Stochastic resonance in a linear system: An exact solution

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Multistable systems can exhibit stochastic resonance which is characterized by the amplification of small periodic signals by additive noise. Here we consider a nonmultistable linear system with a multiplicative noise forced by an external periodic signal. The noise is the sum of a colored noise of mean value zero and a noise with a definite sign. We show that the system exhibits stochastic resonance through the numerical study of an exact analytical expression for the mean value obtained by functional integral techniques. This is proof of the effect for a very general kind of noise which can even have a definite sign.

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I. INTRODUCTION

The term Stochastic resonance (SR) was introduced by Benzi, Sutera, Vulpiani, and Nicolis [1–3] in an attempt to explain the periodic recurrence of ice ages on Earth. This effect consists in that when a periodic signal and noise are applied simultaneously as an input, the system can be tuned to obtain resonance. Typically this phenomenon is found in nonlinear dynamical systems driven by a combination of a random and periodic force [4,5]. Although an external periodic forcing can be replaced by some internal source of a periodic nature, nonlinearity seems to be the necessary ingredient for the occurrence of the SR. Nevertheless, it has been remarked recently [6–11] that SR can also occur in linear systems subject to a multiplicative rather than additive noise. It must be emphasized that the effect takes place only for colored noise, whereas it disappears for white noise.

In this paper we report, to the best of our knowledge, an exact result for a linear model under a multiplicative noise forced with a periodic term. The noise is a linear combination of a colored noise and its square and we make an exact analytic calculation of the mean value of the variable using functional integral techniques. A maximum of the amplitude of the signal is found when we vary the parameters thus showing the appearance of SR. This result confirms and generalizes previous results reported in the literature. Since we obtain an exact analytic solution we can conclude that SR exists in linear systems forced with a variety of multiplicative colored noise. In Sec. II we show the calculation of the analytical solution using path integrals. In Sec. III we show the results of the numerical work performed on the analytical solution found in Sec. II.

II. THE LINEAR SYSTEM: ANALYTICAL SOLUTION

The problem we shall consider is described by a linear model subjected to a linear and a quadratic multiplicative colored noise and driven by a periodic sine wave signal of amplitude A and frequency Ω. It can be written as

$$\dot{x}(t) = -[a_o + a_1\xi(t) + a_2\xi^2(t)]x(t) + A\sin(\Omega t), \tag{1}$$

where a_o, a_1, a_2 are free parameters and $\xi(t)$ is a colored noise. In this paper we will take $\xi(t)$ as the Ornstein-Uhlenbeck (OU) process defined by

$$\dot{\xi}(t) = -\gamma\xi(t) + \sqrt{c}\,\eta(t), \tag{2}$$

where $\gamma > 0$ is the reciprocal of the correlation time of the OU process and $c > 0$ measures the intensity of the Gaussian white noise $\eta(t)$ with mean value $\langle\eta(t)\rangle = 0$ and correlation function $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$. Linear systems with only linear terms in the noise have been considered in Refs. [7–11]. In [7,8] the authors find the SR phenomena in a linear system subjected to a multiplicative color noise of the dichotomous type and with a short autocorrelation time. They examine the stationary mean value of the process where they identify a maximum as a function of the noise intensity. In Refs. [9,10] the same model but with a multiplicative Gaussian type noise is solved, and the phenomena of SR are found as a maximum of the signal-to-noise ratio. In Ref. [11] a linear model with a multiplicative colored noise of the OU type is solved, also finding the phenomena of SR as a maximum of the stationary mean value. We point out that systems driven by a quadratic noise, which modelizes a noise with a constant sign, but without an external periodic signal, have been considered previously in [12–14]. In [12] a functional derivative technique is developed to calculate averages and correlation functions for linear systems subjected to a general Gaussian noise. This method provides partial differential equations for the correlation functions and is different from our approach here which allows, through the use of path integrals, the direct calculation of the correlation functions starting from the solution of the differential equation (1) expressed as a functional of the colored noise. In [13] the author constructs a clever method to obtain evolution equations for the one-dimensional probability distribution $p(x,t)$ of the process generated from Eq. (1), which he points out represents the simplest exactly solvable stochastic system with a quadratic noise. Then he derives evolution equations for the moment. Finally in [14] a linear system is studied under an additive quadratic noise, i.e., $\dot{x}(t) = -\alpha x(t) + \xi^2(t)$. An exact master

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equation for the probability distribution $p(x,t)$ of the process is found and its Kramers-Moyal expansion is studied.

Equation (1) can be seen as a linear first order differential equation for the variable $x(t)$. A formal integration of Eq. (1) leads to

$$x(t) = x_0 e^{-a_0(t-t_0)}\psi[\xi(\cdot);t,t_0] + A\int_{t_0}^{t} d\tau \sin(\Omega\tau) e^{-a_0(t-\tau)}\psi[\xi(\cdot);t,\tau], \quad (3)$$

where

$$\psi[\xi(\cdot);t_2,t_1] = e^{-\int_{t_1}^{t_2} d\tau[a_1\xi(\tau)+a_2\xi^2(\tau)]} \quad (4)$$

is a functional of $\xi(\cdot)$. We perform now the average $\langle\cdots\rangle$ over the realizations of the colored noise $\xi(\cdot)$. We get

$$\langle x(t)\rangle = x_0 e^{-a_0(t-t_0)}\varphi(t,t_0) + A\int_{t_0}^{t} d\tau \sin(\Omega\tau) e^{-a_0(t-\tau)}\varphi(t,\tau), \quad (5)$$

where

$$\varphi(t,t') = \langle e^{-\int_{t'}^{t} d\tau[a_1\xi(\tau)+a_2\xi^2(\tau)]}\rangle, \quad t > t' \quad (6)$$

To calculate the previous mean value we search a functional integral representation using the methods developed in Ref. [15]. We put $\tau_j = t' + j\epsilon$, $\tau_{N+1}=t$, $j=0,1\ldots,N+1$, $\epsilon=(t-t')/(N+1)$. Then the discretized expression is

$$\varphi(t,t') = \left\langle \exp\left[-\epsilon\sum_{j=1}^{N+1}(a_1\xi_j + a_2\xi_j^2)\right]\right\rangle = \int\prod_{j=0}^{N+1} d\xi_j W_{N+1}\exp\left[-\epsilon\sum_{j=1}^{N+1}(a_1\xi_j+a_2\xi_j^2)\right], \quad (7)$$

where

$$W_{N+1} = W_{N+1}(\xi_{N+1},\tau_{N+1};\cdots;\xi_0,\tau_0) = \prod_{j=0}^{N+1} P(\xi_j,\tau_j|\xi_{j-1},\tau_{j-1})W_1(\xi_0,\tau_0) \quad (8)$$

is the joint probability density of the OU process. Here $P(\xi,\tau|\xi',\tau')$, $\tau>\tau'$, is the conditional probability density and $W_1(\xi_0,\tau_0)$ is the one time probability density at time $\tau_0=t'$. Since we are interested in the long time behavior of $\langle x(t)\rangle$ we take the joint probability in the stationary state and then we must take for W_1 the time independent stationary probability of the process which is given by

$$W_1(\xi_0,\tau_0) = P_{st}(\xi_0) = \sqrt{\frac{\gamma}{\pi c}} e^{-\gamma/c\,\xi_0^2}. \quad (9)$$

For small ϵ (big N) we obtain from (2) that the short time conditional probability density is given with sufficient accuracy [16] by

$$P(\xi_j,\tau_j|\xi_{j-1},\tau_{j-1}) = \frac{1}{\sqrt{2\pi\epsilon c}} e^{-\epsilon/2c(\Delta\xi_j/\epsilon + \gamma\xi_{j-1})^2}, \quad (10)$$

where we have defined $\Delta\xi_j \equiv \xi_j - \xi_{j-1}$, $j=1,2\ldots,N+1$. Replacing (8)–(10) into the discretized functional integral (7) and taking into account that due to (10) one is in the prepoint discretization, we have to change to the midpoint discretization in order to be able to do a partial integration in the continuous version of (7), which will give the factor $e^{1/2\gamma(t-t_0)}$ in the next formula [17–19]. After some algebra we obtain

$$\varphi(t,t') = e^{1/2\gamma(t-t_0)}\sqrt{\frac{\gamma}{\pi c}}\int_{-\infty}^{\infty} d\xi_0 d\xi e^{-\gamma/2c(\xi_0^2+\xi^2)} I(\xi,t;\xi_0,t'), \quad (11)$$

where

$$I(\xi,t;\xi_0,t') = \frac{1}{\sqrt{2\pi\epsilon c}}\int\prod_{j=1}^{N}\frac{d\xi_j}{\sqrt{2\pi\epsilon c}}\exp\left\{-\frac{\epsilon}{2c}\sum_{j=1}^{N+1}\left[\left(\frac{\Delta\xi_j}{\epsilon}\right)^2 + \lambda^2\xi_j^2 + 2ca_1\xi_j\right]\right\} \quad (12)$$

and this last functional integral is independent of the discretization. We have defined the real positive number $\lambda = \sqrt{\gamma^2+2ca_2} > 0$ and an obvious condition is that we must have $\gamma^2+2ca_2>0$. If this is not the case we shall not have a stationary state for our process as it will be clear from what it follows. Completing squares on the argument of the exponential the previous expression can be written formally as a discretized independent functional integral in the continuous which corresponds to the limit $N\to\infty$ $(\epsilon\to 0)$

$$I = e^{ca_1^2/2\lambda^2(t-t')}\int D\xi\exp\left\{-\frac{1}{2c}\int_{t'}^{t} d\tau\left[\dot{\xi}^2 + \lambda^2\left(\xi + \frac{ca_1}{\lambda^2}\right)^2\right]\right\} \times\delta(\xi(t)-\xi)\delta(\xi(t')-\xi_0) \quad (13)$$

where the Dirac's deltas are used as a notation to indicate that the values of ξ in t and t' are fixed for the present integration and $\dot{\xi}$ represents the derivative $d\xi(\tau)/d\tau$.

Now performing the transformation $\chi(\cdot) = \xi(\cdot) + ca_1/\lambda^2$, which does not change the measure of integration $D\xi = D\chi$ and any other factor in the integration because the Jacobian is equal to 1, we obtain

$$I = e^{ca_1^2/2\lambda^2(t-t')}\int D\chi e^{-1/2c\int_{t'}^{t} d\tau(\dot{\chi}^2+\lambda^2\chi^2)} \times\delta(\chi(t)-\chi)\delta(\chi(t')-\chi_0). \quad (14)$$

This functional integral can be calculated exactly using the method introduced in [20] and we obtain

$$I = e^{ca_1^2/2\lambda^2(t-t')}\sqrt{\frac{\lambda}{2\pi c\sinh\rho}}\exp\left(\frac{-\lambda}{2c}\frac{\cosh\rho}{\sinh\rho}\chi^2 + \frac{\lambda}{c\sinh\rho}\chi\chi_0 - \frac{\lambda}{2c}\frac{\cosh\rho}{\sinh\rho}\chi_0^2\right), \quad (15)$$

where $\rho \equiv \lambda(t-t')$, $\chi = \xi + ca_1/\lambda^2$, $\chi_0 = \xi_0 + ca_1/\lambda^2$. In this formula we can see that if $\gamma^2+2ca_2<0$ we have to change the hyperbolic functions by trigonometric functions and then the expression is not defined for any ρ.

We replace (15) into expression (11) and use transformation $\chi(\cdot)$ defined above. Since $d\xi_0 d\xi = d\chi_0 d\chi$ we obtain a Gaussian integral with the value

$$\varphi(t,t') = \sqrt{\frac{1-(\beta/\alpha)^2}{1-(\beta/\alpha)^2 e^{-2\rho}}}\exp\left\{-\left[\frac{\beta}{2}-2c\left(\frac{a_1}{\alpha+\beta}\right)^2\right]\times(t-t')-\left(\frac{\alpha-\beta}{\alpha+\beta}\right)\left(\frac{c}{\alpha}\right)\times\left(\frac{2a_1}{\alpha+\beta}\right)^2\left(\frac{1-e^{-\rho}}{1-(\beta/\alpha)e^{-\rho}}\right)\right\}. \quad (16)$$

where $\alpha=\lambda+\gamma$ and $\beta=\lambda-\gamma$. Going back to Eq. (4) we can see that in order to have a stationary state $\langle x(t)\rangle_{st}$ it is necessary to analyze $e^{-a_0(t-\tau)}\varphi(t,\tau)$ in the limiting case $\tau\to-\infty$, since finally we shall have to take the limit $t_0\to-\infty$. From Eq. (16) we can see that this implies the condition $a_0+\beta/2-2ca_1^2/(\alpha+\beta)^2>0$, or in terms of the original parameters

$$a_0+\frac{\sqrt{\gamma^2+2ca_2}}{2}>\frac{1}{2}\left(\gamma+\frac{ca_1^2}{\gamma^2+2ca_2}\right). \quad (17)$$

Then it is direct to see that

$$\langle x(t)\rangle_{st} = A\int_{-\infty}^{t} ds\,\sin(\Omega s)e^{-a_0(t-s)}\varphi(t,s). \quad (18)$$

From (16) we see that $\varphi(t,s)$ depends only on $\lambda(t-s)$. Performing the change of variables $u=\lambda(t-s)$ in formula (18) we can write

$$\langle x(t)\rangle_{st} = \frac{A}{\lambda}\int_0^{\infty} du\,\sin\left(\Omega t-\frac{\Omega}{\lambda}u\right)e^{-a_0u}\varphi(u) \quad (19)$$

and finally by simple algebraic manipulations we arrive to the exact result

$$\langle x(t)\rangle_{st} = \chi_{st}\sin(\Omega t-\phi), \quad (20)$$

$$\chi_{st}=\frac{A}{\lambda}\sqrt{B_1^2+B_2^2},\qquad \phi=\arctan\left(\frac{B_1}{B_2}\right),$$

where

$$B_1=\int_0^{\infty} du\,\cos\left(\frac{\Omega}{\lambda}u\right)e^{-a_0u}\varphi(u), \quad (21)$$

$$B_2=\int_0^{\infty} du\,\sin\left(\frac{\Omega}{\lambda}u\right)e^{-a_0u}\varphi(u).$$

For future reference we write explicitly

$$\varphi(u) = \sqrt{\frac{1-(\beta/\alpha)^2}{1-(\beta/\alpha)^2 e^{-2u}}}\exp\left\{-\frac{1}{\lambda}\left[\frac{\beta}{2}-2c\left(\frac{a_1}{\alpha+\beta}\right)^2\right]u-\left(\frac{\alpha-\beta}{\alpha+\beta}\right)\left(\frac{c}{\alpha}\right)\left(\frac{2a_1}{\alpha+\beta}\right)^2\left(\frac{1-e^{-u}}{1-(\beta/\alpha)e^{-u}}\right)\right\}. \quad (22)$$

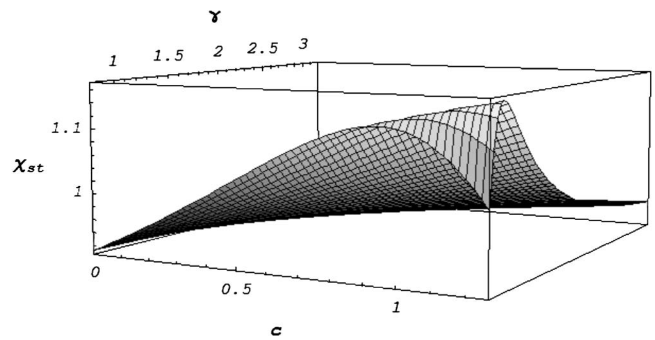

FIG. 1. A surface of resonance $\chi_{st}=\chi_{st}(c,\gamma)$.

III. RESULTS

The following are the main results of the numerical work performed on the above analytical expressions we have arrived at. First, we plotted what we called the surfaces of resonance. The result, which we show in Fig. 1 is the surface plot of the stationary amplitude, χ_{st} versus c and γ, the parameters of the OU process.

Then, now setting c constant, we plot families of χ_{st} varying γ and Ω. We can distinguish three distinct cases, as follows.

A. The general case

This is the solution of the general problem with terms linear and quadratic in the noise. Assigning to the parameters the following values: $a_0=a_1=A=1$; $a_2=-1$; $c=0.98$, Fig. 2 is obtained which shows the stationary amplitude χ_{st} as a function of the parameter γ. Note that we have found a family of curves, each member being identified by a particular frequency belonging to the set {0.25; 0.30; 0.35; 0.40; 0.45}.

B. The linear case

This is the solution of a problem with a linear term in the noise [7,11] and is simply a particular case of the general solution found in Sec. II, namely that with $a_2=0$. For this particular case, from condition (17) we obtain $a_0>ca_1^2/2\gamma^2$. In this case our scheme reproduces the results of [9]. Taking the following values for the parameters: $a_0=a_1=A=1$;

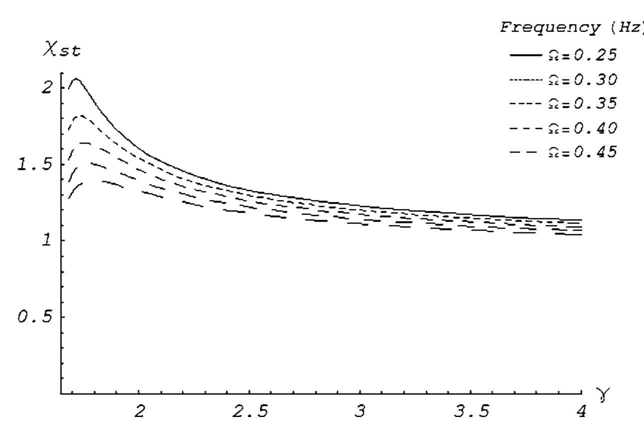

FIG. 2. The general case. Stationary amplitude χ_{st} versus γ.

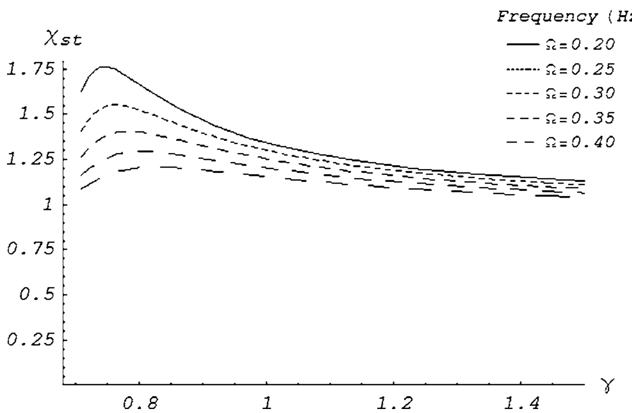

FIG. 3. The linear case. Stationary amplitude χ_{st} versus γ.

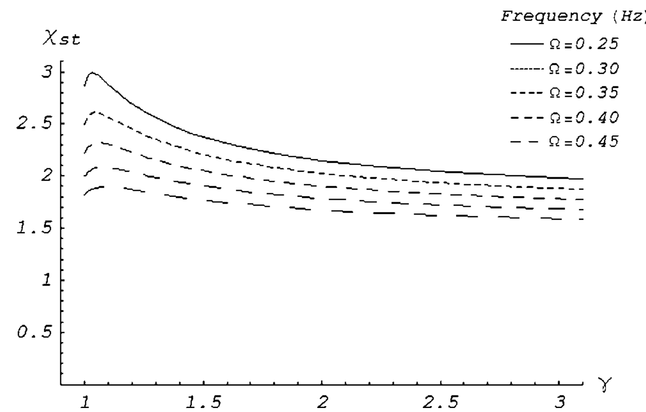

FIG. 4. The pure quadratic case. Stationary amplitude χ_{st} versus γ.

$c=0.98$, we obtain Fig. 3 which shows the stationary amplitude χ_{st} as a function of the parameter γ. The set of frequencies examined in this case is {0.20; 0.25; 0.30; 0.35; 0.40}. As in the previous case we have plotted a family of curves, each member being identified by a particular frequency.

C. The pure quadratic case

This is the solution of a problem with only a quadratic term in the noise and it also corresponds to the particular case of the general solution found in Sec. II, namely that with $a_1=0$. Inequality (17) reduces to the expression $a_0+\sqrt{\gamma^2+2ca_2}/2>1/2\gamma$. In contrast to the previous cases, the parameters here are given by the following values: $a_0=0.525$; $a_2=-1$; $A=1$; $c=0.50$. The frequencies studied are the same as for the general case. Figure 4 shows the stationary amplitude χ_{st} as a function of the parameter γ.

As can be seen in all the previous figures, the stationary amplitude of the mean value decreases monotonically as γ increases which implies that the SR phenomena disappears in the white noise limit, as remarked in the literature.

The reason for assigning to some parameters the value 1 in each of the cases is due to the possibility of a rescaling of the original problem given by Eqs. (1) and (2), a procedure which reduces the number of independent parameters of the original problem. Finally due to its simplicity, the linear equation (1) admits the exact solution (20) for the mean value of $x(t)$, from where one can analyze the diverse possibilities to obtain stochastic resonance varying the parameters that determine the properties of the amplitude of the mean value.

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