it also holds in the C -viscosity sense. Then by Lemma 2.2 we find

$$-\mathcal{M}^+(D^2(u - U)^+) + (|u|^{s-1}u - U^s)^+ \leq 0$$

from where we get

$$-\mathcal{M}^+(D^2(u - U)^+) \leq 0 \quad \text{in } B_R.$$

We observe that the function $u - U$ is negative in the set $R - \delta \leq |x| < R$, for some sufficiently small $\delta > 0$. Then by Alexandroff-Bakelman-Pucci maximum principle $(u - U)^+ = 0$ which implies $u - U \leq 0$ in B_R . From here, taking point-wise limit as $R \rightarrow \infty$ we find that $u \leq 0$.

In case $f \geq 0$ we proceed similarly, but relying in Lemma 2.2 with the operator \mathcal{M}^- , to obtain that $u + U \geq 0$ in B_R . From here the result follows. \square

Proof of Theorem 1.1 (Uniqueness). If u_1 and u_2 are solutions of (1.1) then the continuous function $w = u_1 - u_2$ satisfies

$$-\mathcal{M}^+(D^2w) + |u_1|^{s-1}u_1 - |u_2|^{s-1}u_2 \leq 0$$

in the C -viscosity sense, see for example Proposition 2.1 in Da Lio and Sirakov [8]. Next we use Lemma 2.2 to obtain

$$-\mathcal{M}^+(D^2w^+) + (|u_1|^{s-1}u_1 - |u_2|^{s-1}u_2)^+ \leq 0$$

and using that

$$||a|^{s-1}a - |b|^{s-1}b| \geq \delta|a - b|^s, \quad \forall a, b \in \mathbb{R},$$

for certain $\delta > 0$ we conclude that

$$-\mathcal{M}^+(D^2w^+) + \delta(w^+)^s \leq 0. \quad (2.19)$$

Using Lemma 2.5 we obtain that $u_1 - u_2 \leq 0$. Interchanging the roles of u_1 and u_2 we complete the proof. \square

Next we give an existence theorem for explosive solutions, whose proof follows easily from the estimate given in Lemma 2.4. We keep it in the simplest form, but we think it may be extended to more general situations.

Proof of Theorem 1.2. We first consider an increasing sequence of smooth functions $\{f_n\} \subset L^N(\Omega)$ such

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f|^N = 0.$$

Then we find u_n a solution to the problem

$$-F(D^2u_n) + |u_n|^{s-1}u_n = f_n \quad \text{in } \Omega, \quad (2.20)$$

$$u_n = n \quad \text{in } \partial\Omega. \quad (2.21)$$

Letting $w_n = u_{n+1} - u_n$, we see that w_n satisfies

$$-\mathcal{M}^+(D^2w_n) + |u_{n+1}|^{s-1}u_{n+1} - |u_n|^{s-1}u_n \geq f_{n+1} - f_n$$

the we may use Alexandrof-Bakelman-Pucci inequality to obtain $u_{n+1} \geq u_n$ in Ω for all $n \in \mathbb{N}$. By arguments similar to those given in the proof of Theorem 1.1 (Existence), using Lemma 2.4, we obtain a subsequence, we keep calling $\{u_n\}$ so that u_n converges uniformly to a solution u of (1.2). Moreover $u \geq u_n$ in Ω for all n so that $\liminf_{x \rightarrow \partial\Omega} u \geq n$, for all n , so that u also satisfies (1.3). \square

Remark 2.2. *The argument given above allows to prove the existence of a solution to equation (1.2) assuming that $f \in L_{loc}^N(\Omega)$. Naturally, in this case we do not know about the behavior of the solution on the boundary.*

3. THE RADIAL CASE

In this section we are going to consider only the particular case when F is a maximal Pucci operator \mathcal{M}^+ . This operator can be written in a very simple way when we are dealing with radially symmetric functions. Since the eigenvalues of D^2u are u'' of multiplicity one and u'/r with multiplicity $N - 1$ and defining $\theta(s) = \Lambda$ if $s \geq 0$ and $\theta(s) = \lambda$ if $s < 0$, then we easily see that for every u radially symmetric,

$$\mathcal{M}^+(D^2u)(r) = \theta(u''(r))u''(r) + \theta(u'(r))(N - 1)\frac{u'(r)}{r}.$$

Then we see that equation (1.1) in the classical sense becomes

$$-\theta(u''(r))u'' - \theta(u'(r))(N - 1)\frac{u'}{r} + |u|^{s-1}u = f(r), \quad (3.1)$$

for a radial function f . In order to write this equation in a more simple form, we make some definitions. First we observe that for solutions of (3.1) we have

$$\theta(u''(r)) = \theta\{-\theta(u'(r))(N - 1)\frac{u'}{r} + |u|^{s-1}u - f(r)\},$$

which is more convenient as we see. We define

$$\Theta(r, u(r), u'(r)) = \theta\{-\theta(u'(r))(N - 1)\frac{u'}{r} + |u|^{s-1}u - f(r)\},$$

the 'dimension'

$$N(r, u(r), u'(r)) = \frac{\theta(u'(r))}{\Theta(r, u(r), u'(r))}(N - 1) + 1$$

and the weights

$$\rho(r, u(r), u'(r)) = e^{\int_1^r \frac{N(\tau, u(\tau), u'(\tau)) - 1}{\tau} d\tau}$$

and

$$\tilde{\rho}(r, u(r), u'(r)) = \frac{\rho(r, u(r), u'(r))}{\Theta(r, u(r), u'(r))}.$$

If we define

$$N_+ = \frac{\lambda}{\Lambda}(N - 1) + 1, \quad \text{and} \quad N_- = \frac{\Lambda}{\lambda}(N - 1) + 1$$

we see that $N_+ \leq N(r, u(r), u'(r)) \leq N_-$ and also,

$$r^{N_- - 1} \leq \rho(r, u(r), u'(r)) \leq r^{N_+ - 1} \text{ if } 0 \leq r \leq 1$$

and

$$\frac{\rho}{\Lambda} \leq \tilde{\rho} \leq \frac{\rho}{\lambda}.$$

With these definitions we find that (3.1) is equivalent to

$$-(\rho u')' + \tilde{\rho}|u|^{s-1}u = \tilde{\rho}f(r). \quad (3.2)$$

When no confusion arises we omit the arguments in the functions ρ and $\tilde{\rho}$, in particular when we write $\rho v'$ we mean $\rho(r, v(r), v'(r))v'(r)$ and so on. What is interesting about equation (3.2) is that it allows to define a weaker notion of solution which extends the L^N -viscosity sense to more general f . With this new notion we can prove a theorem for the existence of radial solutions of (1.1) with a weaker condition on f than in the non-radial case of Section §2. See Remark 3.1.

We consider the set of test functions defined as

$$H = \{\varphi : [0, \infty) \rightarrow \mathbb{R} / \exists \phi \in W_0^{1,\infty}(\mathbb{R}^N) \text{ such that } \phi(x) = \varphi(|x|)\},$$

where $W_0^{1,\infty}(\mathbb{R}^N)$ denotes the space of functions in $W^{1,\infty}(\mathbb{R}^N)$ with compact support.

Definition 3.1. *We say that $u : [0, R] \rightarrow \mathbb{R}$ is a weak solution of (3.2) with Dirichlet boundary condition at $r = R$, if u is absolutely continuous in $(0, R]$, $u(R) = 0$,*

$$\int_0^R \rho|u|^s dr < \infty, \quad \int_0^R \rho|u'| dr < \infty \quad (3.3)$$

and

$$\int_0^R \rho u' \varphi' + \tilde{\rho}|u|^{s-1}u \varphi dr = \int_0^R \tilde{\rho} f \varphi dr \quad \forall \varphi \in H. \quad (3.4)$$

Now we state our theorem precisely which is a more complete version of Theorem 1.3

Theorem 3.1. *Assume $s > 1$ and f is a radial function satisfying for all $R > 0$,*

$$\int_0^R r^{N+1}|f(r)| dr < \infty. \quad (3.5)$$

Then equation (3.2) has a unique weak solution u and if f is nonnegative then u is also nonnegative.

Additionally, for any $1 < q < 2s/(s+1)$

$$\int_0^r \rho|u'|^q dr < \infty \quad \text{for all } R > 0. \quad (3.6)$$

Moreover, the function $\rho u'$ is differentiable a.e. in $(0, \infty)$ and consequently satisfies

$$\lim_{r \rightarrow 0} (\rho u')(r) = 0, \quad \lim_{r \rightarrow 0} \int_0^r \rho|u'| dr = 0. \quad (3.7)$$

In order to prove the theorem above we will perform an approximation procedure as in the general case. Because the problem is radial and has a divergence form formulation we can get better estimates and pass to the limit, under weaker assumptions on f .

By regularizing f and using a diagonal procedure we may find a sequence of radial smooth functions $\{f_n\}$ such that for all $0 < R$

$$\lim_{n \rightarrow \infty} \int_0^R r^{N+1} |f_n(r) - f(r)| dr = 0. \quad (3.8)$$

Moreover, we may assume that there exists a function $g : (0, \infty) \rightarrow \mathbb{R}$ such that $|f_n(r)| \leq g(r)$ for all $r > 0$ and $\int_0^R r^{N+1} |g(r)| dr < +\infty$, for all $R > 0$.

First we have an existence result for the approximate problems.

Lemma 3.1. *For every n there is solution u_n in $C^2[0, n]$ satisfying $u_n(n) = 0$, (3.3) with $R = n$ and*

$$\int_0^n \rho_n u_n' \varphi' + \tilde{\rho}_n (c_n u_n + |u_n|^{s-1} u_n) \varphi = \int_0^n \rho_n f_n \varphi, \quad \forall \varphi \in H. \quad (3.9)$$

where $\rho_n(r) = \rho(r, u_n(r), u_n'(r))$ (similarly for $\tilde{\rho}_n$). and $\{c_n\}$ is a positive sequence converging to zero.

Proof. We may use the same argument of Lemma 2.1 together with Da Lio and Sirakov symmetry result [8]. \square

Now we get some estimates following the ideas of Boccardo, Gallouet and Vazquez in [2].

Lemma 3.2. *Let $\{u_n\}$ be the sequence of solutions found in Lemma 3.1. Then, for all $0 < R$ and $m \in (0, s - 1)$ there is a constant C depending on R, m, s, N, λ and Λ , but not on f nor n , such that for all $n \in \mathbb{N}$ we have*

$$\int_0^R \rho_n |u_n|^s ds \leq C(1 + \int_0^{2R} r^{N+1} |f| dr) \quad (3.10)$$

and

$$\int_0^{2R} \frac{\rho_n |u_n'|^2 dr}{(1 + |u_n|)^{m+1}} \leq C(1 + \int_0^R r^{N+1} |f| dr). \quad (3.11)$$

Proof. We consider the function ϕ defined as

$$\phi(t) = \int_0^t \frac{dt}{(1 + s)^{m+1}}, \quad t \geq 0$$

and extended as an odd function to negative t , which is smooth and bounded. We also consider a cut-off function $\theta : [0, \infty) \rightarrow \mathbb{R}$ being

smooth, with support in $[0, 2R]$, equal 1 in $[0, R]$, $0 \leq \theta \leq 1$ and $|\theta'| \leq 2/R$.

We define $v = \phi(u)\theta^\alpha$, where $\alpha > 2s/(s-1-m)$. Omitting the index n in what follows, using v as a test function we obtain

$$\begin{aligned} & \int_0^{2R} \frac{m\rho|u'|^2\theta^\alpha}{(1+|u|)^{1+m}} + \int_0^{2R} \tilde{\rho}|u|^{s-1}u\phi(u)\theta^\alpha dr \\ & \leq \int_0^{2R} \tilde{\rho}f\phi(u)\theta^\alpha dr - \alpha \int_0^{2R} \rho u' \phi(u)\theta^{\alpha-1}\theta' dr \end{aligned} \quad (3.12)$$

$$\leq C\left(\int_0^{2R} r^{N_+-1}|f|dr + \int_0^{2R} \rho|u'|\theta^{\alpha-1}dr\right), \quad (3.13)$$

where we drop the term with c_n in the first inequality. Using Young inequality, for some $\varepsilon > 0$, we have

$$\begin{aligned} \int_0^{2R} \rho|u'|\theta^{\alpha-1}dr & \leq \varepsilon \int_0^{2R} \frac{m\rho|u'|^2\theta^\alpha}{(1+|u|)^{1+m}} dr \\ & \quad + \frac{1}{4\varepsilon} \int_0^{2R} \rho(1+|u|)^{1+m}\theta^{\alpha-2}dr \end{aligned} \quad (3.14)$$

and again

$$\begin{aligned} \int_0^{2R} \rho(1+|u|)^{1+m}\theta^{\alpha-2}dr & \leq \varepsilon^2 \int_0^{2R} \rho(1+|u|)^s\theta^\alpha dr \\ & \quad + \frac{C}{\varepsilon^2} \int_0^{2R} \rho\theta^{\frac{\alpha(s-m-1)-2s}{s-m-1}} dr \\ & \leq C(\varepsilon^{-2} + \varepsilon^2 \int_0^{2R} \rho|u|^s\theta^\alpha dr), \end{aligned} \quad (3.15)$$

where C is a generic constant independent of ε . Here we used our choice of α .

Next we observe that $|t|^s \leq |t|^{s-1}t\phi(t)/\phi(1) + 1$ for all $t \in \mathbb{R}$. Using this in (3.15) and then using what one gets and (3.14) in (3.13), with the choice of a sufficiently small ε we finally obtain the desired inequalities.

□

Corollary 3.1. *For all $q \in (1, 2s/(s+1))$ and for every $0 < R$ there is a constant as in Lemma 3.2, such that*

$$\int_0^R \rho_n|u'_n|^q dr \leq C\left(1 + \int_0^{2R} r^{N_+-1}|f|dr\right). \quad (3.16)$$

Proof. By Hölder inequality we find

$$\int_0^R \rho|u'|^q dr \leq \left(\int_0^R \frac{\rho|u'|^2 dr}{(1+|u|)^{1+m}}\right)^{\frac{q}{2}} \left(\int_0^R \rho(1+|u|)^{\frac{q(1+m)}{2-q}} dr\right)^{\frac{2-q}{2}},$$

then by our choice of m in Lemma 3.2 it is possible to choose $q > 1$ such that $(m + 1)q/(2 - q) < s$ and then from Lemma 3.2 we obtain the result. With the adequate choice of m we can cover the range of q . \square

Proof of Theorem 3.1 (Existence). We consider the sequence of $\{u_n\}$ of solution found in Lemma 3.1 satisfying (3.9). In what follows we show that this sequence converges to a weak solution of (3.1).

Now, considering the estimates in Lemma 3.2, we see that the function $\rho_n u'_n$ has weak derivatives in any interval of the form (r_0, R_0) with $0 < r_0 < R_0$. Since the function ρ_n is differentiable a.e., we obtain then that u_n is twice differentiable a.e. and u''_n is in $L^1(r_0, R_0)$, because of the equation satisfied by u_n and estimates in Lemma 3.2. From here we conclude that u'_n and u_n are uniformly bounded in (r_0, R_0) . Using the equation again we conclude then that u''_n is bounded by an L^1 function in (r_0, R_0) , which implies that u'_n is equicontinuous. By the Arzela-Ascoli Theorem there exists a differentiable function u in the interval (r_0, R_0) such that, up to a subsequence, u_n and u'_n converges to u and u' respectively, in a uniform way in the interval (r_0, R_0) .

We may repeat this argument for any interval (r_0, R_0) , so that by a diagonal procedure, we can prove that up to a subsequence, $\{u_n\}$ and $\{u'_n\}$ converge point-wise to a differentiable function $u : (0, \infty) \rightarrow \mathbb{R}$. Notice that $\{\rho_n\}$ converges point-wise to $\rho(r) = \rho(r, u(r), u'(r))$.

Next we use the estimate (3.16), to prove that the sequence $\{\rho_n u'_n\}$ is equi-integrable in $[0, R]$ and then it converges in $L^1[0, R]$ to $\rho u'$, for all $R > 0$. It is only left to prove that $\{\tilde{\rho}_n |u_n|^s\}$ converges in $L^1[0, R]$. For this purpose we introduce, as in [2], a new function ϕ in \mathbb{R} defined as $\phi(\nu) = \min\{\nu - t, 1\}$ if $\nu \geq 0$ and extended as an odd function to all \mathbb{R} , for a parameter $t > 0$. Then we consider inequality (3.12) with the cut-off function $\phi(u_n)\theta$ to get

$$\int_{E_n^{t+1} \cap (0, R)} \tilde{\rho}_n |u_n|^s dr \leq \int_{E_n^t \cap (0, 2R)} \tilde{\rho}_n |f_n| dr + C \int_{E_n^t \cap (0, 2R)} \rho_n |u'_n| dr,$$

where $E_n^t = \{r > 0 / |u_n(r)| > t\}$. From (3.10) and (3.16) it follows that the second integral approaches zero if $t \rightarrow \infty$. From here the equi-integrability of $\rho_n |u_n|^s$ follows and we conclude.

Finally (3.7) is consequence of the integrability properties just proved for u_n that also hold for u . This finishes the proof. \square

Now we prove the remaining part of Theorem 3.1, that is uniqueness and non-negativity of weak solutions. For this purpose it would be natural to use comparison arguments, however those are a bit delicate in this case. In fact, in a natural way we may define the notion of weak

subsolutions (supersolution) by writing \leq and use only nonnegative test functions in (3.4). It happens that, if u is a weak subsolution and v is a weak supersolution, we cannot be sure that $w = u - v$ is a weak subsolution, since we do not have good control of $\rho w'$ at the origin.

We first consider the no-negativity of solutions of when f is nonnegative. For that purpose we need to find appropriate test functions.

Lemma 3.3. *If u is a solution of (3.1) in the weak sense and $f \leq 0$ a.e. in $[0, \infty)$ then $u \leq 0$ for all $r > 0$.*

Proof. As in the general case, we consider the function U given in the proof of Lemma 2.5, which satisfies (2.18) in B_R . On the other hand by the regularity of u given above, we have that $u(x) = u(r)$ satisfies equation (1.1) a.e. We may subtract the equations for U and u and get

$$-\mathcal{M}^+(D^2(u - U)) + |u|^{s-1}u - U^s \leq 0 \quad \text{a.e. in } B_R.$$

If we write $w = u - U$ then we see that

$$-(\rho w')' + \tilde{\rho}(|u|^{s-1}u - U^s) \leq 0 \quad \text{in } (0, R) \text{ a.e.} \quad (3.17)$$

Here the function ρ and $\tilde{\rho}$ are defined in the natural way with $\theta(r) = \theta(w'(r))$ and Θ given by

$$\Theta(r) = \theta(w''(r)) \quad \text{in } (0, R) \text{ a.e.}$$

We see that the function w is negative near R . If there exists $0 < r_1 < r_2 < R$ such that $w > 0$ in (r_1, r_2) and $w(r_1) = w(r_2) = 0$ then we may choose the function φ , defined as $\varphi = w$ in (r_1, r_2) and $\varphi \equiv 0$ elsewhere, as a test function in (3.17) to get

$$\int_{r_0}^R \rho |w'|^2 + \tilde{\rho}(|u|^{s-1}u - U^s)w dr \leq 0.$$

But each term in the left hand side is positive, then $w = 0$ in (r_1, r_2) .

Thus, either $w(r) \leq 0$ in $(0, R)$ or there is $r_0 \in (0, R)$ such that $w > 0$ in $(0, r_0)$ and $w(r_0) = 0$. To see that the second case is impossible we just need to prove that

$$\int_0^{r_0} \rho(w)|w'| dr < \infty \quad \text{and} \quad \lim_{r \rightarrow 0} (\rho(w)w')(r) = 0, \quad (3.18)$$

since in this case we may use the function φ , defined as $\varphi = w$ in (\bar{r}, r_0) and $\varphi \equiv w(\bar{r})$ in $(0, \bar{r})$, as a test function in (3.17) and get a contradiction.

Assuming (3.18) for the moment, we see that $u \leq U$ in $[0, R]$ and this is true for all $R > 0$. Taking limit as R goes to infinity, keeping r fixed, we conclude that $u \leq 0$ in $[0, \infty)$.

To complete the proof we show (3.18). To see this, we first observe that there is $\bar{r} \in (0, r_0)$ such that $w'(\bar{r}) < 0$ and then from inequality (3.17) we find that $w''(r) > 0$ a.e and $w'(r) < 0$ in $r \in (0, \bar{r})$. A posteriori we see that $w''(r) > 0$ a.e and $w'(r) < 0$ in $r \in (0, r_0)$ and consequently $\rho(w) = r^{N_+-1}$ there. Next we assume that u' is negative at some point in $(0, r_0)$, because otherwise the functions u and u' would be bounded and then w and w' are bounded, yielding (3.18). Since $u'' > U'' > 0$ in $(0, r_0)$ we see then that $u' < 0$ near the origin and consequently $\rho(u) = r^{N_+-1}$. Since (3.7) holds we see that (3.18) holds. \square

Proof of Theorem 1.1 (Uniqueness). Let u_1 and u_2 be two solutions of equation (3.1) in the weak sense, then they satisfies (1.1) a.e. in \mathbb{R}^N , with abuse of notation $u_i(x) = u_i(|x|)$, $i = 1, 2$. Then we define $w = u_1 - u_2$ and proceed as in the Proof of Theorem 1.1 to obtain that w satisfies (2.19) a.e. in \mathbb{R}^N . Now we follows the proof of Lemma 3.3. \square

Remark 3.1. Let us consider a continuous function f in $\mathbb{R}^N \setminus \{x_i / i = 1, \dots, k\}$, such that near each singularity

$$f(x) \sim \frac{c_i}{|x - x_i|^{\alpha_i}}, \quad x \sim x_i, \quad i = 1, \dots, k.$$

In order to apply Theorem 1.1 we need $\alpha_i < 1$ for all $i = 1, \dots, k$. In contrast, assuming that f is radially symmetric with a singularity at the origin of the form

$$f(r) \sim \frac{c}{r^\alpha}, \quad r \sim 0, r > 0,$$

in order to apply Theorem 3.1, we only need $\alpha < N_+$. We observe that if $\lambda/\Lambda \rightarrow 0$ then $N_+ \rightarrow 1$, while if $\lambda/\Lambda = 1$ then $N_+ = N$.

When we have a radial function f being in $L_{loc}^p(\mathbb{R}^N)$ with $p > N/N_+$ then f satisfies our hypothesis (3.5) and we may apply Theorem 3.1. This is particularly interesting if N and N_+ are close to each other.

Remark 3.2. Let f be a function in \mathbb{R}^N and define

$$g(r) = \max\{|f(x)| / |x| = r\}$$

and assume that g satisfies (3.5). This will be the case if f has a singularity of the form $r^{-\alpha}$ with $\alpha < N_+$.

Then we may construct a solution of (3.2). This solution is a 'candidate' for a supersolution for equation (1.1) with f as a right hand side. However, since the two notion of solutions are not compatible, this is not possible.

Remark 3.3. *In this section we have considered only the case of the Pucci operator \mathcal{M}^+ , however these results can be adapted for the operator \mathcal{M}^- as well.*

Acknowledgements: The second author was partially supported by Fondecyt Grant # 1070314 and FONDAP de Matemáticas Aplicadas and the third author was partially supported by Fondecyt Grant # 1040794 and Proyecto Interno USM # 12.05.24 . This research was partially supported by Ecos-Conicyt project C05E09.

REFERENCES

- [1] H. Brezis, Semilinear equations in \mathbb{R}^N without condition at infinity, Appl. Math. Optim. **12** (1984), p. 271-282.
- [2] L. Boccardo, T. Gallouet and J. L. Vazquez, Nonlinear Elliptic Equations in R^N without Growth Restrictions on the Data, Journal of Differential Equations, Volume **105**(2) (1993), p. 334-363.
- [3] L. Boccardo, T. Gallouet and J. L. Vazquez, Solutions of nonlinear parabolic equations without growth restrictions on the data, Electronic Journal of Differential Equations, (60) (2001).
- [4] L. Caffarelli, M. Crandall, M. Kocan and A. Swiech, On viscosity solutions of fully nonlinear equations with measurable ingredients, Communications on Pure and Applied Mathematics **49**(4) (1996), p. 365-398.
- [5] X. Cabré, L.A. Caffarelli, Fully Nonlinear Elliptic Equation, American Mathematical Society, Colloquium Publication, **43** (1995).
- [6] L.A. Caffarelli, Interior a Priori Estimates for Solutions of Fully Non-Linear Equations. The Annals of Mathematics, 2nd Ser., **130**(1) (1989), p. 189-213
- [7] M. Crandall, H. Ishii and P.L. Lions. User's guide to viscosity solutions of second order partial differential equations. Bulletin of the AMS, **27**(1) (1992), p. 1-67.
- [8] F. Da Lio, B. Sirakov, Symmetry results for viscosity solutions of fully nonlinear uniformly elliptic equations. J. Eur. Math. Soc. (JEMS) **9**(2) (2007), p. 317-330.
- [9] M. Del Pino, R. Letelier, The influence of domain geometry in boundary blow-up elliptic problems. Nonlinear Analysis: Theory, Methods & Applications, **48**(6) (2002), p. 897-904.
- [10] G. Diaz, J.I. Diaz, Uniqueness of the boundary behavior for large solutions to a degenerate elliptic equation involving the ∞ -Laplacian. Rev. R. Acad. Cien. Serrie A Mat. **97**(3) (2003), p. 455-460.
- [11] G. Díaz, R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, Nonlinear Analysis: Theory, Methods & Applications, **20**(2) (1993), p. 97-125.
- [12] D. Gilbarg, N. Trudinger, *Elliptic partial differential equation of second order*, 2nd ed., Springer-verlag 1983.
- [13] T. Kato, Schrödinger operators with singular potentials, Israel J. Math., **13** (1972), p. 135-148
- [14] J.B. Keller, On solutions of $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), p. 503-510.

- [15] V. Kondrat'ev, V. Nikishkin, Asymptotics, near the boundary of a solution of a singular boundary value problems for semilinear elliptic equations, *Differential Equations* **26** (1990), p. 345-348.
- [16] D. Labutin, Removable singularities for Fully Nonlinear Elliptic Equations, *Arch. Rational Mech. Anal.* **155** (2000) 201-214.
- [17] F. Leoni, Nonlinear elliptic equations in R^N with "absorbing" zero order terms, *Adv. Differential Equations*, **5** (2000), p. 681-722.
- [18] C. Loewner, L. Nirenberg, Partial differential equations invariant under conformal projective transformations, in *Contributions to Analysis (a collection of papers dedicated to Lipman Bers)*, Academic Press, New York, 1974, p. 245-272.
- [19] M. Marcus, L. Veron, Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, *Ann. Inst. H. Poincaré* **14** (1997), p. 237-274.
- [20] R. Osserman, On the inequality $\Delta u \geq f(u)$. *Pacific J. Math.* **7** (1957), p. 1641-1647.
- [21] C. Pucci, Maximum and minimum first eigenvalue for a class of elliptic operators, *Proc. Amer. Math. Soc.* **17** (1966), p. 788-795.

M.J. ESTEBAN, CEREMADE UMR CNRS 7534, UNIVERSITÉ PARIS DAUPHINE,
75775 PARIS CEDEX 16, FRANCE

P. FELMER, DEPARTAMENTO DE INGENIERÍA MATEMÁTICA, AND CENTRO DE
MODELAMIENTO MATEMÁTICO, UMR2071 CNRS-UCHILE, UNIVERSIDAD DE
CHILE, CASILLA 170 CORREO 3, SANTIAGO, CHILE.

A. QUAAS, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDAD TÉCNICA FED-
ERICO SANTA MARÍA CASILLA: V-110, AVDA. ESPAÑA 1680, VALPARAÍSO,
CHILE.