



On the identification of a rigid body immersed in a fluid: A numerical approach [☆]

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ABSTRACT

This work deals with the study of an inverse geometric problem in fluid mechanics. In particular, we are interested in the numerical reconstruction of a rigid body which is immersed in a cavity, filled with a fluid, by means of measurements of the Cauchy forces and the velocity of the fluid on one part of the exterior boundary. This problem was studied in [Alvarez C, Conca C, Friz L, Kavian L, Ortega JH. Identification of immersed obstacles via boundary measurements. *Inverse Problems* 2005; **21**:1531–52], where the authors proved the identifiability and stability for this problem. In this work we present a numerical method for the reconstruction of the rigid body for some particular geometries.

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1. Setting of the problem and previous results

This work deals with the study of a geometrical inverse boundary problem arising in fluid mechanics. Geometrical inverse problems are frequent models in several applied areas, such as medical imaging and non-destructive evaluation of materials. In this work we are interested in the numerical reconstruction of an inaccessible rigid body D immersed in a viscous fluid, in such a way that D plays the role of an obstacle around which the fluid is flowing in a greater bounded domain $\Omega \subset \mathbb{R}^N$. Our problem is to determine D or some geometrical information (i.e. its shape, volume, etc.) via a single boundary measurement on the boundary $\partial\Omega$.

Let $\phi \in H^{1/2}(\partial\Omega)^N$ be a non-homogeneous Dirichlet boundary data satisfying the standard compatibility condition

$$\int_{\partial\Omega} \phi \cdot n \, ds = 0, \quad (1)$$

and let $(v, p) \in H^1(\Omega \setminus \bar{D}) \times L^2(\Omega \setminus \bar{D})$ be the unique solution of the Stokes system of equations:

$$\begin{cases} \operatorname{div} \sigma(v, p) = 0 & \text{in } \Omega \setminus \bar{D}, \\ \operatorname{div} v = 0 & \text{in } \Omega \setminus \bar{D}, \\ v = \phi & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial D; \end{cases} \quad (2)$$

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σ being the stress tensor defined as follows $\sigma(v, p) = -p\mathbf{I} + 2\nu e(v)$, where \mathbf{I} is the identity matrix, $\nu > 0$ is a given constant representing the *kinematic viscosity* of the fluid and $e(v)$ is the linear strain tensor defined by $e(v) = \frac{1}{2}(\nabla v + {}^t\nabla v)$.

The classical inverse boundary problem is the well known electrical impedance tomography problem proposed by A.P. Calderon in 1980. In this case, the boundary map is the so-called voltage to current map, also called the Dirichlet to Neumann map; that is, the map assigns the voltage potential on the boundary of a medium to a corresponding current flux at the boundary. Calderon's inverse problem is to reconstruct the conductivity of the medium from this boundary map. This classical problem was at the middle of the 1980s the starting point for the mathematical analysis of inverse problems. The interested reader is referred to the review by Uhlmann [1] for key historical remarks in this subject and to the pioneering works by Kohn and Vogelius [2] or to Sylvester and Uhlmann [3] for early results on this theory.

In this work we are dealing with a geometrical inverse boundary problem. We will look for the unknown D in the following set of admissible geometries

$$\mathcal{D}_{ad} = \{D \subset \subset \Omega : D \text{ is a smooth connected open set in } \Omega, \text{ such that, } \Omega \setminus \bar{D} \text{ is connected}\},$$

Then, we can define the boundary map, which we will refer to as the velocity to stress force map, as follows:

$$A : D \longrightarrow A_D, \quad A_D(\phi) = \sigma(v, p)n \text{ on } \partial\Omega,$$

acting from $H^{1/2}(\partial\Omega)^N$ to $H^{-1/2}(\partial\Omega)^N$, where (v, p) is the unique solution of the stationary Stokes system (2).

Our inverse problem is to develop a numerical algorithm which allow us to recover D from the above boundary map.

Firstly we will recall an identifiability result for this problem, that is, given a fixed non-homogeneous Dirichlet boundary data, two any different geometries $D_0 \neq D_1 \in \mathcal{U}_{ad}$, there correspond different boundary stress force measurements. This result does not only give a positive answer to the above question, but it also proves that for the identifiability of D , one measurement of the velocity to stress boundary map is enough. It is worth noting that once the identifiability result has been proved, the boundary map $D \rightarrow A_D$ has an inverse A^{-1} acting from the range of A to \mathcal{D}_{ad} . Then, the result reads as follows.

Theorem 1 (Alvarez et al. [4]). *Let $T > 0$ and $\Omega \subseteq \mathbb{R}^N$, $N = 2$ or $N = 3$, be a bounded, $C^{1,1}$ domain, and Γ be a non-empty open subset of $\partial\Omega$. Let $D_0, D_1 \in \mathcal{D}_{ad}$ and $\phi \in C^1([0, T]; H^{3/2}(\partial\Omega)^N)$ with $\phi \neq 0$, satisfying the flux condition (1). For $\varepsilon_* = 0$ or $\varepsilon_* = 1$, let (v_j, p_j) for $j = 0, 1$, be solution of*

$$\begin{cases} \frac{\partial v_j}{\partial t} - \operatorname{div}(\sigma(v_j, p_j)) \\ \quad + \varepsilon_* \operatorname{div}(v_j \otimes v_j) = 0 & \text{in } (\Omega \setminus \bar{D}_j) \times (0, T), \\ \operatorname{div}(v_j) = 0 & \text{in } (\Omega \setminus \bar{D}_j) \times (0, T), \\ v_j(x, 0) = 0 & \text{for } x \in \Omega \setminus \bar{D}_j, \\ v_j(s, t) = \phi(s, t) & \text{for } (s, t) \in \partial\Omega \times (0, T), \\ v_j(s, t) = 0 & \text{for } (s, t) \in \partial D_j \times (0, T). \end{cases} \quad (3)$$

Assume that (v_j, p_j) are such that

$$\sigma(v_0, p_0)n = \sigma(v_1, p_1)n \quad \text{on } \Gamma \times (0, T).$$

Then $D_0 \equiv D_1$.

The same identification result holds for the stationary problem:

Theorem 2 (Alvarez et al. [4]). *Let $\Omega \subseteq \mathbb{R}^N$, $N = 2$ or $N = 3$, be a bounded, Lipschitz domain, and Γ a non-empty open subset of $\partial\Omega$. Let $D_0, D_1 \in \mathcal{D}_{ad}$ and $\phi \in H^{3/2}(\partial\Omega)^N$ with $\phi \neq 0$, satisfying the flux condition (1). For $\varepsilon_* = 0$ or $\varepsilon_* = 1$, let (v_j, p_j) for $j = 0, 1$, be solution of*

$$\begin{cases} -\operatorname{div}(\sigma(v_j, p_j)) + \varepsilon_* \operatorname{div}(v \otimes v) = 0 & \text{in } \Omega \setminus \bar{D}_j, \\ \operatorname{div}(v_j) = 0 & \text{in } \Omega \setminus \bar{D}_j, \\ v_j(s) = \phi(s) & \text{for } s \in \partial\Omega, \\ v_j(s) = 0 & \text{for } s \in \partial D_j. \end{cases} \quad (4)$$

Assume that (v_j, p_j) are such that

$$\sigma(v_0, p_0)n = \sigma(v_1, p_1)n \quad \text{on } \Gamma.$$

Then $D_0 \equiv D_1$.

On the other hand, the stability of our inverse geometrical boundary problem corresponds to study continuity properties of the inverse of the boundary map. In [4], the authors obtained a partial answer to this problem, that is, they proved a directional continuity of the inverse boundary map.

Let us consider the change of variables

$$\Psi : \Omega_0 \rightarrow \Omega_\tau := \Psi_\tau(\Omega_0) = I + \tau\Psi_1(\Omega_0),$$

and let $(v_\tau, p_\tau) \in H^1(\Omega_\tau)^N \times L^2(\Omega_\tau)$ be the unique solution of the Stokes system in the deformed domain, that is,

$$\begin{cases} -\operatorname{div}(\sigma(v_\tau, p_\tau)) + \varepsilon_* \operatorname{div}(v \otimes v) = 0 & \text{in } \Omega_\tau, \\ \operatorname{div}(v_\tau) = 0 & \text{in } \Omega_\tau, \\ v_\tau(s) = \phi(s) & \text{for } s \in \partial\Omega, \\ v_\tau(s) = 0 & \text{for } s \in \partial D_\tau. \end{cases} \quad (5)$$

(For $\tau = 0$ note that $\Psi_0 = I$ and $\Omega_\tau = \Omega_0$.) Then, under suitable assumptions on the function Ψ_1 , the mapping

$$\Psi_\tau \mapsto \sigma(v_\tau, p_\tau)n =: A_{\Psi_\tau(\Omega_0)}$$

is analytic in an open subset of $W^{1,\infty}$ -“diffeomorphisms” (by an abuse of language we shall say that Ψ is a $W^{1,\infty}$ -diffeomorphism when both Ψ and Ψ^{-1} are in $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$), and restricted, respectively, to the domains $\Omega \setminus \bar{D}_0$ and $\Psi(\Omega \setminus \bar{D}_0)$. In particular, it can be differentiated with respect to Ψ ; as a matter of fact, the corresponding derivative is the so-called *shape derivative* of the solution of (5) with respect to the geometry.

Theorem 3 (Alvarez et al. [4]). *Let $D_0 \in \mathcal{D}_{ad}$, $\Psi_0 := I$ and $\Psi_1 \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ such that $\Psi_1 \equiv 0$ in a neighbourhood of the boundary $\partial\Omega$ and $\Psi_1 \neq 0$ on D_0 . Denote $\Psi_\tau = I + \tau\Psi_1$ and (v_τ, p_τ) the solution of (5) (including $\tau = 0$), and by $\tau_1 > 0$ a positive number such that $\tau \mapsto \sigma(v_\tau, p_\tau)n$ is analytic on $(-\tau_1, \tau_1)$. Assume that for some $\tau_* \in (-\tau_1, \tau_1)$ one has $\Psi_{\tau_*}(D_0) \neq D_0$. Then there exists a strictly positive constant $C = C(\Psi_1, \Omega, D_0, \phi)$ and a positive integer $m = m(\Psi_1, \Omega, D_0, \phi)$ such that for some $\tau_0 > 0$ and all $\tau \in [-\tau_0, \tau_0]$, we have*

$$\|A_{D_0}(\phi) - A_{\Psi_\tau(D_0)}(\phi)\|_{H^{-1/2}(\Gamma)^N} \geq C|\tau|^m,$$

where $A_{\Psi_\tau(D_0)}(\phi) := \sigma(v_\tau, p_\tau)n$ on Γ .

2. Asymptotic expansion of the solutions of the Stokes problem with respect to the domain

2.1. Shape differentiation

In this work we will consider a different approach to the one used in [4] for the study of the stability by using the so-called *shape differentiation*. To do this, we will present the so-called *shape differentiation* due to Murat and Simon, for more details about this technique we refer to [5,6] and the bibliographies therein. These results are useful in shape optimization theory and the study of generic results for partial differential equations.

Given a subset $\Omega \subset \mathbb{R}^n$ and a function $u : \Omega \rightarrow \mathbb{R}^m$, we define the new domain $\Omega + u$, by

$$\Omega + u = \{y \in \mathbb{R}^n : y = x + u(x), x \in \Omega\}. \quad (6)$$

Let us to consider perturbations u in the space $W^{k,\infty}(\Omega, \mathbb{R}^N)$ with norm

$$\|u\|_{k,\infty} = \sup_{0 \leq |\alpha| \leq k, x \in \Omega} \operatorname{ess} |D^\alpha u(x)|.$$

Remark 4. Now, we must note that if Ω is of class C^j , we can choose $k = j + 1$ (and therefore the perturbation space $W^{k,\infty}(\Omega, \mathbb{R}^N)$), such that our new domain $\Omega_0 + u$ is also of class C^j .

Definition 1 (Simon [6]). Let $k \geq m \geq 1, 1 \leq r < \infty$. We say that the function $u \rightarrow v(u)$ has a first order local variation at $u = 0$ on $W_{loc}^{m-1,r}(\Omega)$, if $v(u) \in W^{m,r}(\Omega + u)$ for all $u \in W^{k,\infty}(\Omega, \mathbb{R}^n)$ and there exists a linear map $v'(\Omega; u)$ defined from $u \in W^{k,\infty}(\Omega, \mathbb{R}^n)$ to $W_{loc}^{m-1,r}(\Omega)$ such that, for all open set $\omega \subset \subset \Omega$,

$$v(u) = v(0) + v'(\Omega; u) + \hat{\theta}(u) \quad \text{in } \omega,$$

when $\|u\|_{k,\infty}$ is small enough and

$$\frac{\hat{\theta}(u)}{\|u\|_{k,\infty}} \rightarrow 0 \quad \text{in } W^{m-1,r}(\omega) \text{ as } \|u\|_{k,\infty} \rightarrow 0.$$

Remark 5. From Definition 1 it follows that the first local variation can be defined as

$$v'(\Omega; u) = \lim_{t \rightarrow 0} \frac{v(tu)|_\omega - v(0)|_\omega}{t} \quad \text{in } \omega, \quad (7)$$

where $\omega \subset \subset \Omega$ and $v(tu)|_\omega, v(0)|_\omega$ are the restrictions of the functions $v(tu)$ and $v(0)$ to ω .

Remark 6. In what follows, to simplify matters, we will write $v'(u)$ instead of $v'(\Omega; u)$.

On the other hand (see [6]) we note that if for each $u \in W^{k,\infty}(\Omega, \mathbb{R}^n)$ small enough,

$$v(u) = 0 \quad \text{on } \partial(\Omega + u),$$

the first local derivative at $u = 0$, in the direction u , denoted by $v'(u)$, verifies

$$v'(u) = -(u \cdot n) \frac{\partial v(0)}{\partial n} \quad \text{on } \partial\Omega,$$

where n is the unit outward normal vector to Ω .

2.2. Computation of the first local derivatives

Assume that the deformed domain has the form $\Omega^* + u = (\Omega \setminus \bar{D}) + u$. Note that we are interested in obtaining the asymptotic expansion of A_{D+u} around $u = 0$.

Firstly, for every $u \in \mathcal{W}$, we can write down the following Stokes equation

$$\begin{cases} -\nu \Delta v_u + \nabla p_u = 0 & \text{in } \Omega^* + u, \\ \operatorname{div} v_u = 0 & \text{in } \Omega^* + u, \\ v_u = \phi & \text{on } \partial\Omega, \\ v_u = 0 & \text{on } \partial D + u, \\ \int_{\Omega^*} p_u \circ (I + u) \, dx = 0, \end{cases} \quad (8)$$

where $\phi \in H^{3/2}(\partial\Omega)^N$ satisfies the compatibility condition $\int_{\partial\Omega} \phi \cdot n \, ds = 0$.

We define the following cost functional. Let σ_m be a given measurement of the normal component of the stress tensor on $\Gamma_m \subset \partial\Omega$, corresponding to the target obstacle (Fig. 1)

$$J(u) = \int_{\Gamma_m} |\sigma(v_u, p_u)n - \sigma_m|^2. \quad (9)$$

Then our inverse problem corresponds to the minimization problem

$$\min_{u \in \mathcal{U}_{ad}} J(u),$$

where $\mathcal{U}_{ad} \subset W^{1,\infty}(\Omega, \mathbb{R}^n)$ is the set of deformations such that $D + u \in \mathcal{D}_{ad}$, that is, $D + u$ is an admissible geometry. We can note that thanks to the identifiability result (cf. Theorem 2), the functional J has a unique global minimum, which is achieved when the cost function J vanishes. Moreover, due to the analyticity of the solutions of (8) with respect to the perturbation parameter u , we have that the functional $J(u)$ is analytic in a neighborhood of $u = 0$. Thus, we are interested in obtaining an asymptotic expansion with respect to u in order to compute the derivatives. This will also be useful to develop a numerical scheme based in gradient methods.

We recall that if we consider a regular function $u \rightarrow z_u$, it is enough to consider a differentiable function, then we have for

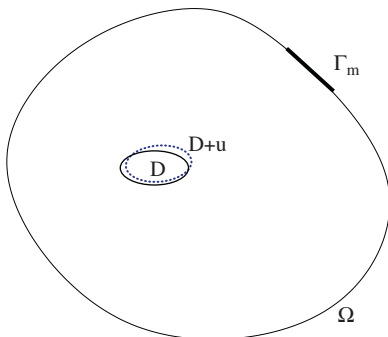


Fig. 1. Boundary measurement.

a set $S \subset \mathbb{R}^n$,

$$\begin{aligned} \int_{\partial S+u} z(u) \, ds &= \int_{\partial S} z(0) \, ds + \int_{\partial S} z'(u) \, ds \\ &+ \int_{\partial S} (u \cdot \operatorname{grad} z(0) + z(0) \operatorname{div}_{\partial S} u) \, ds + o(u), \end{aligned}$$

where $o(u)/\|u\|_{1,\infty} \rightarrow 0$ as $\|u\|_{1,\infty} \rightarrow 0$, $z'(u)$ being the first local variation of z_u at $u = 0$ in the direction u (see [6]). Moreover, since $\operatorname{supp} u \subset \subset \Omega$, $u = 0$ in a neighborhood of $\partial\Omega$, and considering $S = \Gamma_m$ we have

$$z(u) = |\sigma(v_u, p_u)n - \sigma_m|^2 \quad \text{on } \Gamma_m, \quad (10)$$

and

$$\begin{aligned} \int_{\Gamma_m} |\sigma(v_u, p_u)n - \sigma_m|^2 \, ds &- \int_{\Gamma_m} |\sigma(v_0, p_0)n - \sigma_m|^2 \, ds \\ &= \int_{\Gamma_m} 2(\sigma(v_0, p_0)n - \sigma_m) \cdot \sigma(v'(u), p'(u))n \phi \, ds + o(u), \end{aligned} \quad (11)$$

where $(v'(u), p'(u))$ is solution of

$$\begin{cases} -\operatorname{div} \sigma(v'(u), p'(u)) = 0 & \text{in } \Omega^*, \\ \nabla \cdot v'(u) = 0 & \text{in } \Omega^*, \\ v'(u) = 0 & \text{on } \partial\Omega, \\ v'(u) = -(u \cdot n) \frac{\partial v_0}{\partial n} & \text{on } \partial D. \end{cases} \quad (12)$$

Note that (11) corresponds to the first order expansion of the cost functional

$$\begin{aligned} J(u) &= J(0) + \int_{\Gamma_m} 2(\sigma(v_0, p_0)n - \sigma_m) \cdot \sigma(v'(u), p'(u))n \phi \, ds \\ &+ o(u). \end{aligned}$$

On the other hand, let us to consider the auxiliary problem

$$\begin{cases} -\operatorname{div} \sigma(\varphi, q) = 0 & \text{in } \Omega^*, \\ \nabla \cdot \varphi = 0 & \text{in } \Omega^*, \\ \varphi = 0 & \text{on } \partial\Omega \setminus \Gamma_m, \\ \varphi = 0 & \text{on } \partial D, \\ \varphi = 2(\sigma(v_0, p_0)n - \sigma_m)n & \text{on } \Gamma_m, \end{cases} \quad (13)$$

then multiplying the first equation of (12) by φ and integrating by parts, we obtain

$$\begin{aligned} \int_{\partial\Omega \cup \partial D} \sigma(v'(u), p'(u))n \varphi \, ds \\ = \int_{\partial\Omega \cup \partial D} v'(u) \cdot \sigma(\varphi, q)n \, ds. \end{aligned} \quad (14)$$

Thus, we have

$$\int_{\Gamma_m} \sigma(v'(u), p'(u))n \varphi \, ds = \int_{\partial D} v'(u) \cdot \sigma(\varphi, q)n \, ds, \quad (15)$$

that is

$$\begin{aligned} \int_{\Gamma_m} \sigma(v'(u), p'(u))n \cdot (2(\sigma(v_0, p_0)n - \sigma_m)n) \, ds \\ = \int_{\partial D} \left(-(u \cdot n) \frac{\partial v_0}{\partial n} \right) \cdot \sigma(\varphi, q)n \, ds. \end{aligned} \quad (16)$$

Therefore, we conclude that

$$J(u) = J(0) - \int_{\partial D} \left((u \cdot n) \frac{\partial v_0}{\partial n} \right) \cdot \sigma(\varphi, q)n \, ds + o(u), \quad (17)$$

which give us the first order expansion of the cost function. In an analogous way it is possible to compute the higher order terms by considering the higher order local variations of the solution (v_u, p_u) of the problem (8). A similar computation of the derivative for the Navier–Stokes equation was obtained in [7].

3. A numerical approach

This section deals with the development of numerical algorithms to recover geometric information about an object immersed in a region filled by a viscous fluid governed by the Stokes system. Additional information is provided by measurements of the internal forces (stress forces) density around the region boundaries. In particular we are interested in recovering information about the position, geometry and volume of the object.

The numerical strategy we have followed to solve our inverse geometric problem is based in the following general observation: The problems of optimal and inverse design can both be systematically treated within the mathematical theory for the control of systems governed by partial differential equations, by regarding the design problem as a control problem in which the control is the shape of the boundary. This approach requires a well established shape differentiation theory. The inverse geometric problem then becomes a special case of the optimal design problem in which the shape changes are driven by the discrepancy between the current and target pressure distributions.

Search techniques used in the control process are gradient based method, which requires the formulation of the Jacobian. By using techniques coming from domain differentiation one can formulate the gradient in terms of adjoint variables, which greatly improve the efficiency of the process against finite difference methods.

The objective function, even for simple geometries as circles, shows to be non convex which requires the development of heuristic method to reach the global minimum. Standard methods like Simulated Annealing (no need of gradient) requires high number of function evaluations and proves themselves quite unsatisfactory. However a heuristic method based on similar concept was developed. The idea is to use a re-scaling of the parameters and to allow for “up-hills” moves within the search. The factoring involved in the re-scaling plays the same role as the annealing temperature, where the transition from higher to lower scales allows the jumps from local to global minimum.

In this work we consider the following 2D problem, which corresponds to a channel with open ends, the velocity on the inlet satisfies a parabolic profile, while the boundary conditions in the outlet are those of a free boundary. The object satisfies the non-slip boundary condition (null velocity) on the rest of the boundary. Immersed in the fluid there is a circular obstacle with unknown center and radius. Our goal is a three (ball shaped case), five (elliptic obstacle) or even a more general situation where the geometry of the obstacle depends on a finite number of degrees of freedom (polygonal geometries) (Figs. 2–4).

To simplify matters, we start considering the simplest case of a ball shaped obstacle, which consists in recovering the center (a, b) and the radius r of the obstacle by means of exterior measurements, namely the normal component of the stress tensor in the upper side, later we will show some examples also considering also ellipse shaped obstacle. That is, let $\Omega^* = \Omega \setminus \overline{B_{a,b,r}}$, where Ω is the rectangle $(-10, 10) \times (-5, 5)$ and $B_{a,b,r} = B((a, b), r)$ (see Fig. 5) and let consider the Stokes system

$$\begin{cases} \operatorname{div} \sigma(v, p) = 0 & \text{in } \Omega^*, \\ \operatorname{div} v = 0 & \text{in } \Omega^*, \\ v = \phi & \text{on } \Gamma_{in}, \\ \sigma(v, p)n = 0 & \text{on } \Gamma_{out}, \\ v = 0 & \text{on } \Gamma_m \cup \Gamma, \\ v = 0 & \text{on } \partial B_{a,b,r}. \end{cases} \quad (18)$$

In order to do that, let us introduce

$$S = \{(a, b, r) \in \mathbb{R}^3 : \overline{B((a, b), r)} \subset \Omega\}.$$

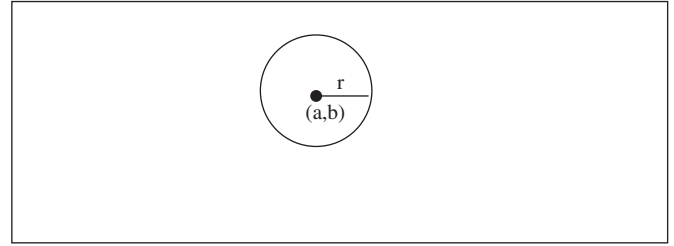


Fig. 2. Three degrees of freedom, which correspond to the center (a, b) and the radius r of the ball.

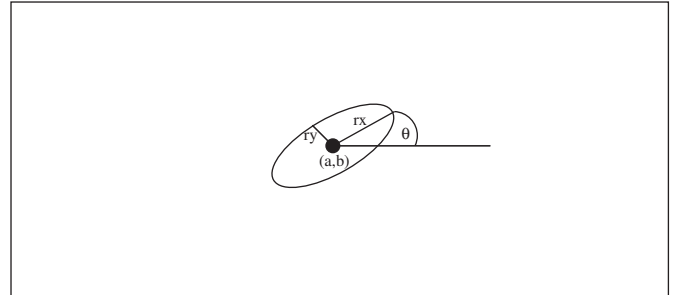


Fig. 3. Five degrees of freedom, which correspond to the center (a, b) , the angle θ and the semi-axes r_x and r_y of the ellipse.

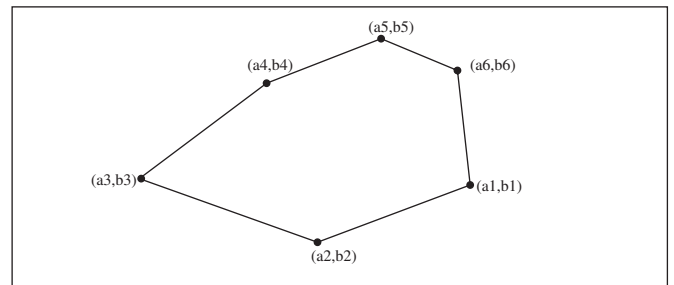


Fig. 4. Finite number of degrees of freedom, which correspond to the edges (a_i, b_i) of the polygon.

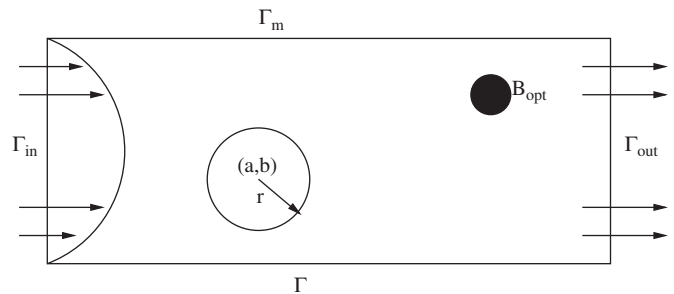


Fig. 5. Domain of study.

And let σ_m be a given measurement of the normal component of the stress tensor on $\Gamma_m \subset \partial\Omega$, corresponding to the target obstacle. Then we define the functional

$$J : S \subset \mathbb{R}^3 \rightarrow \mathbb{R}$$

as

$$J(a, b, r) = \int_{\Gamma_m} |\sigma(v, p)n - \sigma_m|^2 ds, \quad (19)$$

where (v, p) is the unique solution of the Stokes system (18).

Thus, our problem can be equivalently formulated as the following minimization problem:

$$\min_{(a,b,r) \in S} J(a, b, r). \tag{20}$$

It is clear, thanks to the identifiability result (cf. Theorem 2), that the functional J has a unique global minimum, which is achieved when the cost function J vanishes. At first glance this would mean that the minimization problem (20) can be solved trivially. This is however far from being true since the solution corresponds to the unknown rigid body B_{opt} (see Fig. 5), which is our actual goal. But setting up this minimization problem is not a completely void idea because it provides a strategy or algorithm to recover B_{opt} , numerically, or at least to get closer to its own position and shape: starting from an initial configuration B_0 we look for a sequence of coordinates $(a_n, b_n, r_n) \in S$ in such a way that the objective function J decreases as n goes to infinity, and furthermore that $J(a_n, b_n, r_n) \rightarrow 0$ as $n \rightarrow \infty$.

On the other hand, from the regularity of the solutions of the Stokes system with respect to the obstacle, the functional J is a regular function from $S \subset \mathbb{R}^3$ to \mathbb{R} , moreover it is a differentiable function. Thus, we can see that the derivative at the point (a, b, r) in the direction $(\tilde{a}, \tilde{b}, \tilde{r})$ is given by

$$J'((a, b, r); (\tilde{a}, \tilde{b}, \tilde{r})) = \int_{\partial B_{a,b,r}} [(\tilde{a} - a, \tilde{b} - b) \cdot n + (\tilde{r} - r) \left[\frac{\partial v}{\partial n} \cdot \sigma(\varphi, q)n \right]] ds, \tag{21}$$

where (φ, q) is the unique solution of the so-called adjoint problem

$$\begin{cases} -\text{div}(\sigma(\varphi, q)) = 0 & \text{in } \Omega^*, \\ \text{div}(\varphi) = 0 & \text{in } \Omega^*, \\ \sigma(\varphi, q)n = 0 & \text{on } \Gamma_{out}, \\ \varphi = 0 & \text{on } \Gamma_{in} \cup \Gamma, \\ \varphi = 2[\sigma(v, p) - \sigma_m]n & \text{on } \Gamma_m, \\ \varphi = 0 & \text{on } \partial B_{a,b,r}. \end{cases} \tag{22}$$

Now, we give further details about how such a numerical algorithm can be practically implemented to recover simple particular geometries of rigid bodies (essentially spheres and ellipsoids). In this case, we implement the steepest descent (SD) and non linear conjugate gradient (NLCG) method as shown in [8] (these algorithms are subsequently referred as NR methods). The main idea is to update the parameter space (x) at

each iteration (i) by

$$x_i = x_{i-1} + \alpha d,$$

where d is the search direction defined accordingly for SD and CG methods, and α is the step size, which is chosen such that the maximum function reduction is attained at each iteration. The latter is achieved by a standard line search method (brent as defined in [8]) which does not use gradient information but function evaluation only.

It is a well known fact that gradient search methods perform better if appropriate scaling is used for the gradient. The idea is to choose a scaling such that all the components of the gradient are of the same magnitude. For instance if we take our geometry with the target ball of center $(a_{opt}, b_{opt}) = (0, 0)$ and radius $r_{opt} = 0.8$ and the initial guess is the ball with center $(a_0, b_0) = (1, 0)$ and radius $r_0 = 0.6$, then the gradient of J at (a_0, b_0, r_0) is

$$\begin{aligned} \nabla J(a_0, b_0, r_0) &= \left(\frac{\partial J}{\partial x}, \frac{\partial J}{\partial y}, \frac{\partial J}{\partial r} \right) (1, 0, 0.6) \\ &= (1.18226, 0.14909, -35.5256). \end{aligned}$$

Thus an initial scaling for this problem would be to reduce the radius component by a factor of 100 and leave the others unchanged. Lets define the re-scaling by the new set of variables:

$$(X, Y, R) = (x, y, \text{fact}R * r),$$

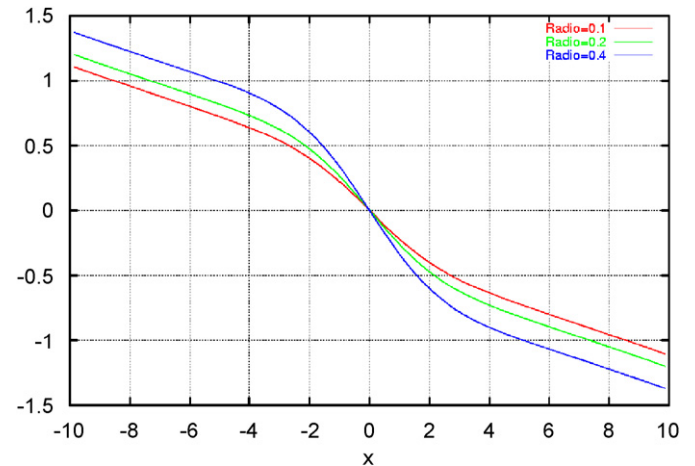


Fig. 7. Pressure measurements on the upper boundary for same center and different radius ($r = 0.1, 0.2$ and 0.4).

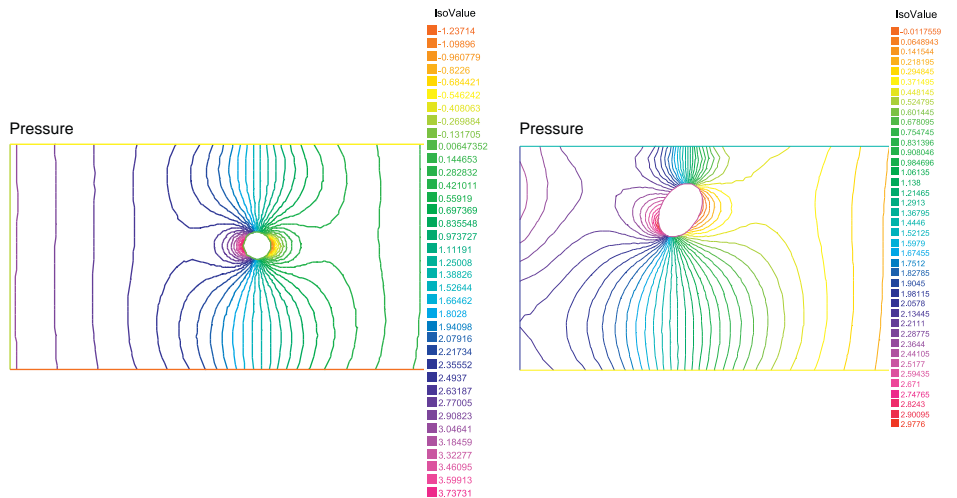


Fig. 6. Isobar corresponding to a ball and to an ellipse.

that means that for a gradient search over the parameters X, Y and R the methods goes as

$$R = R_0 + \alpha \partial_R J \tag{23}$$

but $grad_R = (1/factR)\partial_R J$, where J is our objective function.

Since $R = factR * r$, (23) is written as

$$r = r_0 + \frac{\alpha}{(factR)^2} \partial_r J.$$

The meaning of $factR$ is to attenuate (or enlarge) the contribution of R against the other variables. In our above example to achieve a 100 reduction, $factR$ need to be set to 10.

The point is that starting at $factR = 0.1$ makes the radius contribution dominant, which takes the solution to a local minimum. Then by increasing $factR$, the contribution of r decreases which allows to find another local minimum. The idea

is to eventually reach the global minimum. The heuristic is quite analogous to the SA method, where the choice and setting of $factR$ is as much arbitrary as the choice of temperature in SA.

Finally, we can note that the choice of the initial guess it is not completely by hazard, in fact from the pressure measurements on the upper boundary Γ_m , we can see that the obstacle introduce a perturbation on the pressure field as it is shown in Fig. 6; and we can see that the x -component of the inflection point of the pressure measured on the upper side of the boundary corresponds to the x -component of the center, and it is independent of the radius which allow us to choose a good initial guess (see Fig. 7).

Now, we will show some numerical results for two cases. In both of them we consider a Poiseuille flow and the container is the rectangle $(-10, 10) \times (-5, 5)$ (Figs. 8–11).

For the first one, we consider the target as the ball of center $(a_{opt}, b_{opt}) = (5, -3)$ with radius $r_{opt} = 0.4$.

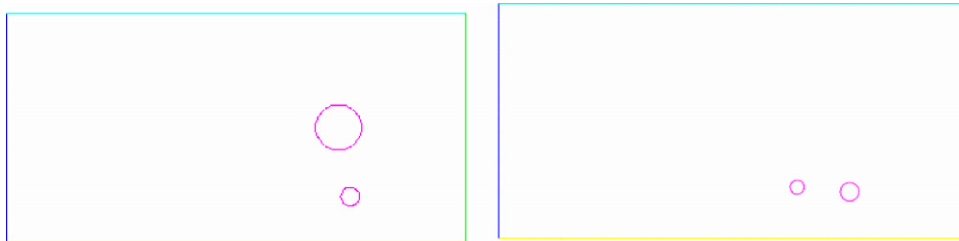


Fig. 8. Iterations 1 and 10 $(a_1, b_1, r_1) = (4.5, 0, 1)$ and $(a_{10}, b_{10}, r_{10}) = (2.7910, -2.7989, 0.3)$.

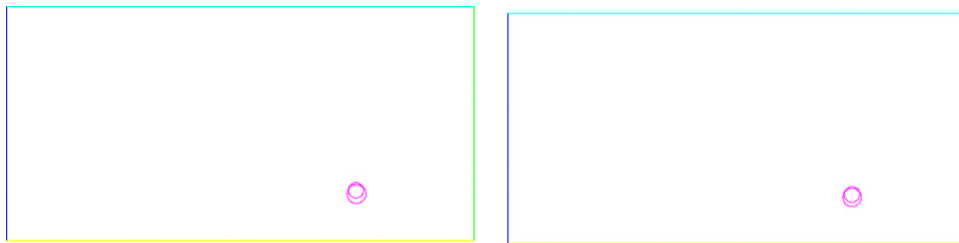


Fig. 9. Iterations 25 and 45 $(a_{25}, b_{25}, r_{25}) = (4.9746, -2.8437, 0.3271)$ and $(a_{45}, b_{45}, r_{45}) = (4.9930, -2.8727, 0.3363)$.

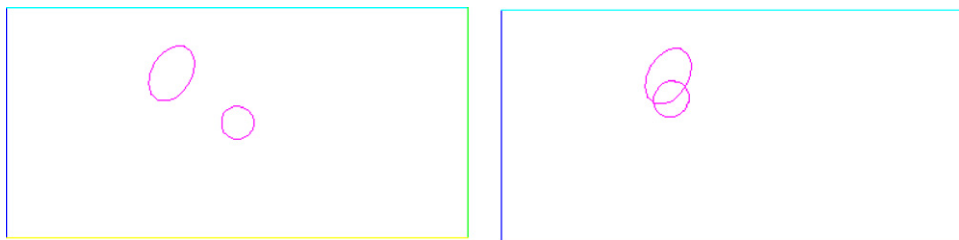


Fig. 10. Iterations 1 and 20 $(a_1, b_1, r_{x,1}, r_{y,1}, \theta_1) = (0, 0, 1, 1, 0)$ and $(a_{20}, b_{20}, r_{x,20}, r_{y,20}, \theta_{20}) = (-3.8205, 1.63942, 1.12443, 1.03204, 0.922812)$.

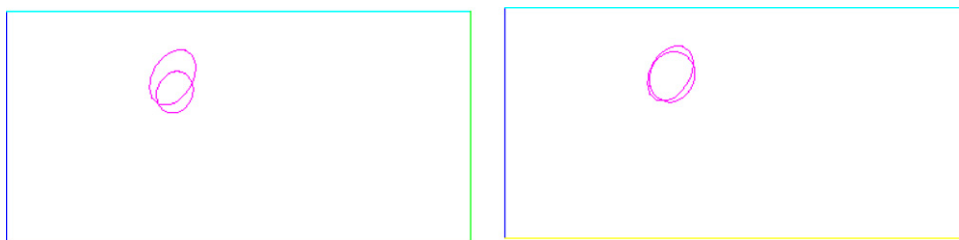


Fig. 11. Iterations 40 and 70 $(a_{40}, b_{40}, r_{x,40}, r_{y,40}, \theta_{40}) = (-3.87028, 2.08849, 1.27299, 1.07465, 1.20775)$ and $(a_{70}, b_{70}, r_{x,70}, r_{y,70}, \theta_{70}) = (-3.89335, 2.78944, 1.5613, 1.35159, 1.20772)$.

A second numerical test corresponds to the target as the ellipse of center $(a_{opt}, b_{opt}) = (-4, 3)$ with semiaxes $r_{x,opt} = 1.8, r_{y,opt} = 1.2$ and rotation angle $\theta_{opt} = \pi/3$.

In summary, the global method seems quite robust to deal with detecting a Stokes fluid, independent of the choice of the initial guess. Moreover, we can note that this numerical approach can be extended to another geometries as ellipses or polygonal geometries as well (finite number of parameters).

References

- [1] Uhlmann G. Inverse boundary value problems for partial differential equations, Doc Math J 1998; Extra volume ICM: 1–10.
- [2] Kohn R, Vogelius M. Determining conductivity by boundary measurements. Commun Pure Appl Math 1984;37:289–98.
- [3] Sylvester J, Uhlmann G. A uniqueness theorem for an inverse boundary problem in electrical prospection. Commun Pure Appl Math 1986;39:91–112.
- [4] Alvarez C, Conca C, Friz L, Kavian L, Ortega JH. Identification of immersed obstacles via boundary measurements. Inverse Problems 2005;21:1531–52.
- [5] Bello JA, Fernández-Cara E, Lemoine J, Simon J. The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier–Stokes flow. SIAM J Control Optim 1997;35(2):626–40.
- [6] Simon J. Differentiation with respect to the domain in boundary value problems. Num Funct Anal Optim 1980;2:649–87.
- [7] Doubova A, Fernández-Cara E, Ortega JH. On the identification of a single body immersed in a Navier–Stokes fluid. Eur J Appl Math 2007;17:1–24.
- [8] Press WH, Teukolsky SA, Vetterling WT, Flannery BP. Numerical recipes in C: the art of scientific computing. 2nd ed. Cambridge: Cambridge University Press; 1997.