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HERMITIAN QUADRATIC EIGENVALUE PROBLEMS OF RESTRICTED RANK

CARLOS CONCA AND HEINRICH PUSCHMANN
Departamento de Ingeniería Matemática
Universidad de Chile, Santiago, Chile

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Abstract—We consider a quadratic eigenvalue problem such that the second order term is a Hermitian matrix of rank r, the linear term is the identity matrix, and the constant term is an arbitrary Hermitian matrix $A \in \mathbb{C}_{nn}$. Of the $n+r$ solutions that this problem admits, we show at least $n-r$ to be real and located in specific intervals defined by the eigenvalues of A, whence at most $2r$ are nonreal occuring in possibly repeated conjugate pairs.

1. INTRODUCTION

Consider the following quadratic eigenvalue problem: for $A \in \mathbb{C}_{nn}, B \in \mathbb{C}_{nn}$ Hermitian matrices with $(B) = r \leq n$, find $\lambda \in \mathbb{C}, u \in \mathbb{C}_{n1}$ such that

$$(A - \lambda I + \lambda^2 B)u = 0, \qquad u \neq 0. \tag{1.1}$$

It should be remarked that even if A and B are Hermitian, the solutions of (1.1) may not be real. Our study of this problem is directed towards finding accurate estimates for the maximum number of nonreal solutions that (1.1) can possess and comparing the real eigenvalues of (1.1) with the solutions $\alpha_1, \dots, \alpha_n$ of the corresponding linear problem.

Equation (1.1) admits $n + \operatorname{rank}(B) = n + r$ solutions $\lambda \in \mathbb{C}$. Let the eigenvalues of A be $\alpha_1 \leq \cdots \leq \alpha_n$. Theorem 2.5 and its Corollary 2.6 show that at least $n-r$ of the solutions of (1.1) are real and located in each of the intervals

$$[\alpha_1, \alpha_r],\ [\alpha_2, \alpha_{r+1}],\ \dots,\ [\alpha_{n-r}, \alpha_n], \tag{1.2}$$

while at most $2r$ of these solutions are nonreal and occur in (possibly repeated) conjugate pairs.

By means of an example, we show that the above results can fail for a non-Hermitian second order term: not only may the number of solutions be less than $n+r$, but it is also possible that all of them might be nonreal.

Quadratic eigenvalue problems frequently arise in nonlinear vibration theory. For example, it can be seen in J. Planchard [1] that the study of the vibratory eigenmodes of fluid-solid structures leads naturally to the spectral analysis of some differential problems which involve linear operators in infinite-dimensional Hilbert spaces. These problems are usually quadratic eigenvalue problems, as can be seen in the above reference, or in the papers [2,3]. The second order terms in these eigenvalue problems are positive semidefinite Hermitian operators of finite rank, while the zero order terms are coercitive operators of dense image. For other eigenvalue problems of higher degree the reader is referred to [4, pp. 149–150, 5].

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Mailing address: Heinrich Puschmann Depto. Ing. Mat., P.O. Box 170-3 Correo 3, Santiago, Chile.

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In practice, eigenvalues $\lambda \in \mathbb{C}$ such that $\mathfrak{Im}(\lambda) \neq 0$ play a very important role in this kind of problem, since they imply the existence of unstable vibratory eigenmodes. Their corresponding eigenfunctions can be written in the form

$$\psi(x,t) = \exp(i\lambda t)\varphi(x), \tag{1.3}$$

where $\varphi(x)$ depends only on the space variables. Thus, for $\mathfrak{Im}(\lambda) \neq 0$ either the eigenmotion $\psi(x,t)$ or its conjugate will diverge as the time t tends to infinity. This situation cannot arise if $\lambda \in \mathbb{R}$, since in that case $|\exp(i\lambda t)| = 1$ and therefore the amplitude of the corresponding eigenmotion remains bounded (actually constant) as t goes to infinity. This is the main reason that it is so important to have sharp *a priori* bounds for the number of nonreal eigenvalues.

Frequently, the discretization of a quadratic differential spectral problem leads to (1.1), which is one of the simplest quadratic eigenvalue problems one can imagine. Though it does not have the degree of complexity of nonlinear vibration theory, its study is a convenient step towards tackling more complex models.

2. THE NATURE OF HERMITIAN QUADRATIC EIGENVALUES

We shall designate the set of nonnegative integers by $\mathbb{Z}_{\oplus}$ and the set of nonnegative real numbers by $\mathbb{R}_{\oplus}$. We also define $\dim(\{0\}) = 0$ for any zero-subspace $\{0\} \subset \mathbb{C}_{n1}$. We designate by $A^* \in \mathbb{C}_{nn}$ the conjugate transpose of $A \in \mathbb{C}_{nn}$.

For any finite matrix sequence $A_0, \dots, A_t$, $A_i \in \mathbb{C}_{nn}$ such that $\det(\sum_{i=0}^{t} \lambda^i A_i)$ is a nonvanishing polynomial in the indeterminate λ, we define the *geometric multiplicity* of an $\alpha \in \mathbb{C}$ by

$$\operatorname{geo}(\alpha \mid A_0, \dots, A_t) \stackrel{\text{def}}{=} \dim \left\{ u \in \mathbb{C}_{n1} \models \left(\sum_{i=0}^{t} \alpha^i A_i \right) u = 0 \right\} \tag{2.1}$$

and the *polynomial multiplicity* of an $\alpha \in \mathbb{C}$ by

$$\operatorname{pol}(\alpha \mid A_0, \dots, A_t) \stackrel{\text{def}}{=} \max \left\{ m \in \mathbb{Z}_{\oplus} \models (\lambda - \alpha)^m \text{ divides } \det \left(\sum_{i=0}^{t} \lambda^i A_i \right) \right\}. \tag{2.2}$$

We refrained from calling $\operatorname{pol}(\alpha| A_0, \dots, A_t))$ the *algebraic multiplicity* of α only because of alternative ways to generalize this concept [6, pp. 34–43] to a problem of higher degree.

The following results 2.1 through 2.3 can easily be proven by means of the classical theory of linear algebra and of matrix polynomials; for instance, see [4, pp. 149–159, Theorem 1; 5,7–9] Detailed proofs can be found in [10].

THEOREM 2.1. *Let $A_0, \dots, A_t$, $A_i \in \mathbb{C}_{nn}$, be such that $\det(\sum_{i=0}^{t} \lambda^i A_i) \not\equiv 0$ is a nonvanishing polynomial in the indeterminate λ. Then the following inequality holds:*

$$\forall \alpha \in \mathbb{C} \qquad \operatorname{geo}(\alpha \mid A_0, \dots, A_t) \leq \operatorname{pol}(\alpha \mid A_0, \dots, A_t). \tag{2.3}$$

PROPOSITON 2.2. *Let $A \in \mathbb{C}_{nn}$, $B \in \mathbb{C}_{nn}$, $B^* = B$. Then the characteristic polynomial of the quadratic eigenvalue problem (1.1),*

$$(A - \lambda I + \lambda^2 B)u = 0, \qquad u \neq 0$$

is of degree exactly $n + \operatorname{rank}(B)$, whence

$$\sum_{\gamma \in \mathbb{C}} \operatorname{geo}(\gamma \mid A, -I, B) \;\leq\; \sum_{\gamma \in \mathbb{C}} \operatorname{pol}(\gamma \mid A, -I, B) \;=\; n + \operatorname{rank}(B). \tag{2.4}$$

PROPOSITION 2.3. *Let $A \in \mathbb{C}_{nn}$, $A^* = A$, $B \in \mathbb{C}_{nn}$, $B^* = B$. For any $\gamma \in \mathbb{C}$ and its complex conjugate $\bar{\gamma}$ we have*

$$\operatorname{geo}(\bar{\gamma} \mid A, -I, B) = \operatorname{geo}(\gamma \mid A, -I, B), \tag{2.5}$$
$$\operatorname{pol}(\bar{\gamma} \mid A, -I, B) = \operatorname{pol}(\gamma \mid A, -I, B). \tag{2.6}$$

Statements 2.1 and 2.2 establish the existence of $n + \operatorname{rank}(B)$ eigenvalues for problem (1.1); we now proceed to localize some of them on the real axis. Let us first recall the following classical result:

PROPOSITION 2.4. *Let $A \in \mathbb{C}_{nn}$, $A^* = A$, $B \in \mathbb{C}_{nn}$, $B^* = B$. Let the eigenvalues of A be $\alpha_1 \le \cdots \le \alpha_n$, and define $\forall\, i < 1$ $\alpha_i = -\infty$, $\forall\, i > n$ $\alpha_i = \infty$. Let B have k (possibly repeated) negative eigenvalues and l (possibly repeated) positive eigenvalues, whence $k + l = \operatorname{rank}(B)$. Then the eigenvalues of $(A + B) \in \mathbb{C}_{nn}$, designated by $\lambda_1 \le \cdots \le \lambda_n$, satisfy*

$$\forall\, i = 1, \ldots, n \qquad \alpha_{i-k} \le \lambda_i \le \alpha_{i+l}. \tag{2.7}$$

This result is due to Weyl and Courant and well-documented in the literature; for instance, see Riesz and Nagy [11, pp. 236–237, Section 95]. A possible proof goes by

(a) diagonalizing A instead of B,
(b) proving the proposition in the case of $\operatorname{rank}(B) = 1$, and
(c) applying mathematical induction on the rank of the perturbation matrix B. ∎

In the proof of the following main theorem of our note we use rather familiar tools found in [5,6, pp. 62–73; 11, pp. 368–373].

THEOREM 2.5. *Let $A \in \mathbb{C}_{nn}$, $A^* = A$, $B \in \mathbb{C}_{nn}$, $B^* = B$, and let the eigenvalues of A be $\alpha_1 \le \cdots \le \alpha_n$. Let B have k (possibly repeated) negative eigenvalues and l (possibly repeated) positive eigenvalues, whence $k + l = \operatorname{rank}(B)$. Then the quadratic eigenvalue problem (1.1),*

$$(A - \lambda I + \lambda^2 B)u = 0, \qquad u \neq 0$$

has at least $n - \operatorname{rank}(B)$ (possibly repeated) real solutions $\rho_{1+k}, \ldots, \rho_{n-l}$ that satisfy

$$\forall\, i = 1 + k, \ldots, n - l, \qquad \alpha_{i-k} \le \rho_i \le \alpha_{i+l}. \tag{2.8}$$

Repeated solutions correspond to eigenspaces of higher dimension, whence

$$\sum_{\rho \in \mathbb{R}} \operatorname{pol}(\rho \mid A, -I, B) \ \ge\ \sum_{\rho \in \mathbb{R}} \operatorname{geo}(\rho \mid A, -I, B) \ \ge\ n - \operatorname{rank}(B). \tag{2.9}$$

PROOF. Since $(A + \varepsilon B) \in \mathbb{C}_{nn}$ is Hermitian for all $\varepsilon \in \mathbb{R}$ and its coefficients depend analytically on ε, there are real-valued analytical functions

$$\tilde{\lambda}_s : \mathbb{R} \longrightarrow \mathbb{R}$$

such that $\tilde{\lambda}_s(0) = \alpha_s$ and each $\tilde{\lambda}_s(\varepsilon)$ is an eigenvalue of $(A + \varepsilon B)$ (see [6, pp. 62–73] or [11, pp. 368–373], for instance). Overlapping function values $\tilde{\lambda}_s(\varepsilon) = \tilde{\lambda}_t(\varepsilon)$, $s \neq t$, correspond to multiple eigenvalues of $(A + \varepsilon B)$.

For each $\varepsilon \ge 0$, we may now order the indexed set $\{\tilde{\lambda}_1(\varepsilon), \ldots, \tilde{\lambda}_n(\varepsilon)\}$ according to its function values, thereby defining a new indexed set $\{\lambda_1(\varepsilon) \le \cdots \le \lambda_n(\varepsilon)\}$. As functions of $\varepsilon \ge 0$, these

$$\lambda_i : \mathbb{R}_{\oplus} \longrightarrow \mathbb{R}, \qquad \lambda_1 \le \cdots \le \lambda_n \tag{2.10}$$

are real-valued continuous (though not necessarily differentiable) functions, such that $\lambda_i(0) = \alpha_i$ and $\lambda_i(\varepsilon)$ is the i^{th} eigenvalue of $(A + \varepsilon B)$. Because of Proposition 2.4, the indexed set of ordered eigenvalues $\{\lambda_1(\varepsilon) \le \cdots \le \lambda_n(\varepsilon)\}$ satisfies

$$\forall i = 1, \ldots, n, \qquad \alpha_{i-k} \le \lambda_i(\varepsilon) \le \alpha_{i+l},$$

where, as usual, we define $\forall\, i < 1$ $\alpha_i = -\infty$, $\forall\, i > n$ $\alpha_i = \infty$.

For any of the $n - \operatorname{rank}(B)$ indices $i = 1+k, \dots, n-l$ together with $\lambda \in \mathbb{R}$, the following function

$$\Lambda_i : \mathbb{R} \longrightarrow \mathbb{R}, \qquad \Lambda_i(\lambda) \overset{\text{def}}{=} \lambda - \lambda_i(\lambda^2) \tag{2.11}$$

is a real-valued continuous function that satisfies

$$\lambda - \alpha_{i+l} \le \Lambda_i(\lambda) \le \lambda - \alpha_{i-k}, \qquad \lim_{\lambda \to -\infty} \Lambda_i(\lambda) = -\infty, \quad \lim_{\lambda \to \infty} \Lambda_i(\lambda) = \infty.$$

Thus there is always some $\rho \in \mathbb{R}$ such that $\Lambda_i(\rho) = 0$; choose ρ_i to be the largest of all such ρ. Now

$$\Lambda_i(\rho_i) = 0 \Longrightarrow \rho_i = \lambda_i(\rho_i^2) \Longrightarrow \alpha_{i-k} \le \rho_i \le \alpha_{i+l}$$

and

$$\rho_i = \lambda_i(\rho_i^2) \Longrightarrow \det(A - \rho_i I + \rho_i^2 B) = \det\left((A + \rho_i^2 B) - \lambda_i(\rho_i^2) I\right) = 0$$

according to the claim of the theorem.

If $\rho_i = \rho_j$, $i \ne j$, then the functions $\lambda_i(\rho_i^2) = \rho_i = \rho_j = \lambda_j(\rho_j^2) = \lambda_j(\rho_i^2)$ overlap at ρ_i^2. Now overlapping function values $\lambda_i(\varepsilon) = \lambda_j(\varepsilon)$, $i \ne j$ of these bounded eigenvalue functions correspond to multiple eigenvalues of $(A + \varepsilon B)$, whence $\lambda_i(\rho_i^2) = \rho_i$ is a geometrically multiple eigenvalue of the Hermitian matrix $A + \rho_i^2 B$. But then

$$\operatorname{geo}(\rho_i \mid A, -I, B) = \operatorname{geo}(\rho_i \mid A + \rho_i^2 B, -I) > 1,$$

so ρ_i has an eigenspace of higher dimension. ∎

COROLLARY 2.6. *Let $A \in \mathbb{C}_{nn}$, $A^* = A$, $B \in \mathbb{C}_{nn}$, $B^* = B$. Then the quadratic eigenvalue problem (1.1),*

$$(A - \lambda I + \lambda^2 B)u = 0, \qquad u \ne 0$$

has at most $2\operatorname{rank}(B)$ nonreal solutions λ, occurring in (possibly repeated) conjugate pairs.

3. COMMENTS

For arbitrary matrices $A \in \mathbb{R}_{nn}$ and $B \in \mathbb{R}_{nn}$, the polynomial $\det(A - \lambda I + \lambda^2 B)$ may well vanish, as shown in

$$\det\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \lambda I + \lambda^2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \equiv 0. \tag{3.1}$$

Thus there is a need for some additional condition, such as A invertible, B invertible, $A = A^*$, or $B = B^*$, in order to rule out this possibility. With any of the above conditions, the number of solutions will be less than or equal to $n + \operatorname{rank}(B)$. In [10] we provide an example (with a nonvanishing characteristic polynomial) showing that a non-Hermitian quadratic term B of only rank 1 may cause the number of solutions to be strictly less than $n + \operatorname{rank}(B)$ and prevent any real solutions of $\det(A - \lambda I + \lambda^2 B) = 0$.

Theorem 2.5 remains true if we impose the additional condition $\rho_{1+k} \le \cdots \le \rho_{n-l}$. However, we must by no means assume that the eigenvalues $\rho_{1+k}, \dots, \rho_{n-l}$ are *consecutive* real eigenvalues. For a counterexample, choose

$$A = \begin{bmatrix} 1 & & & \\ & 101 & 99 & \\ & 99 & 101 & \\ & & & 201 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & & & \\ & b & & \\ & & 0 & \\ & & & 0 \end{bmatrix}. \tag{3.2}$$

Since

$$\begin{aligned} \det(A - \lambda I) &= (1 - \lambda)\left[(101 - \lambda)^2 - (99)^2\right](201 - \lambda) \\ &= (1 - \lambda)\left[(2 - \lambda)(200 - \lambda)\right](201 - \lambda), \end{aligned}$$

the eigenvalues of A are $\alpha_1 = 1$, $\alpha_2 = 2$, $\alpha_3 = 200$, $\alpha_4 = 201$. Adding the extra term corresponding to the quadratic problem, we get

$$p(\lambda) \stackrel{\text{def}}{=} \det(A - \lambda I + \lambda^2 B) = (1-\lambda)\left[(2-\lambda)(200-\lambda) + b\lambda^2(101-\lambda)\right](201-\lambda),$$

whence for $b = 0.1$ we obtain $p(2) > 0, p(10) < 0, p(60) > 0, p(200) < 0$. This implies that equation $\det(A - \lambda I + \lambda^2 B) = 0$ has 5 solutions, all of them real, satisfying

$$1 \;= \rho_1 < \;2\; < \rho_4 < \rho_5 < \rho_2 < \;200\; < \rho_3 = \;201.$$

In the case of this example, it is interesting to observe the development of all real solutions to $\det(A - \lambda I + \lambda^2 B) = 0$ in terms of $b \geq 0$. Since

$$p(\lambda) = 0 \qquad \Longleftrightarrow \qquad \left(\lambda = 1 \quad \vee \quad b = \frac{(\lambda-2)(\lambda-200)}{\lambda^2(\lambda-101)} \quad \vee \quad \lambda = 201\right), \tag{3.3}$$

this can easily be accomplished by drawing the graph of b in terms of $\lambda \in \mathbb{R}$. We may then observe that there is no way of defining real-valued *continuous* functions $\rho_i(b)$ that yield solutions to $\det(A - \lambda I + \lambda^2 B) = 0$ for all $b \geq 0$.

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