Comment on "Symmetric path integrals for stochastic equations with multiplicative noise"

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We recall our approach through discretizations for path integrals and its general results for representations of probability densities. It is shown that the result of Arnold [P. Arnold, Phys. Rev. E **61**, 6099 (2000)] is a particular case of our work.

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In Ref. [1] the author comes back to the problem of the determination of path integral representations for some probability densities. His central result is the Lagrangian in formula (1.13) that corresponds to midpoint discretization. This result, which is certainly of interest, is a special case of general results obtained by Langouche, Roekaerts, and Tirapegui (LRT in what follows) in Ref. [2], Sec. 7.6 (1982), where one can also find references to the original papers of the authors and to other previous works. We summarize the LRT work that concerns the study of Langevin equations of the form (sum from 1 to n over repeated indices should be understood from now on)

$$\frac{dq^{\nu}}{dt}+a^{\nu}[q(t)]=\sqrt{\eta}\sigma^{\nu}_{\mu}[q(t)]\xi^{\mu}(t), \quad \nu=1,2,\ldots,n, \tag{1}$$

where $[\xi''(l),\ \mu=1,2,\ldots,n]$ are Gaussian white noises with zero mean and correlations $\langle\xi^{\mu}(t)\xi^{\nu}(t')\rangle=\delta^{\mu\nu}\delta(t-t')$ and η measures the intensity of the noise. A discretized version of Eq. (1) is

$$\Delta q^{\nu}_{j}+\varepsilon a^{\nu}(q^{(r)}_{j-1})=\sqrt{\eta}\sigma^{\nu}_{\mu}(q^{(s)}_{j-1})\Delta w^{\mu}_{j}, \quad \nu=1,2,\ldots,n,$$
$$j=1,2,\ldots,(N+1). \tag{2}$$

One considers the Markov process defined by Eqs. (1) and (2) in the interval $[t_0,T]$; and we use the notations $t_j=t_0+j\varepsilon$, $j=1,2.\ldots\ (N+1)$; $t_{N+1}=T$; $q^{\mu}_{j}=q^{\mu}(t_j)$; and $q^{(\alpha)}_{j-1}=q_{j-1}+\alpha\Delta q_j$, where the number α stands for r or s in Eq. (2); $\Delta q^{\mu}_{j}=q^{\mu}_{j}-q^{\mu}_{j-1}$; $\Delta w^{\mu}_{j}=w^{\mu}_{j}-w^{\mu}_{j-1}$, and $w^{\mu}_{j}=w^{\mu}(t_j)$, where $w^{\mu}(t)$ is the Wiener process defined by $dw^{\mu}/dt=\xi^{\mu}(t)$. From Eq. (2) and using three different methods (two of them are used in Ref. [1]) LRT obtained the functional integral representation for the probability density as the limit $P(Q,T|Q_0,t_0)=I_N|_{N\to\infty}$, with [formula (7.83) in Ref. [2]]

$$I_N=\int\frac{1}{\sqrt{(2\pi\varepsilon\eta)^n\det g^{\mu\nu}(q^{(s)}_{N})}}\prod_{j=1}^{j=N}\prod_{\mu=1}^{\mu=n}\frac{dq^{\mu}_{j}}{\sqrt{(2\pi\varepsilon\eta)^n\det g^{\mu\nu}(q^{(s)}_{j-1})}}\exp\Bigg\{-\varepsilon\sum_{j=1}^{j=N+1}\Bigg[\frac{1}{2\eta}g_{\mu\nu}(q^{(s)}_{j-1})\left(\frac{\Delta q^{\mu}_{j}}{\varepsilon}+a^{\mu}(q^{(r)}_{j-1})\right)$$
$$\times\left(\frac{\Delta q^{\nu}_{j}}{\varepsilon}+a^{\nu}(q^{(r)}_{j-1})\right)-s\sigma^{\nu}\rho(q^{(s)}_{j-1})\partial_{\nu}\sigma_{\gamma\rho}(q^{(s)}_{j-1})\left(\frac{\Delta q^{\gamma}_{j}}{\varepsilon}+a^{\gamma}(q^{(r)}_{j-1})\right)-r\partial_{\mu}a^{\mu}(q^{(r)}_{j-1})$$
$$+\frac{1}{2}\eta s^2\partial_{\mu}\sigma^{\nu}_{\rho}(q^{(s)}_{j-1})\partial_{\nu}\sigma^{\mu}_{\rho}(q^{(s)}_{j-1})\Bigg]\Bigg\}, \tag{3}$$

where we have defined $\sigma_{\mu\nu}(q)$, $g_{\mu\nu}(q)$, and $g^{\mu\nu}(q)$, through $\sigma^{\mu}{}_{\rho}(q)\sigma_{\mu\nu}(q)=\delta_{\rho\nu}$, $g_{\mu\nu}(q)=\sigma_{\mu\rho}\sigma_{\nu\rho}$, $g^{\mu\nu}g_{\nu\rho}=\delta_{\mu\rho}$, and one has then that $\sigma^{\nu}{}_{\rho}\sigma_{\gamma\rho}=\delta_{\nu\gamma}$. The path integral for $P(Q,T|Q_0,t_0)$ is written formally as

$$P(Q,T|Q_0,t_0)=\int_{\gamma(r,s)}D\left(\frac{q}{\sqrt{\det g_{\mu\nu}}}\right)\exp\left[-\int dt\,L^{\gamma(r,s)}\right.$$
$$\left.\times\left(q,\frac{dq}{dt}\right)\right]\delta[q(T)-Q]\delta[q(t_0)-Q_0], \tag{4}$$

where $L^{\gamma(r,s)}(q,dq/dt)$ can be read from Eq. (3) which is the definition of Eq. (4). After using the identity $\sigma^{\nu}_{\rho}\partial_{\nu}\sigma_{\gamma\rho}=-\sigma_{\gamma\rho}\partial_{\nu}\sigma^{\nu}{}_{\rho}$, we have

$$L^{\gamma(r,s)}\left(q,\frac{dq}{dt}\right)=\frac{1}{2\eta}q_{\mu\nu}\left(\frac{dq^{\mu}}{dt}+a^{\mu}(q)\right)\left(\frac{dq^{\nu}}{dt}+a^{\nu}(q)\right)$$
$$+s\sigma_{\gamma\rho}\partial_{\nu}\sigma^{\nu}\rho\left(\frac{dq^{\gamma}}{dt}+a^{\gamma}(q)\right)-r\partial_{\mu}a^{\mu}(q)$$
$$+\frac{1}{2}\eta s^2\partial_{\mu}\sigma^{\nu}{}_{\rho}\partial_{\nu}\sigma^{\mu}\rho. \tag{5}$$

The index $\gamma(r,s)$ in Eq. (4) stands for the discretization involved in the definition of the path integral that depends here on two parameters (r,s) and is completely specified by Eq. (3). The Lagrangian is also labeled with γ since it obviously depends on the discretization, one is using to define the functional integral [2]. Expression (5) reduces for $r=s=\frac{1}{2}$ (midpoint discretization) to the result in Ref. [1]. We make now several remarks that are important in a paper dealing with discretization problems in path integrals:

(1) The Markov process defined by Eqs. (1) and (2) depends on s but not on r [the reason for this is the well-known relation $(\Delta q_j)^2 \approx O(\epsilon)$ of diffusion processes]. This means that when one calculates the functional integral in Eq. (4) all dependence in r must cancel. This property can be verified at each order in a perturbation expansion of Eq. (4) (see, Ref. [3]) and rigorous results confirming the work of LRT can be found in Ref. [4]. This means that, we can put $r=0$ in the discretized form (3) of Eq. (4) and the result will be the conditional probability density of the Markov process defined by Eqs. (1),(2) with $\sigma_\nu^\rho(q)$ discretized in $q_{j-1}^{(s)}$. We recall that $s=0$ and $s=\frac{1}{2}$ correspond to the Ito and Statonovic interpretation, respectively (see Ref. [5]).

(2) The discretization dependence, of functional integrals, was exhaustively studied by LRT (see, Ref. [2]) and, in particular, the relation to the ordering of operators [6] when one introduces an operator formalism as it is done for example in quantum mechanics.

(3) It is stated in Ref. [1] that the midpoint discretization is "natural" since it allows the use of standard calculus in the action. This is true for a term linear in the derivatives dq^μ/dt of a path but not in the dominant term that is quadratic in the derivatives. Once again this is due to $(\Delta q_j)^2 \approx O(\epsilon)$ that is also related to the fact that nondifferentiable paths have probability one. If one looks to point transformations in the action $\int L^\gamma(q,dq/dt)dt$, one immediately concludes that one cannot use the usual rules of calculus in the action [2,7,8,9] and that the transformation must be done in the discrete version (see, Chap. VI of Ref. [2] and, especially, Sec. 6.4 for covariant discretizations). One has in general that a discretization associated with a correspondence rule cannot be the same before and after a formal change of variables in the path integral, a statement that is a translation of the well-known result of quantization theory that states that the relation between an operator theory and a c-number theory cannot be given by one and the same correspondence rule before and after a nonlinear canonical transformation is done. From this last point of view there is nothing "natural" in the midpoint discretization (see Ref. [8] for a discussion of this specific point), and this is also the case of perturbation expansions where the prepoint discretization gives often simpler ways of calculation [10].

(4) Functional integrals are calculated in most cases of interest as perturbative expansions. These expansions are not well defined since the value of some equal time contractions is not defined [2,3,11,12]. The solution of this problem is the use of the concept of discretization as defined by LRT that tells us how to give definite values to the ambiguous quantities.

(5) Covariant definitions of path integrals, which are not based on discretizations, have been proposed by Feynman [13,14] and by Graham [15]. However, if one wants to make perturbative calculations it follows from the work of LRT [2,3] that it is necessary to determine the discretization that corresponds to the Lagrangian appearing in the configuration space path integral (the problems of undefined quantities that appear in configuration space are solved in Chap. X of Ref. [2]). We remark that a related problem, which also leads to a Lagrangian, is the determination of the most probable path for a diffusion process [16,17]. This Lagrangian, sometimes called the Onsager-Machlup function, is such that its Euler-Lagrange equations determine the most probable path and it is known to coincide with the Lagrangian of Graham's definition of path integrals that is different from the Feynman Lagrangian in curved spaces by a factor $\frac{1}{6}R$, where R is the curvature determined by the metric $g_{\mu\nu}(q)$ defined after Eq. (3).

(6) The preference for a given discretization can be dictated by the simplifications obtained for a given problem: this is the case of the exact calculation by LRT of the propagator on the sphere S^3 (Chap. XII of Ref. [2]). It is also important to recall that the use of the discretization approach allowed LRT to calculate for the first time the higher order corrections to the WKB approximation in curved spaces [18]. We finally point out that the use of Fourier series to calculate functional integrals as it has been proposed in Ref. [19] does not eliminate the discretization problems that just appear in a different form [20].

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