

Approximation of Young measures by functions and application to a problem of optimal design for plates with variable thickness

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Given a parametrised measure and a family of continuous functions (φ_n) , we construct a sequence of functions (u_k) such that, as $k \rightarrow \infty$, the functions $\varphi_n(u_k)$ converge to the corresponding moments of the measure, in the weak * topology. Using the sequence (u_k) corresponding to a dense family of continuous functions, a proof of the fundamental theorem for Young measures is given.

We apply these techniques to an optimal design problem for plates with variable thickness. The relaxation of the compliance functional involves three continuous functions of the thickness. We characterise a set of admissible generalised thicknesses, on which the relaxed functional attains its minimum.

1. Introduction

Usual strategies to prove existence of solutions for PDEs, and the problems of the calculus of variations, consist in studying the limit of an approximating sequence. When the problem is linear, this limit will usually satisfy the PDE or be an admissible candidate for a minimum. However, the limit of nonlinear expressions does not in general coincide with the nonlinear expression of the limit. The following theorem, known as the fundamental theorem for Young measures, enables us to represent limits of nonlinear expressions of a sequence of functions by a parametrised family of measures.

THEOREM 1.1. *Let Q be a closed boundary set in \mathbb{R}^p , Ω an open bounded set in \mathbb{R}^r .*

(1) *Let (u_k) be a sequence of measurable functions satisfying*

$$u_k : \Omega \rightarrow \mathbb{R}^p, \quad u_k \in Q \quad \text{a.e. } x \in \Omega,$$

and consider a continuous function $f \in \mathcal{C}(Q, \mathbb{R})$. There exists a subsequence, still denoted

(u_k) and a family of Borel probability measures $(v_x)_{x \in \Omega}$ such that

$$\begin{aligned} \text{supp}(v_x) &\subset Q \quad \text{a.e. } x \in \Omega, \\ f(u_k) &\rightharpoonup \bar{f}(x) \quad \text{weakly } * \text{ in } L^\infty(\Omega), \end{aligned}$$

where $\bar{f}(x) = \langle v_x, f \rangle$. (For a Borel measure μ with support in \mathbb{R}^p , $\langle \mu, f \rangle$ denotes $\int_{\mathbb{R}^p} f(\lambda) d\mu(\lambda)$.)

(2) Conversely, given (v_x) as above, there exists a sequence of measurable functions $u_k: \Omega \rightarrow Q$, such that

$$\forall f \in \mathcal{C}(Q, \mathbb{R}), \quad f(u_k) \rightharpoonup \bar{f}(x) = \langle v_x, f \rangle \quad \text{weakly } * \text{ in } L^\infty(\Omega). \quad (1.1)$$

Parametrised measures were introduced by L. C. Young [20] as a means of studying problems of the calculus of variations that did not admit solutions in the classical sense. Subsequent developments and applications to problems of optimal control have been made by McShane [14].

The above version of the fundamental theorem is due to Tartar [19], who applied it to show existence of solutions to a single scalar hyperbolic equation [9, 17]. Di Perna [15] proved existence of solutions in the more difficult case of a system of two hyperbolic equations in one space variable. James and Kinderlehrer [13], and Chipot and Kinderlehrer [8] have used Young measures in the context of variational problems of continuum mechanics. Kinderlehrer and Pedregal [11] have addressed the problem of identifying those Young measures which are weak * limits of a sequence of gradients.

Tartar's proof is based on the Radon–Nykodym Theorem, and on the closedness and convexity properties of the set of measures associated with a measurable function, i.e. measures μ such that there exists a measurable function $u: \Omega \rightarrow Q$, with $\langle \mu, f \rangle = \int_\Omega f(x, u(x)) dx$, for all continuous functions $f \in \mathcal{C}(\Omega \times Q, \mathbb{R})$.

In this paper, we are mainly interested in proving the converse part of the fundamental theorem in a more constructive way: given a parametrised measure (v_x) , we exhibit a sequence (u_k) that satisfies (1.1).

Our construction can be sketched as follows. We consider a dense family of functions $(\varphi_m) \in \mathcal{C}(Q, \mathbb{R})$. For each n , we construct a sequence of functions $(u_{k,n}): \Omega \rightarrow Q$, that satisfies (1.1) for the n first φ_m . These functions are obtained using a result in measure theory stated in Section 2 of this paper, Theorem 2.2, which yields an approximation of the parametrised measure v_x by convex sums of Dirac masses. A diagonal process, as n tends to infinity, yields the desired sequence (u_k) .

The sequence of functions we construct, have ‘‘rapid variations’’, and can be interesting in the characterisation of minimising sequences in problems of the calculus of variations. As an example, we apply our method to a problem of optimal design of orthotropic plates with parallel stiffeners. The admissible half-thicknesses $h \in L^\infty$ depend on one variable only. We would like to minimise the compliance (the work done by the load) under the constraint of a prescribed volume. However, this minimisation problem does not have a solution in the set of admissible thicknesses, because the non-zero coefficients of the thickness matrix are proportional to h^3 . We show that, with our approximation Theorem 2.2, the set of admissible thicknesses and the definition of the compliance can be extended, so that the minimisation problem has a solution.

This paper is organised as follows: in Section 2, we state our approximation

Theorem 2.2: given a parametrised measure $(\nu_x)_{x \in \Omega}$, there exists a measurable convex sum of Dirac masses for a.e. x , whose moments, with respect to a finite number of continuous functions (φ_n) , coincide with those of ν_x . This result is related to measurable multifunctions and to the selection theorem of convex analysis [3, 5], although we do not use the same techniques here. We prove this first for $p = 1$, i.e. when the support of ν_x lies in an interval, and when the continuous functions are the monomials $(\lambda, \lambda^2, \lambda^3, \dots)$. This case is related to the Gauss–Jacobi mechanical quadrature and, when ν_x is the Lebesgue measure, to the Gauss–Lobatto quadrature formula.

In Section 3, we consider the general case $p \geq 1$, with sufficient hypotheses on the family (φ_n) . The proof of the fundamental theorem for Young measures is given in Section 4, and Section 5 is devoted to the application to optimal design of plates with stiffeners.

2. Approximation of the moments of a measure by those of convex sums of Dirac masses

Let $Q = [0, 1]^p$ and let $(\varphi_k)_{k \in \mathbb{N}}$ be a sequence of continuous functions defined on Q , that satisfy the following hypotheses:

- (H1) the functions $(\varphi_k)_{k \in \mathbb{N}}$ form a dense set of linearly independent functions in $\mathcal{C}(Q, \mathbb{R})$;
- (H2) the functions $(\varphi_k)_{k \in \mathbb{N}}$ are positive and bounded on Q , uniformly with respect to k , for example

$$\forall \lambda \in Q, \quad 0 \leq \varphi_k(\lambda) \leq 1;$$

- (H3) there exists $N \geq 1$, such that for $n \geq N$, no point $(\varphi_1(\lambda), \dots, \varphi_n(\lambda))$, with $\lambda \in Q$, can be written as a finite convex combination of points of the same form. That is, if

$$\begin{pmatrix} \varphi_1(\lambda) \\ \vdots \\ \varphi_n(\lambda) \end{pmatrix} = \sum_{i=1}^m \theta_i \begin{pmatrix} \varphi_1(\lambda_i) \\ \vdots \\ \varphi_n(\lambda_i) \end{pmatrix}$$

with

$$\lambda, \lambda_i \in Q \quad \theta_i \geq 0 \quad 1 \leq i \leq m, \quad \sum_{i=1}^m \theta_i = 1,$$

then there exists $i_0, 1 \leq i_0 \leq m$, such that

$$\begin{cases} \lambda_{i_0} = \lambda, \\ \theta_{i_0} = 1. \end{cases}$$

Remark 2.1. Hypothesis (H3) implies that the functions $(\varphi_k)_{k \in \mathbb{N}}$ separate the measures δ_λ . When $p = 1$, and when the functions φ_k are the polynomials λ^k , assumptions (H1)–(H3) are clearly satisfied. In particular, (H3) is satisfied since the curve $\lambda \rightarrow (\lambda, \lambda^2, \dots, \lambda^n)$ is a twisted curve in \mathbb{R}^n .

Our construction of a sequence of functions satisfying (1.1), in the converse part

of the fundamental theorem for Young measures, is based on the following result, which we sometimes refer to as the “approximation theorem”.

THEOREM 2.2. *Let Ω be an open bounded set in \mathbb{R}^r , and $(\mu_x)_{x \in \Omega}$ be a family of positive Borel measures such that*

$$\begin{aligned} \text{supp } (\mu_x) &\subset Q \\ \langle \mu_x, 1 \rangle &= 1 \end{aligned} \quad \text{a.e. } x \in \Omega.$$

Assume that $x \rightarrow \mu_x$ is measurable, i.e. that

$$\forall f \in \mathcal{C}(Q, \mathbb{R}), \quad x \rightarrow \langle \mu_x, f \rangle$$

is Lebesgue measurable on Ω .

Then there exist $2(n + 1)$ measurable functions $\theta_j(x), a_j(x)$, defined on Ω ,

$$\begin{cases} \theta_j(x) \in [0, 1] \\ a_j(x) \in Q \\ \sum_{j=1}^{n+1} \theta_j(x) = 1 \end{cases} \quad \text{a.e. } x \in \Omega,$$

such that

$$\forall 1 \leq m \leq n, \quad \langle \mu_x, \varphi_m \rangle = \sum_{j=1}^{n+1} \theta_j(x) \varphi_m(a_j(x)) \quad \text{a.e. } x \in \Omega.$$

If, moreover, $p = 1$, and $\varphi_k = \lambda^k$, then there exist $2n$ measurable functions $\theta_j(x), a_j(x)$, defined on Ω , with value in $[0, 1]$, such that

$$\forall 1 \leq m \leq 2n - 1 \quad \langle \mu_x, \lambda^m \rangle = \sum_{j=1}^n \theta_j(x) a_j^m(x) \quad \text{a.e. } x \in \Omega. \tag{2.1}$$

The rest of this section is devoted to the proof of this theorem when $p = 1$, and when $\varphi_k = \lambda^k$.

Orthogonal polynomials

Let us fix a point $x \in \Omega$, and let us consider the distribution function $\alpha(\lambda)$ associated with the measure ν_x , i.e. the unique increasing function of bounded variation, defined on $[0, 1]$, such that

$$\forall f \in \mathcal{C}([0, 1]), \quad \langle \nu_x, f(\lambda) \rangle = \int_0^1 f(\lambda) d\alpha(\lambda),$$

where the integral on the right-hand side is defined in the sense of Lebesgue–Stieltjes integrals [2]. Since $L^2_\alpha(0, 1)$ is a Hilbert space, the following result holds:

THEOREM 2.3. *Assume that α has at least $l + 1$ points of increase. There exists a unique set of orthonormal polynomials with real coefficients, $p_0(\lambda), p_1(\lambda), \dots, p_l(\lambda)$, such that*

$$(a) \quad \int_0^1 p_n(\lambda) p_m(\lambda) d\alpha(\lambda) = \delta_{nm} \quad 0 \leq n, m \leq l, \quad \text{and}$$

(b) for $n \leq l$, the degree of $p_n(\lambda)$ is exactly n , and the coefficient of λ^n is positive.

Let us list some of the properties of these polynomials. The proofs (and more properties) can be found in Szegő's book [18].

PROPOSITION 2.4. *The quantity*

$$c_n = \int_0^1 \lambda^n d\alpha(\lambda) = \langle v_{x_0}, \lambda^n \rangle$$

is the n -th moment of the distribution function α . The polynomial $p_n(\lambda)$ has the following explicit representation:

$$p_n(\lambda) = (D_{n-1} D_n)^{-\frac{1}{2}} \begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_n \\ c_1 & c_2 & c_3 & \dots & c_{n+1} \\ \vdots & & & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \dots & c_{2n-1} \\ 1 & \lambda & \lambda^2 & \dots & \lambda^n \end{bmatrix},$$

where the brackets denote the determinant and where, for $n \geq 0$,

$$D_n = [c_{i+j}]_{0 \leq i, j \leq n}.$$

Let $\mathcal{X}_n(\lambda)$ denote the set of polynomials with complex coefficients, of degree less than or equal to n . The polynomials $\{p_k\}$, $0 \leq k \leq n-1$, form a family of increasing degree, hence a basis of $\mathcal{X}_{n-1}(\lambda)$. Any polynomial in the latter space can be written as a linear combination of the p_k , and the orthogonality properties yield the following proposition:

PROPOSITION 2.5. $\forall q \in \mathcal{X}_{n-1}(\lambda), \int_0^1 q(\lambda) p_n(\lambda) d\alpha(\lambda) = 0$.

We also need some properties of the zeros of orthogonal polynomials:

PROPOSITION 2.6. *The zeros of $p_n(\lambda)$ are real and distinct. They are located in the interior of the interval $(0, 1)$.*

Proof. Let a_0 be an arbitrary root of $p_n(\lambda)$. As its coefficients are real, it is divisible by the polynomial $(\lambda - \bar{a}_0)$. The polynomial $(p_n(\lambda))/(\lambda - \bar{a}_0)$ is in \mathcal{X}_{n-1} , and is thus orthogonal to $p_n(\lambda)$:

$$0 = \int_0^1 \frac{p_n(\lambda)}{\lambda - a_0} p_n(\lambda) d\alpha(\lambda) = \int_0^1 (\lambda - a_0) \left(\frac{p_n(\lambda)}{|\lambda - a_0|} \right)^2 d\alpha(\lambda),$$

i.e.

$$a_0 \int_0^1 \left(\frac{p_n(\lambda)}{|\lambda - a_0|} \right)^2 d\alpha(\lambda) = \int_0^1 \lambda \left(\frac{p_n(\lambda)}{|\lambda - a_0|} \right)^2 d\alpha(\lambda).$$

Since the integrand on the right-hand side is positive, a_0 is real and

$$0 < a_0 < 1.$$

If a_0 were a multiple root, we would have

$$0 = \int_0^1 \frac{p_n(\lambda)}{(\lambda - a_0)^2} p_n(\lambda) d\alpha(\lambda) = \int_0^1 \left(\frac{p_n(\lambda)}{\lambda - a_0} \right)^2 d\alpha(\lambda),$$

which is impossible, since α is positive. \square

The following theorem is known as the Gauss–Jacobi mechanical quadrature:

THEOREM 2.7. *Let (a_{n1}, \dots, a_{nn}) denote the zeros of $p_n(\lambda)$. There exists a unique set of real numbers $(\theta_{n1}, \dots, \theta_{nn}) \in \mathbb{R}^n$, such that*

$$\forall p \in \mathcal{X}_{2n-1}(\lambda), \quad \int_0^1 p(\lambda) d\alpha(\lambda) = \sum_{i=1}^n \theta_{ni} p(a_{ni}). \quad (2.2)$$

The coefficients θ_{ni} are called the *Cristoffel numbers*, and satisfy

$$\theta_{ni} = \int_0^1 \left(\frac{p_n(\lambda)}{p'_n(\lambda)(\lambda - a_{ni})} \right)^2 d\alpha(\lambda) > 0. \quad (2.3)$$

Proof. Let p be a polynomial of degree less than $2n - 1$, and let Lp denote the Lagrange interpolation polynomial of degree $n - 1$, such that

$$Lp(a_{ni}) = p(a_{ni}), \quad 1 \leq i \leq n,$$

i.e.

$$Lp(\lambda) = \sum_{i=1}^n p(a_{ni}) \frac{p_n(\lambda)}{(\lambda - a_{ni})p'_n(a_{ni})}.$$

The polynomial $p(\lambda) - Lp(\lambda)$ is divisible by $p_n(\lambda)$:

$$\exists r \in \mathcal{X}_{n-1}(\lambda)/p(\lambda) - Lp(\lambda) = p_n(\lambda)r(\lambda).$$

Integrating with respect to α and applying Proposition 2.5 yields

$$\begin{aligned} \int_0^1 p(\lambda) d\alpha(\lambda) &= \int_0^1 Lp(\lambda) d\alpha(\lambda) \\ &= \sum_{i=1}^n p(a_{ni}) \int_0^1 \frac{p_n(\lambda)}{(\lambda - a_{ni})p'_n(a_{ni})} d\alpha(\lambda), \end{aligned}$$

which is formula (2.2) with

$$\theta_{ni} = \int_0^1 \frac{p_n(\lambda)}{(\lambda - a_{ni})p'_n(a_{ni})} d\alpha(\lambda).$$

To get (2.3), choose

$$p(\lambda) = \left(\frac{p_n(\lambda)}{(\lambda - a_{ni})p'_n(a_{ni})} \right)^2,$$

and (2.2) becomes

$$\int_0^1 \left(\frac{p_n(\lambda)}{(\lambda - a_{ni})p'_n(a_{ni})} \right)^2 d\alpha(\lambda) = \sum_{j=1}^n \left(\frac{p_n(a_{nj})}{(a_{nj} - a_{ni})p'_n(a_{ni})} \right) \theta_{nj} = \theta_{ni}. \quad \square$$

Proof of the second part of the approximation theorem

Let $C_k, k \geq 0$ be the set of points x in Ω , such that the distribution function associated to ν_x has k points of increase, i.e. ν_x is a sum of k Dirac masses. Let $\mathcal{O}_{n+1} = (\Omega \setminus \cup_{k=1}^n C_k)$.

As x varies in \mathcal{O}_{n+1} , we can define for $1 \leq j \leq n$

$$\theta_j(x) = \theta_{nj}, \quad a_j(x) = a_{nj},$$

to be the weights and roots associated to the measure μ_x .

If $x \in C_k$, we can only define k orthonormal polynomials, because the functions $\{1, \lambda, \lambda^2, \dots, \lambda^{k+1}\}$ are not independent. In this case, we define $a_i(x), \theta_i(x)$ as the roots and Cristoffel numbers associated with $p_k(\lambda)$, i.e.

$$\theta_j(x) = \theta_{kj}, \quad a_j(x) = a_{kj}, \quad 1 \leq j \leq k,$$

and we set

$$\theta_j(x) = 0, \quad a_j(x) = 0, \quad j > k.$$

With these definitions and using Theorem 2.7, we see that relations (2.1) are satisfied.

To complete the proof of Theorem 2.2, we have to show that these functions are measurable.

PROPOSITION 2.8. *Assume that $\{\nu_x\}_{x \in \Omega}$ is a family of probability measures with support in $[0, 1]$ such that, for all continuous functions f defined on $[0, 1]$, the function*

$$\langle \nu_x, f(\lambda) \rangle : \Omega \rightarrow \mathbb{R}$$

is measurable. Then the functions θ_j, a_j , as defined above, are measurable on $\Omega \times [0, 1]$ with respect to Lebesgue measure.

Proof. Step 1: Consider the function

$$G_{n+1} : x \in \mathcal{O}_{n+1} \rightarrow (a_{n1}(x), \dots, a_{nn}(x)).$$

By Proposition 2.6, G_{n+1} is the composition of the following functions:

$$g_1 : x \in \mathcal{O}_{n+1} \rightarrow p_n(\lambda) \in \mathcal{K}_n^*(\lambda),$$

$$g_2 : p \in \mathcal{K}_n^*(\lambda) \rightarrow (\lambda_i/p(\lambda_i) = 0) \in \mathbb{R}^n,$$

where $\mathcal{K}_n^*(\lambda)$ is the subset of elements of $\mathcal{K}_n(\lambda)$ which have n distinct roots. The Implicit Function Theorem ensures that g_2 is continuous on \mathcal{K}_n^* . On the other hand, Proposition 2.4 gives us explicit expressions for the coefficients of $p_n(\lambda)$ in terms of the moments $\langle \nu_x, \lambda^p \rangle$. By assumption, these are measurable functions. Hence, the composition $g_2 \circ g_1$ is measurable.

Theorem 2.7 provides explicit formulae for the Cristoffel numbers in terms of the moments and the roots of $p_n(\lambda)$. Hence, the mapping

$$x \in \mathcal{O}_{n+1} \rightarrow (\theta_{n1}(x), \dots, \theta_{nn}(x)) \in \mathbb{R}^n$$

is also measurable.

The same argument shows the measurability of the mappings

$$x \in C_k : \rightarrow (a_{k1}(x), \dots, a_{kk}(x), 0, \dots, 0),$$

$$x \in C_k : \rightarrow (\theta_{k1}(x), \dots, \theta_{kk}(x), 0, \dots, 0).$$

Step 2: We now show that the sets $C_k, 0 \leq k \leq n$, and \mathcal{O}_{n+1} , are measurable. Let \mathcal{M} be the set of Borel probability measures defined on $[0, 1]$. To each element of \mathcal{M} , there corresponds a unique increasing distribution function of bounded variation. Consider the function H which associates to an element in \mathcal{M} , the number of points of increase of its distribution function. Since the elements of \mathcal{M} are positive measures, the set $\mathcal{M}_n = H^{-1}([0, n])$ consists of the convex sums of at most n Dirac masses.

LEMMA 2.9. *The set \mathcal{M}_n is a closed set for the weak * topology on \mathcal{M} .*

Proof. Assuming that

$$\mu_k \rightarrow \mu \text{ weakly * in } \mathcal{M},$$

$$\mu_k = \sum_{i=1}^n \theta_{ki} \delta_{b_{ki}}, \quad \sum_{i=1}^n \theta_{ki} = 1, \quad \text{and } b_{ki} \in [0, 1],$$

we can extract a subsequence such that, as k tends to infinity,

$$\begin{cases} b_{ki} \rightarrow b_i \\ \theta_{ki} \rightarrow \theta_i \end{cases} \quad 1 \leq i \leq n.$$

Then, $\sum_{i=1}^n \theta_i = 1$, and for $\varphi \in \mathcal{C}([0, 1])$ we have

$$\langle \mu_k, \varphi \rangle = \sum_{i=1}^n \theta_{ki} \varphi(b_{ki}) \rightarrow \sum_{i=1}^n \theta_i \varphi(b_i) = \left\langle \sum_{i=1}^n \theta_i \delta_{b_i}, \varphi \right\rangle.$$

Thus, $\mu = \sum_{i=1}^n \theta_i \delta_{b_i}$ belongs to \mathcal{M}_n . \square

Now consider the mapping:

$$F : \Omega \rightarrow (\mathcal{M}, \text{weak * topology}),$$

$$x \rightarrow \nu_x.$$

By assumption F is measurable, and we have:

$$\begin{cases} \mathcal{O}_{n+1} = F^{-1}(\mathcal{M} \setminus \mathcal{M}_n), \\ C_k = F^{-1}(\mathcal{M}_k \setminus \mathcal{M}_{k-1}), \quad 1 \leq k \leq n. \end{cases}$$

Since the sets $(\mathcal{M} \setminus \mathcal{M}_n), (\mathcal{M}_k \setminus \mathcal{M}_{k-1})$ are Borel sets, \mathcal{O}_{n+1}, C_k are measurable. \square

3. The proof for $p \geq 1$

In this part we consider the general case. The proof is based on techniques of convex analysis, since polynomials of several variables are much more difficult to handle. However, our result will not be as sharp as in the case $p = 1$, since we need $n + 1$ Dirac masses to represent n moments of the original measure μ_x , for a.e. x .

Properties of the moment space

Let \mathcal{M} denote the space of (Borel) probability measures with supports in Q . We define the moment space D_n by

$$D_n = \{y \in \mathbb{R}^n \text{ such that } \exists \mu \in \mathcal{M} / y_m = \langle \mu, \varphi_m \rangle \ 1 \leq m \leq n\}.$$

When a point $y \in D_n$ is defined by a Dirac measure $\delta_{\hat{\lambda}}$, we say that y is an *image* of $\hat{\lambda}$.

PROPOSITION 3.1. *The set D_n is a closed, convex, bounded body in \mathbb{R}^n .*

Proof. Our proof and the proof of Theorem 3.2 extend the work of Karlin and Shapley [8]. For simplicity, we assume that $\varphi_m(0) = 0$ for all m .

(1) Since \mathcal{M} is a convex set, we have

$$\forall 1 \leq m \leq n, \quad \forall \theta \in [0, 1], \quad \forall \mu_1, \mu_2 \in \mathcal{M},$$

$$\theta \langle \mu_1, \varphi_m \rangle + (1 - \theta) \langle \mu_2, \varphi_m \rangle = \langle \theta \mu_1 + (1 - \theta) \mu_2, \varphi_m \rangle.$$

It follows from (H2) that D_n is bounded: indeed for $y = (\langle \mu, \varphi_1 \rangle, \dots, \langle \mu, \varphi_m \rangle) \in D_n$ we have

$$\forall 1 \leq m \leq n, \quad |y_m| = |\langle \mu, \varphi_m \rangle| \leq \sup |\varphi_m| \langle \mu, 1 \rangle \leq \sup |\varphi_m|.$$

(2) Let us consider a sequence of points $(y_k)_{k \geq 1}$ in D_n , that converges to some y_∞ . A measure μ_k is associated to each point, such that

$$\forall 1 \leq m \leq n, \quad y_{k,m} = \langle \mu_k, \varphi_m \rangle.$$

Since the sequence (μ_k) is relatively compact for the weak convergence in \mathcal{M} [2], we can extract a subsequence that converges weakly to a measure $\mu_\infty \in \mathcal{M}$. In particular μ_∞ satisfies

$$\forall 1 \leq m \leq n, \quad y_{k,m} \rightarrow \langle \mu_\infty, \varphi_m \rangle.$$

It follows that D_n is closed.

(3) To show that D_n is a body, i.e. that it contains an n -dimensional manifold, it suffices to prove that it contains a n -simplex. We claim that, since the functions $(\varphi_1, \dots, \varphi_n)$ are linearly independent on Q , there exist n distinct points $(\hat{\lambda}_1, \dots, \hat{\lambda}_n) \in Q^n$, such that

$$\det \begin{vmatrix} \varphi_1(\hat{\lambda}_1) & \dots & \varphi_1(\hat{\lambda}_n) \\ \vdots & & \vdots \\ \varphi_n(\hat{\lambda}_1) & \dots & \varphi_n(\hat{\lambda}_n) \end{vmatrix} \neq 0. \tag{3.1}$$

This claim is easily proved by induction: let us suppose that it is true up to $n - 1$. Were it not true for n , developing the determinant (3.1) about its last column would yield

$$\sum_{i=1}^n \varphi_i(\hat{\lambda}_n) G_i(\hat{\lambda}_1, \dots, \hat{\lambda}_{n-1}) = 0, \quad \forall (\hat{\lambda}_1, \dots, \hat{\lambda}_{n-1}) \in Q_n, \tag{3.2}$$

where the functions G_i are minor determinants. By the induction hypothesis, there exist $n - 1$ points $(\underline{\eta}_1, \dots, \underline{\eta}_{n-1})$ such that

$$G_n(\underline{\eta}_1, \dots, \underline{\eta}_{n-1}) \neq 0.$$

Relation (3.2) yields

$$\forall \underline{\lambda} \in Q, \quad \sum_{i=1}^n g_i \varphi_i(\underline{\lambda}) = 0,$$

with $g_i = G_i(\eta_1, \dots, \eta_{n-1})$, which is contradictory to the assumption of linear independence of the (φ_m) . Hence, there exist $(\underline{\lambda}_1, \dots, \underline{\lambda}_n) \in Q^n$, which satisfy (3.2). It follows that the images of $(0, \underline{\lambda}_1, \dots, \underline{\lambda}_n)$, i.e. the points

$$\underline{Q}, (\varphi_1(\underline{\lambda}_1), \dots, \varphi_n(\underline{\lambda}_1)), \dots, (\varphi_1(\underline{\lambda}_n), \dots, \varphi_n(\underline{\lambda}_n)),$$

form a n -simplex of D_n . \square

Characterisation of the extreme points of D_n

We recall that a point x in a convex set C is called an extreme point if x does not lie in the interior of any segment of points of C . We say that $x \in C$ is *spanned* by points $(x_i)_{1 \leq i \leq k} \in C^k$, if x is a convex combination of these points.

We are now ready to describe the extreme points of D_n : let C_n denote the points of D_n which are images of points of Q , i.e.

$$\begin{aligned} C_n &= \{ \langle \delta_{\underline{\lambda}}, \varphi_1 \rangle, \dots, \langle \delta_{\underline{\lambda}}, \varphi_n \rangle; \underline{\lambda} \in Q \} \\ &= \{ (\varphi_1(\underline{\lambda}), \dots, \varphi_n(\underline{\lambda})); \underline{\lambda} \in Q \}. \end{aligned}$$

THEOREM 3.2. *For $n \geq N$ given by hypothesis (H3), the set of extreme points of D_n is exactly C_n .*

Proof. Let $\tilde{\mathcal{M}}$ denote the subset of \mathcal{M} , consisting of finite convex combinations of Dirac masses with support in Q , and let \tilde{D}_n be the moment space of elements of $\tilde{\mathcal{M}}$. Clearly, \tilde{D}_n is spanned by C_n . Since C_n is closed and bounded, so is \tilde{D}_n [17]. Moreover, $\tilde{\mathcal{M}}$ is dense in \mathcal{M} for the weak topology [2]: for each $\mu \in \mathcal{M}$, there exists a sequence $\tilde{\mu}_k$ of elements of $\tilde{\mathcal{M}}$, such that

$$\forall f \in \mathcal{C}(Q, \mathbb{R}), \quad \langle \tilde{\mu}_k, f \rangle \rightarrow \langle \mu, f \rangle.$$

In particular, for $f = \varphi_m$, $1 \leq m \leq n$, we can approximate the moments of μ by the moments of $\tilde{\mu}_k$, for k large enough. Hence, D_n is contained in the closure of \tilde{D}_n , and since the latter is closed,

$$D_n = \tilde{D}_n.$$

Thus, C_n spans D_n . Assumption (H3) implies that no point in C_n can be spanned by other points of C_n . It follows that C_n is the set of extreme points of D_n [17]. \square

EXAMPLE 3.3. We consider the family of homogeneous monomials in $[0, 1]^p$, classified by increasing total order,

$$P_k = x_1^{k_1} \dots x_p^{k_p}, \quad k = (k_1, \dots, k_p).$$

We are going to show that this family satisfies hypotheses (H1)–(H3). The first two follow from standard results on polynomials, so we focus only on (H3).

Let

$$r \geq 2, \quad n = \binom{r+p}{r}.$$

We consider the n polynomials $K_n = \{P_k / |k| \leq r\}$, and the associated moment space D_n .

We denote by $k(i)$ the p -upple $(\delta_{i,j})_{1 \leq j \leq p}$, while $2k(i)$ stands for $(2\delta_{i,j})_{1 \leq j \leq p}$. Thus, the polynomials $P_{k(i)}$ and $P_{2k(i)}$ are simply x_i and x_i^2 respectively, and they belong to K_n .

Given a point $\underline{\eta} \in Q$, we consider the linear form $H_{\underline{\eta}}$ defined on \mathbb{R}^n by

$$\begin{aligned} H_{\underline{\eta}}(y) &= \sum_{i=1}^p (y_{2k(i)} - 2y_{k(i)}P_{k(i)}(\underline{\eta}) + P_{2k(i)}(\underline{\eta})) \\ &= \sum_{i=1}^p (y_{2k(i)} - 2y_{k(i)}\eta_i + \eta_i^2), \end{aligned}$$

where y_j denotes the j -th component of y , and η_i the i -th component of $\underline{\eta}$.

Let $y(\underline{\lambda})$ denote the image of a point $\underline{\lambda} \in Q$, in the moment space D_n , i.e.

$$y(\underline{\lambda})_k = P_k(\underline{\lambda}).$$

For $\underline{\lambda} \neq \underline{\eta}$, we have

$$\begin{aligned} H_{\underline{\eta}}(y(\underline{\lambda})) &= \sum_{i=1}^p (\lambda_i^2 - 2\lambda_i\eta_i + \eta_i^2) \\ &= \|\underline{\lambda} - \underline{\eta}\|^2 > 0, \\ H_{\underline{\eta}}(y(\underline{\eta})) &= 0. \end{aligned}$$

It follows that $y(\underline{\eta})$ cannot be written as a convex combination of other points of the form $y(\underline{\lambda})$, for we would have

$$\begin{aligned} 0 &= H_{\underline{\eta}}(y(\underline{\eta})) \\ &= \sum_k t_k H_{\underline{\eta}}(y(\underline{\lambda}_k)) \\ &= \sum_k t_k \|\underline{\lambda}_k - \underline{\eta}\|^2 > 0. \end{aligned}$$

Approximation of the n first moments of a measure

From Caratheodory's Theorem [17], we conclude that for $n \geq N$,

$$\forall \mu \in \mathcal{M}, \quad \begin{cases} \exists (\eta_1, \dots, \eta_{n+1}) \in Q^{n+1} \\ \exists (t_1, \dots, t_{n+1}) \in [0, 1]^{n+1} \end{cases}$$

such that

$$\forall 1 \leq m \leq n, \quad \langle \mu, \varphi_m \rangle = \sum_{i=1}^{n+1} t_i \varphi_m(\eta_i), \tag{3.3}$$

$$\sum_{i=1}^{n+1} t_i = 1. \tag{3.4}$$

Of course, for a given measure $\mu \in \mathcal{M}$, such $(n+1)$ -upples of points η_i and weights t_i are not unique (in fact, if the moments of μ are in the interior of D_n , there are infinitely many ways of spanning these moments). We are going to construct an application that will select *one* set of points η_i , and the associated weights:

$$\mu \in \mathcal{M} \rightarrow (\underline{\eta}_1(\mu), \dots, \eta_{n+1}(\mu), t_1(\mu), \dots, t_{n+1}(\mu)).$$

Let us fix $\mu \in \mathcal{M}$. We define $\mathcal{M}_n(\mu)$ as the set of convex combinations of at most $n + 1$ Dirac masses, and whose n first moments coincide with those of μ :

$$\mathcal{M}_n(\mu) = \left\{ \begin{array}{l} E = (\eta_1, \dots, \eta_{n+1}) \in Q^{n+1} \\ T = (t_1, \dots, t_{n+1}) \in [0, 1]^{n+1} \\ \mu(E, T) = \sum_{i=1}^{n+1} t_i \delta_{\eta_i} \\ \sum_{i=1}^{n+1} t_i = 1 \\ \langle \mu, \varphi_m \rangle = \langle \mu(E, T), \varphi_m \rangle \quad 1 \leq m \leq n \end{array} \right\}.$$

We also define a function $\Phi_\mu : \mathcal{M} \rightarrow \mathbb{R}^+$, by

$$\Phi_\mu(v) = \sum_{k=1}^{\infty} 2^{-k} \langle \mu - v, \varphi_k \rangle^2.$$

Hypothesis (H2) ensures that Φ_μ is well defined, and is a bounded positive function. The same argument as in the proof of Lemma 2.9, shows that $\mathcal{M}_n(\mu)$ is compact for the weak * topology on \mathcal{M} .

LEMMA 3.4. *The function Φ_μ is continuous on \mathcal{M} , and strictly convex.*

Proof. (a) Since \mathcal{M} is metrisable [16], it suffices to show that Φ_μ is sequentially continuous. Let (v_k) be a sequence of elements of \mathcal{M} that converge weakly to a measure v , and let $\varepsilon > 0$.

$$\exists L/\forall l \geq L, \quad 2^{-l+1} < \varepsilon/4.$$

The weak convergence of (v_k) implies that

$$\exists N/\forall n \geq N, \quad \sum_{i=1}^L 2^{-i} |\langle v_k - \mu, \varphi_i \rangle^2 - \langle v - \mu, \varphi_i \rangle^2| \leq \varepsilon/2.$$

For $k \geq N$, it follows that

$$|\Phi_\mu(v_k) - \Phi_\mu(v)| \leq \sum_{i=1}^L 2^{-i} |\langle v_k - \mu, \varphi_i \rangle^2 - \langle v - \mu, \varphi_i \rangle^2| + 2^{-L+2} \leq \varepsilon.$$

(b) Let $\theta \in [0, 1]$, and consider $v_1, v_2 \in \mathcal{M}$.

$$\begin{aligned} \Phi_\mu(\theta v_1 + (1 - \theta)v_2) &\leq \sum_{k=1}^{\infty} 2^{-k} \langle \theta v_1 + (1 - \theta)v_2 - \mu, \varphi_k \rangle^2 \\ &\leq \sum_{k=1}^{\infty} 2^{-k} (\theta \langle v_1 - \mu, \varphi_k \rangle^2 + (1 - \theta) \langle v_2 - \mu, \varphi_k \rangle^2), \end{aligned}$$

with equality if and only if

$$\forall k \in \mathbb{N}, \quad \langle v_1, \varphi_k \rangle = \langle v_2, \varphi_k \rangle,$$

i.e. if and only if $v_1 = v_2$, since (φ_k) is a dense family in $\mathcal{C}(Q, \mathbb{R})$. Thus, Φ_μ is strictly convex. \square

Remark 3.5. A slight modification in part (a) of the above proof gives a stronger

result: if (μ_k, ν_k) is a sequence of elements in $\mathcal{M} \times \mathcal{M}_n$ that converges to (μ, ν) , then

$$\Phi_{\mu_k}(v_k) \rightarrow \Phi_\mu(v) \quad \text{as } k \rightarrow \infty.$$

THEOREM 3.6. *There exists a unique element $\xi_\mu \in \mathcal{M}_n(\mu)$, that minimises Φ_μ over $\mathcal{M}_n(\mu)$.*

Proof. Let us consider a minimising sequence $\mu(E_k, T_k)$ in $\mathcal{M}_n(\mu)$. Since $\mathcal{M}_n(\mu)$ is compact, we can extract a subsequence that converges weakly to some ξ_μ . Since Φ_μ is continuous,

$$\Phi_\mu(\xi_\mu) = \inf_{\mathcal{M}_n(\mu)} \Phi_\mu.$$

The minimum is unique due to the strict convexity of Φ_μ . \square

We can therefore define a mapping

$$X : \mu \in \mathcal{M} \rightarrow \xi_\mu \in \mathcal{M}_n.$$

Measurability of the mapping X

In this part, we show that the mapping X defined above is measurable. This result will enable us to conclude that the functions θ_j, a_j are measurable, as advertised in the statement of Theorem 2.2.

To this effect, we first approximate X in order to deal with a minimisation problem defined on the whole \mathcal{M}_n , rather than on $\mathcal{M}_n(\mu)$. We define, for $\nu \in \mathcal{M}_n$,

$$\Phi_\mu^s(\nu) = s \left(\sum_{k=1}^n 2^{-k} \langle \mu - \nu, \varphi_k \rangle^2 \right) + \sum_{k=n+1}^{+\infty} 2^{-k} \langle \mu - \nu, \varphi_k \rangle^2.$$

If (μ_k, ν_k) , sequence of elements of $\mathcal{M} \times \mathcal{M}_n$, converges weakly to (μ, ν) , it follows from Lemma 3.4 and from Remark 4.2, that

$$\Phi_{\mu_k}^s(\nu_k) \rightarrow \Phi_\mu^s(\nu) \quad \text{as } k \rightarrow \infty. \tag{3.5}$$

The function Φ_μ^s is therefore continuous and strictly convex. We remark that, for $\nu \in \mathcal{M}_n(\mu)$,

$$\Phi_\mu^s(\nu) = \Phi_\mu(\nu). \tag{3.6}$$

By an argument similar to that in Theorem 3.6, there exists a unique $\xi_\mu^s \in \mathcal{M}_n$, which minimises Φ_μ^s over \mathcal{M}_n . We denote by X_s the mapping that associates ξ_μ^s to μ .

LEMMA 3.7. $\xi_\mu^s \rightarrow \xi_\mu$, as $s \rightarrow \infty$.

Proof. By compactness of \mathcal{M}_n , we can extract a subsequence $(\xi_\mu^s)_s$, that converges to some $\xi_\mu^\infty \in \mathcal{M}_n$. We first show that $\xi_\mu^\infty \in \mathcal{M}_n(\mu)$, and then that $\xi_\mu^\infty = \xi_\mu$.

(a) Suppose that for some $m, 1 \leq m \leq n$, we have

$$\langle \xi_\mu^\infty, \varphi_m \rangle - \langle \xi_\mu, \varphi_m \rangle = \alpha \neq 0.$$

Then

$$\Phi_\mu^s(\xi_\mu^s) \geq s 2^{-k} \langle \mu - \xi_\mu^s, \varphi_k \rangle^2 \sim 2^{-k} \alpha^2 s \quad \text{as } s \rightarrow \infty.$$

But $\Phi_\mu^s(\xi_\mu^s)$ has to be bounded uniformly with respect to s , since we have by (3.6), for all s and for $\nu \in \mathcal{M}_n(\mu)$

$$\Phi_\mu^s(\xi_\mu^s) \leq \Phi_\mu^s(\nu) = \Phi_\mu(\nu) < \infty.$$

Thus we must have

$$\langle \xi_\mu^\infty, \varphi_m \rangle = \langle \mu, \varphi_m \rangle \quad \forall 1 \leq m \leq n, \quad \text{i.e. } \xi_\mu^\infty \in \mathcal{M}_n(\mu).$$

(b) By definition of ξ_μ^s , and since $\xi_\mu^\infty \in \mathcal{M}_n(\mu)$, we have

$$\Phi_\mu^s(\xi_\mu^s) \leq \Phi_\mu^s(\xi_\mu) = \Phi_\mu(\xi_\mu) \leq \Phi_\mu(\xi_\mu^\infty).$$

On the other hand,

$$\Phi_\mu^s(\xi_\mu^s) \geq \sum_{k=n+1}^{\infty} 2^{-k} \langle \mu - \xi_\mu^s, \varphi_k \rangle^2,$$

and the argument used in Lemma 3.4 shows that the right-hand side converges, as $s \rightarrow \infty$, to

$$\sum_{k=n+1}^{\infty} 2^{-k} \langle \mu - \xi_\mu^\infty, \varphi_k \rangle^2 = \Phi_\mu(\xi_\mu^\infty).$$

Thus, $\Phi_\mu(\xi_\mu^\infty) = \Phi_\mu(\xi_\mu)$, and the lemma follows from the strict convexity of Φ_μ . \square

LEMMA 3.8. *For s fixed, X_s is continuous.*

Proof. Here again, it suffices to show that X_s is sequentially continuous. Let (μ_k) be a sequence in \mathcal{M} , that converges to $\mu \in \mathcal{M}$. We can extract a subsequence, such that $(\xi_{\mu_k}^s)$ converges weakly to some $\xi^s \in \mathcal{M}_n$. Relation (3.5) implies

$$\Phi_{\mu_k}^s(\xi_{\mu_k}^s) \rightarrow \Phi_\mu^s(\xi^s) \quad \text{as } k \rightarrow \infty. \quad (3.7)$$

The definition of $\xi_{\mu_k}^s$ implies

$$\Phi_{\mu_k}^s(\xi_{\mu_k}^s) \leq \Phi_{\mu_k}^s(\xi_{\mu_k}^s). \quad (3.8)$$

By (3.5), the right-hand side of (3.8) converges to $\Phi_\mu^s(\xi_\mu^s)$. Thus, it follows from (3.7)–(3.8) that

$$\Phi_\mu^s(\xi^s) \leq \Phi_\mu^s(\xi_\mu^s),$$

and so $\xi^s = \xi_\mu^s$, by strict convexity of Φ_μ^s . Thus, $X_s(\mu_k)$ converges weakly to $X_s(\mu)$ as $k \rightarrow \infty$. \square

As an immediate consequence of Lemma 3.8, we have

COROLLARY 3.9. *The mapping $X = \liminf_{s \rightarrow \infty} X_s$ is a Borel function.*

Definition of the functions $x \rightarrow \theta_j(x)$, $a_j(x)$

Let $E = (\eta_1, \dots, \eta_{n+1})$ and $T = (t_1, \dots, t_{n+1})$ be the points and weights associated with ξ_μ . They are only defined up to a permutation of the points associated with a non-zero weight. Let $b(\xi_\mu)$ be the number of such points and

$$C_m = \{ \mu / b(\xi_\mu) \leq m \}, \quad 1 \leq m \leq n.$$

For $\mu_x \in C_m$, we define

$$\begin{cases} \theta_j(x) = t_j(\xi_{\mu_x}) \\ a_j(x) = \eta_j(\xi_{\mu_x}) \end{cases} \text{ for } 1 \leq j \leq m,$$

$$\begin{cases} \theta_j(x) = 0 \\ a_j(x) = 0 \end{cases} \text{ for } j > m,$$

where the points η_j are classified by lexicographic order.

From the measurability of the application which associates to an element of $\mathcal{M}_m \setminus \mathcal{M}_{m-1}$ its points, classified by lexicographic order and its weights, from Corollary 3.9, which implies that

$$C_m = X^{-1}((\mathcal{M}_m \setminus \mathcal{M}_{m-1}) \cap \mathcal{M}_m(\mu))$$

is a Borel set, and from the measurability of $x \rightarrow \mu_x$, it follows that the functions $\theta_j(x), a_j(x)$ are measurable.

By (3.3)–(3.4), these functions also satisfy the desired property on the moments:

$$\forall 1 \leq m \leq n, \quad \langle \mu_x, \varphi_m \rangle = \sum_{i=1}^{n+1} \theta_i(x) \varphi_m(a_i(x)),$$

$$\sum_{i=1}^{n+1} \theta_i(x) = 1.$$

This concludes the proof of Theorem 2.2. \square

4. A version of the proof of the fundamental theorem for Young measures

Representation of the weak * limit by parametrised measures

In this section we prove the direct part of Theorem 1.1. For a proof of a more general statement, we refer to the article of J. Ball [1].

We can always assume that $Q = [0, 1]^p$. Let $\mathcal{K}_Q(\lambda)$ denote the space of polynomials with rational coefficients. Since the sequence of functions $(u_k)_k$ is bounded in Q , after a countable number of extractions, we can extract a subsequence such that for each polynomial $p \in \mathcal{K}_Q$, there exists a function $\bar{p} \in L^\infty(\Omega)$ such that

$$p(u_k) \rightarrow \bar{p} \in L^\infty(\Omega) \text{ weakly } * \text{ as } k \rightarrow \infty.$$

We denote by ω the set of those $x \in \Omega$, such that $p \in \mathcal{K}_Q \rightarrow \bar{p}(x) \in \mathbb{R}$ is a positive mapping and

$$\forall \lambda, \mu \in Q, \quad \forall p, q \in \mathcal{K}_Q \quad \overline{\lambda p + \mu q}(x) = \lambda \bar{p}(x) + \mu \bar{q}(x)$$

$$\forall p \in \mathcal{K}_Q, \quad |\bar{p}(x)| \leq \| \bar{p} \|_{L^\infty(\omega)}. \tag{4.1}$$

Since \mathcal{K}_Q is countable, $(\Omega \setminus \omega)$ is a set of measure 0.

We extend the mapping $p \rightarrow \bar{p}(x)$ to all continuous functions: by the Stone–Weierstrass Theorem, a function $f \in \mathcal{C}(Q, \mathbb{R})$ can be uniformly approximated by a sequence $(p_n)_n$ of polynomials, and these polynomials can be chosen in \mathcal{K}_Q . We define then for $x \in \omega$,

$$\tilde{f}(x) = \lim_{n \rightarrow \infty} \bar{p}_n(x).$$

This limit exists since by definition of ω we have

$$\begin{aligned} \forall x \in \omega, \quad |\bar{p}_n(x) - \bar{p}_m(x)| &\leq |\overline{p_n - p_m}(x)| \\ &\leq \|\overline{p_n - p_m}\|_{L^\infty(\Omega)} \\ &\leq \|p_n - p_m\|_{L^\infty(0,1)}, \end{aligned}$$

hence, $\bar{p}_n(x)$ is a Cauchy sequence. Letting m tend to zero in the previous inequality yields

$$\forall x \in \omega, \quad |\bar{f}(x) - \bar{p}_n(x)| \leq \|f - p_n\|_{L^\infty(0,1)}. \tag{4.2}$$

The same argument shows that if $(q_n)_n$ is another sequence of polynomials in \mathcal{X}_Q , that converges uniformly to f ,

$$\forall x \in \omega, \quad \lim_{n \rightarrow \infty} \bar{q}_n(x) = \lim_{n \rightarrow \infty} \bar{p}_n(x) = \bar{f}(x),$$

i.e. \bar{f} is well defined on ω .

We claim that $f(u_k)$ converges to \bar{f} weakly $*$ in $L^\infty(\Omega)$ as $k \rightarrow \infty$. Indeed, let $\psi \in L^1(\Omega)$, and consider

$$\begin{aligned} \left| \int_{\Omega} (f(u_k(x)) - \bar{f}(x))\psi \, dx \right| &\leq \left| \int_{\Omega} (f - p_n)(u_k(x))\psi \, dx \right| + \left| \int_{\Omega} (\bar{f}(x) - \bar{p}_n(x))\psi \, dx \right| \\ &\quad + \left| \int_{\Omega} (p_n(u_k(x)) - \bar{p}_n(x))\psi \, dx \right|. \end{aligned}$$

The first two terms on the right-hand side go to 0 by the uniform convergence of p_n to f which, by (4.2), implies uniform convergence of \bar{p}_n towards \bar{f} since $(\Omega \setminus \omega)$ has measure 0. The last term tends to 0 by the definition of \bar{p}_n , the weak $*$ limit of $p_n(u_k)$.

For $x \in \omega$, we can consider the mapping

$$f \in \mathcal{C}(Q, \mathbb{R}) \rightarrow \bar{f}(x) \in \mathbb{R}.$$

Using the properties of ω , it is easily seen that this mapping is linear and positive. By the Riesz–Markov Representation Theorem [14], there exists a unique Borel measure ν_x , such that

$$\forall f \in \mathcal{C}([0, 1]) \quad \bar{f}(x) = \langle \nu_x, f(\lambda) \rangle.$$

Taking $f \equiv 1$, we see that ν_x is a probability measure a.e. on Ω .

Proof of the converse

For simplicity, we assume that Ω is an interval in \mathbb{R} and that $p = 1$, but the proof can be extended to the general case. Let ν_x be a family of measures parametrised by the elements of Ω , that satisfies the assumptions of Theorem 1.1. Let K be the set of monomials λ^m , $m \geq 0$.

We first construct a sequence $u_{n,k}$, for which we can pass to the limit in a finite number of nonlinear expressions, corresponding to the $2n - 1$ first functions of K :

$$\forall 0 \leq m \leq 2n - 1, \quad u_{n,k}^m \rightarrow \langle \nu_x, \lambda^m \rangle \quad \text{as } k \rightarrow \infty.$$

According to Theorem 2.2, there exist $2n$ measurable functions $\theta_j(x), a_j(x)$, such that

$$\forall 0 \leq m \leq 2n - 1, \quad \langle \mu_x, \lambda^m \rangle = \sum_{i=1}^n \theta_i(x) a_i^m(x).$$

Let $A_i(x)$ be the interval

$$A_i(x) = \left] \sum_{j=1}^{i-1} \theta_j(x), \sum_{j=1}^i \theta_j(x) \right].$$

Note that the (Lebesgue) measures of $A_i(x)$ is $\theta_i(x)$. Let

$$u_n(x, y) = \sum_{i=1}^n a_i(x) 1_{A_i(x)}(y).$$

This is a measurable function on $\Omega \times [0, 1]$, with respect to Lebesgue measure. Also, it satisfies

$$0 \leq \inf a_i(x) \leq u_n(x, y) \leq \sup a_i(x) \leq 1 \quad \text{a.e. on } \Omega \times [0, 1].$$

Let us consider the following sequence of functions:

$$u_{n,k}(x) = u_n(x, (kx) - [kx]), \quad k \in \mathbb{N},$$

where the brackets $[z]$ denote the highest integer less than z . We remark that $(u_{n,k}(x))^m$ is also a function of the variables x and $kx - [kx]$. This form of dependence on the second variable introduces a rapidly oscillating behaviour around a mean value that depends on x . By a lemma of [4], we can compute weak * limits of such functions:

$$v_k(x) = v(x, (kx) - [kx]) \rightharpoonup \int_0^1 v(x, y) dy \quad \text{weakly * in } L^\infty(\Omega), \quad \text{as } k \rightarrow \infty.$$

If we apply that lemma to our sequence $(u_{n,k})_{k \in \mathbb{N}}$, we obtain as $k \rightarrow \infty$:

$$\begin{aligned} (u_{n,k})^m &\rightharpoonup \int_0^1 \sum_{i=1}^n a_i^m(x) 1_{A_i(x)}(y) dy \quad \forall 0 \leq m \leq 2n - 1, \\ &= \sum_{i=1}^n a_i^m(x) \theta_i(x) \\ &= \langle v_x, \lambda^m \rangle. \end{aligned} \tag{4.3}$$

A diagonal process yields a sequence, the limit of whose moments coincides with the moments of v_x . Consider a dense countable family $\{\psi_r\}_{r \in \mathbb{N}}$ in $L^1(\Omega)$. Let $R \geq 0, \varepsilon > 0$. From (4.3) we have

$$\begin{aligned} \forall n \geq 0 \quad \forall 0 \leq m \leq 2n - 1 \quad \forall 0 \leq r \leq R \\ \exists K(R, \varepsilon, n) / \forall k > K(R, \varepsilon, n), \quad \left| \int_{\Omega} (u_{n,k}^m - \langle v_x, \lambda^m \rangle) \psi_r dx \right| < \varepsilon. \end{aligned}$$

PROPOSITION 4.1. *The sequence $v_n(x) = u_{n, K(n, 1/n, n)}(x)$ satisfies condition (1.1).*

Proof. Let $r \geq 0, p \geq 0$ and $\varepsilon > 0$. Choosing n greater than the supremum of $p, r, 1/\varepsilon$,

we have

$$\left| \int_{\Omega} (v_n^p - \langle v_x, \lambda^p \rangle) \psi_r dx \right| < \varepsilon.$$

This completes the proof of the fundamental theorem. \square

Remarks 4.2. (a) The extension to the case $p > 1$ is straightforward: we only need to replace the monomials λ^m by the family $(\lambda_1^{m_1}, \dots, \lambda_p^{m_p})$, since by Example 3.3 this family satisfies Hypotheses (H1)–(H3). The functions $u_{n,k}$ are then defined in the same way.

(b) When Ω is in \mathbb{R}^r , with $r > 1$, the functions $u_{n,k}$ can be defined in a similar way, for instance,

$$u_n(x_1, \dots, x_r, y) = \sum_{i=1}^n a_i(x_1, \dots, x_r) 1_{A_i(x_1, \dots, x_r)}(y) \quad \text{for } (x, y) \in \Omega \times [0, 1],$$

$$\text{and } u_{n,k}(x_1, \dots, x_r) = u_n(x_1, \dots, x_r, (kx_1) - [kx_1]).$$

5. An application to a problem of optimal design

Preliminaries

As an application of Theorem 2.2, we establish the relative compactness for the weak * topology of some subsets of $L^\infty(\Omega)^n$.

THEOREM 5.1. *Let Ω be a bounded domain in \mathbb{R}^r , $Q = [0, 1]^p$, $p \geq 1$, and consider a family of n linearly independent functions $(\varphi_1, \dots, \varphi_n) \in \mathcal{C}(Q, \mathbb{R})^n$, that satisfy (H2) and (H3).*

Let H be the set of functions $h = (h_1, \dots, h_n) \in (L^\infty(\Omega))^n$, such that there exist

$$\begin{cases} \theta_i \in L^\infty(\Omega, [0, 1]) \\ a_i \in L^\infty(\Omega, Q)^p \end{cases}, \quad 1 \leq i \leq n + 1,$$

with

$$\sum_{i=1}^{n+1} \theta_i(x) \varphi_m(a_i(x)) = h_m(x), \quad 1 \leq m \leq n,$$

$$\sum_{i=1}^{n+1} \theta_i(x) = 1.$$

*Then H is compact for the weak * topology.*

Proof. We only need to show that H is weakly closed. Let $(h_k)_k = (h_{1,k}, \dots, h_{n,k})_k \subset H$ be a sequence that converges to some $h = (h_1, \dots, h_n)$, for the weak * topology, and let $\theta_{i,k}, a_{i,k}$ be the weights and points associated to h_k .

From the fundamental theorem on Young measures, we deduce the existence of a parametrised measure $(\mu_x)_{x \in \Omega}$, with support in $[0, 1]^{n+1} \times Q^{n+1}$, such that, after

extraction of a subsequence:

$$\begin{cases} \theta_{i,k} \rightharpoonup \langle \mu_x, \lambda_i \rangle \\ a_{i,k} \rightharpoonup \langle \mu_x, \Lambda_i \rangle \end{cases} \text{ weakly } * \text{ as } k \rightarrow \infty,$$

$$h_{m,k} = \sum_{i=1}^{n+1} \theta_{i,k} \varphi_m(a_{i,k}) \rightharpoonup \left\langle \mu_x, \sum_{i=1}^{n+1} \lambda_i \varphi_m(\Lambda_i) \right\rangle \quad (4.4)$$

where, for convenience, we have denoted by $(\lambda_1, \dots, \lambda_{n+1}, \Lambda_1, \dots, \Lambda_{n+1})$ the elements of $\mathbb{R}^{n+1} \times (\mathbb{R}^p)^{n+1}$.

Let $(v_x)_{x \in \Omega}$ denote the parametrised measure defined by

$$\forall \varphi \in \mathcal{C}(Q, \mathbb{R}), \quad \langle v_x, \varphi \rangle = \left\langle \mu_x, \sum_{i=1}^{n+1} \lambda_i \varphi(\Lambda_i) \right\rangle.$$

We verify easily that $\text{supp}(v_x) \subset Q$, for a.e. x , and that

$$\langle v_x, 1 \rangle = \left\langle v_x, \sum_{i=1}^{n+1} \lambda_i \right\rangle = w * \lim_{k \rightarrow \infty} \sum_{i=1}^{n+1} \theta_{i,k}(x) = 1.$$

Thus, $v_x \in \mathcal{M}$, for a.e. x , and satisfies the hypothesis of Theorem 2.2. Moreover, it follows from (4.4), that

$$h_{m,k} \rightharpoonup \langle v_x, \varphi_m \rangle = h_m \quad \text{as } k \rightarrow \infty.$$

Theorem 2.2 yields a measurable convex combination of $n + 1$ Dirac masses whose moments with respect to $(\varphi_1, \dots, \varphi_n)$ coincide with those of (v_x) , i.e. there exist

$$\begin{cases} \theta_i \in L^\infty(\Omega, [0, 1]) \\ a_i \in L^\infty(\Omega, Q)^p \end{cases}, \quad 1 \leq i \leq n + 1, \quad \text{such that for a.e. } x \in \Omega,$$

$$\sum_{i=1}^{n+1} \theta_i(x) \varphi_m(a_i(x)) = \langle v_x, \varphi_m \rangle = h_m(x), \quad \sum_{i=1}^{n+1} \theta_i(x) = 1.$$

In other words, $h = (h_1, \dots, h_n) \in H$. \square

COROLLARY 5.2.

$$H_3 = \left\{ (h, k, l) \in (L^\infty(\Omega))^3 / \begin{cases} h(x) = \sum_{i=1}^4 \theta_i(x) h_i(x) \\ k(x) = \sum_{i=1}^4 \theta_i(x) h_i^3(x) \\ l(x) = \sum_{i=1}^4 \theta_i(x) h_i^{-3}(x) \end{cases} \right\}$$

is compact for the weak * topology.

Relaxation of a plate optimisation problem

Introduction. Let Ω be a smooth domain in \mathbb{R}^2 . We consider a Kirchhoff model for pure bending of symmetric plates with midplane Ω . The deflection w satisfies an equation of the form

$$\partial_{\alpha\beta}(M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} w) = F \quad \text{in } \Omega, \quad (4.5)$$

where the tensor $M_{\alpha\beta\gamma\delta}$ depends on the half-thickness $h(x, y)$ of the plate

$$M_{\alpha\beta\gamma\delta} = 2/3h^3(x, y)\bar{B}_{\alpha\beta\gamma\delta},$$

and \bar{B} is a constant tensor that depends only on material constants. We are only interested in thicknesses that depend on one variable, $h = h(x)$, and we denote

$$(a, b) = \{x/\exists y \text{ with } (x, y) \in \Omega\}.$$

Also, we only consider orthotropic materials, for which the non-zero elements of $\bar{B}_{\alpha\beta\gamma\delta}$ are

$$\begin{aligned} \bar{B}_{1111} = \bar{B}_{2222} &= \frac{E}{1 - \nu^2}, & \bar{B}_{1122} = \bar{B}_{2211} &= \frac{E\nu}{1 - \nu^2}, \\ \bar{B}_{1212} = \bar{B}_{1221} = \bar{B}_{2112} = \bar{B}_{2121} &= \frac{E}{2(1 + \nu)}, \end{aligned}$$

where E, ν , denote respectively Young’s modulus and Poisson’s ratio. We also assume that the plate is clamped, i.e. that w satisfies the boundary conditions

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{4.6}$$

which makes w the minimiser of the following energy functional:

$$E(w) = \frac{1}{2} \int_{\Omega} M_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} w \partial_{\gamma\delta} w - \int_{\Omega} Fw.$$

We assume that the load F is sufficiently smooth, and we denote $H_0^2(\Omega)$ the set of functions in $H^2(\Omega)$, which satisfy (4.6).

We define the compliance of the plate to be the work done by the load F ,

$$L = \int_{\Omega} Fw.$$

The value of this functional can be viewed as a measure of the overall rigidity of the plate under F . We consider $L = L(h)$ as a functional of the half-thickness h , and we seek to minimise $L(h)$ among certain plates with prescribed volume. The set of admissible thicknesses is

$$\mathcal{H} = \{h \in L^\infty(a, b) / h_{\min} \leq h(x) \leq h_{\max}, \int_{\Omega} h = V_0\},$$

where h_{\min}, h_{\max}, V_0 are positive constants that satisfy

$$0 < h_{\min} \text{ meas } (\Omega) < V_0 < h_{\max} \text{ meas } (\Omega).$$

Numerical experiments [6,7] have shown that the optimisation problem may have no solution in \mathcal{H} . To overcome this difficulty, Kohn and Vogelius, and Bonnetier and Vogelius suggested some relaxation of L [4,12]. This amounts to introducing a set of generalised thicknesses $\bar{\mathcal{H}}$, which contains \mathcal{H} , and to defining an extension \bar{L} of L to $\bar{\mathcal{H}}$, such that:

(P1) for each $\bar{h} \in \bar{\mathcal{H}}$, there exists a sequence $(h_n)_n$ of elements of \mathcal{H} such that

$$\bar{L}(\bar{h}) = \lim_{n \rightarrow \infty} L(h_n); \tag{4.7}$$

(P2) \bar{L} attains its minimum in $\bar{\mathcal{H}}$.

The previous authors call a couple $(\bar{\mathcal{H}}, \bar{L})$ a partial or a full relaxation, whether it satisfies (P1) only, or both (P1)–(P2) respectively. They have also shown [4], that the following choice of admissible thicknesses and generalised compliance led to a partial relaxation:

$$\bar{\mathcal{H}} = \left\{ \begin{array}{l} h_{\min} \leq h_s(x) \leq h_{\max} \\ 0 \leq \theta(x) \leq 1 \\ (\theta, h_s) \in (L^\infty(a, b))^2 / \int_{\Omega} (\theta h_{\max} + (1 - \theta)h_s) = V_0 \end{array} \right\},$$

$$\bar{L}(\theta, h_s) = \int_{\Omega} F \bar{w},$$

where \bar{w} is the solution to

$$\partial_{\alpha\beta}(\bar{M}_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} \bar{w}) = F \quad \text{in } \Omega, \tag{4.8}$$

with the boundary conditions (4.6). The non-zero elements of \bar{M} are

$$\left\{ \begin{array}{l} \bar{M}_{1111} = \frac{2}{3}c(x) \frac{E}{1 - \nu^2}, \\ \bar{M}_{2222} = \frac{2}{3}m(x)E + \frac{2}{3}c(x) \frac{E\nu^2}{1 - \nu^2}, \\ \bar{M}_{1122} = \bar{M}_{2211} = \frac{2}{3}c(x) \frac{E\nu}{1 - \nu^2}, \\ \bar{M}_{1212} = \bar{M}_{1221} = \bar{M}_{2112} = \bar{M}_{2121} = \frac{1}{3}m(x) \frac{E}{1 + \nu}, \end{array} \right. \tag{4.9}$$

where m and c denote respectively the ‘‘cubic-average’’ and ‘‘harmonic cubic-average’’ of \bar{h} :

$$m(x) = \theta(x)h_{\max}^3 + (1 - \theta(x))h_s^3(x),$$

$$c(x)^{-1} = \theta(x)h_{\max}^{-3} + (1 - \theta(x))h_s^{-3}(x).$$

The function θ represents the density of fine scale stiffeners, which appear naturally in the original optimisation problem [12].

However, property (P2) could not be verified, although numerical experiments indicated that $(\bar{\mathcal{H}}, \bar{L})$ could be a full relaxation for particular choices of the load F .

A full relaxation. In this paragraph, we extend $\bar{\mathcal{H}}, \bar{L}$, to a full relaxation. We first recall a H -convergence result, proved in [4].

We call a tensor M orthotropic if only the coefficients $M_{1111}, M_{2222}, M_{1122} = M_{2211}, M_{1212} = M_{1221} = M_{2112} = M_{2121}$, are different from 0. We say that M is bounded by positive constants (d, D) if, for any symmetric second-order tensor $t_{\alpha\beta}$,

we have for a.e. $x \in (a, b)$,

$$d|t|^2 \leq M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta}, \quad |M_{\alpha\beta\gamma\delta} t_{\alpha\beta}| \leq D|t| \quad \forall \gamma, \delta.$$

LEMMA 5.3. Let $(M_{\alpha\beta\gamma\delta}^k)$, $1 \leq k \leq \infty$, be a family of orthotropic tensors which are uniformly bounded by (d, D) for $k < \infty$. We also assume that

$$\left\{ \begin{array}{l} (M_{1111}^k)^{-1} \rightarrow (M_{1111}^\infty)^{-1}, \\ M_{1122}^k (M_{1111}^k)^{-1} \rightarrow M_{1122}^\infty (M_{1111}^\infty)^{-1}, \\ M_{2222}^k - (M_{1122}^k)^2 (M_{1111}^k)^{-1} \rightarrow M_{2222}^\infty - (M_{1122}^\infty)^2 (M_{1111}^\infty)^{-1}, \\ M_{1212}^k \rightarrow M_{1212}^\infty, \end{array} \right. \quad (4.10)$$

weakly * in $L^\infty(a, b)$. Let w_k , $1 \leq k \leq \infty$, denote the solution to

$$\partial_{\alpha\beta} (M_{\alpha\beta\gamma\delta}^k \partial_{\gamma\delta} w^k) = F \quad \text{in } \Omega, \quad (4.11)$$

with the boundary conditions (4.6). Then

$$w_k \rightharpoonup w_\infty \quad \text{weakly * in } H_0^2(\Omega) \text{ as } k \rightarrow \infty.$$

This lemma and relations (4.9) imply that if the cubic-averages and harmonic cubic-averages of a sequence $\bar{h}_k = (\theta_k, h_{s,k}) \in \mathcal{H}$ converge to the cubic-average and harmonic cubic-average of some $\bar{h} \in \mathcal{H}$, for the weak * topology, then

$$\bar{L}(\bar{h}_k) \rightarrow \bar{L}(\bar{h}).$$

Thus, in order to satisfy (P2), we would like to select as admissible thicknesses, those whose averages (because of the volume constraint), cubic-averages and harmonic cubic-averages, form a relatively compact set for the weak * topology. Recalling Corollary 5.2, we consider

$$\mathcal{H} = \left\{ (T, H) = (\theta_1, \theta_2, \theta_3, \theta_4, h_1, h_2, h_3, h_4) \in (L^\infty(a, b))^8 \quad \text{such that} \right. \\ \left. \begin{array}{l} \theta_i(x) \in [0, 1], \\ h_{\min} \leq h_i(x) \leq h_{\max}, \quad \text{a.e. } x \in (a, b), \quad 1 \leq i \leq 4, \\ \sum_{i=1}^4 \theta_i = 1, \quad \int_{\Omega} \sum_{i=1}^4 \theta_i h_i = V_0. \end{array} \right\}$$

We define a generalised compliance \tilde{L} by

$$\tilde{L}(T, H) = \int_{\Omega} F \tilde{w},$$

where \tilde{w} is a solution to

$$\partial_{\alpha\beta} (\tilde{M}_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} \tilde{w}_1) = F \quad \text{in } \Omega, \quad (4.12)$$

with the boundary conditions (4.6). The tensor \tilde{M} is defined as in (4.9), but m and c are respectively replaced by

$$m(x) = \sum_{i=1}^4 \theta_i(x) h_i^3(x), \quad c(x)^{-1} = \sum_{i=1}^4 \theta_i(x) h_i^{-3}(x).$$

THEOREM 5.4. (\mathcal{H}, \tilde{L}) is a full relaxation of (H, L) .

Remark 5.5. In this optimisation problem we have to consider the limits of three expressions depending on the thickness, and we know from Theorem 2.2 that we can represent them through the mixture of four families of thicknesses. However, we have not been able to prove yet that \mathcal{H} was the smallest set leading to a full relaxation. We believe, though, that this smallest set is bigger than \mathcal{H} .

If the three moments corresponded to the polynomials $\lambda, \lambda^2, \lambda^3$, the second part of Theorem 2.2 would suggest that one could write the moments of ν_x as those of a convex combination of two distinct Dirac masses, in the general case.

That would suggest that the smallest set for a full relaxation should be

$$\mathcal{G} = \left\{ \begin{array}{l} (T, H) = (\theta_1, \theta_2, h_1, h_2) \in (L^\infty(a, b))^4 \text{ such that} \\ \theta_1(x), \theta_2(x) \in [0, 1], \\ h_{\min} \leq h_1(x), h_2(x) \leq h_{\max}, \quad \text{a.e. } x \in (a, b) \\ \theta_1 + \theta_2 = 1 \\ \int_{\Omega} \theta_1 h_1 + \theta_2 h_2 = V_0 \end{array} \right\}.$$

This set is larger than \mathcal{H} , since the latter imposes one of the h_i to be equal to h_{\max} .

Proof of Theorem 5.4. (a) The property of partial relaxation (P1) is obtained using Lemma 5.3, as in [4], considering elements of \mathcal{H} of the form

$$h_\varepsilon(x) = \begin{cases} h_1(x) & \text{for } x/\varepsilon - [x/\varepsilon] \leq \theta_1(x), \\ h_2(x) & \text{for } \theta_1(x) < x/\varepsilon - [x/\varepsilon] \leq \theta_1(x) + \theta_2(x), \\ h_3(x) & \text{for } \theta_1(x) + \theta_2(x) < x/\varepsilon - [x/\varepsilon] \leq \theta_1(x) + \theta_2(x) + \theta_3(x), \\ h_4(x) & \text{for } \theta_1(x) + \theta_2(x) + \theta_3(x) < x/\varepsilon - [x/\varepsilon] \leq 1, \end{cases}$$

where $[x/\varepsilon]$ denotes the integer part of x/ε .

(b) For (P2), we show that there exists an element of \mathcal{H} , which attains the infimum of \tilde{L} . Let $(T_k, H_k) \in \mathcal{H}$ be a minimising sequence for \tilde{L} . We can extract a subsequence such that

$$\sum_{i=1}^4 \theta_{i,k} h_{i,k} = V_0, \quad \sum_{i=1}^4 \theta_{i,k} h_{i,k}^3 \rightharpoonup f, \quad \sum_{i=1}^4 \theta_{i,k} h_{i,k}^{-3} \rightharpoonup g,$$

weakly * in $L^\infty(a, b)$, as $k \rightarrow \infty$.

Thus, the triple of functions (V_0, f, g) belongs to the adherence of the set H_3 : by Corollary 5.2, there exists $(T, H) \in \mathcal{H}$, $T = (\theta_1, \dots, \theta_4)$, $H = (h_1, \dots, h_4)$, such that

$$\sum_{i=1}^4 \theta_i h_i = V_0, \quad \sum_{i=1}^4 \theta_i h_i^3 = f, \quad \sum_{i=1}^4 \theta_i h_i^{-3} = g.$$

As the averages, cubic-averages and harmonic cubic-averages of (T_k, H_k) converge to those of (T, H) , it follows from Lemma 5.3 that

$$\tilde{w}(T_k, H_k) \rightharpoonup \tilde{w}(T, H) \text{ weakly * in } H_0^2(\Omega) \text{ as } k \rightarrow \infty,$$

i.e.

$$\tilde{L}(T_k, H_k) \rightarrow \tilde{L}(T, H) = \inf_{\mathcal{H}} (\tilde{L}).$$

Thus, \tilde{L} attains its minimum on \mathcal{H} , (P2) is satisfied. \square

References

- 1 J. Ball. A version of the fundamental theorem for Young measures. In *PDE's and Continuum Models of Phase Transitions*, eds. M. Rascle, D. Serre and M. Slemrod, Lecture Notes in Physics 344, 207–215 (Berlin: Springer, 1989).
- 2 H. Bauer. *Probability Theory and Elements of Measure Theory*, eds. Z. W. Birnbaum and E. Lukacs (New York: Academic Press, 1981).
- 3 H. Berliocchi and J. M. Lasry. Intégrales normales et mesures paramétrées en calcul des variations. *Bull. Soc. Math. France* **101** (1973), 129–184.
- 4 E. Bonnetier and M. Vogelius. Relaxation of a compliance functional for a plate optimization problem. In *Applications of Multiple Scaling in Mechanics*, eds. P. G. Ciarlet and E. Sanchez-Palencia (Paris: Masson, 1987).
- 5 C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics **580** (Berlin: Springer, 1977).
- 6 K. T. Cheng and N. Olhoff. An investigation concerning optimal design of solid elastic plates. *Int. J. Solids Structures* **17** (1981) 305–323.
- 7 K. T. Cheng and N. Olhoff. Regularized formulation for optimal design of axisymmetric plates. *Int. J. Solids Structures* **18** (1982) 153–169.
- 8 M. Chipot and D. Kinderlehrer. Equilibrium configurations of crystals. *Arch. Rational Mech. Anal.* **103** (1988), 237–277.
- 9 B. Dacorogna. *Weak Continuity and Weak Semicontinuity of Nonlinear Functionals*, Lecture Notes in Mathematics 922 (Berlin: Springer, 1982).
- 10 S. Karlin and L. S. Shapley. Geometry of moment spaces. *Mem. Amer. Math. Soc.* **12** (1953), 1–93.
- 11 D. Kinderlehrer and P. Pedregal. Characterization of Young measures generated by gradients. *Arch. Rational Mech. Anal.* **115** (1991), 329–365.
- 12 R. V. Kohn and M. Vogelius. Thin plates with varying thicknesses and their relation to structural optimization. In *Homogenization and Effective Moduli of Materials and Media*, IMA Vol. Math. Apl. 1, eds. J. L. Ericksen, D. Kinderlehrer, R. Kohn, J. L. Lions, 126–149 (Berlin: Springer, 1986).
- 13 R. D. James and D. Kinderlehrer. Theory of diffusionless phase transitions. In *PDE's and Continuum Models of Phase Transitions*, eds. M. Rascle, D. Serre and M. Slemrod, Lecture Notes in Physics 344 (Berlin: Springer, 1989).
- 14 E. J. McShane. Generalized curves. *Duke Math. J.* **6** (1940), 513–536.
- 15 R. Di Perna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.* **82** (1983), 27–70.
- 16 M. Reed and B. Simon. *Functional Analysis I* (New York: Academic Press, 1980).
- 17 A. W. Roberts and D. E. Varberg. *Convex Functions* (New York: Academic Press, 1973).
- 18 G. Szegő. *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications 23 (Providence, RI: American Mathematical Society, 1959).
- 19 L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear Analysis and Mechanics: Heriot Watt Symposium, Vol. IV*, ed. R. Knops, Research Notes in Mathematics 39 (London: Pitman, 1979).
- 20 L. C. Young. *Lectures on the Calculus of Variations and Optimal Control Theory* (Philadelphia: Chelsea, 1969).

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