Integrating short turning and deadheading in the optimization of transit services

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ABSTRACT

Urban transit demand exhibits peaks in time and space, which can be efficiently served by means of different fleets, increasing frequencies in those groups of stops with larger passenger inflow. In this paper we develop a model that combines short turning and deadheading in an integrated strategy for a single transit line, where the optimization variables are both of a continuous and discrete nature: frequencies within and outside the high demand zone, vehicle capacities, and those stations where the strategy begins and ends. We show that closed solutions can be obtained for frequencies in some cases, which resembles the classical "square root rule". Unlike the existing literature that compares different strategies with a given normal operation (no strategy – single frequency), we use an optimized base case, in order to assess the potential benefits of the integrated strategy on a fair basis. We found that the integrated strategy can be justified in many cases with mixed load patterns, where unbalances within *and* between directions are observed. In general, the short turning strategy may yield large benefits in terms of total cost reductions, while low benefits are associated with deadheading, due to the extra cost of running empty vehicles in some sections.

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1. Introduction

In urban systems, the demand for public transport generally presents different shapes in space and time. This poses different design problems in order to provide a reasonable level of service to users. A major difficulty is how to deal with both spatial and temporal peak periods of demand in their daily operation, which results in inefficient schemes when the unbalances are improperly treated by offering the same bus supply along the entire route and during a long period. If we focus the analysis on the spatial dimension, the specialized literature shows several strategies to better assign the available fleet by increasing the frequency on the most demanded route segments in order to adjust the demand to the effective capacity of buses.

Among the spatial-type of fleet assignment strategies, the most relevant operational schemes proposed in recent years are short turning (Furth, 1987; Delle Site and Filippi, 1998; Ceder, 1989, 2003b), deadheading (Furth, 1985; Eberlein et al., 1998, 1999; Fu et al., 2003) and expressing (Jordan and Turnquist, 1979; Furth, 1986; Eberlein et al., 1999). Short turning consists in selecting a portion of the fleet to serve short cycles on those route sections exhibiting high demand; deadheading consists in increasing the frequency in the most demanded direction by allowing some of the buses to skip the stops in the low demand direction; and express services operate just stopping at a subset of the normal service stops. Studies and applications of such strategies suggest that the concentration of demand on one direction would motivate the use of deadheading, while

the concentration of journeys over a specific sector of the route probably would require the implementation of a short turn. There are cases, however, in which the demand configuration indeed presents unbalances but does not suggest *a priori* the type of strategy to follow in order to optimize the use of the fleet. In some portions of the route a short turning strategy could be useful, while in the next segment either expressing or deadheading might be better. In these cases a mixed or integrated strategy is worth exploring.

Let us consider a linear bus corridor. Particularly interesting are those cases where in one direction the demand is larger than in the other and, besides, local demand concentrations in different sectors of the route are observed. Therefore, if these mixed patterns appeared it would not be clear which strategy is better in order to efficiently distribute the bus supply along the entire corridor. Our research group has analyzed both strategies individually, namely short turning and deadheading, based on a microeconomic approach to model public transport systems, accounting for all costs involved, users' and operators', and considering station-to-station demand information (Tirachini and Cortés, 2006; Tirachini et al., 2011). Under the same microeconomic principles, in this paper we develop a new model that integrates short turning and deadheading for a linear bus line and a single period of operation, where the variables to be optimized are the frequencies within and outside the high demand zone and vehicle sizes (continuous variables), and the stations where the strategy begins and ends (discrete variables). In the proposed model, we clearly identify all components of both users and operators costs – whose sum is minimized – to investigate how the integrated strategy affects both parties.

Unlike the existing literature that compares different strategies with a given normal operation or base case (no strategy is applied, resulting in a single frequency along the entire route), in our model the normal operation scheme and the integrated strategy are both optimized, i.e. aimed at minimizing total costs, to perform the comparison on a fair basis. The new model – which is formulated using spatially disaggregated demand information – allows us to better characterize the effects of the integrated strategy on the different agents of the system (users differentiated by origin–destination and operators). This permits the identification at a tactical level of those demand patterns that are good candidates for the combined strategy. Under some assumptions, optimal frequencies in cases with and without the strategy are analytically obtained, which permits the study of the influence of the parameters involved in the final solution. In other cases, results are obtained numerically.

The remainder of this section contains a brief review of the literature regarding the most interesting formulations and studies related to deadheading, expressing and short turning, within an operation planning context. The integrated strategy is formally defined in Section 2, together with the analytical formulation of the problem, highlighting the conditions under which the hybrid scheme recovers the basic strategies, namely deadheading and short turning. In Section 3 some numerical examples are shown and Section 4 concludes.

Deadheading as a pre-planned strategy has been studied by a limited number of authors. Furth (1985) proposes to apply this concept to corridors with unbalanced demand between directions (in which deadheaded buses skip the entire low demand direction). According to his findings, the strategy reduces both the operator cost – by means of savings in fleet size – and the user cost – by reducing the waiting time of passengers. Three objective functions are explored: the minimization of the fleet size by imposing a maximum headway condition into the formulation; the minimization of waiting times for a fixed fleet size; and the sum of operator and users costs. Ceder and Stern (1981) and Ceder (2003a, 2004) consider the insertion of an empty trip between two terminals (*deadheading trip*) in order to reduce the fleet size subject to satisfying a given schedule of bus departures from the terminals. Eberlein et al. (1998, 1999) and Fu et al. (2003) consider a different problem: deadheaded vehicles start their trip empty until a station to be determined and, from that stop, these buses start their normal service until the end of their route. Under this scheme, vehicles save time and reduce the headway with respect to the bus ahead. Both works formulate mixed integer non-linear problems, quadratic in the continuous variables (headway between vehicles). Eberlein et al. (1998) only accounts for the waiting time in the objective function, obtaining a very complex formulation. They then resolve a simplified version of their original problem. On the other hand, Fu et al. (2003) assume that if a vehicle is going to skip stations, necessarily the next bus has to serve the entire route, which simplifies the problem and provides a minimum level of service for passengers waiting at skipped stations. The authors minimize the costs associated with both waiting an in-vehicle travel time. We also find interesting studies of expressing (Jordan and Turnquist, 1979; Furth, 1986; Leiva et al., 2010), in which a public transport corridor is divided into sectors, where some vehicles serve selected segments or bus stops while others serve the entire corridor.

Short turning has been studied by many authors using different assumptions and solution methodologies. Furth (1987) focuses on the efficient use of short turning, constraining the frequency of vehicles performing the short turn (hereafter, fleet B) to be a multiple *n* of the frequency of those vehicles serving the entire route (hereafter, fleet A), which is equivalent to assume a scheduled operation scheme; the parameter *n* is known as the "scheduling mode". The author formulates three optimization problems, pursuing the minimization of the fleet size (through the maximization of vehicle headway), the minimization of passenger waiting times for a fixed fleet size, and a combination of both. The optimization considers as exogenous the number of short turns, the limit stations for the short turn and the bus capacities. By using an aggregated demand model, Ceder (1989, 2003b) proposes a two-stage optimization approach. In a first stage, the fleet size is minimized for a given demand level while in a second stage, the number of trips using the short turn is minimized to reduce the effect of the strategy on passenger times. Vijayaraghavan and Anantharmaiah (1995) propose the insertion of express services as well as short turns for the service of some trips pursuing the reduction of fleet size. The authors report benefits when applying their approach in terms of fleet and crew utilization and users' performance in terms of passenger travel times. Delle Site and Filippi (1998) develop a multi-period short turning using both elastic and inelastic demand and extending the work by Furth (1987) to the case of Poisson bus arrivals. They formulate a maximization of social benefits (elastic demand case), and a

minimization of user plus operator costs (inelastic demand case). The decision variables are frequency, fare and limit stations of the short turn (start and end) plus the capacity of vehicles (both treated parametrically). They show with an example that the strategy turns out to be beneficial only if demand patterns exhibit pronounced peaks.

We can finally mention some efforts to formulate problems combining several strategies, like the works by Eberlein (1995) and Eberlein et al. (1999), who formulate a real-time control model encompassing holding, deadheading and expressing, aimed at solving problems such as bunching or service disruptions, i.e. as a tool to make real-time *operational* decisions according to the conditions of the system. Bus holding seems to be the best strategy. With a similar real-time purpose, Fu et al. (2003) also find through simulation that the applications of a combined version of holding and deadheading turns out to be more effective than the application of each strategy separately, in terms of reduction of waiting and in-vehicle times. Note that this research is not aimed at an *ex-ante* design of the system, which is the objective of ours.

2. Characterization, modeling and optimization of the integrated strategy

Fig. 1 below will help illustrate the elements of the integrated strategy. It shows a typical load profile of a real public transport linear corridor in Santiago, Chile, during a morning peak hour, in both directions 1 (East–West) and 2 (West–East), comprising 24 stations (E1–E24).

In direction 1 we could establish that the load is concentrated between stations E4 and E19, while in direction 2 the inflexion points could be stations E21 and E11. Therefore, one can envision a possibly suitable integrated strategy that could consider a fleet serving the entire corridor from terminal to terminal (hereafter fleet A) as a traditional service, while another group of buses (hereafter fleet B) starts serving at station E4 and finishes at station E19 in direction 1. When these buses operate in direction 2, they run without passengers as fast as feasible from stations E19 to E21 (deadheading), where they turn and start serving passengers again until E11 in direction 2. From E11, fleet B-buses run again with no passengers (deadheading) moving as fast as feasible until station E4 to start a new short cycle. We will call this an *integrated deadheading-short-turning strategy* (IDS).

Let us define the elements of the IDS in a general case of a single linear transit line. Following Fig. 2, the system contains N stations in one direction (N–1 segments). Direction 1 refers to vehicles moving from stations 1 to N and direction 2 when they move from stations N to 1. Frequencies f_A and f_B are associated with fleets A and B respectively; the former serves the whole route while the latter serves just the section where the strategy is applied. All vehicles have the same size K. Under normal operation (case without strategy of any kind) the single frequency of the fleet is denoted as f (not in the figure). To implement such a strategy we have to optimally choose four relevant stations, corresponding to the points where fleet B starts and ends executing IDS. Let us call these stations s_0, s_1, s_2 and s_3. Stations s_0 and s_1 indicate the start and end of the fleet B service in direction 1. The service starts again in station s_3 until s_2 in the other direction. Note that in the example of Fig. 1 we assumed these stations to be E4, E19, E11 and E21, respectively.

We develop a model that minimizes total cost for two situations: normal operation, where all vehicles operate along the whole route, and operation with IDS. The model is used to measure potential benefits of the integrated scheme under several demand configurations. Note that the election of the four limit stations is not straightforward. For example, the load profile
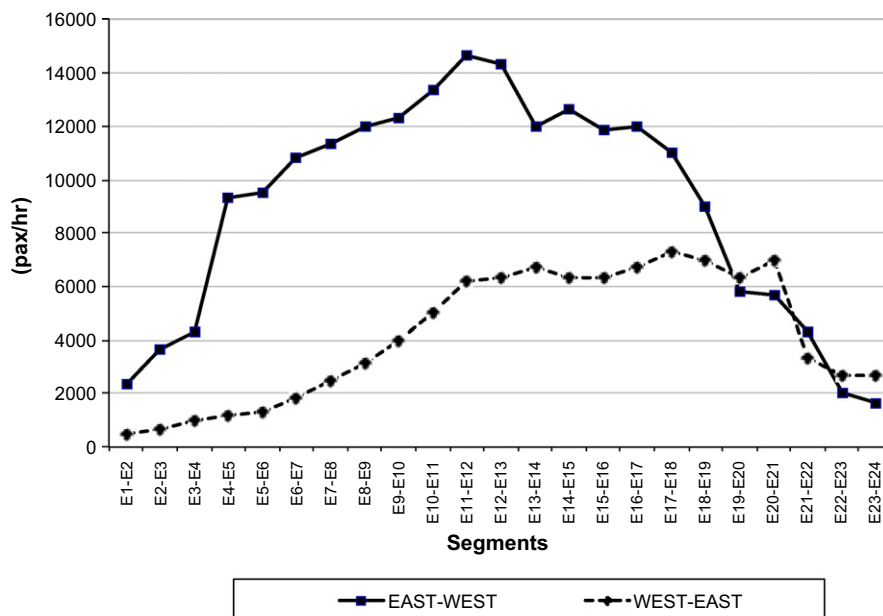

Fig. 1. Load profile of a corridor with spatial concentration of the demand.
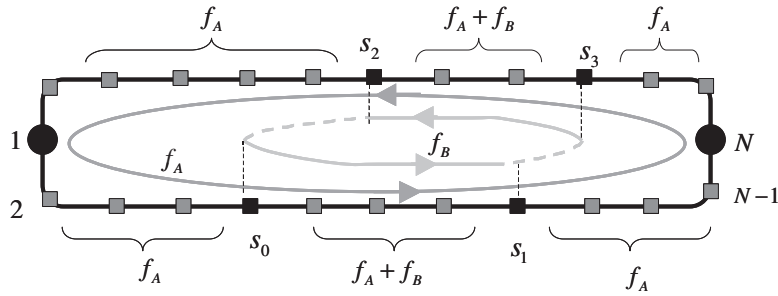
Fig. 2. Operational scheme of Integrated Strategy deadheading-short-turning (IDS).

in Fig. 1 suggests various candidate stations for implementing the IDS strategy. Indeed, note that in the zone around station E11 the difference in load between stations is quite small. Therefore, we postulate that the limit stations are decision variables of our model, subject to potential physical/operational constraints that limit the choice set. In summary, the optimization variables of the minimization problem are s_0, s_1, s_2, s_3, f_A, f_B, f and K. The first four are discrete while the last four are continuous.

The "pure" strategies deadheading and short turning are particular cases of the IDS strategy due to the definition of the four limit stations. Implementing deadheading requires defining only one limit station, s_0, which corresponds to the point where the service starts again after a portion of the route where deadheaded buses run without passengers. On the other hand, the definition of short turning requires obtaining two limit stations, which are the start and end of the short loop, operated by buses belonging to fleet B in our nomenclature. Therefore, if $s_2 = s_3$ and $s_1 = N$, IDS becomes deadheading and s_0 is the only station that remains as a decision variable, and if $s_0 = s_2$ and $s_1 = s_3$, IDS becomes short turning. In any case, the limit stations s_0 for deadheading and s_1 and s_2 for short turning are treated as discrete variables. Then, the formulation and mathematical solution of a general microeconomic model for the IDS strategy will allow us to study the convenience of implementing any of the three strategies involved: IDS, short turning or deadheading.

In the formulation the known parameters are L, length of the corridor [km]; R_k, bus running time under normal service between stations k and $k + 1$ including acceleration and deceleration at bus stops [min]; b, marginal passenger boarding time [seg/pax]; and λ_{kl}, the trip rate between stations k and l [pax/hour]. This disaggregated demand is assumed fixed (steady state) over the studied period, defining a trip matrix.

In addition, the following functions utilized in our formulations are defined:

- Passenger boarding rate at station k, whose destination is among stations l_1 and l_2 inclusive [pax/hour]: $\lambda_k^+(l_1, l_2) = \sum_{l=l_1}^{l_2} \lambda_{kl}$.
- Passenger alighting rate at station k, whose origin is among stations l_1 and l_2 inclusive [pax/hour]: $\lambda_k^-(l_1, l_2) = \sum_{l=l_1}^{l_2} \lambda_{lk}$.

From these functions, we can establish the following quantities:

- Passenger boarding rate at station k, direction 1: $\lambda_k^{1+} \equiv \lambda_k^+(k+1, N) = \sum_{l=k+1}^{N} \lambda_{kl}$.
- Passenger alighting rate at station k, direction 1: $\lambda_k^{1-} \equiv \lambda_k^-(1, k-1) = \sum_{l=1}^{k-1} \lambda_{lk}$.
- Passenger boarding rate at station k, direction 2: $\lambda_k^{2+} \equiv \lambda_k^+(1, k-1) = \sum_{l=1}^{k-1} \lambda_{kl}$.
- Passenger alighting rate at station k, direction 2: $\lambda_k^{2-} = \lambda_k^-(k+1, N) = \sum_{l=k+1}^{N} \lambda_{lk}$.

We assume that at stations boarding and alighting process are simultaneous and that boarding dominates alighting (i.e. the boarding process is slower than the alighting process); therefore, total dwell time at stops is considered through the boarding parameter b.

Now we can formally write the cost functions associated with both operational schemes: with and without the implementation of the IDS strategy. The total cost of running a bus system has two components: one accounting for the operator cost (buses) and another reflecting the user costs through access, waiting and in-vehicle time values.[1] The implementation of the IDS strategy should have an effect on all components with respect to the normal case: on the operator cost through a better use of the fleet due to the flexibility of the strategy; on the user cost through larger frequency on high demand sectors and less dwell times for passengers on board since supply and demand are better adjusted. After finding the cost components we will obtain expressions for the optimal frequencies, which in the case of the IDS strategy will depend parametrically on the stations that define the boundaries of the strategy. This will be the basis for a two-stage process in order to find overall optimal frequencies, optimal vehicle sizes and optimal limit stations.

The analytical expression for the waiting time depends on the vehicle and passenger arrival processes. We assume that passengers arrive at stations uniformly at a fixed rate, which is a reasonable assumption in cases of high frequency services.

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[1] In our models the distance between stops is fixed, therefore access time is not considered in the optimization process.

Buses are assumed to arrive either Poisson or regularly spaced; the average waiting time at each station is equal to the entire headway in the former case and to half of the headway in the second. This can be handled generally using an auxiliary binary variable x that is equal to 1 in the Poisson case and 0 in the regularly spaced case. The waiting time cost (C_w) is the total waiting time multiplied by the waiting time value (P_w). Recalling that the headway is the inverse of frequency and that total ridership is the sum of the demand in all stations, C_w for the normal operation is:

$$C_w = P_w \frac{1+x}{2} \left\{ \sum_{k=1}^{N} \frac{\lambda_k^{1+}}{f} + \sum_{k=1}^{N} \frac{\lambda_k^{2+}}{f} \right\} \tag{1}$$

When the IDS strategy is applied, users observe different frequencies depending on their position along the route. Passengers traveling between stations s_0 and s_1 in direction 1, or between s_2 and s_3 in direction 2 observe a frequency $f_A + f_B$, whereas the other passengers only board vehicles that belong to fleet A (i.e., we assume that users do not transfer between vehicles A and B). Then, the waiting time cost after applying the IDS strategy, namely C_w^{ds}, can be calculated as

$$C_w^{ds} = P_w \frac{1+x}{2} \left\{ \sum_{k=1}^{s_0-1} \frac{\lambda_k^{1+}}{f_A} + \sum_{k=s_0}^{s_1-1} \left(\frac{\lambda_k^+(k+1,s_1)}{f_A+f_B} + \frac{\lambda_k^+(s_1+1,N)}{f_A} \right) + \sum_{k=s_1}^{N} \frac{\lambda_k^{1+}}{f_A} + \sum_{k=s_3+1}^{N} \frac{\lambda_k^{2+}}{f_A} \right.$$
$$\left. + \sum_{k=s_2+1}^{s_3} \left(\frac{\lambda_k^+(1,s_2-1)}{f_A} + \frac{\lambda_k^+(s_2,k-1)}{f_A+f_B} \right) + \sum_{k=1}^{s_2} \frac{\lambda_k^{2+}}{f_A} \right\} \tag{2}$$

Regarding the in-vehicle time cost (C_v) for the normal operation, the travel time t_{kl} between stations k and l (not necessarily consecutive) is calculated as

$$\begin{cases} \sum_{i=k}^{l-1} \left(R_i + b \frac{\lambda_i^{1+}}{f} \right) & \text{if } k < l \\ \sum_{i=l+1}^{k} \left(R_{i-1} + b \frac{\lambda_i^{2+}}{f} \right) & \text{if } l < k \end{cases} \tag{3}$$

Multiplying expression (3) by the demand rate between k and l, λ_{kl}, and adding over all OD pairs, the total in-vehicle time cost C_v can be obtained as follows:

$$C_v = P_v \left\{ \sum_{k=1}^{N} \sum_{l=k+1}^{N} \left[\sum_{i=k}^{l-1} \left(R_i + b \frac{\lambda_i^{1+}}{f} \right) \right] \lambda_{kl} + \sum_{k=1}^{N} \sum_{l=1}^{k-1} \left[\sum_{i=l+1}^{k} \left(R_{i-1} + b \frac{\lambda_i^{2+}}{f} \right) \right] \lambda_{kl} \right\} \tag{4}$$

where P_v represents the in-vehicle time value. The analytical formulation for C_v when the IDS strategy is applied turns out very complicated, since a trip may encompass sections where users board vehicles at rates λ_i^+/f_A or $\lambda_i^+/(f_A+f_B)$, depending on the case. Synthetically, the in-vehicle time cost for the case with strategy C_v^{ds} can be expressed as:

$$C_v^{ds} = P_v \left\{ 2 \sum_{k=1}^{N-1} R_k + b \frac{g_5(\vec{s})}{f_A+f_B} + b \frac{g_6(\vec{s})}{f_A} \right\} \tag{5}$$

The set of functions $g_i(\vec{s})$ are defined and explained in Appendix A. They are second order expressions of the trip rates λ_{kl} specified in terms of the limit stations $\vec{s} = (s_0, s_1, s_2, s_3)$.

Regarding the operator cost C_o, we consider two components (temporal and spatial). The former includes items such as personnel costs (crew), while the latter comprises running costs, such as fuel consumption, lubricants, tires, and maintenance. Following Jansson (1980) and Oldfield and Bly (1988), the unit operator costs are expressed as a linear function of the vehicle capacity K, where $c(K)$ is the vehicle–hour cost (expressed in \$/vh) and $c'(K)$ corresponds to the vehicle–kilometer cost (expressed in \$/vkm). Analytically

$$c(K) = c_0 + c_1 K \quad c'(K) = c_0' + c_1' K \tag{6}$$

$$C_o = c(K)F + c'(K)vF \tag{7}$$

where v is the bus system commercial speed (including dwell times) and F represents the fleet size. F is computed as the product of the frequency and the cycle time t_c (considering not only the running time but also the time spent in transferring passengers at stations), that is $F = ft_c$. Thus, (7) can be rewritten as a function of t_c, f and L (length of the corridor) as shown in (8) and (9) for the case of normal operation

$$C_o = c(K)ft_c + 2c'(K)fL \tag{8}$$

$$C_o = f \left\{ c(K) \left[\sum_{k=1}^{N-1} \left(R_k + b \frac{\lambda_k^{1+}}{f} \right) + \sum_{k=2}^{N} \left(R_{k-1} + b \frac{\lambda_k^{2+}}{f} \right) \right] + 2c'(K)L \right\} \tag{9}$$

The operator cost after applying the IDS strategy is analyzed separately for fleets A and B. In the case of fleet A, we have to consider two groups of passengers, boarding the buses at rates either λ_i^+/f_A or $\lambda_i^+/(f_A+f_B)$ depending on the case, which af-

fects the cycle time. Moreover, we have to pay special attention to the cost of vehicles belonging to fleet B that are deadhead-ed in certain sectors, as in general the running costs are a function of both the running speed and the vehicle detentions, which might occur if vehicles accept alighting. Therefore, the second equation in (6) becomes

$$c'_d(K) = c'_{0d} + c'_{1d}K \tag{10}$$

Finally, the operator cost is the sum of the expenditures associated with each fleet A and B. As fleet A operates normally, the associated cost function C_{oA}^{ds} is similar to (9), but recognizing the influence of fleet B in the cycle time by means of the number of passengers who can choose to take either a bus A or B. Analytically

$$C_{oA}^{ds} = f_A \left\{ c(K) \left[\sum_{k=1}^{s_0-1} \left(R_k + b\frac{\lambda_k^{1+}}{f_A} \right) + \sum_{k=s_0}^{s_1-1} \left(R_k + b\left\{ \frac{\lambda_k^+(k+1,s_1)}{f_A+f_B} + \frac{\lambda_k^+(s_1+1,N)}{f_A} \right\} \right) + \sum_{k=s_1}^{N-1} \left(R_k + b\frac{\lambda_k^{1+}}{f_A} \right) \right. \right.$$
$$\left. \left. + \sum_{k=s_3+1}^{N} \left(R_{k-1} + b\frac{\lambda_k^{2+}}{f_A} \right) + \sum_{k=s_2+1}^{s_3} \left(R_{k-1} + b\left\{ \frac{\lambda_k^+(1,s_2-1)}{f_A} + \frac{\lambda_k^+(s_2,k-1)}{f_A+f_B} \right\} \right) + \sum_{k=2}^{s_2} \left(R_{k-1} + b\frac{\lambda_k^{2+}}{f_A} \right) \right] + 2c'(K)L \right\} \tag{11}$$

Moreover, note that the buses of type B are deadheaded between stations s_1 and s_3 in direction 1, and between stations s_2 and s_0 in direction 2, while they operate normally between stations s_0 and s_1 in direction 1, and between stations s_3 and s_2 in direction 2. As the running costs are different under normal service and deadheading (expressions (6) and (10)), the oper-ational cost associated with fleet B, namely C_{oB}^{ds}, must differentiate the operation per kilometer in these two cases, taking into account the fraction of the cycle time t_{cB} spent by B-vehicles operating under each scheme. Besides, if we assume that the N stations are equally spaced in both directions, the fractions of cycle time that the B-buses are deadheaded and run under normal operation (considering both directions) will be, respectively

$$\frac{|s_2 - s_0|}{N-1} + \frac{|s_3 - s_1|}{N-1} \quad \text{and} \quad \frac{s_1 - s_0}{N-1} + \frac{s_3 - s_2}{N-1} \tag{12}$$

Then

$$C_{oB}^{ds} = f_B \left\{ c(K) \left[\sum_{k=min(s_0,s_2)}^{max(s_0,s_2)-1} R_k + \sum_{k=s_0}^{s_1-1} \left(R_k + b\frac{\lambda_k^+(k+1,s_1)}{f_A+f_B} \right) + \sum_{k=min(s_1,s_3)}^{max(s_1,s_3)-1} R_k + \sum_{k=s_2+1}^{s_3} \left(R_{k-1} + b\frac{\lambda_k^+(s_2,k-1)}{f_A+f_B} \right) \right] \right.$$
$$\left. + \left[C'(K)\frac{s_1 - s_0 + s_3 - s_2}{N-1} + c'(K)\frac{|s_2 - s_0| + |s_3 - s_1|}{N-1} \right]L \right\} \tag{13}$$

Finally, the total operator cost with the IDS strategy, C_o^{ds}, is the sum $C_{oA}^{ds} + C_{oB}^{ds}$ of (11) and (13).

With the detailed expressions for waiting time cost, in-vehicle time cost and operator cost, we can now minimize total cost under both the normal and IDS regimes. First we should note that vehicle capacities are determined by the largest flow across the line segments, namely q_{max} (obtained from the OD matrix), the frequency and the design maximum vehicle occu-pancy rate η (for example, $\eta = 0.85$), whose purpose is to keep extra capacity for absorbing the intrinsic randomness of the demand (not explicitly considered in the model). Thus, under normal operation the capacity can be obtained as $K = q_{max}/\eta f$.

For the normal operation case, after introducing the expression of K in (6), the optimal value of frequency f is obtained applying first order conditions (FOC), which yields

$$f^* = \sqrt{\frac{P_w\frac{1+x}{2}\left(\sum_{k=1}^{N} \lambda_k^{1+} + \sum_{k=1}^{N} \lambda_k^{2+} \right) + P_v b\left(\sum_{k=1}^{N} \sum_{l=k+1}^{N} \lambda_{kl} \sum_{i=k}^{l-1} \lambda_i^{1+} + \sum_{k=1}^{N} \sum_{l=1}^{k-1} \lambda_{kl} \sum_{i=l+1}^{k} \lambda_i^{2+} \right) + c_1\frac{q_{max}}{\eta}b\left(\sum_{k=1}^{N} \lambda_k^{1+} + \sum_{k=1}^{N} \lambda_k^{2+} \right)}{2\left(c_0 \sum_{k=1}^{N-1} R_k + c'_0 L \right)}} \tag{14}$$

Expression (14) corresponds to an expansion of the classical "square root formula" (Mohring, 1972; Jansson, 1980; Jara-Díaz and Gschwender, 2003), which now considers detailed, station-to-station demand (Jara-Diaz et al., 2008).

Let us analyze now the total cost for the IDS strategy, which can be synthesized as

$$C_t^{ds}\left(f_A, f_B, K\,\vec{s} \right) = f_A \left\{ c(K) \left(g_0 + b\frac{g_1(\vec{s})}{f_A+f_B} + b\frac{g_2(\vec{s})}{f_A} \right) + 2c'(K)L \right\}$$
$$+ f_B \left\{ c(K)\left(g_3(\vec{s}) + b\frac{g_1(\vec{s})}{f_A+f_B} \right) + \left[c'(K)\frac{s_1 - s_0 + s_3 - s_2}{N-1} + c'_d(K)\frac{|s_2 - s_0| + s_3 - s_1}{N-1} \right]L \right\} + P_w$$
$$\times \frac{1+x}{2}\left(\frac{g_1(\vec{s})}{f_A+f_B} + \frac{g_2(\vec{s})}{f_A} \right) P_v\left(g_0 + b\frac{g_5(\vec{s})}{f_A+f_B} + b\frac{g_6(\vec{s})}{f_A} \right) \tag{15}$$

The complex expression of C_t^{ds} can be visualized for a given set of limit stations as in the example of Fig. 3. From the figure and Eq. (15), it is clear that the function is undefined at $f_A = 0$ as the user cost for those passengers not favored with the strategy becomes infinite (there is no bus service for them). Besides, we observe that for relatively small values of the frequencies (f_A and f_B between 0 and 1 veh/min), the total cost function is decreasing, while for values close to 3 veh/min the total cost increases, concluding that there must be values of f_A^* and f_B^* within the drawn range, for which the IDS strategy reaches its minimum cost.

For the IDS strategy the expression for the maximum load K is not as straightforward as in the normal case. Using recursively the boarding and alighting rates at stations λ_k^+ and λ_k^-, it is possible to obtain the load of the buses along their route for each segment considering the application of the strategy. From that, the capacity can be set as the maximum load, which is of the form

$$K = \frac{1}{\eta}\left(\frac{\sigma_0(\vec{s})}{f_A} + \frac{\sigma_1(\vec{s})}{f_A + f_B}\right) \tag{16}$$

where $\sigma_0(\vec{s})$ and $\sigma_1(\vec{s})$ represent the terms that yield the maximum load, which depends on the selection of stations s_0, s_1, s_2 and s_3 (see Appendix B for details). Let us analyze two bus arrival schemes.

2.1. Bus arrivals distributed Poisson

The first case we analyze is when the bus arrivals are distributed Poisson, for both the normal case and the IDS strategy. In the latter case, the assumption applies for all segments, i.e. those with and without the strategy (operating either normally or *deadheading*). This is equivalent to set $x = 1$ in expressions (14) and (15).

The problem is solved in two-stages. First parametric in the value of $\vec{s} = (s_0, s_1, s_2, s_3)$. In such a case, from the first order conditions we do not obtain closed expressions for the continuous variables f_A^* and f_B^* (since K was already set as in (16)) and the problem must be solved numerically. However, the particular case when the operator cost does not depend on the capacity K, that is $c(K) = c$, $c'(K) = c_0'$ and $c_d'(K) = c_d'$, yields analytical values for the optimal frequencies, conditional to the limit stations $\vec{s} = (s_0, s_1, s_2, s_3)$. Analytically

$$f_A^*(\vec{s}) = \sqrt{\frac{P_w g_2(\vec{s}) + P_v b g_6(\vec{s})}{c(g_0 - g_3(\vec{s})) + c'\left(2 - \frac{s_1 - s_0 + s_3 - s_2}{N-1}\right)L - c_d'\frac{|s_2 - s_0| + |s_3 - s_1|}{N-1}L}} \tag{17}$$

$$f_B^*(\vec{s}) = \sqrt{\frac{P_w g_1(\vec{s}) + P_v b g_5(\vec{s})}{c(g_3(\vec{s}) + c'\frac{s_1 - s_0 + s_3 - s_2}{N-1}L + c_d'\frac{|s_2 - s_0| + |s_3 - s_1|}{N-1}L}} - \sqrt{\frac{P_w g_2(\vec{s}) + P_v b g_6(\vec{s})}{c(g_0 - g_3(\vec{s})) + c'\left(2 - \frac{s_1 - s_0 + s_3 - s_2}{N-1}\right)L - c_d'\frac{|s_2 - s_0| + |s_3 - s_1|}{N-1}L}} \tag{18}$$

From a second order analysis, it can be proven that the values in (17) and (18) correspond to a global minimum of $C_t^{ds}(f_A, f_B, \vec{s})$, $\forall (f_A, f_B) \in R_+^2$.

The second stage is to evaluate the objective function $C_t^{ds}(f_A^*(\vec{s}), f_B^*(\vec{s}), \vec{s}) \equiv \bar{C}_t^{ds}(\vec{s})$ for all possible combinations and select the set $\vec{s}^* = (s_0^*, s_1^*, s_2^*, s_3^*)$ such that $\vec{s} = \arg\min \bar{C}_t^{ds}(\vec{s})$ Frequency f_A^* in (17) resembles the optimal frequency under normal operation (14) and corresponds to the optimal frequency for serving the stations outside the operation zone of fleet B,
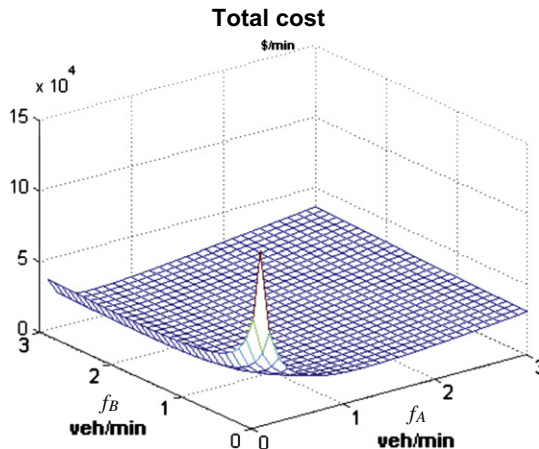

Fig. 3. Total cost function for a given IDS strategy.

i.e., as if the segment between $min\{s_0, s_2\}$ and $max\{s_1, s_3\}$ did not exist for the calculation of the operator cost. On the contrary, the optimal frequency of the fleet B, which serves only the high demand area (between s_0 and s_1 in direction 1, between s_2 and s_3 in direction 2), is computed as the difference between the optimal frequency to serve the high demand area only (first term of 18) and the frequency already provided by the entire route services, f_A^* (second term of 18).

By observing the expression for $f_B^*(\vec{s})$, we note that if the demand favored by the strategy were small with respect to the rest of the trips, that is those trips with origin and destination between s_0 and s_1 in direction 1, and between s_2 and s_3 in direction 2, it is unlikely that the strategy would be applicable, as the value of $f_B^*(\vec{s})$ is reduced, even to the point of becoming negative, in cases where the limit stations s_0, s_1, s_2 and s_3 are not properly chosen.

If the second stage (election of limit stations s^*) is solved by explicit enumeration of feasible solutions, the number of options to be evaluated would be $N^2 (N-1)^2/4$, since the number of discrete variables are 4. As an example, recalling the conceptual (but real) example of Fig. 1, the line comprises 24 stations, which represents 76,176 possible combinations of s_0, s_1, s_2 and s_3. Nevertheless, a proper processing of the load profiles permits to considerably bind the feasible solution space. Thus, for example, observing Fig. 1, we can predict that s_0 will be between E4 and E6 (three stations), s_1 between E17 and E19 (three stations), s_2 between E11 and E7 (five stations), and s_3 will be E21 as that station represents a big jump in the load profile. With these considerations, the number of options reduces to 45, saving computation time due to the information provided by the shape of the curves. Note that the particular cases of short turning or deadheading only are much easier to deal with, as they have just two and one limit stations to search for, respectively, and explicit enumeration becomes a very effective method, even for the analysis of long bus corridors (for the short turning case, see Tirachini et al., 2011).

2.2. Scheduled bus arrivals

For the case of a scheduled service, we consider $x = 0$ and $f_B = nf_A$, where n is the "scheduling mode" introduced by Furth (1987) as explained in Section 1. In this case, the vehicle capacity can be expressed as:

$$K = \frac{1}{\eta}\left[\frac{\sigma_0(\vec{s})}{f_A} + \frac{\sigma_1(\vec{s})}{(n+1)f_A}\right] = \frac{1}{\eta f_A}\left[\sigma_0(\vec{s}) + \frac{\sigma_1(\vec{s})}{(n+1)}\right] \equiv \frac{\sigma_{max}(n, (\vec{s}))}{\eta f_A} \tag{19}$$

This expression is introduced into the total cost, leaving the cost expression as a function of n, the stations s_0, s_1, s_2 and s_3, and the only continuous variable f_A, whose optimal condition is

$$\frac{\partial C_t^{ds}}{\partial f_A} = c_0\left[g_0(\vec{s}) + ng_3(\vec{s})\right] + 2c_0'L + nL\left[c_0'\frac{s_1 - s_0 + s_3 - s_2}{N-1} + c_{0d}'\frac{|s_2 - s_0| + |s_3 - s_1|}{N-1}\right]$$
$$- \frac{1}{f_A^2}\left\{P_w\left[\frac{g_1(\vec{s})}{2(n+1)} + \frac{g_2(\vec{s})}{2}\right] + P_v b\left[\frac{g_5(\vec{s})}{n+1} + g_6(\vec{s})\right] + c_1 b\frac{\sigma_{max}(n, \vec{s})}{\eta}[g_1(\vec{s}) + g_2(\vec{s})]\right\} = 0 \tag{20}$$

Then, the optimal frequency is expressed as (21)

$$f_A^*(n, \vec{s}) = \sqrt{\frac{P_w\left[\frac{g_1(\vec{s})}{2(n+1)} + \frac{g_2(\vec{s})}{2}\right] + P_v b\left[\frac{g_5(\vec{s})}{(n+1)} + g_6(\vec{s})\right] + c_1 b\frac{\sigma_{max}(n, \vec{s})}{\eta}[g_1(\vec{s}) + g_2(\vec{s})]}{C_0[g_0(\vec{s}) + ng_3(\vec{s})] + 2c_0', L + nL\left[c_0', \frac{s_1 - s_0 + s_3 - s_2}{N-1} + \frac{|s_2 - s_0| + |s_3 - s_1|}{N-1}\right]}} \tag{21}$$

The term (21) can be evaluated in the objective function, $C_t^{ds}(f_A^*(n, \vec{s}), n, \vec{s}) \equiv \bar{C}_t^{ds}(n, \vec{s})$. Thus, the global optimum $(n^*, \vec{s}^*)$ is the one that minimizes $\bar{C}_t^{ds}(n, \vec{s})$, $(n^*, \vec{s}^*) = \arg\min_{n, \vec{s}} \bar{C}_t^{ds}(n, \vec{s})$. Finally, with all these elements we can compute the capacity of the vehicles as

$$K^* = \frac{\sigma_{max}(n^*, \vec{s}^*)}{\eta f_A^*(n^*, \vec{s}^*)} \tag{22}$$

3. Numerical applications

The methodology developed in the previous section is applied to a bus corridor comprising 10 stations, with four different demand scenarios that reproduce various trips' concentration patterns, typical of real urban public transport systems. The matrices and demand profiles are shown in Figs 4 and 5, where Example 1 is a real case (corresponding to the Los Pajaritos corridor in Santiago, Chile, with the matrix estimated from data in MTT, 1998), while the other examples are hypothetical. Example 1 represents a radial corridor during the morning peak period, where 82% of demand is in direction 1 (towards the Central Business District). Example 2 shows a typical cross-town peak-demand route pattern, as illustrated in the resulting load profile of Fig. 2b, while Examples 3 and 4 are variations of Example 2. Note that the optimal IDS frequencies f_A^* and f_B^* for the case with random arrivals are found for the general case with parameters c_1, c_1' and c_{1d}' different from zero, which is solved numerically using the Newton method. The particular case of $c_1 = c_1' = c_{1d}' = 0$, which results in equations (17) and (18) as closed forms for frequencies, is taken as initial solution in the numerical method.
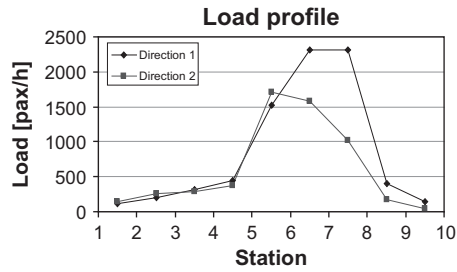
O\D	1	2	3	4	5	6	7	8	9	10
1		600	189	165	64	44	342	605	726	395
2	3620		11	10	4	3	20	35	42	23
3	790	38		5	2	1	10	18	22	12
4	1585	75	82		0	0	2	4	5	3
5	281	13	14	14		2	13	24	29	16
6	186	9	10	9	8		13	22	27	15
7	264	13	14	13	12	9		12	14	8
8	2631	125	135	130	117	86	107		36	19
9	337	16	17	17	15	11	14	18		67
10	4425	211	228	218	197	144	180	232	200	

O\D	1	2	3	4	5	6	7	8	9	10
1		5	8	9	13	10	17	41	4	2
2	7		7	8	11	8	15	35	4	2
3	40	35		13	17	13	23	55	6	3
4	20	18	22		24	19	33	77	8	4
5	17	15	19	34		150	265	624	67	35
6	15	13	16	29	348		267	630	67	36
7	20	18	23	40	484	131		529	56	30
8	21	18	23	40	494	134	130		52	28
9	3	3	3	6	72	19	19	5		5
10	1	1	1	2	23	6	6	2	3	
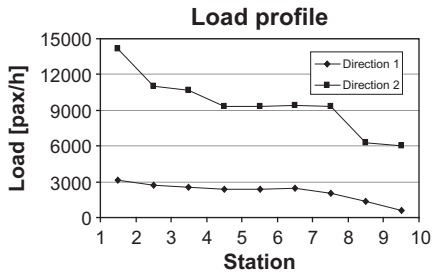
a. Example 1

b. Example 2

Fig. 4. O/D matrices and load profiles, Examples 1 and 2.

O\D	1	2	3	4	5	6	7	8	9	10
1		5	8	9	13	10	17	41	4	2
2	7		7	8	11	8	15	35	4	2
3	40	35		13	17	13	23	55	6	3
4	2	1	2		24	19	33	77	8	4
5	23	20	27	350		150	265	624	67	35
6	32	28	36	487	131		267	630	67	36
7	30	27	34	454	122	119		529	56	30
8	5	5	6	79	21	21	79		52	28
9	3	3	3	46	12	12	46	5		5
10	1	1	1	15	4	4	15	2	3	

O\D	1	2	3	4	5	6	7	8	9	10
1		5	8	9	13	10	17	41	4	2
2	7		7	8	11	8	15	35	4	2
3	6	6		13	17	13	23	55	6	3
4	32	28	360		24	19	33	77	8	4
5	44	39	537	131		150	265	624	67	35
6	41	36	467	122	119		267	630	67	36
7	5	5	59	15	15	58		529	56	30
8	3	3	36	9	9	35	29		52	28
9	3	3	36	9	9	35	29	5		5
10	1	1	11	3	3	11	9	2	3	
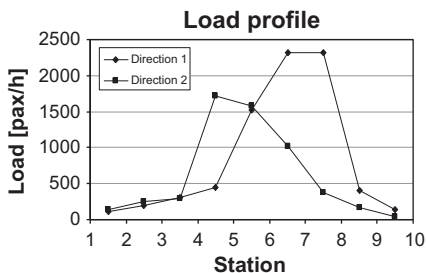
a. Example 3
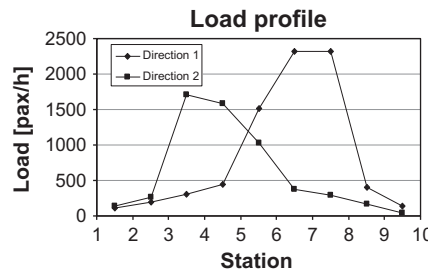
b. Example 4

Fig. 5. O/D matrices and load profiles, Examples 3 and 4.

The representative values assigned to the parameters were: $R = 1.2$ min, $R' = 0.7$ min $\forall k$, $L = 5$ km (for Example 1: $R = 3.5$ min, $R' = 2.0$ min $\forall k$, $L = 9$ km), $P_w = 2700$ [\$/h], $P_v = 900$ [\$/h], $c_0 = 1800$ [\$/vh], $c_1 = 30$ [\$/h-pax], $c_0' = 400$ [\$/vkm], $c_{0d}' = 300$ [\$/vkm], $c_1' = c_{1d}' = 1$ [\$/h-pax], $b = 5$ [s/pax] and $\eta = 0.9$ (\$: CLP-Chilean peso, USD $1 \approx$ CLP 600). For the case of scheduled arrivals we took $n \in \{1, 2, 3, 4\}$, as greater scheduling modes resulted in higher total costs.

Table 1 shows the most interesting results obtained from Example 1, for both random and regular headways, under normal operation (single frequency) and IDS. The minimum cost with the strategy is reached when fleet B skips the whole direction 1 ($s_0 = 1$, $s_1 = N$, $s_2 = s_3$), i.e. a pure deadheading strategy, directly benefiting the most demanded direction 2, yet total cost savings are negligible, around 0.5%. The operators are able to save 2 or 5 vehicles if the strategy is applied, whilst the maximum expected load per vehicle is also reduced, from 90 to 86 pax/veh with random headways, and from 94 to 85 pax/veh under a regular headway scheme; this suggests the use of smaller vehicles depending on commercial availability. Considering random headways, 131 veh/h serve the whole route while 51 veh/h are assigned to serve direction 2 only, whereas under

Table 1
Results Example 1.

Operation		f_A (veh/h)	f_B (veh/h)	F (veh)	K (pax/veh)	C_w ($/pax)	C_v ($/pax)	C_o ($/pax)	C_t ($/pax)	ΔC_w (%)	ΔC_v (%)	ΔC_o (%)	ΔC_t (%)
Random	Normal	175 (f)		213	90	15.4	326.9	121.6	463.9	3.25	−0.18	−1.64	−0.45
Headways	Strategy	131	51	208	86	15.9	326.3	119.6	461.8				
Regular	Normal	166 (f)		203	94	8.1	330.0	117.8	455.9	6.17	−0.70	−0.76	−0.59
Headways	Strategy	92	92	201	85	8.6	327.7	116.9	453.2				

Table 2
Results Example 2.

Operation		f_A (veh/h)	f_B (veh/h)	F (veh)	K (pax/veh)	C_w ($/pax)	C_v ($/pax)	C_o ($/pax)	C_t ($/pax)	ΔC_w (%)	ΔC_v (%)	ΔC_o (%)	ΔC_t (%)
Random	Normal	75 (f)		36	34	35.6	66.4	74.7	176.7	−3.37	−12.05	−10.31	−9.56
Headways	Strategy	39	91	33	31	34.4	58.4	67.0	159.8				
Regular	Normal	64 (f)		31	41	21.1	71.0	65.3	157.4	−1.90	−11.83	−12.71	−10.86
Headways	Strategy	34	68	29	38	20.7	62.6	57.0	140.3				

regular headways the optimal scheduling mode is $n = 1$ and $f_A = f_B = 92$ veh/h. Nevertheless, this causes nearly no savings on average waiting and in-vehicle time costs, which means that the advantage for passengers in direction 2 (82% of total demand) of a larger frequency (184 veh/h with strategy against 166 veh/h under normal operation, in the case of regular headways) is offset by a loss associated with passengers who utilize the service in direction 1 (18% of total demand), with an observed frequency reduction from 166 to 92 veh/h. The high demand (around 14,000 pax/h in the most loaded section, see Fig. 4a) explains the extremely large frequencies and low waiting time costs in Table 1.

In Example 2 (Table 2), the optimal strategy is a short turn between stations 5 and 8. The unbalance in the demand is lower than in Example 1, as now 73% of the demand is moving between stations 5 and 8, while the demand benefited in Example 1 is 82% of the total. Nonetheless, the strategy induces total cost savings around 10%, much better than the deadheading case. There are benefits for both users and operators. First, the waiting and in-vehicle time loss by users outside the short turn (where frequency drops from 75 to 39 veh/h under random headways) is compensated by the savings of passengers traveling inside the short-turn (frequency raises from 75 to 130 veh/h). Second, operators require a smaller fleet (three vehicles less than under normal operation) to provide the service. The optimal scheduling mode is $n = 2$ with regular headways, which resembles the results obtained in the Poisson arrivals' case, where f_B is slightly larger than the double of f_A.

Examples 3 and 4 were generated from Example 2, with slight changes on the demand pattern in direction 2. Thus, in Example 3 the demand between stations 5 and 8 is moved to the area between stations 4 and 7 (see Fig. 5a), while in Example 4 the demand peak in this direction is shifted even more, namely between stations 3 and 6 (Fig. 5b). Note that the demand in direction 1 is not modified, keeping total demand constant in the three examples (3335 pax/h). In Tables 3 and 4 only the limit stations and cost savings for the case of random headways are shown.

The first row of Table 3 (Example 3) shows the results with the original value of the parameters; the outcome is a pure short turn between stations 4 and 8 causing 7% of total savings. Thus, even though the resulting loads between stations 4 and 5 along direction 1 and between stations 7 and 8 in direction 2 are quite low, the strategy includes those segments inside the high frequency area. Nevertheless, this result depends on the relative difference between the parameters under deadheading and normal operation. Note that only if the operator cost with deadheading were very low, the configuration of the strategy would change, as seen in the third row of Table 3. A drastic (and likely unreal) reduction of 40% in the value of c'_{0d} would be needed for that to happen: when c'_{0d} is reduced from 300[2] to 170 $/bus-km, the vehicles of fleet B are deadheaded between stations 4 and 5 and operate normally (picking up and dropping passengers) only between stations 5 and 8. In station 8 they turn back and are deadheaded until station 7, to start the passengers' service again until station 4. There, they turn back again and start a new cycle. Note that the configuration $s_0 = 5$, $s_1 = 8$, $s_2 = 4$ and $s_3 = 7$ will remain unchanged even if c'_{0d} is further reduced, which is explained by the concentration of the demand, which happens on those segments (see Fig. 5a).

We can see that only when the unitary cost for operators associated with deadheading is around half of the unitary cost of vehicles performing normal service (a rather unlikely situation), it would be convenient that some vehicles were deadheaded in some sections of the route. If c'_{0d} was similar to c'_0 the model would suggest that a longer short turn is the most cost-effective alternative, as the operator cost saving by skipping stations in a low demand section would be outweighed by the increased cost of these users, who face a lower frequency f_A. As a conclusion, we can say that even though in some cases deadheading seems intuitively attractive, our analysis concluded that a short turning strategy turned out to be better in-

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[2] This value, 75% of the cost under normal operation, is estimated as the fuel efficiency gain of operating buses at a higher speed and no stops to get passengers on and off.

Table 3
Results Example 3.

c'_{0d}	s_0	s_1	s_2	s_3	Operation	C_w ($/pax)	C_v ($/pax)	C_o ($/pax)	C_t ($/pax)	ΔC_w (%)	ΔC_v (%)	ΔC_o (%)	ΔC_t (%)
					Normal	35.7	66.3	74.6	177				
300	4	8	4	8	Strategy	34.7	59.5	69.6	164	−2.80	−10.26	−6.70	−7.25
170	5	8	4	7	Strategy	35.1	59.5	69.1	164	−1.68	−10.26	−7.37	−7.30

Table 4
Results Example 4.

c'_{0d}	s_0	s_1	s_2	s_3	Operation	C_w ($/pax)	C_v ($/pax)	C_o ($/pax)	C_t ($/pax)	ΔC_w (%)	ΔC_v (%)	ΔC_o (%)	ΔC_t (%)
					Normal	35.5	66.3	74.5	176				
300	3	8	3	8	Strategy	34.9	60.7	71.9	168	−1.69	−8.45	−3.49	−4.99
250	3	8	3	7	Strategy	35.1	60.7	71.5	167	−1.13	−8.45	−4.03	−5.10
200	4	8	3	6	Strategy	35.4	60.5	70.7	167	−0.28	−8.75	−5.10	−5.50
170	5	8	3	6	Strategy	35.3	60.3	70.4	166.0	−0.56	−9.05	−5.50	−5.84

stead. Moreover, we have supported this quantitatively by illustrating numerically the unrealistic conditions under which deadheading could become a good alternative. We can say that IDS allow us to identify conditions for implementing a flexible fleet management strategy under subtle differences in demand profiles, even in cases where the analysis is focused on deciding a single scheme, such as short turning or deadheading.

In Table 4 (Example 4), if c'_{0d} = 300$/bus-km the optimal strategy would still be a pure short turn between stations 3 and 8 (Fig. 5b), with total savings of 5%, smaller than the savings in Example 3, which in turn were smaller than savings in Example 2. This clearly suggests that, *ceteris paribus*, the more separated the demand peaks between directions are, the less benefits the strategy provides, since a larger section with relatively low demand is included in the high frequency area.

On the other hand, when the relative operator cost difference between deadheading and in service vehicles is increased, the configuration of the strategy is spatially closer to the most demanded sections of the corridor, as shown in Table 4 where the optimal limit stations change after successive decreases in the value of c'_{0d}, after the value c'_{0d} = 170$/bus-km, when the limit stations are exactly those in which the demand peaks occur, namely, $s_0 = 5$, $s_1 = 8$, $s_2 = 3$, $s_3 = 6$. Note that even when the deadheading operator cost is low enough to skip stations, more benefits are still associated to the load profile of Example 2 than to its deviations on Examples 3 and 4, which reinforces the conclusion that the more separated the high demand sections are in both directions, the less attractive the application of the strategy is, in this case, because the distance under deadheading operation is larger and consequently the operator costs are increased.

Note that in real settings there are stations where it is not feasible for vehicles to turn around and restart the service in the opposite direction, due to physical or operational constraints. This issue is ignored in these examples.

4. Conclusions

In this paper we have developed a model to set the optimal value of frequencies and capacity of vehicles serving a linear transit corridor, combining deadheading and short turning into an integrated fleet management strategy (IDS). The potential benefits of IDS are compared against an optimized normal operation scenario (single frequency). We claim that the IDS strategy is able to handle demand configurations not properly tractable by implementing just a single strategy. The optimization variables are the limit stations that define the IDS, the frequencies inside and outside such limit stations and the capacity of the vehicles. The IDS strategy allows offering higher frequency in the most loaded sectors of the route, by permitting a number of vehicles to skip some segments with low demand to serve spatially shifted demand peaks in different directions more efficiently.

Several experiments were conducted to test the model for different situations. The analysis was focused on the key results: the optimal values of the decision variables under normal operation and with the IDS strategy, and the impact on waiting time cost, in-vehicle time cost and operator cost. Let us highlight the most important findings of this work.

- Regarding the analytical consistency of the model, the final form for the optimal frequencies with the IDS strategy generalizes the square root formula, handling a disaggregated (station-to-station) demand description.
- When the general IDS formulation is applied, there are four possible results regarding the optimal configuration of the service: no strategy, deadheading, short turning, or IDS. Moreover, the relative advantages of one configuration over any other can be examined through costs, waiting times, in-vehicle times and so on.
- The deadheading strategy in its pure version intuitively becomes a reasonable tool to adjust the frequency to the demand in corridors where the flows are quite unbalanced, by means of sending buses as fast as they can through the low demand direction. However, this feature makes the strategy inflexible and very limited in the sense of reducing the cost of users and operators as vehicles operate a considerable portion of the corridor without serving passengers, which means that the effective frequency (observed by passengers) is smaller than the frequency actually provided, incurring in unavoidable operational costs along the entire route. Previous experiments showed that even under very attractive demand configuration scenarios, the benefits of deadheading were always of the order of 2% (Tirachini and Cortés, 2006; Tirachini, 2007).

- The IDS strategy is a good option to deal with mixed load patterns, where unbalances within and between directions are observed, taking advantage of the potential benefits of deadheading and short turning in the different portions of the route where each pure strategy could be useful. Considering the empirical limitations of deadheading in the sense explained in the previous comment, the idea of implementing a combined strategy is very attractive to somehow compensate the apparent inefficiency of deadheading with the benefits of the integrated scheme when facing demand configurations not able to be handled either by a pure short turning or a pure deadheading.
- In general, since the IDS scheme is a combination of short turning and deadheading, the potential benefits of the integration are somewhat in the middle. This observation does not mean that the strategy should be replaced by a pure short turning, as when the relative deadheading costs are sufficient low, the combined configuration allows taking advantage of the portions of the route in which some vehicles are deadheaded as a response to the low demand levels observed there, which does not justify the extra fleet to serve normally on such segments (in the way a pure short turning would work).
- Some of the numerical examples suggest that in real contexts does not seem easy to find situations in which it is optimal to skip stations, unless the relative cost of deadheading is too low or the demand peaks are too separated between directions. Importantly, this conclusion is subject to the conditions modeled and objective pursued; yet deadheading may be useful in other contexts, for example in real-time optimization of frequencies.
- As the demand concentrations (peaks) along each direction become more spatially separated, the IDS strategy starts becoming less effective (higher costs of deadheading since sections with no passenger service become longer).
- The configuration of the sectors where the frequency is increased by the IDS strategy is strongly influenced by the value of the cost of deadheading with respect to that associated with normal service. Thus, the less the operational cost of buses traveling empty is, the more differentiated the limit stations are in one direction with respect to the other (that is, s_0 different from s_2 and s_1 different from s_3). On the contrary, the less attractive the corridor is for deadheading, the more similar to a pure short turning the IDS scheme becomes.
- In all the numerical examples, the optimal vehicle size for IDS was smaller than that obtained under normal operation (base case). This result shows how the IDS scheme allows providing a larger frequency on the section with the largest load along the corridor, value that is inversely related to the vehicle capacity. This result applies even though the vehicles belonging to fleet A are more loaded than those belonging to fleet B. This result implies that the more realistic model, in which the operational cost is a function of the vehicle capacity, reports larger benefits than the simpler one considering the unitary operational costs as independent of vehicle size.

As interesting extensions, some elements not considered in this model could be added, as the following examples:

- Crowding inside vehicles, which affects waiting time (as done by Tirachini and Cortés, 2007), and in-vehicle time value (Kraus, 1991; Jara-Díaz and Gschwender, 2003).
- At the beginning of the operation, deadheaded vehicles could be asymmetrically distributed between terminals, concentrating more buses to start normal operation (in the high demand direction) to reduce some movements in deadheading and save operational costs of transporting empty vehicles. If this feature were properly considered in the implementation of the IDS, it could eventually report increased benefits.
- Extension of the model to a network (or at least, two routes connected by a transfer station).

Acknowledgements

Appendix A. Definition OF g_i functions

- g_0 is the total running time:

$$g_0 = 2 \sum_{k=1}^{N-1} R_k \tag{A.1}$$

- g_1 and g_2 are in the waiting and cycle times, g_1 is the demand benefited by the strategy (origin and destination between s_0 and s_1 in direction 1, and between s_2 and s_3 in direction 2), while g_2 are the passenger whose origin or destination are outside the limit stations.

$$g_1(\vec{s}) = \sum_{k=s_0}^{s_1-1} \lambda_k^+(k+1, s_1) + \sum_{k=s_2+1}^{s_3} \lambda_k^+(s_2, k-1) \tag{A.2}$$

$$g_2(\vec{s}) = \sum_{k=1}^{s_0-1} \lambda_k^{1+} + \sum_{k=s_0}^{s_1-1} \lambda_k^+(s_1+1, N) + \sum_{k=s_1}^{N} \lambda_k^{1+} + \sum_{k=s_3+1}^{N} \lambda_k^{2+} + \sum_{k=s_2+1}^{s_3} \lambda_k^{2+}(1, s_2-1) + \sum_{k=1}^{s_2} \lambda_k^{2+} \tag{A.3}$$

- g_3 is the running time for vehicles of fleet B

$$g_3(\vec{s}) = \sum_{k=min(s_0,s_2)}^{max(s_0,s_2)-1} R'_k + \sum_{k=s_0}^{s_1-1} + \sum_{k=min(s_1,s_3)}^{max(s_1,s_3)-1} R'_k + \sum_{k=s_2}^{s_3-1} R_k \tag{A.4}$$

- g_4 is the in-vehicle time experienced by passengers

$$g_4 = \sum_{k=1}^{N} \sum_{l=1}^{N} \lambda_{kl} \sum_{i=k}^{l-1} R_i \tag{A.5}$$

g_5 and g_6 are the factors to calculate the total dwell time; g_5 refers to the passengers that are better off (frequency $f_A + f_B$) after the frequency is applied, and g_6 is for those users that are worse off (frequency f_A). They accommodate quadratic terms on demand that arise as all travelers that board buses impose a time cost on all passengers already on board. To understand how g_5 and g_6 are obtained, it is necessary to look at the shape of the in-vehicle time cost function. For any origin destination pair (k, l), in-vehicle time of demand λ_{kl} includes the running time between stops $\sum_{i=k}^{l-1} R_i$ (included in factor g_4) and the time spent at bus stops transferring passengers, which depends on all passengers that board the bus between stops k and l, which consequently makes this term quadratic in demand. However, as different frequencies f_A and $f_A + f_B$ are provided in different sectors along a route, some passengers board buses at rate λ_k^+/f_A and others at rate $\lambda_k^+/(f_A + f_B)$; in the former case are those users whose origin or destination is outside the short-turning area (between s_0 and s_1 in direction 1), whereas the later case is for passenger whose trip is covered by fleet B (origin and destination between s_0 and s_1). This issue complicates the definition of the in-vehicle time cost, because in each direction we need to distinguish among six cases depending on where origin k and destination l are, as shown in Table A.1 for direction 1.

Then, adding over all origin destination pairs, the in-vehicle time cost in direction 1 can be expressed as:

$$
\begin{aligned}
C_v^{ds1} = P_v \Bigg\{ & \sum_{k=1}^{s_0-1} \sum_{l=k+1}^{s_0} \left[\sum_{i=k}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) \right] \lambda_{kl} + \sum_{k=1}^{s_0-1} \sum_{l=s_0+1}^{s_1} \left[\sum_{i=k}^{s_0-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) + \sum_{i=s_0}^{l-1} \left(R_i + b\left(\frac{\lambda_i^+(i+1,s_1)}{f_A+f_B} + \frac{\lambda_i^+(s_1+1,N)}{f_A} \right) \right) \right] \lambda_{kl} \\
& + \sum_{k=1}^{s_0-1} \sum_{l=s_1+1}^{N} \left[\sum_{i=k}^{s_0-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) + \sum_{i=s_0}^{s_1-1} \left(R_i + b\left(\frac{\lambda_i^+(i+1,s_1)}{f_A+f_B} + \frac{\lambda_i^+(s_1+1,N)}{f_A} \right) \right) + \sum_{i=s_1}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) \right] \lambda_{kl} \\
& + \sum_{k=s_0}^{s_1-1} \sum_{l=k+1}^{s_1} \left[\sum_{i=k}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A+f_B} \right) \right] \lambda_{kl} + \sum_{k=s_0}^{s_1-1} \sum_{l=s_1+1}^{N} \left[\sum_{i=k}^{l-1} \left(R_i + b\left(\frac{\lambda_i^+(i+1,s_1)}{f_A+f_B} + \frac{\lambda_i^+(s_1+1,N)}{f_A} \right) \right) \right. \\
& \left. + \sum_{i=s_1}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) \right] \lambda_{kl} + \sum_{k=s_1}^{N} \sum_{l=k+1}^{N} \left[\sum_{i=k}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) \right] \lambda_{kl} \Bigg\}
\end{aligned}
\tag{A.6}
$$

Table A.1
In-vehicle time according to location of origin and destination.

Location of origin k and destination l with respect to s_0 and s_1	In-vehicle time
$k \quad l \quad s_0 \quad\quad s_1$	$\displaystyle \sum_{i=k}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right)$
$k \quad\quad s_0 \quad l \quad s_1$	$\displaystyle \sum_{i=k}^{s_0-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) + \sum_{i=s_0}^{l-1} \left(R_i + b\left(\frac{\lambda_i^+(i+1,s_1)}{f_A+f_B} + \frac{\lambda_i^+(s_1+1,N)}{f_A} \right) \right)$
$k \quad\quad s_0 \quad\quad s_1 \quad l$	$\displaystyle \sum_{i=k}^{s_0-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right) + \sum_{i=s_0}^{s_1-1} \left(R_i + b\left(\frac{\lambda_i^+(i+1,s_1)}{f_A+f_B} + \frac{\lambda_i^+(s_1+1,N)}{f_A} \right) \right) + \sum_{i=s_1}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right)$
$s_0 \quad k \quad l \quad s_1$	$\displaystyle \sum_{i=k}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A+f_B} \right)$
$s_0 \quad k \quad s_1 \quad l$	$\displaystyle \sum_{i=k}^{s_1-1} \left(R_i + b\left(\frac{\lambda_i^+(i+1,s_1)}{f_A+f_B} + \frac{\lambda_i^+(s_1+1,N)}{f_A} \right) \right) + \sum_{i=s_1}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right)$
$s_0 \quad\quad s_1 \quad k \quad l$	$\displaystyle \sum_{i=k}^{l-1} \left(R_i + b\frac{\lambda_i^{1+}}{f_A} \right)$

The quadratic terms on demand that appear in (A.6) reflect that all passengers that get on buses between stations k and l increase the boarding time of all passengers that travel between those stops, λ_{kl}. Expression (A.6) is formulated in synthetic form as

$$C_v^{ds1} = P_v \left\{ \sum_{k=1}^{N-1} R_k + b\frac{g_5^1(s_0,s_1)}{f_A + f_B} + b\frac{g_6^1(s_0,s_1)}{f_A} \right\} \tag{A.7}$$

where g_5^1 and g_6^1 encompass all the quadratic terms in demand observed in (A.6):

$$g_5^1(s_1,s_2) = \sum_{k=1}^{s_0-1}\sum_{l=s_0+1}^{s_1}\lambda_{kl}\sum_{i=s_0}^{l-1}\lambda_i^+(i+1,s_1) + \sum_{k=1}^{s_0-1}\sum_{l=s_1+1}^{N}\lambda_{kl}\sum_{i=s_0}^{s_1-1}\lambda_i^+(i+1,s_1) + \sum_{k=s_0}^{s_1-1}\sum_{l=k+1}^{s_1}\lambda_{kl}\sum_{i=k}^{l-1}\lambda_i^{1+} + \sum_{k=s_0}^{s_1-1}\sum_{l=s_1+1}^{N}\lambda_{kl}\sum_{i=k}^{s_1-1}\lambda_i^+(i+1,s_1) \tag{A.8}$$

$$\begin{aligned}g_6^1(s_1,s_2) = & \sum_{k=1}^{s_0-1}\sum_{l=k+1}^{s_0}\lambda_{kl}\sum_{i=k}^{l-1}\lambda_i^{1+} + \sum_{k=1}^{s_0-1}\sum_{l=s_0+1}^{s_1}\lambda_{kl}\left[\sum_{i=k}^{s_0-1}\lambda_i^{1+} + \sum_{i=s_0}^{l-1}\lambda_i^+(s_1+1,N)\right] \\ & + \sum_{k=1}^{s_0-1}\sum_{l=s_1+1}^{N}\lambda_{kl}\left[\sum_{i=k}^{s_0-1}\lambda_i^{1+} + \sum_{i=s_0}^{s_1-1}\lambda_i^+(s_1+1,N) + \sum_{i=s_1}^{l-1}\lambda_i^{1+}\right] + \sum_{k=s_0}^{s_1-1}\sum_{l=s_1+1}^{N}\lambda_{kl}\left[\sum_{i=k}^{s_1-1}\lambda_i^+(s_1+1,N) + \sum_{i=s_1}^{l-1}\lambda_i^{1+}\right] \\ & + \sum_{k=s_1}^{N}\sum_{l=k+1}^{N}\lambda_{kl}\sum_{i=k}^{l-1}\lambda_i^{1+}\end{aligned} \tag{A.9}$$

Analogously, for direction 2 we obtain:

$$C_v^{ds2} = P_v\left\{\sum_{k=1}^{N-1}R_k + b\frac{g_5^2(s_2,s_3)}{f_A+f_B} + b\frac{g_6^2(s_2,s_3)}{f_A}\right\} \tag{A.10}$$

where

$$\begin{aligned}g_5^2(s_2,s_3) = & \sum_{k=s_3+1}^{N}\sum_{l=s_2}^{s_3-1}\lambda_{kl}\sum_{i=l+1}^{s_3}\lambda_i^+(s_2,i-1) + \sum_{k=s_3+1}^{N}\sum_{l=1}^{s_2-1}\lambda_{kl}\sum_{i=s_2+1}^{s_3}\lambda_i^+(s_2,i-1) + \sum_{k=s_2+1}^{s_1}\sum_{l=s_2}^{k-1}\lambda_{kl}\sum_{i=l+1}^{k}\lambda_i^{2+} \\ & + \sum_{k=s_2+1}^{s_3}\sum_{l=1}^{s_2-1}\lambda_{kl}\sum_{i=s_2+1}^{k}\lambda_i^+(s_2,i-1)\end{aligned} \tag{A.11}$$

$$\begin{aligned}g_6^2(s_2,s_3) = & \sum_{k=s_3+1}^{N}\sum_{l=s_3}^{k-1}\lambda_{kl}\sum_{i=l+1}^{k}\lambda_i^{2+} + \sum_{k=s_3+1}^{N}\sum_{l=s_2}^{s_3-1}\lambda_{kl}\left[\sum_{i=s_3+1}^{k}\lambda_i^{2+} + \sum_{i=l+1}^{s_3}\lambda_i^+(1,s_2-1)\right] \\ & + \sum_{k=s_3+1}^{N}\sum_{l=1}^{s_2-1}\lambda_{kl}\left[\sum_{i=s_3+1}^{k}\lambda_i^{2+} + \sum_{i=s_2+1}^{s_3}\lambda_i^+(1,s_2-1) + \sum_{i=l+1}^{s_2}\lambda_i^{2+}\right] + \sum_{k=s_2+1}^{s_3}\sum_{l=1}^{s_2-1}\lambda_{kl}\left[\sum_{i=s_2+1}^{k}\lambda_i^+(1,s_2-1) + \sum_{i=l+1}^{s_2}\lambda_i^{1+}\right] \\ & + \sum_{k=1}^{s_2}\sum_{l=1}^{k-1}\lambda_{kl}\sum_{i=l+1}^{k}\lambda_i^{2+}\end{aligned} \tag{A.12}$$

Finally, the total in-vehicle time cost (5) is

$$C_v^{ds} = C_v^{ds1} + C_v^{ds2} = P_v\left\{2\sum_{k=1}^{N-1}R_k + b\frac{g_5(\vec{s})}{f_A+f_B} + b\frac{g_6(\vec{s})}{f_A}\right\} \tag{A.13}$$

with $g_5(\vec{s}) = g_5^1(s_1,s_2) + g_5^2(s_2,s_3)$ and $g_6(\vec{s}) = g_6^1(s_1,s_2) + g_6^2(s_2,s_3)$ given by expressions (A.8), (A.9), (A.11), and (A.12).

Appendix B. Load of vehicles between stations

B.1. Load of vehicles serving the entire corridor (fleet A)

Direction 1

$$\pi_k^1 = \begin{cases} \pi_{k-1}^1 + \frac{\lambda_k^{1+}}{f_A} - \frac{\lambda_k^{1-}}{f_A} & \text{if } 1 \leqslant k \leqslant s_0 - 1 \\ \pi_{k-1}^1 + \frac{\lambda_k^+(k+1,s_1)}{f_A+f_B} + \frac{\lambda_k^+(s_1+1,N)}{f_A} - \frac{\lambda_k^-(1,s_0-1)}{f_A} - \frac{\lambda_k^-(s_0,k-1)}{f_A+f_B} & \text{if } s_0 \leqslant k \leqslant s_1 \\ \pi_{k-1}^1 + \frac{\lambda_k^{1+}}{f_A} - \frac{\lambda_k^{1-}}{f_A} & \text{if } s_1 + 1 \leqslant k \leqslant N \end{cases}$$

Direction 2

$$\pi_k^2 = \begin{cases} \pi_{k+1}^2 + \frac{\lambda_k^{2+}}{f_A} - \frac{\lambda_k^{2-}}{f_A} & \text{if } s_3 + 1 \leqslant k \leqslant N \\ \pi_{k+1}^2 + \frac{\lambda_k^+(s_2,k-1)}{f_A+f_B} + \frac{\lambda_k^+(1,s_2-1)}{f_A} - \frac{\lambda_k^-(s_3+1,N)}{f_A} - \frac{\lambda_k^-(k+1,s_3)}{f_A+f_B} & \text{if } s_2 \leqslant k \leqslant s_3 \\ \pi_{k+1}^2 + \frac{\lambda_k^{2+}}{f_A} - \frac{\lambda_k^{2-}}{f_A} & \text{if } 1 \leqslant k \leqslant s_2 - 1 \end{cases}$$

B.2. Load of vehicles performing IDS (fleet B)

Direction 1

$$\tilde{\pi}_k^1 = \begin{cases} 0 & \text{if } 1 \leq k \leq s_0 - 1 \\ \tilde{\pi}_{k-1}^1 + \frac{\lambda_k^+(k,s_1)}{f_A+f_B} - \frac{\lambda_k^-(s_0,k-1)}{f_A+f_B} & \text{if } s_0 \leq k \leq s_1 - 1 \\ 0 & \text{if } s_1 \leq k \leq N \end{cases}$$

Direction 2

$$\tilde{\pi}_k^2 = \begin{cases} 0 & \text{if } s_3 + 1 \leq k \leq N \\ \tilde{\pi}_{k+1}^2 + \frac{\lambda_k^+(s_2,k-1)}{f_A+f_B} - \frac{\lambda_k^-(k+1,s_3)}{f_A+f_B} & \text{if } s_2 + 1 \leq k \leq s_3 \\ 0 & \text{if } 1 \leq k \leq s_2 \end{cases}$$

After applying the IDS strategy, the maximum load will correspond to some segment, which can belong to either direction 1 or 2. However, by observing the recursive form of the load equations above, we can establish the following:

- If the maximum load occurs along direction 1, then we can say that

$$\eta K = \pi_{\max}^1 = \frac{\varphi_0^1(s_0,s_1)}{f_A} + \frac{\varphi_1^1(s_0,s_1)}{f_A+f_B}$$

- Otherwise, if the maximum load occur along direction 2, then we can say that

$$\eta K = \pi_{\max}^1 = \frac{\varphi_0^2(s_2,s_3)}{f_A} + \frac{\varphi_1^2(s_2,s_3)}{f_A+f_B}$$

Thus, a generic relation between the design capacity of buses and the frequencies f_A and f_B can be established in the way summarized in equation (16):

$$K = \frac{1}{\eta}\left(\frac{\sigma_0(\vec{s})}{f_A} + \frac{\sigma_1(\vec{s})}{f_A+f_B}\right)$$

where, if the maximum load happens on a segment that belongs to direction 1, then $\sigma_0(\vec{s}) = \varphi_0^1(s_0,s_1)$ and $\sigma_1(\vec{s}) = \varphi_1^1(s_0,s_1)$. Otherwise, if the maximum load occurs on a segment that belongs to direction 2, then $\sigma_0(\vec{s}) = \varphi_0^2(s_2,s_3)$ and $\sigma_1(\vec{s}) = \varphi_1^2(s_2,s_3)$.

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