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HADAMARD FUNCTIONS OF INVERSE M-MATRICES*

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Abstract. We prove that the class of generalized ultrametric matrices (GUM) is the largest class of bipotential matrices stable under Hadamard increasing functions. We also show that any power $\alpha \ge 1$, in the sense of Hadamard functions, of an inverse M-matrix is also inverse M-matrix. This was conjectured for $\alpha = 2$ by Neumann in [*Linear Algebra Appl.*, 285 (1998), pp. 277–290], and solved for integer $\alpha \ge 1$ by Chen in [*Linear Algebra Appl.*, 381 (2004), pp. 53–60]. We study the class of filtered matrices, which include naturally the GUM matrices, and present some sufficient conditions for a filtered matrix to be a bipotential.

Key words. M-matrices, Hadamard functions, ultrametric matrices, potential matrices

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1. Introduction and basic notations.

In this article we study stability properties under Hadamard functions for the class of inverse M-matrices and the class of filtered matrices, which includes GUM (generalized ultrametric matrices).

A nonnegative matrix U is said to be a *potential* if it is nonsingular and its inverse satisfies

$$\forall i \neq j,\ U^{-1}_{ij} \le 0, \qquad \forall i,\ U^{-1}_{ii} > 0,$$

$$\forall i,\ \sum_j U^{-1}_{ij} \ge 0,$$

that is, if U^{-1} is an M-matrix which is row diagonally dominant. We denote this class of matrices by $\mathcal{P}$. In addition we say that U is a *bipotential* if U^{-1} is also column diagonally dominant. This class of matrices is denoted by $bi\mathcal{P}$. We note that $\mathcal{P}, bi\mathcal{P}$ are contained in $\mathcal{M}^{-1}$, the class of inverses of M-matrices.

The class of potential matrices play an important role in probability theory. They represent the potential (from which we have taken the name) of a transient continuous time Markov chain $(X_t)_{t\ge0}$, with generator $-U^{-1}$. That is,

$$U_{ij} = \int_0^\infty (e^{-U^{-1}t})_{ij}\, dt = \int_0^\infty \mathbb{P}_i\{X_t = j\} dt$$

is the mean expected time expended at site j when the chain starts at site i. Clearly U is a bipotential if both U and U' are potentials.

To get a discrete time interpretation take $K_0 = \max_i\{U^{-1}_{ii}\}$. For any $K \ge K_0$ the matrix $P_K = \mathbb{I} - \frac{1}{K}U^{-1}$ is nonnegative, substochastic, and verifies

$$U^{-1} = k(\mathbb{I} - P_K).$$

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If we can take $K = 1$, then $U^{-1} = \mathbb{I} - P$ (with $P = P_1$) and U is the mean expected number of visits of a Markov chain $(Y_n)_{n\in\mathbb{N}}$ whose transition probability is given by P. Indeed,

$$U_{ij} = \sum_{n\geq 0} P^n_{ij} = \sum_{n\geq 0} \mathbb{P}_i\{Y_n = j\}.$$

We notice that if U is a potential, then for all i, j we have $U_{ii} \geq U_{ji}$. The probabilistic proof of this fact is based on the so-called strong Markov property which allows us to conclude

$$U_{ji} = f_{ji} U_{ii},$$

where $f_{ji} \leq 1$ is the probability that the Markov process (X_t), starting from j, ever reaches the state i. If U is a bipotential, then $U_{ii} \geq \max\{U_{ij}, U_{ji}\}$.

For any nonnegative matrix U we define the quantity

$$\tau(U) = \inf\{t \geq 0 : \mathbb{I} + tU \notin bi\mathcal{P}\},$$

which is invariant under permutations; that is, $\tau(U) = \tau(\Pi U \Pi')$. We point out that if U is a positive matrix, then $\tau(U) > 0$. We shall study some properties of this function τ. In particular we are interested on matrices for which $\tau(U) = \infty$, generalizing the class $bi\mathcal{P}$ as the next result shows.

PROPOSITION 1.1. *Assume that U is a nonnegative matrix, which is nonsingular and $\tau(U) = \infty$; then $U \in bi\mathcal{P}$.*

Proof. It is direct from the observation that

$$t(\mathbb{I} + tU)^{-1} \underset{t\to\infty}{\rightarrow} U^{-1}. \qquad \square$$

Remark 1.1. Later on, we shall prove that the converse is also true: if U is in class $bi\mathcal{P}$, then $\tau(U) = \infty$.

The following notion will play an important role in this article.

DEFINITION 1.2. *Given a matrix B, a vector μ is said to be a right equilibrium potential if*

$$B\mu = \mathbf{1},$$

where $\mathbf{1}$ is the constant vector of ones. Similarly it is defined the notion of a left equilibrium potential, which is the right equilibrium potential for B'. When B is nonsingular the unique right and left equilibrium potentials are, respectively, denoted by μ_B and ν_B.

We denote by $\bar{\mu} = \mathbf{1}'\mu$ the total mass of μ. In the nonsingular case, it is not difficult to see that $\bar{\nu} = \bar{\mu}$.

Notice that for a matrix $U \in bi\mathcal{P}$ the right and left equilibrium potentials are nonnegative. This is exactly the fact that the inverse of a bipotential matrix is row and column diagonally dominant.

DEFINITION 1.3. *The constant block form (CBF) matrices are defined recursively in the following way: given two CBF matrices A, B of sizes p and $n - p$, respectively, and numbers α, β, we produce the new CBF matrix by*

$$U = \begin{pmatrix} A & \alpha \mathbf{1}_p \mathbf{1}'_{n-p} \\ \beta \mathbf{1}_{n-p} \mathbf{1}'_p & B \end{pmatrix}, \qquad (1.1)$$

where the vector $\mathbf{1}_p$ *is the vector of ones, of size* p. *We also say that* U *is in* increasing *CBF if* $\min\{A, B\} \geq \min\{\alpha, \beta\}$.

The Definitions 1.4 and 1.6 below were introduced in [12] and [15], generalizing Definition 1.5 of ultrametric matrices introduced in [11] (see also [14]).

DEFINITION 1.4. *A nonnegative CBF matrix* U *is in nested block form (NBF) if in* (1.1) A *and* B *are NBF matrices and*

- $0 \leq \alpha \leq \beta$;
- $\min\{A_{ij}, A_{ji}\} \geq \alpha$ *and* $\min\{B_{kl}, B_{lk}\} \geq \alpha$;
- $\max\{A_{ij}, A_{ji}\} \geq \beta$ *and* $\max\{B_{kl}, B_{lk}\} \geq \beta$.

DEFINITION 1.5. *A nonnegative symmetric matrix* U *is said to be an* ultrametric matrix *if*

(1) *for all* $i, j, U_{ii} \geq U_{ij}$,

(2) *for all* i, j, k, *the inequality* $U_{ij} \geq \min\{U_{ik}, U_{kj}\}$ *is satisfied.*

The matrix U *is* strictly *ultrametric if in* (1) *the inequality is strict.*

Remark 1.2. The name ultrametric comes from ultrametric distances. One may think as $U_{ij} = \frac{1}{\delta_{ij}}$ (for $i \neq j$), where δ is an ultrametric distance.

A possible generalization of this concept to the nonsymmetric case is the following.

DEFINITION 1.6. *A nonnegative matrix* U *of size* n *is said to be a GUM if, for all* $i, j, U_{ii} \geq \max\{U_{ij}, U_{ji}\}$ *and, when* $n > 2$, *every three distinct elements* i, j, k *have a* preferred element. *Assume that this element is* i *which means*

- $U_{ij} = U_{ik}$;
- $U_{ji} = U_{ki}$;
- $\min\{U_{jk}, U_{kj}\} \geq \min\{U_{ji}, U_{ij}\}$;
- $\max\{U_{jk}, U_{kj}\} \geq \max\{U_{ji}, U_{ij}\}$.

By definition the transpose of a GUM is also a GUM. We note that an ultrametric matrix is a symmetric GUM. The study of the incidence graph for the inverse of an ultrametric matrix was done in [6] and for a GUM in [7] (this is the one step graph induced by a Markov chain associated with the matrix).

In the next result we summarize the main results obtained in [12] and [15] concerning GUM.

THEOREM 1.7. *Let* U *be a nonnegative matrix.*

- U *is a GUM if and only if it is a permutation similar to a NBF.*
- *If* U *is a GUM, then it is nonsingular if and only if it does not contain a row of zeros and no two rows are the same.*
- *If* U *is a nonsingular GUM, then* $U \in bi\mathcal{P}$.

It is clear that if U is a GUM, then $\mathbb{I} + tU$ is a nonsingular GUM. In particular, $\tau(U) = \infty$.

We introduce a main object of this article.

DEFINITION 1.8. *Given a function* f *and a matrix* U, *the matrix* $f(U)$ *is defined as* $f(U)_{ij} = f(U_{ij})$. *We shall say that* $f(U)$ *is a Hadamard function of* U.

Given two matrices A, B *of the same size, we denote by* $A \odot B$ *the Hadamard product of them, where* $(A \odot B)_{ij} = A_{ij}B_{ij}$.

Given a vector a, we denote by D_a the diagonal matrix whose diagonal is a. We have $D_a D_b = D_a \odot D_b = D_{a \odot b}$.

The class of CBF matrices (and its permutations) is closed under Hadamard functions. Similarly, the class of increasing CBF (and its permutations) is closed under increasing Hadamard functions.

On the other hand, the class of NBF, and therefore also the class of GUM, is stable under Hadamard nonnegative increasing functions. We summarize this result in the following proposition.

PROPOSITION 1.9. *Assume that U is a GUM and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function. Then $f(U)$ is a GUM. In particular, $\tau(f(U)) = \infty$, and if $f(U)$ is nonsingular, then $f(U) \in bi\mathcal{P}$. A sufficient condition for $f(U)$ to be nonsingular is that U is nonsingular and f is strictly increasing.*

Proof. It is clear that $f(U)$ is a GUM, and therefore $\tau(f(U)) = \infty$. Then, from Proposition 1.1 we have that $f(U) \in bi\mathcal{P}$ as long as $f(U)$ is nonsingular. If U is nonsingular, then it does not contain a row (or column) of zeros, and there are not two equal rows (or columns). This condition is stable under strictly increasing nonnegative functions, so the result follows. ∎

One of our main results is a sort of reciprocal of the previous one. We shall prove that if $\tau(f(U)) = \infty$ for all increasing nonnegative functions f, then U must be a GUM (see Theorem 2.4).

Let us introduce the following index.

DEFINITION 1.10. *We say that a nonnegative matrix U is in class $\mathcal{T}$ if*

$$\tau(U) = \inf\{t > 0 : \ (\mathbb{I} + tU)^{-1}\mathbf{1} \ngeq 0 \ \text{or} \ \mathbf{1}'(\mathbb{I} + tU)^{-1} \ngeq 0\},$$

and $\mathbb{I} + \tau(U)\,U$ is nonsingular whenever $\tau(U) < \infty$.

We shall prove that every nonnegative matrix U that is a permutation of an increasing CBF is in class $\mathcal{T}$.

We remark here that our purpose is to study Hadamard functions of matrices and not spectral functions of matrices, which are quite different concepts. For spectral functions of matrices there are deep and beautiful results for the same classes of matrices we consider here. See, for example, the work of Bouleau [3] for filtered operators. For M matrices, see the works of Varga [17], Micchelli and Willoughby [13], Ando [1], Fiedler and Schneider [9], and the recent work of Bapat, Catral, and Neummann [2] for M-matrices and inverse M-matrices.

2. Main results.

THEOREM 2.1. *Assume $U \in \mathcal{P}$ and that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative strictly increasing convex function. Then $f(U)$ is nonsingular and $det(f(U)) > 0$. Also $f(U)$ has a right nonnegative equilibrium potential. Moreover, if $f(0) = 0$, we have that $M = U^{-1}f(U)$ is an M-matrix. If $U \in bi\mathcal{P}$, then $f(U)$ also has a left nonnegative equilibrium potential.*

Note that $H = f(U)^{-1}$ is not necessarily a Z-matrix; that is, for some $i \neq j$ it can happen that $H_{ij} > 0$, as the following example will show. Therefore the existence of a nonnegative right equilibrium potential, which is

$$\forall i, \ \ H_{ii} + \sum_{j \neq i} H_{ij} \geq 0,$$

does not necessarily imply that the inverse is row diagonally dominant, that is,

$$\forall i, \ \ H_{ii} \geq \sum_{j \neq i} |H_{ij}|.$$

Example 2.1. Consider the matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Then $U = (\mathbb{I} - P)^{-1} \in bi\mathcal{P}$. Consider the nonnegative strictly convex function $f(x) = x^2 - \cos(x) + 1$. A numerical computation gives

$$(f(U))^{-1} \approx \begin{pmatrix} 0.3590 & -0.0975 & 0.0027 \\ -0.0975 & 0.2372 & -0.0975 \\ 0.0027 & -0.0975 & 0.3590 \end{pmatrix},$$

which is not a Z-matrix.

We denote by $U^{(\alpha)}$ the Hadamard transformation of U under $f(x) = x^\alpha$. In particular, $U^{(2)} = U \odot U$. It was conjectured by Neumann in [16] that $U^{(2)}$ is an inverse M-matrix if U is so. This was solved by Chen in the beautiful article [4] for any positive integer power of U. Our next result is a generalization of Chen's result. His proof depends on the following interesting result: U is an inverse M-matrix if and only if its adjoint is a Z-matrix, and each proper principal submatrix is an inverse M-matrix. Our technique is entirely different and is based strongly on the idea of an equilibrium potential.

This result has the following probabilistic interpretation. If U is the potential of a transient continuous time Markov process, then $U^{(\alpha)}$ is also the potential of a transient continuous time Markov process. In Theorem 2.3 we show that the same is true for a potential of a Markov chain. An interesting open question is what is the relation between the Markov chain associated with U and that associated with $U^{(\alpha)}$.

THEOREM 2.2. *Assume $U \in \mathcal{M}^{-1}$ and $\alpha \geq 1$. Then $U^{(\alpha)} \in \mathcal{M}^{-1}$. If $U^{-1} \in \mathcal{P}$, then $(U^{(\alpha)})^{-1} \in \mathcal{P}$. If $U \in bi\mathcal{P}$, then $U^{(\alpha)} \in bi\mathcal{P}$.*

THEOREM 2.3. *Assume that $U^{-1} = \mathbb{I} - P$, where P is a sub-Markov kernel, that is, $P \geq 0$, $P\mathbf{1} \leq \mathbf{1}$. Then for all $\alpha \geq 1$ there is a sub-Markov kernel $Q(\alpha)$ such that $(U^{(\alpha)})^{-1} = \mathbb{I} - Q(\alpha)$. Moreover, if $P'\mathbf{1} \leq \mathbf{1}$, then $Q(\alpha)'\mathbf{1} \leq \mathbf{1}$.*

The next result establishes that the class of GUM is the largest class of potentials stable under increasing Hadamard functions.

THEOREM 2.4. *Let U be a nonnegative matrix such that $\tau(f(U)) = \infty$ for all increasing nonnegative functions f. Then, U must be a GUM.*

Example 2.2. Given $a, b, c, d \in \mathbb{R}_+$, consider the nonsingular matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}.$$

For all increasing nonnegative functions f and all $t > 0$, $(\mathbb{I} + tf(U))^{-1}$ is an M-matrix, while U is not a GUM. Moreover, U is not a permutation of an increasing CBF. This shows that the last theorem does not hold if, in the definition of τ, we replace the class $bi\mathcal{P}$ by the class $\mathcal{M}^{-1}$.

THEOREM 2.5. *Let $U \in bi\mathcal{P}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing convex function. $f(U)$ is in $bi\mathcal{P}$ if and only if $f(U)$ belongs to the class $\mathcal{T}$.*

THEOREM 2.6. *If U is a nonnegative increasing CBF matrix, then U is in the class $\mathcal{T}$.*

As a corollary of the two previous theorems we obtain the following important result.

THEOREM 2.7. *Assume that $U \in bi\mathcal{P}$ is an increasing CBF matrix and that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative strictly increasing convex function. Then $f(U) \in bi\mathcal{P}$.*

3. Proofs of Theorems 2.1, 2.2, 2.3, and 2.5. Let us start with a useful lemma.

LEMMA 3.1. *Assume $U \in \mathcal{M}^{-1}$. Then for all $t \geq 0$, $(\mathbb{I} + tU) \in \mathcal{M}^{-1}$. Moreover, if $U \in \mathcal{P}$, then $(\mathbb{I} + tU) \in \mathcal{P}$ and its right equilibrium potential is strictly positive. In particular if $U \in bi\mathcal{P}$, then so is $\mathbb{I} + tU$, and its equilibrium potentials are strictly positive. Similarly, let $0 \leq s < t$ and assume $\mathbb{I} + tU \in bi\mathcal{P}$; then $\mathbb{I} + sU \in bi\mathcal{P}$, and its equilibrium potentials are strictly positive.*

Proof. For some $K > 0$ large enough, $U^{-1} = K(\mathbb{I} - N)$, where $N \geq 0$ (and $N\mathbf{1} \leq \mathbf{1}$ for the row diagonally dominant case). In what follows we can assume that $K = 1$. (It is enough to consider the matrix KU instead of U.)

From the equality $(\mathbb{I} - N)(\mathbb{I} + N + N^2 + \cdots + N^p) = \mathbb{I} - N^{p+1}$ we get that

$$\mathbb{I} + N + N^2 + \cdots + N^p = U(\mathbb{I} - N^{p+1}) \leq U.$$

We deduce that the series $\sum_{l=1}^{\infty} N^l$ is convergent and its limit is U.

Consider now the matrix

$$N_t = t\left(\left(\mathbb{I} - \frac{1}{1+t}N\right)^{-1} - \mathbb{I}\right) = t\sum_{l=1}^{\infty}\left(\frac{1}{1+t}\right)^l N^l.$$

We have $N_t \geq 0$. In the case $N\mathbf{1} \leq \mathbf{1}$, since N is a nonnegative matrix we deduce that $N^l\mathbf{1} \leq \mathbf{1}$. This allows us to prove

$$N_t\mathbf{1} = t\sum_{l=1}^{\infty}\left(\frac{1}{1+t}\right)^l N^l\mathbf{1} \leq t\sum_{l=1}^{\infty}\left(\frac{1}{1+t}\right)^l \mathbf{1} = \mathbf{1}.$$

Therefore the matrix $\mathbb{I} - N_t$ is a Z-matrix (which is row diagonally dominant when U^{-1} is). On the other hand, we have

$$\mathbb{I} + tU = \mathbb{I} + t(\mathbb{I} - N)^{-1} = (t\mathbb{I} + \mathbb{I} - N)(\mathbb{I} - N)^{-1} = (1+t)\left(\mathbb{I} - \frac{1}{1+t}N\right)(\mathbb{I} - N)^{-1},$$

and we deduce that $\mathbb{I} + tU$ is nonsingular and its inverse is

$$\begin{aligned}
(\mathbb{I} + tU)^{-1} &= \frac{1}{1+t}(\mathbb{I} - N)\left(\mathbb{I} - \frac{1}{1+t}N\right)^{-1} \\
&= \frac{1}{1+t}\left(\left(\mathbb{I} - \frac{1}{1+t}N\right)^{-1} - N\left(\mathbb{I} - \frac{1}{1+t}N\right)^{-1}\right) \\
&= \frac{1}{1+t}\left(\sum_{l=0}^{\infty}(1+t)^{-l}N^l - \sum_{l=0}^{\infty}(1+t)^{-l}N^{l+1}\right) \\
&= \frac{1}{1+t}(\mathbb{I} - N_t).
\end{aligned}$$

This shows that the inverse of $\mathbb{I} - N_t$ is nonnegative, and therefore this matrix is an M-matrix. We conclude $\mathbb{I} + tU \in \mathcal{M}^{-1}$.

The only thing left to prove is that $N_t\mathbf{1} < \mathbf{1}$ in the row diagonally dominant case, that is, when $N\mathbf{1} \leq \mathbf{1}$. Notice that from the convergence of the series $\sum_{l\geq 0} N^l$ we deduce that $N^l \to 0$ as $l \to \infty$. Then for large l, say $l > l_0$, we have $N^l\mathbf{1} \leq \frac{1}{2}\mathbf{1}$. Thus

$$N_t\mathbf{1} = t\sum_{l=1}^{\infty}\left(\frac{1}{1+t}\right)^l N^l\mathbf{1} \leq t\left(\sum_{l=1}^{l_0}\left(\frac{1}{1+t}\right)^l + \frac{1}{2}\sum_{l=l_0+1}^{\infty}\left(\frac{1}{1+t}\right)^l\right)\mathbf{1} < \mathbf{1}.$$

For a general $K > 0$ we have the equality $(\mathbb{I}+tU)^{-1} = \frac{K}{t+K}(\mathbb{I}-\frac{t}{K}\sum_{l=1}^{\infty}(\frac{K}{t+K})^l N^l)$, where $N = \mathbb{I} - \frac{1}{K}U^{-1}$.

Finally, assume that $\mathbb{I} + tU \in bi\mathcal{P}$. Hence $\mathbb{I} + \beta(\mathbb{I} + tU) \in bi\mathcal{P}$ for all $\beta \geq 0$. This implies that

$$\mathbb{I} + \frac{\beta}{1+\beta} t U \in bi\mathcal{P}.$$

Now it is enough to take $\beta \geq 0$ such that $s = \frac{\beta}{1+\beta}t$. ∎

This lemma has two immediate consequences.

COROLLARY 3.2. *If $U \in bi\mathcal{P}$, then $\tau(U) = \infty$.*

COROLLARY 3.3. *Let U be a nonnegative matrix; then*

$$\tau(U) = \sup\{t \geq 0 : \ \mathbb{I} + tU \in bi\mathcal{P}\}.$$

Proof. It is clear that $\tau(U) \leq \sup\{t \geq 0 : \ \mathbb{I} + tU \in bi\mathcal{P}\}$. On the other hand, if $\mathbb{I} + tU \in bi\mathcal{P}$, we get $\mathbb{I} + sU \in bi\mathcal{P}$ for all $0 \leq s \leq t$. This fact and the definition of $\tau(U)$ imply the result. ∎

Proof of Theorem 2.1. We first assume that $f(0) = 0$. We have that $U^{-1} = K(\mathbb{I} - P)$ for some $K > 0$ and P a substochastic matrix. Without loss of generality we can assume $K = 1$, because it is enough to consider KU instead of U and $\tilde{f}(x) = f(x/K)$ instead of f.

Consider $M = (U^{-1}f(U))$. For $i \neq j$ let us compute

$$M_{ij} = (U^{-1}f(U))_{ij} = (1 - p_{ii})f(U_{ij}) - \sum_{k \neq i} p_{ik} f(U_{kj}).$$

Since $1 - p_{ii} - \sum_{k \neq i} p_{ik} \geq 0$ (which is equivalent to $\sum_k p_{ik} \leq 1$) and f is convex, we obtain

$$\left(1 - \sum_k p_{ik}\right) f(0) + \sum_k p_{ik} f(U_{kj}) \geq f\left(\sum_k p_{ik} U_{kj}\right) = f(U_{ij}).$$

The last equality follows from the fact that $U^{-1} = \mathbb{I} - P$. This shows that $M_{ij} \leq 0$. Consider now a positive vector x such that $y' = x'U^{-1} > 0$ (for its existence, see [10, Theorem 2.5.3]). Then

$$x'M = x'U^{-1}f(U) = y'f(U) > 0,$$

which implies, by the same cited theorem in [10], that M is an M-matrix. In particular, M is nonsingular and $\det(M) > 0$. So $f(U)$ is nonsingular and $\det(f(U)) > 0$. Now consider ρ the right equilibrium potential of $f(U)$. We have

$$M\rho = U^{-1}f(U)\rho = U^{-1}\mathbf{1} = \mu_U \geq 0,$$

then $\rho = M^{-1}\mu_U \geq 0$, because M^{-1} is a nonnegative matrix. This means that $f(U)$ possesses a nonnegative right equilibrium potential. Since $f(U)$ is nonsingular, we also have a left equilibrium potential, but we do not know whether it is nonnegative. Then the first part is proven under the extra hypothesis that $f(0) = 0$.

Assume now $a = f(0) > 0$, and consider $g(x) = f(x) - a$, which is a strictly increasing convex function. Obviously $f(U) = g(U) + a\mathbf{1}\mathbf{1}'$, so

$$\mu_{f(U)} = \frac{1}{1 + a\bar{\mu}_{g(U)}}\mu_{g(U)} \geq 0, \qquad \nu_{f(U)} = \frac{1}{1 + a\bar{\mu}_{g(U)}}\nu_{g(U)},$$

where $\bar{\mu}_{g(U))} = \mathbf{1}'\mu_{g(U))} > 0$. We have used the fact that $\bar{\mu}_{g(U))} = \bar{\nu}_{g(U))}$. Thus $f(U)$ has a nonnegative right equilibrium potential and a left equilibrium potential. We need to prove that $f(U)$ is nonsingular and $\det(f(U)) > 0$. This follows immediately from the equality

$$f(U) = g(U)(\mathbb{I} + a\mu_{g(U)}\mathbf{1}'),$$

which implies

$$f(U)^{-1} = g(U)^{-1} - \frac{a}{1 + a\bar{\mu}_{g(U)}}\mu_{g(U)}(\nu_{g(U)});$$
$$\det(f(U)) = \det(g(U))(1 + a\bar{\mu}_{g(U)}).$$

Then the first part of the result is proven.

In the bipotential case use U' instead of U to obtain the existence of a nonnegative left equilibrium potential for $f(U)$. ∎

Proof of Theorem 2.5. Using the same ideas as above, we can assume that $f(0) = 0$. Also we have that $U^{-1}(\mathbb{I} + tf(U)) = M_t$ is an M-matrix for all $t \geq 0$. Therefore $\mathbb{I} + tf(U)$ is nonsingular for all t, and we denote by μ_t and ν_t the equilibrium potentials for $\mathbb{I} + tf(U)$.

Assume first that $f(U)$ is in class $\mathcal{T}$ (see Definition 1.10), which means that

$$\tau(f(U)) = \min\{t > 0 : \ \mu_t \ngeqq 0 \text{ or } \nu_t \ngeqq 0\}.$$

We prove that for all $t \geq 0$, μ_t, ν_t are nonnegative. Since

$$M_t\mu_t = U^{-1}\mathbf{1} = \mu_U,$$

we obtain that $\mu_t = M_t^{-1}\mu_U \geq 0$, because M_t^{-1} is a nonnegative matrix. Thus, $\tau(f(U)) = \infty$, and since $f(U)$ is nonsingular we get from Proposition 1.1 that $f(U) \in bi\mathcal{P}$. Conversely if $f(U) \in bi\mathcal{P}$, then $\tau(f(U)) = \infty$, and the result follows. ∎

For the rest of the section n denotes the size of U.

LEMMA 3.4. *Assume that $U \in \mathcal{P}$. Then any principal square submatrix A of U is also in class $\mathcal{P}$. The same is true if we replace $\mathcal{P}$ by $bi\mathcal{P}$.*

Proof. By induction and a suitable permutation the restriction of U to $\{1, \dots, n-1\} \times \{1, \dots, n-1\}$ is enough to prove the result for A. Assume that

$$U = \begin{pmatrix} A & b \\ c' & d \end{pmatrix} \quad \text{and} \quad U^{-1} = \begin{pmatrix} \Lambda & -\zeta \\ -\varrho' & \theta \end{pmatrix}.$$

Since $A^{-1} = \Lambda - \frac{1}{\theta}\zeta\varrho'$ we get that the off-diagonal elements of A^{-1} are nonpositive. It is quite easy to see that the result will follow as soon as $A^{-1}\mathbf{1} \geq 0$.

Since $U \in \mathcal{P}$ we have that $\Lambda\mathbf{1} - \zeta \geq 0$ and $\theta \geq \varrho'\mathbf{1}$. Therefore,

$$A^{-1}\mathbf{1} = \Lambda\mathbf{1} - \frac{1}{\theta}\zeta\varrho'\mathbf{1} = \Lambda\mathbf{1} - \frac{\varrho'\mathbf{1}}{\theta}\,\zeta \geq \Lambda\mathbf{1} - \zeta \geq 0. \qquad ∎$$

Recall that for a vector a, D_a is the associated diagonal matrix.

LEMMA 3.5. *Assume $U \in bi\mathcal{P}$ and $\alpha \geq 1$. If*

$$U = \begin{pmatrix} A & b \\ c' & d \end{pmatrix},$$

then there exists a nonnegative vector η *such that*

$$A^{(\alpha)}\eta = b^{(\alpha)}.$$

Proof. We first perturb the matrix U to have a positive matrix. Consider $\epsilon > 0$ and the positive matrix $U_\epsilon = U + \epsilon \mathbf{1}\mathbf{1}'$. It is direct to prove that

$$U_\epsilon^{-1} = U^{-1} - \frac{\epsilon}{1+\epsilon\bar{\mu}_U}\mu_U(\nu_U)',$$

where $\bar{\mu}_U = \mathbf{1}'\mu_U$ is the total mass of μ_U. Then $U_\epsilon \in bi\mathcal{P}$, and its equilibrium potentials are given by

$$\mu_{U_\epsilon} = \frac{1}{1+\epsilon\bar{\mu}_U}\mu_U, \qquad \nu_{U_\epsilon} = \frac{1}{1+\epsilon\bar{\nu}_U}\nu_U.$$

We decompose the inverse of U_ϵ as

$$U_\epsilon^{-1} = \begin{pmatrix} \Lambda_\epsilon & \zeta_\epsilon \\ \varrho_\epsilon' & \theta_\epsilon \end{pmatrix},$$

and we notice that $A_\epsilon\zeta_\epsilon + \theta_\epsilon b_\epsilon = 0$, which implies that

$$b_\epsilon = A_\epsilon\lambda_\epsilon,$$

with $\lambda_\epsilon = -\frac{1}{\theta_\epsilon}\zeta_\epsilon \geq 0$. Also we mention here that λ_ϵ is a subprobability vector, that is, $\mathbf{1}'\lambda_\epsilon \leq 1$. This follows from the fact that U_ϵ^{-1} is column diagonally dominant.

Take now the matrix $V_\epsilon = D_{b_\epsilon}^{-1}A_\epsilon$. It is direct to check that $V_\epsilon \in \mathcal{M}^{-1}$ and that its equilibrium potentials are

$$\mu_{V_\epsilon} = \lambda_\epsilon, \qquad \nu_{V_\epsilon} = D_{b_\epsilon}\nu_{A_\epsilon}.$$

Thus $V_\epsilon \in bi\mathcal{P}$, and we can apply Theorem 2.1 to get that the matrix $V_\epsilon^{(\alpha)}$ possesses a right equilibrium potential $\eta_\epsilon \geq 0$; that is, for all i,

$$\sum_j (V_\epsilon^{(\alpha)})_{ij}(\eta_\epsilon)_j = 1,$$

which is equivalent to

$$\sum_j \frac{(A_\epsilon)_{ij}^\alpha}{(b_\epsilon)_i^\alpha}(\eta_\epsilon)_j = 1.$$

Hence

$$A_\epsilon^{(\alpha)}\eta_\epsilon = b_\epsilon^{(\alpha)}.$$

Recall that the matrix $A^{(\alpha)}$ is nonsingular. Since obviously $A_\epsilon^{(\alpha)} \to A^{(\alpha)}$ as $\epsilon \to 0$, we get

$$\eta_\epsilon \to \eta = (A^{(\alpha)})^{-1}b^{(\alpha)},$$

and the result follows. ∎

Proof of Theorem 2.2. Consider first the case where $U \in bi\mathcal{P}$. We already know that $U^{(\alpha)}$ is nonsingular and that it has left and right nonnegative equilibrium potentials. Therefore, in order to prove $U^{(\alpha)} \in bi\mathcal{P}$, it is enough to prove that $(U^{(\alpha)})^{-1}$ is a Z-matrix; that is, we need to show $((U^{(\alpha)})^{-1})_{ij} \le 0$ for $i \ne j$. An argument based on permutations shows that it is enough to prove the claim for $i = 1, j = n$.

Decompose $U^{(\alpha)}$ and its inverse as follows:

$$U^{(\alpha)} = \begin{pmatrix} A^{(\alpha)} & b^{(\alpha)} \\ (c^{(\alpha)})' & d^{\alpha} \end{pmatrix} \quad \text{and} \quad (U^{(\alpha)})^{-1} = \begin{pmatrix} \Omega & -\beta \\ -\alpha' & \delta \end{pmatrix}.$$

We will show $\beta \ge 0$. We notice that $\delta = \frac{\det(A^{(\alpha)})}{\det(U^{(\alpha)})} > 0$ and $-A^{(\alpha)}\beta + \delta b^{(\alpha)} = 0$, and we deduce

$$b^{(\alpha)} = A^{(\alpha)} \left(\frac{\beta}{\delta}\right).$$

Therefore, $\frac{\beta}{\delta} = \eta \ge 0$, where η is the vector given in Lemma 3.5. Thus $\beta \ge 0$, and the result is proven for the case $U \in bi\mathcal{P}$.

Now, consider $U = M^{-1}$ the inverse of the M-matrix M. Using Theorem 2.5.3 in [10], we get the existence of two positive diagonal matrices D, E such that DME is a strictly row and column diagonally dominant M-matrix. Thus $V = E^{-1}UD^{-1}$ is in $bi\mathcal{P}$, from which it follows that $V^{(\alpha)} \in bi\mathcal{P}$. Hence, $U^{(\alpha)} = E^{(\alpha)}V^{(\alpha)}D^{(\alpha)}$ is the inverse of an M-matrix. The rest of the result is proven in a similar way. ∎

Proof of Theorem 2.3. By hypothesis we have $U = \mathbb{I} - P$, where $P \ge 0$ and $P\mathbf{1} \le \mathbf{1}$. We notice that U is diagonally dominant on each column, which means that for all i, j

$$U_{ii} \ge U_{ji}.$$

Also we notice that $U = \mathbb{I} + PU$ and therefore $U_{ii} \ge 1$.

According to Theorem 2.2 we know that $H = (U^{(\alpha)})^{-1}$ is a row diagonally dominant M-matrix. The only thing left to prove is that the diagonal elements of H are dominated by one: $H_{ii} \le 1$ for all i. We will prove it for $i = n$.

Consider the following decompositions:

$$U = \begin{pmatrix} A & b \\ c' & d \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \Lambda & -\omega \\ -\eta' & \gamma \end{pmatrix}, \quad (U^{(\alpha)})^{-1} = \begin{pmatrix} \Omega & -\beta \\ -\alpha' & \delta \end{pmatrix},$$

$$U^{-1}U^{(\alpha)} = \begin{pmatrix} \Xi & -\zeta \\ -\chi' & \rho \end{pmatrix}.$$

A direct computation gives that

$$\gamma = \rho\delta + \chi'\beta \ge \rho\delta.$$

We need to show that $\delta \le 1$. By hypothesis, $\gamma \le 1$; then it is enough to prove that $\rho \ge 1$. On the one hand, we have

$$\rho = (1 - p_{nn})U^{\alpha}_{nn} - \sum_{j \ne n} p_{nj} U^{\alpha}_{jn} = U^{\alpha}_{nn} - \sum_{j} p_{nj} U^{\alpha}_{jn} = U^{\alpha}_{nn} - \sum_{j} p_{nj} U_{jn} U^{\alpha-1}_{jn}.$$

On the other hand, we also have $U_{jn}^{\alpha-1} \leq U_{nn}^{\alpha-1}$ and $\sum_j p_{nj} U_{jn} = U_{nn} - 1$. Hence we deduce

$$\rho \geq U_{nn}^{\alpha-1} \geq 1.$$

This finishes the first part of the theorem . The rest of the result is proven by using U' instead of U. ∎

4. Proof of Theorem 2.4. Notice that U is a GUM if and only if $n \leq 2$ or every principal submatrix of size 3 is a GUM.

Since by hypothesis the matrix $\mathbb{I} + tU$ is a bipotential, it is diagonally dominant,

$$1 + tU_{ii} \geq tU_{ij},$$

and by taking $t \to \infty$, we find $U_{ii} \geq U_{ij}$. This proves the result when $n \leq 2$. So, in what follows we assume $n \geq 3$.

Consider A any principal submatrix of U, of size 3×3. Since $\mathbb{I}+tf(A)$ is a principal submatrix of $\mathbb{I} + tf(U)$, we deduce that $\mathbb{I} + tf(A) \in bi\mathcal{P}$ (as long as $\mathbb{I} + tf(U) \in bi\mathcal{P}$). If the result holds for the 3×3 matrices, we deduce that A is a GUM, implying that U is also a GUM.

Thus, in the rest of the proof we can assume that U is a 3×3 matrix that verifies the hypothesis of the theorem. After a suitable permutation we can further assume that

$$U = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d & \alpha \\ c_2 & \beta & e \end{pmatrix},$$

where $\alpha = \min\{U_{ij} : \ i \neq j\} = \min\{U\}$ and $\beta = \min\{U_{ji} : \ U_{ij} = \alpha, \ i \neq j\}$.

Since U is diagonally dominant we have $\min\{a, d, e\} \geq \alpha$. Take f increasing such that $f(\alpha) = 0$ and $f(x) > 0$ for $x > \alpha$. Then,

$$\mathbb{I} + f(U) = \begin{pmatrix} 1 + f(a) & f(b_1) & f(b_2) \\ f(c_1) & 1 + f(d) & 0 \\ f(c_2) & f(\beta) & 1 + f(e) \end{pmatrix}$$

is a $bi\mathcal{P}$-matrix whose inverse we denote by

$$\begin{pmatrix} \delta & -\rho_1 & -\rho_2 \\ -\theta_1 & \gamma_1 & -\gamma_2 \\ -\theta_2 & -\gamma_3 & \gamma_4 \end{pmatrix}.$$

In particular we obtain

$$\begin{pmatrix} 1 + f(d) & 0 \\ f(\beta) & 1 + f(e) \end{pmatrix}^{-1} = \begin{pmatrix} \gamma_1 & -\gamma_2 \\ -\gamma_3 & \gamma_4 \end{pmatrix} - \frac{1}{\delta} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}',$$

and we deduce that

$$0 = \gamma_2 = \theta_1 \rho_2. \tag{4.1}$$

- *Case* $\rho_2 = 0$. We get $f(b_2) = 0$, which implies further

$$b_2 = \alpha \quad \text{and} \quad c_2 \geq \beta. \tag{4.2}$$

The last conclusion follows from the definition of β. Therefore,

$$U = \begin{pmatrix} a & b_1 & \alpha \\ c_1 & d & \alpha \\ c_2 & \beta & e \end{pmatrix}. \tag{4.3}$$

We must prove that U is GUM.

Consider another increasing function g such that $g(\beta) = 0$ and $g(x) > 0$ for $x > \beta$. Then,

$$\mathbb{I} + g(U) = \begin{pmatrix} 1 + g(a) & g(b_1) & 0 \\ g(c_1) & 1 + g(d) & 0 \\ g(c_2) & 0 & 1 + g(e) \end{pmatrix}.$$

Its inverse is of the form

$$\begin{pmatrix} \tilde{\delta} & -\tilde{\rho}_1 & 0 \\ -\tilde{\theta}_1 & \tilde{\gamma}_1 & 0 \\ -\tilde{\theta}_2 & -\tilde{\gamma}_3 & \tilde{\gamma}_4 \end{pmatrix}.$$

As before, we deduce that $0 = \tilde{\gamma}_3 = \tilde{\theta}_2 \tilde{\rho}_1$.

  - *Subcase* $\tilde{\theta}_2 = 0$. We have $g(c_2) = 0$, which implies $c_2 = \beta$. In this situation we have

$$U = \begin{pmatrix} a & b_1 & \alpha \\ c_1 & d & \alpha \\ \beta & \beta & e \end{pmatrix}.$$

By permuting rows and columns $1, 2$, if necessary, we can assume that $b_1 \leq c_1$. Consider the situation where $c_1 < \beta$; of course, implicitly we should have $\alpha < \beta$. Under a suitable increasing transformation h, we get

$$\mathbb{I} + h(U) = \begin{pmatrix} 1 + h(a) & 0 & 0 \\ 0 & 1 + h(d) & 0 \\ h(\beta) & h(\beta) & 1 + h(e) \end{pmatrix}$$

and its inverse

$$\begin{pmatrix} \frac{1}{1+h(a)} & 0 & 0 \\ 0 & \frac{1}{1+h(d)} & 0 \\ -\frac{h(\beta)}{(1+h(a))(1+h(e))} & -\frac{h(\beta)}{(1+h(d))(1+h(e))} & \frac{1}{1+h(e)} \end{pmatrix}.$$

The sum of the third row is then

$$\frac{1}{1 + h(e)} \left(1 - h(\beta) \left(\frac{1}{1 + h(a)} + \frac{1}{1 + h(d)} \right) \right),$$

and this quantity can be made negative by choosing an appropriate function h. The idea is to make $h(\beta) \to \infty$ and

$$\frac{h(\beta)}{\max\{h(a), h(d)\}} \to 1.$$

Therefore, $c_1 \geq \beta$ and U is a GUM.

– *Subcase* $\tilde{\rho}_1 = 0$. We have $g(b_1) = 0$ and then $b_1 \le \beta$. Take again an increasing function ℓ such that

$$\mathbb{I} + \ell(U) = \begin{pmatrix} 1+\ell(a) & 0 & 0 \\ \ell(c_1) & 1+\ell(d) & 0 \\ \ell(c_2) & 0 & 1+\ell(e) \end{pmatrix}$$

and its inverse

$$\begin{pmatrix} \frac{1}{1+\ell(a)} & 0 & 0 \\ -\frac{\ell(c_1)}{(1+\ell(a))(1+\ell(d))} & \frac{1}{1+\ell(d)} & 0 \\ -\frac{\ell(c_2)}{(1+\ell(a))(1+\ell(e))} & 0 & \frac{1}{1+\ell(e)} \end{pmatrix}.$$

The sum of the first column is

$$\frac{1}{1+\ell(a)}\left(1 - \frac{\ell(c_1)}{(1+\ell(d))} - \frac{\ell(c_2)}{(1+\ell(e))}\right),$$

which can be made negative by repeating a similar argument as before if both $c_1 > \beta$ and $c_2 > \beta$.

Therefore if we assume that $c_1 > \beta$, we necessarily have $c_2 \le \beta$. On the other hand, from (4.2) we know $c_2 \ge \beta$, proving that $c_2 = \beta$. The conclusion is $\alpha \le b_1 \le \beta < c_1$ and

$$U = \begin{pmatrix} a & b_1 & \alpha \\ c_1 & d & \alpha \\ \beta & \beta & e \end{pmatrix},$$

which is a GUM.

Therefore we can continue under the hypothesis $c_1 \le \beta \le c_2$.

∗ *Subsubcase* $b_1 < \beta$. Again we must have $\alpha < \beta$. Under this condition we have that $c_2 > \alpha$. Using an increasing function ω, we get

$$\mathbb{I} + \omega(U) = \begin{pmatrix} 1+\omega(a) & 0 & 0 \\ \omega(c_1) & 1+\omega(d) & 0 \\ \omega(c_2) & \omega(\beta) & 1+\omega(e) \end{pmatrix},$$

and its inverse is

$$\begin{pmatrix} \frac{1}{1+\omega(a)} & 0 & 0 \\ -\frac{\omega(c_1)}{(1+\omega(a))(1+\omega(d))} & \frac{1}{1+\omega(d)} & 0 \\ -\frac{\omega(c_2)(1+\omega(d))-\omega(\beta)\omega(c_1)}{(1+\omega(a))(1+\omega(d))(1+\omega(e))} & -\frac{\omega(\beta)}{(1+\omega(d))(1+\omega(e))} & \frac{1}{1+\omega(e)} \end{pmatrix}.$$

The sum of the third row is

$$\frac{1}{(1+\omega(e))}\left(1 - \frac{\omega(c_2)}{1+\omega(a)} + \frac{\omega(\beta)\omega(c_1)}{(1+\omega(a))(1+\omega(d))} - \frac{\omega(\beta)}{1+\omega(d)}\right). \tag{4.4}$$

If $c_1 < \beta$, we can assume $\omega(c_1) = 0$. With this choice we can make the sum in (4.4) negative by a suitable selection of ω as we did before. Thus we must have $c_1 = \beta$, in which case the sum under study is proportional to

$$1 - \frac{\omega(c_2)}{1+\omega(a)} + \frac{\omega(\beta)^2}{(1+\omega(a))(1+\omega(d))} - \frac{\omega(\beta)}{1+\omega(d)}. \tag{4.5}$$

If $c_2 = \beta$, then

$$U = \begin{pmatrix} a & b_1 & \alpha \\ \beta & d & \alpha \\ \beta & \beta & e \end{pmatrix}$$

is a GUM. So, we must analyze the case where $c_2 > \beta$ in (4.5). We will arrive at a contradiction by taking an asymptotic as before. Consider a fixed number $\lambda \in (0,1)$. Choose a family of functions $(\omega_r)_{r\in\mathbb{N}}$ such that, as $r \to \infty$,

$$\omega_r(\beta) \to \infty, \quad \frac{\omega_r(\beta)}{\omega_r(c_2)} \to \lambda, \quad \frac{\omega_r(c_2)}{\omega_r(a)} \to 1, \quad \frac{\omega_r(d)}{\omega_r(a)} \to \phi,$$

where $\phi = 1$ if $d > \beta$, and $\phi = \lambda$ if $d = \beta$. The asymptotic of (4.5) is then

$$1 - 1 + \frac{\lambda^2}{\phi} - \frac{\lambda}{\phi}.$$

This quantity is strictly negative for the two possible values of ϕ, which is a contradiction, and therefore $c_2 = \beta$.

To finish with the Subcase $\tilde{\rho}_1 = 0$, which will in turn finish with Case $\rho_2 = 0$, we consider a further subcase.

∗ *Subsubcase* $b_1 = \beta$. We recall that we are under the restrictions $c_1 \le \beta \le c_2$ and

$$U = \begin{pmatrix} a & \beta & \alpha \\ c_1 & d & \alpha \\ c_2 & \beta & e \end{pmatrix}.$$

Notice that if $c_2 = \beta$, then U is GUM. So, we may assume in this part that $c_2 > \beta$. If $c_1 = \alpha$, we can permute 1 and 2 to get

$$\Pi U \Pi' = \begin{pmatrix} d & \alpha & \alpha \\ \beta & a & \alpha \\ \beta & c_2 & e \end{pmatrix},$$

which is also in NBF, and U is a GUM. Thus we can assume $c_1 > \alpha$, and again we have $\alpha < \beta$.

Take an increasing function m such that

$$\mathbb{I} + m(U) = \begin{pmatrix} 1+m(a) & m(\beta) & 0 \\ m(c_1) & 1+m(d) & 0 \\ m(c_2) & m(\beta) & 1+m(e) \end{pmatrix}.$$

We take the asymptotic under the following restrictions:

$$\frac{m(\beta)}{m(a)} \to \lambda \in (0,1), \ \frac{m(c_1)}{m(a)} \to \lambda, \ \frac{m(e)}{m(a)} \to 1, \ \frac{m(c_2)}{m(a)} \to 1, \ \frac{m(d)}{m(a)} \to \phi,$$

where $\phi = 1$ if $d > \beta$, and $\phi = \lambda$ if $d = \beta$. The limiting matrix for $\frac{1}{m(a)}(\mathbb{I} + m(U))$ is

$$V = \begin{pmatrix} 1 & \lambda & 0 \\ \lambda & \phi & 0 \\ 1 & \lambda & 1 \end{pmatrix},$$

whose determinant is $\Delta = \phi - \lambda^2 > 0$. Therefore it is nonsingular, and as the limit of matrices in $bi\mathcal{P}$, V itself must belong to $bi\mathcal{P}$. On the other hand, the inverse of V is given by

$$V^{-1} = \frac{1}{\Delta} \begin{pmatrix} \phi & -\lambda & 0 \\ -\lambda & 1 & 0 \\ -(\phi - \lambda^2) & 0 & \phi - \lambda^2 \end{pmatrix},$$

and the sum of the first column is

$$\frac{\lambda^2 - \lambda}{\Delta} < 0,$$

which is a contradiction.

This finishes with the subcase $\rho_2 = 0$, and we return to (4.1) to consider now the following case.

- *Case* $\theta_1 = 0$. Under this condition we get $c_1 = \alpha$ and

$$U = \begin{pmatrix} a & b_1 & b_2 \\ \alpha & d & \alpha \\ c_2 & \beta & e \end{pmatrix}.$$

Consider the transpose of U and permute on it 2 and 3, to obtain the matrix

$$\tilde{U} = \begin{pmatrix} a & c_2 & \alpha \\ b_2 & e & \alpha \\ b_1 & \beta & d \end{pmatrix},$$

where now $b_1 \geq \beta$. Clearly the matrix $\tilde{U}$ verifies the hypothesis of the theorem and has the shape of (4.3); that is, we are in the "case $\rho_2 = 0$," which, we already know, implies that $\tilde{U}$ is a GUM. Then U itself is a GUM, and the theorem is proven. ∎

5. Filtered matrices and sufficient conditions for classes $bi\mathcal{P}$ and $\mathcal{T}$. The class of filtered matrices, which turn out to be a generalization of GUM, gives a good framework to study a potential theory of matrices. They were introduced as operators in [8] to generalize the class of self-adjoint operators whose spectral decomposition is written in terms of conditional expectations (see, for instance, [3], [5], and [11]).

The basic tool to construct these matrices is partitions of $\mathcal{J}_n = \{1, \dots, n\}$. The components of a partition $\mathcal{R}$ are called atoms, and we denote by $\overset{\mathcal{R}}{\sim}$ the equivalence relation induced by $\mathcal{R}$. Then i, j are in the same atom of $\mathcal{R}$ if and only if $i \overset{\mathcal{R}}{\sim} j$.

A partition $\mathcal{R}$ is coarser than or equal to a partition $\mathcal{Q}$ if the atoms of $\mathcal{Q}$ are contained in the atoms of $\mathcal{R}$. This (partial) order relation is denoted by $\mathcal{R} \preccurlyeq \mathcal{Q}$. It is also said that $\mathcal{Q}$ is finer than $\mathcal{R}$. For example, in $\mathcal{J}_4$ we have $\mathcal{R} = \{\{1, 2\}, \{3, 4\}\} \preccurlyeq$

$\mathcal{Q} = \{\{1\}, \{2\}, \{3, 4\}\}$. The coarsest partition is the trivial one $\mathcal{N} = \{\mathcal{J}_n\}$, and the finest one is the discrete partition $\mathcal{F} = \{\{1\}, \{2\}, \dots, \{n\}\}$.

Definition 5.1. *A* filtration $\mathbb{F} = \{\mathcal{R}_0 \prec \mathcal{R}_1 \prec \cdots \prec \mathcal{R}_k\}$ *is a strictly increasing sequence of comparable partitions.* $\mathbb{F}$ *is said to be* dyadic *if each nontrivial atom of* $\mathcal{R}_s$ *is divided into two atoms of* $\mathcal{R}_{s+1}$.

A filtration in the wide sense is an increasing sequence of comparable partitions $\mathbb{G} = \{\mathcal{R}_0 \preccurlyeq \mathcal{R}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{R}_k\}$.

The difference between a filtration and a filtration in the wide sense is that in the latter case repetition of partitions is allowed.

Each partition $\mathcal{R}$ induces an incidence matrix $F =: F(\mathcal{R})$ given by

$$F_{ij} = \begin{cases} 1 & \text{if } i \overset{\mathcal{R}}{\sim} j, \\ 0 & \text{otherwise} . \end{cases}$$

A vector $v \in \mathbb{R}^n$ is said to be $\mathcal{R}$-measurable if v is constant on the atoms of $\mathcal{R}$, that is,

$$i \overset{\mathcal{R}}{\sim} j \Rightarrow v_i = v_j.$$

This can be expressed in terms of standard matrix operations as

$$F(\mathcal{R})v = D_{w_{\mathcal{R}}} v,$$

where $w_{\mathcal{R}} = F(\mathcal{R})\mathbf{1}$ is the vector constant on each atom, and this constant is the size of the respective atom (recall that D_z is the diagonal matrix associated with the vector z). The set of $\mathcal{R}$-measurable vectors is a linear subspace of $\mathbb{R}^n$. Notice that if the partition is $\mathcal{F}$, then the associated incidence matrix is the identity and the subspace of measurable vectors is just $\mathbb{R}^n$. On the other hand, if the partition is the trivial one $\mathcal{N}$, then the incidence matrix is $\mathbf{1}\mathbf{1}'$ and the measurable vectors in this case are the constant ones.

Definition 5.2. *A matrix U is said to be* filtered *if there exists a filtration in the wide sense* $\mathbb{G} = \{\mathcal{Q}_0 \preccurlyeq \mathcal{Q}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{Q}_l\}$, *vectors* $\mathfrak{a}_0, \dots, \mathfrak{a}_l$, $\mathfrak{b}_0, \dots, \mathfrak{b}_l$ *with the restriction that* $\mathfrak{a}_s, \mathfrak{b}_s$ *are* $\mathcal{Q}_{s+1}$*-measurable (we take* $\mathcal{Q}_{l+1} = \mathcal{F}$ *the discrete partition), and*

$$U = \sum_{s=0}^{\ell} D_{\mathfrak{a}_s} F(\mathcal{Q}_s) D_{\mathfrak{b}_s}. \tag{5.1}$$

There is no loss of generality if we assume that $\mathcal{Q}_0 = \mathcal{N}$ and $\mathcal{Q}_\ell = \mathcal{F}$, that is, $F(\mathcal{Q}_0) = \mathbf{1}\mathbf{1}'$ and $F(\mathcal{Q}_\ell) = \mathbb{I}$. Let us see that (5.1) can be simply written in terms of a filtration. Indeed, notice that if $\mathfrak{a}_s$ and $\mathfrak{b}_s$ are $\mathcal{Q}_s$-measurable, then

$$D_{\mathfrak{a}_s} F(\mathcal{Q}_s) D_{\mathfrak{b}_s} = D_{\mathfrak{a}_s} D_{\mathfrak{b}_s} F(\mathcal{Q}_s) = D_{\mathfrak{a}_s \odot \mathfrak{b}_s} F(\mathcal{Q}_s),$$

where the vector $\mathfrak{a}_s \odot \mathfrak{b}_s$ is the Hadamard product of $\mathfrak{a}_s$ and $\mathfrak{b}_s$, which is also $\mathcal{Q}_s$-measurable. Hence a sum of terms of the form

$$D_{\mathfrak{a}_s} F(\mathcal{Q}_s) D_{\mathfrak{b}_s} + D_{\mathfrak{a}_{s+1}} F(\mathcal{Q}_{s+1}) D_{\mathfrak{b}_{s+1}} + \cdots + D_{\mathfrak{a}_{s+r}} F(\mathcal{Q}_{s+r}) D_{\mathfrak{b}_{s+r}},$$

with $\mathcal{R} = \mathcal{Q}_s = \cdots = \mathcal{Q}_{s+r}$, can be reduced to the sum of two terms as

$$D_C F(\mathcal{R}) + D_{\mathfrak{a}_{s+r}} F(\mathcal{R}) D_{\mathfrak{b}_{s+r}},$$

where $C = \sum_{h=0}^{r-1} \mathfrak{a}_{s+h} \odot \mathfrak{b}_{s+h}$ is $\mathcal{R}$-measurable. In this way the representation (5.1) can be written as

$$U = \sum_{s=0}^{k} D_{C_s} F(\mathcal{R}_s) + D_{\mathfrak{m}_s} F(\mathcal{R}_s) D_{\mathfrak{n}_s}, \tag{5.2}$$

where $\mathbb{F} = \{\mathcal{R}_0 \prec \mathcal{R}_1 \prec \cdots \prec \mathcal{R}_k\}$ is a filtration, $\mathcal{N} = \mathcal{R}_0$, $\mathcal{F} = \mathcal{R}_k$, C_s is $\mathcal{R}_s$-measurable, $\mathfrak{m}_s, \mathfrak{n}_s$ are $\mathcal{R}_{s+1}$-measurable, and $\mathfrak{m}_k = 0$ (again we assume that $\mathcal{R}_{k+1} = \mathcal{F}$). We shall always consider this reduced representation of (5.1), and we shall say that U is *filtered* with respect to the filtration $\mathbb{F}$.

If all $\mathfrak{m}_s, \mathfrak{n}_s$ are $\mathcal{R}_s$-measurable, then (5.1) reduces to the form

$$U = \sum_{s=0}^{k} D_{C_s + \mathfrak{m}_s \odot \mathfrak{n}_s} F(\mathcal{R}_s), \tag{5.3}$$

and U is a symmetric matrix.

We are mainly interested in a decomposition like (5.2) with the vectors $\mathfrak{m}_s, \mathfrak{n}_s$ having the following special structure:

$$\mathfrak{m}_s = \Gamma_s \odot p_s, \qquad \mathfrak{n}_s = q_s, \tag{5.4}$$

where Γ_s is $\mathcal{R}_s$-measurable and $\{p_s, q_s\}$ is an $\mathcal{R}_{s+1}$-measurable partition; that is, $\{p_s, q_s\}$ are $\mathcal{R}_{s+1}$-measurable $\{0,1\}$-valued vectors with disjoint support $p_s \odot q_s = 0$ and $p_s + q_s = 1$. If this is the case, U is said to be a special filtered matrix (SFM),

$$U = \sum_{s=0}^{k} D_{C_s}\; F(\mathcal{R}_s) + D_{\Gamma_s} D_{p_s}\; F(\mathcal{R}_s)\; D_{q_s}. \tag{5.5}$$

Notice that $\Gamma_k = 0$.

It is not difficult to see that every CBF matrix is filtered. This is done by induction. Assume that

$$U = \begin{pmatrix} A & \alpha \mathbf{1}_p \mathbf{1}'_{n-p} \\ \beta \mathbf{1}_{n-p} \mathbf{1}'_p & B \end{pmatrix}.$$

Define $\mathcal{R}_0 = \mathcal{N}$ and $\mathcal{R}_1 = \{\{1, \ldots, p\}, \{p+1, \ldots, n\}\}$. Take

$$C_0 = \alpha \mathbf{1}_n, \quad \Gamma_0 = (\beta - \alpha)\mathbf{1}_n, \quad p_0 = (\mathbf{0}_p, \mathbf{1}_{n-p})', \quad q_0 = (\mathbf{1}_p, \mathbf{0}_{n-p})';$$

then we obtain

$$D_{C_0} F(\mathcal{R}_0) + D_{\Gamma_0} D_{p_0} F(\mathcal{R}_0) D_{q_0} = \begin{pmatrix} \alpha \mathbf{1}_p \mathbf{1}'_p & \alpha \mathbf{1}_p \mathbf{1}'_{n-p} \\ \beta \mathbf{1}_{n-p} \mathbf{1}'_p & \alpha \mathbf{1}_{n-p} \mathbf{1}'_{n-p} \end{pmatrix}.$$

The key step is that $A - \alpha, B - \alpha$ are also in CBF. We have that C_0, Γ_0 are $\mathcal{R}_0$-measurable and p_0, q_0 is an $\mathcal{R}_1$-measurable partition. We also notice that if $0 \le \alpha \le \beta$, then $C_0 \ge 0, \Gamma_0 \ge 0$.

The induction also shows that U can be decomposed as in (5.5), where $\mathbb{F} = \{\mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k\}$ is a dyadic filtration; C_s, Γ_s are $\mathcal{R}_s$-measurable; and $\{p_s, q_s\}$ is a $\mathcal{R}_{s+1}$-measurable partition.

We now summarize the representation form for the class of CBF, NBF, and GUM matrices.

PROPOSITION 5.3. *V is a permutation of a CBF matrix if and only if there exists a dyadic filtration $\mathbb{F} = \{\mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k\}$; a sequence of vectors $C_0, \ldots, C_k,\ \Gamma_0, \ldots, \Gamma_k$ verifying C_s, Γ_s are $\mathcal{R}_s$-measurable, and a sequence $\{p_s, q_s\}$ of $\mathcal{R}_{s+1}$-measurable partitions such that*

$$V = \sum_{s=0}^{k} D_{C_s} F(\mathcal{R}_s) + D_{\Gamma_s} D_{p_s} F(\mathcal{R}_s) D_{q_s}.$$

That is V is an SFM.

Also V is a permutation of an increasing CBF matrix if and only if there is a decomposition where $\Gamma_0, C_s, \Gamma_s,\ s = 1, \ldots, k$, are nonnegative. Furthermore, V is a nonnegative matrix if and only if C_0 is nonnegative.

Moreover, V is a GUM if and only if $C_s, \Gamma_s,\ s = 0, \ldots, k$, are nonnegative and for $s = 0, \ldots, k-1$ it holds that

$$\Gamma_s \le C_{s+1} + \Gamma_{s+1}. \tag{5.6}$$

Finally, V is an ultrametric matrix if and only if there is a decomposition with $\Gamma_s = 0$ for all s.

Remark 5.1. We can assume without loss of generality that each p_s, q_s is obtained as follows. The nontrivial atoms $\mathcal{A}_1, \ldots, \mathcal{A}_r$ of $\mathcal{R}_s$ are divided into the new atoms

$$\mathcal{A}_{1,1}, \mathcal{A}_{1,2}, \ldots, \mathcal{A}_{r,1}, \mathcal{A}_{r,2}$$

of $\mathcal{R}_{s+1}$. Consider $\mathcal{B}_1, \ldots, \mathcal{B}_r$ the set of trivial atoms in $\mathcal{R}_s$ (that is, the atoms which are singletons). Let q_s be the indicator of $\mathcal{A}_{1,1} \cup \cdots \cup \mathcal{A}_{r,1}$, p_s be the indicator of $\mathcal{A}_{1,2} \cup \cdots \cup \mathcal{A}_{r,2} \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$, and $\Gamma_s = 0$ on the $\mathcal{R}_s$-measurable set $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_r$. We point out that the partition $\mathcal{R}_{s+1}$ is obtained from $\mathcal{R}_s$ refined by p_s. The following consistency relation,

$$D_{p_s} F(\mathcal{R}_s) p_s = D_{p_s} F(\mathcal{R}_{s+1}) \mathbf{1}, \tag{5.7}$$

will be used further in order to give sufficient treatable conditions for an SFM to be a bipotential.

Example 5.1. Consider the CBF matrix

$$U = \begin{pmatrix} a & \alpha_2 & \alpha_1 & \alpha_1 \\ \beta_2 & b & \alpha_1 & \alpha_1 \\ \beta_1 & \beta_1 & c & \hat{\alpha}_2 \\ \beta_1 & \beta_1 & \hat{\beta}_2 & d \end{pmatrix}.$$

U is an NBF matrix if the constraints $\alpha_1 \le \beta_1,\ \alpha_1 \le \min\{\alpha_2, \hat{\alpha}_2\},\ \beta_1 \le \min\{\beta_2, \hat{\beta}_2\}$, $\alpha_2 \le \beta_2,\ \hat{\alpha}_2 \le \hat{\beta}_2$ are verified and finally the diagonal elements dominate on each row and column, that is, $\beta_2 \le \min\{a, b\},\ \hat{\beta}_2 \le \min\{c, d\}$.

U is filtered with respect to the dyadic filtration $\mathcal{R}_0 = \{1, 2, 3, 4\} \prec \mathcal{R}_1 = \{\{1, 2\}, \{3, 4\}\} \prec \mathcal{R}_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ and can be written as

$$U = D_{C_0} F(\mathcal{R}_0) + D_{\Gamma_0} D_{p_0} F(\mathcal{R}_0) D_{q_0} + D_{C_1} F(\mathcal{R}_1) + D_{\Gamma_1} D_{p_1} F(\mathcal{R}_1) D_{q_1} + D_{C_2} F(\mathcal{R}_2), \tag{5.8}$$

where

$$C_0 = \begin{pmatrix} \alpha_1 \\ \alpha_1 \\ \alpha_1 \\ \alpha_1 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} \beta_1 - \alpha_1 \\ \beta_1 - \alpha_1 \\ \beta_1 - \alpha_1 \\ \beta_1 - \alpha_1 \end{pmatrix}, \quad p_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad q_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \alpha_2 - \alpha_1 \\ \alpha_2 - \alpha_1 \\ \hat{\alpha}_2 - \alpha_1 \\ \hat{\alpha}_2 - \alpha_1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} \beta_2 - \alpha_2 \\ \beta_2 - \alpha_2 \\ \hat{\beta}_2 - \hat{\alpha}_2 \\ \hat{\beta}_2 - \hat{\alpha}_2 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and

$$C_2 = \begin{pmatrix} a - \alpha_2 \\ b - \alpha_2 \\ c - \hat{\alpha}_2 \\ d - \hat{\alpha}_2 \end{pmatrix}.$$

The decomposition in (5.8) is then

$$U = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \\ \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_1 - \alpha_1 & \beta_1 - \alpha_1 & 0 & 0 \\ \beta_1 - \alpha_1 & \beta_1 - \alpha_1 & 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} \alpha_2 - \alpha_1 & \alpha_2 - \alpha_1 & 0 & 0 \\ \alpha_2 - \alpha_1 & \alpha_2 - \alpha_1 & 0 & 0 \\ 0 & 0 & \hat{\alpha}_2 - \alpha_1 & \hat{\alpha}_2 - \alpha_1 \\ 0 & 0 & \hat{\alpha}_2 - \alpha_1 & \hat{\alpha}_2 - \alpha_1 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta_2 - \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{\beta}_2 - \hat{\alpha}_2 & 0 \end{pmatrix} + \begin{pmatrix} a - \alpha_2 & 0 & 0 & 0 \\ 0 & b - \alpha_2 & 0 & 0 \\ 0 & 0 & c - \hat{\alpha}_2 & 0 \\ 0 & 0 & d - \hat{\alpha}_2 & 0 \end{pmatrix}.$$

The constraints are translated into the positivity of the vectors C and Γ and the ones induced by (5.6). We point out that we can also choose, for example, $\Gamma_1 = (0, \beta_2 - \alpha_2, 0, \hat{\beta}_2 - \hat{\alpha}_2)'$, but in this case Γ_1 is not $\mathcal{R}_1$-measurable. As we will see in subsection (5.1), this measurability condition will play an important role.

Example 5.2. Consider the nonnegative CBF matrix

$$U = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

This matrix is an SFM and can be decomposed as in (5.5). Nevertheless, none of these decompositions can have all its terms nonnegative. In particular, no permutation of U is an increasing CBF matrix.

Remark 5.2. Notice that the class of CBF matrices is stable under Hadamard functions. Nevertheless there are examples of filtered matrices for which $f(U)$ is not filtered. Consider the matrix

$$U = D_\alpha F_1 + D_{\mathfrak{a}} F_1 D_{\mathfrak{b}} + D_\beta F_2,$$

where $F_1 = F(\mathcal{N}) = \mathbf{1}\mathbf{1}'$ and $F_2 = \mathbb{I}$. The vector α is constant, and we confound it with the constant $\alpha \in \mathbb{R}$. The vectors $\mathfrak{a}$, $\mathfrak{b}, \beta$ are all $\mathcal{F}$-measurable. Then U is filtered and, moreover,

$$U = \alpha + \mathfrak{a}\mathfrak{b}' + D_\beta. \tag{5.9}$$

Take $\alpha = \beta = 0$, $\mathfrak{a} = (2,3,5,7)'$, and $\mathfrak{b} = (11,13,17,19)'$. Then all the entries of U are different. As f runs over all possible functions, $f(U)$ runs over all 4×4 matrices. This implies that some of them can not be written as in (5.9), because in this representation we have at most 13 free variables. Still is possible that each $f(U)$ is decomposable as in (5.1), using maybe a different filtration. A more detailed analysis shows that this is not the case. For example, if we choose the filtration $\mathcal{N} \prec \{\{1,2\},\{3,4\}\} \prec \mathcal{F}$, then every matrix V filtered with respect to this filtration verifies that $V_{13} = V_{23} = V_{14} = V_{24}$.

Matrices of the type $F(\mathcal{R})$ are related to conditional expectations (in probability theory). Indeed, let $\mathcal{R} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_r\}$ and $n_\ell = \#(\mathcal{A}_\ell)$ be the size of each atom. It is direct that $w = w_{\mathcal{R}} = F(\mathcal{R})\mathbf{1}$ is an $\mathcal{R}$-measurable vector that verifies $w_i = n_\ell$ for $i \in \mathcal{A}_\ell$. Then

$$\mathbb{E}_{\mathcal{R}} = D_w^{-1}\, F(\mathcal{R}) = F(\mathcal{R})\, D_w^{-1}$$

is the matrix of conditional expectation with respect to the σ-algebra generated by $\mathcal{R}$. This matrix $\mathbb{E} = \mathbb{E}_{\mathcal{R}}$ satisfies

$$\begin{array}{l} \mathbb{E}\mathbb{E} = \mathbb{E},\ \mathbb{E}' = \mathbb{E},\ \mathbb{E}\mathbf{1} = \mathbf{1}; \\ \forall v,\ \mathbb{E}v \text{ is } \mathcal{R}\text{-measurable;} \\ \text{ if } v \text{ is } \mathcal{R}\text{-measurable, then } \mathbb{E}v = v. \end{array}$$

Therefore, $\mathbb{E}$ is the orthogonal projection over the subspace of all $\mathcal{R}$-measurable vectors. In the case of the trivial partition $\mathcal{N}$, one gets $\mathbb{E}_{\mathcal{N}} = \frac{1}{n}\mathbf{1}\mathbf{1}'$ as the mean operator.

Remark 5.3. The L^2 space associated with $\{1,\ldots,n\}$ endowed with the counting measure is identified with $\mathbb{R}^n$ with the standard Euclidean scalar product. In this way each vector of $\mathbb{R}^n$ can be seen as a function in L^2, and $\mathbb{E}$ is an orthogonal projection. The product $D_v\mathbb{E}$ (as matrices) is the product of the operators D_v and $\mathbb{E}$, where D_v is the multiplication by the function v. Notice that $\mathbb{E}D_v$ and $\mathbb{E}(v)$ are quite different. The former is an operator (a matrix), and the latter is a function (vector). They are related by $\mathbb{E}(v) = \mathbb{E}D_v(\mathbf{1})$, where $\mathbf{1}$ is the constant function.

Let $\mathcal{R},\ \mathcal{Q}$ be two partitions; then $\mathcal{R} \preccurlyeq \mathcal{Q}$ is equivalent to $\mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{Q}} = \mathbb{E}_{\mathcal{Q}}\mathbb{E}_{\mathcal{R}} = \mathbb{E}_{\mathcal{R}}$. This commutation relation can be written as a commutation relation for $F(\mathcal{R})$ and $F(\mathcal{Q})$. In fact,

$$\begin{aligned} F(\mathcal{R})F(\mathcal{Q}) &= \mathbb{E}_{\mathcal{R}} D_{w_{\mathcal{R}}} \mathbb{E}_{\mathcal{Q}} D_{w_{\mathcal{Q}}} = \mathbb{E}_{\mathcal{R}}\mathbb{E}_{\mathcal{Q}} D_{w_{\mathcal{R}}} D_{w_{\mathcal{Q}}} \\ &= \mathbb{E}_{\mathcal{R}} D_{w_{\mathcal{R}}} D_{w_{\mathcal{Q}}} = F(\mathcal{R}) D_{w_{\mathcal{Q}}}, \end{aligned}$$

$$F(\mathcal{Q})F(\mathcal{R}) = (F(\mathcal{R})F(\mathcal{Q}))' = D_{w_{\mathcal{Q}}} F(\mathcal{R}).$$

5.1. An algorithm for filtered matrices: Conditions to be in $bi\mathcal{P}$. In this section we introduce a backward algorithm that gives a sufficient condition for a filtered matrix to be in class $bi\mathcal{P}$. For that purpose assume that U has a representation as in (5.1):

$$U = \sum_{s=0}^{\ell} D_{\mathfrak{a}_s} F(\mathcal{Q}_s) D_{\mathfrak{b}_s},$$

where we assume further that $\mathfrak{a}_s, \mathfrak{b}_s$ are all nonnegative. In particular, U is a nonnegative matrix.

We introduce the conditional expectations $\mathbb{E}_s = \mathbb{E}_{\mathcal{Q}_s} = D^{-1}_{F(\mathcal{Q}_s)\mathbf{1}}\, F(\mathcal{Q}_s)$ and the normalized factors $a_s = \mathfrak{a}_s \odot F(\mathcal{Q}_s)\mathbf{1},\ b_s = \mathfrak{b}_s$. Then U can be written as

$$U = \sum_{s=0}^{\ell} D_{a_s}\mathbb{E}_s D_{b_s} = \sum_{s=0}^{\ell} a_s\mathbb{E}_s b_s, \tag{5.10}$$

where we have identified vectors (functions) and the operator of multiplication they induce. We shall use this notation throughout this section. Finally, we recall that $\mathbb{E}_\ell = \mathbb{I}$.

We can now use the algorithm developed in [8] to study the inverse of $\mathbb{I} + U$. In what follows, we take the convention $0 \cdot \infty = 0/0 = 0$. This algorithm is defined by the backward recursion starting with the values $\lambda_\ell = \mu_\ell = \kappa_\ell = 1,\ \sigma_\ell = (1 + a_\ell b_\ell)^{-1}$ and for $s = \ell - 1, \dots, 0$,

$$\begin{aligned} \lambda_s &= \lambda_{s+1}[1 - \sigma_{s+1}a_{s+1}\mathbb{E}_{s+1}(\kappa_{s+1}b_{s+1})],\\ \mu_s &= \mu_{s+1}[1 - \sigma_{s+1}b_{s+1}\mathbb{E}_{s+1}(\kappa_{s+1}a_{s+1})],\\ \kappa_s &= \mathbb{E}_{s+1}(\lambda_s) = \mathbb{E}_{s+1}(\mu_s),\\ \sigma_s &= (1 + \mathbb{E}_s(\kappa_s a_s b_s))^{-1}. \end{aligned} \tag{5.11}$$

We get the recursion

$$\kappa_{s-1} = \mathbb{E}_s(\kappa_s) - \frac{\mathbb{E}_s(\kappa_s a_s)\mathbb{E}_s(\kappa_s b_s)}{1 + \mathbb{E}_s(\kappa_s a_s b_s)}. \tag{5.12}$$

The algorithm continues until some λ or μ is negative; otherwise we arrive at $s = 0$. If this is the case, then $\mathbb{I}+U$ is nonsingular and its inverse is of the form $\mathbb{I}-N$, where

$$N = \sum_{s=0}^{\ell} \sigma_s\lambda_s a_s\mathbb{E}_s b_s\mu_s.$$

We also have that

$$\lambda_{-1} = (\mathbb{I} - N)\mathbf{1} \quad \text{and} \quad \mu_{-1} = (\mathbb{I} - N)'\mathbf{1},$$

where λ_{-1}, μ_{-1} are obtained from the first two formulae in (5.11) for $s = -1$. Therefore, if they are also nonnegative, the matrix $\mathbb{I} + U$ is a $bi\mathcal{P}$-matrix.

In this way we have that a sufficient condition for $\mathbb{I}+U$ to be a $bi\mathcal{P}$-matrix is that the algorithm works for $s = \ell, \dots, 0$ and that all the λ, μ are nonnegative, including λ_{-1}, μ_{-1}. In this situation we have that λ (and μ) is a decreasing nonnegative sequence of vectors. Sufficient treatable conditions on the coefficients of the expansion (5.10) involve the recurrence (5.12). Starting from $\kappa_\ell = 1$, we assume that this recurrence has a solution such that $\kappa_s \in [0, 1]$ for all $s = \ell, \dots, -1$. We shall study closely this recursion for the class of SFM, and we shall obtain sufficient conditions to have $\mathbb{I}+U$ in $bi\mathcal{P}$.

Before studying this problem, we further discuss the algorithm. We have the following relations:

$$\left(\mathbb{I} + \sum_{k=s}^{\ell} a_k\mathbb{E}_k b_k\right)^{-1} = \mathbb{I} - \sum_{k=s}^{\ell} \sigma_k\lambda_k a_k\mathbb{E}_k b_k\mu_k = \mathbb{I} - N_s,$$
$$\lambda_{s-1} = (\mathbb{I} - N_s)\mathbf{1}, \qquad \mu_{s-1} = (\mathbb{I} - N_s)'\mathbf{1}.$$

That is, our condition is to impose that all the matrices

$$\mathbb{I} + a_\ell \mathbb{E}_\ell b_\ell, \dots, \mathbb{I} + \sum_{k=s}^{\ell} a_k \mathbb{E}_k b_k, \dots, \mathbb{I} + \sum_{k=0}^{\ell} a_k \mathbb{E}_k b_k = \mathbb{I} + U$$

are in class $bi\mathcal{P}$.

We now assume that U is an SFM with a decomposition like

$$U = \sum_{s=0}^{k} D_{C_s}\ F(\mathcal{R}_s) + D_{\Gamma_s} D_{p_s}\ F(\mathcal{R}_s)\ D_{q_s},$$

where $\mathbb{F} = \mathcal{R}_0 \prec \cdots \prec \mathcal{R}_k$ is a filtration; C_s, Γ_s are nonnegative $\mathcal{R}_s$-measurable; and $\{p_s, q_s\}$ is a $\mathcal{R}_{s+1}$-measurable partition. Again we set $\mathbb{E}_s = D^{-1}_{F(\mathcal{R}_s)\mathbf{1}}\ F(\mathcal{R}_s)$ and the normalized $\mathcal{R}_s$-measurable factors

$$c_s = C_s \odot F(\mathcal{R}_s)\mathbf{1}, \qquad \gamma_s = \Gamma_s \odot F(\mathcal{R}_s)\mathbf{1}.$$

Since diagonal matrices commute, we get that U has a representation of the form

$$U = \sum_{s=0}^{k} c_s \mathbb{E}_s + \gamma_s p_s \mathbb{E}_s q_s,$$

with $\gamma_k = 0$. In the previous algorithm we can make two steps at each time and consider κ_s in place of κ_{2s}, λ_s instead of λ_{2s+1}, l_s instead of λ_{2s}. We also introduce $d_s = 1/\kappa_s$ to simplify certain formulae (this vector can take the value ∞). We get, starting from $\kappa_k = l_k = 1, \sigma_k = (1 + c_k)^{-1}$, that for $s = k-1, \dots, 0$

$$\begin{aligned} \lambda_s &= \sigma_{s+1} l_{s+1}, \\ l_s &= \lambda_s[1 - \gamma_s p_s \mathbb{E}_s(q_s/(c_{s+1} + d_{s+1}))], \\ \kappa_s &= \mathbb{E}_s(l_s), \\ \sigma_s &= 1/(1 + \kappa_s c_s) = d_s/(c_s + d_s). \end{aligned}$$

Similar recursions hold for μ, m, which are the analogues of λ, l. Relation (5.12) takes the form

$$\frac{1}{d_s} = \mathbb{E}_s\left(\frac{1}{c_{s+1} + d_{s+1}}\right) - \gamma_s\, \mathbb{E}_s\left(\frac{p_s}{c_{s+1} + d_{s+1}}\right) \mathbb{E}_s\left(\frac{q_s}{c_{s+1} + d_{s+1}}\right). \tag{5.13}$$

The inverse of $\mathbb{I} + U$ is $\mathbb{I} - N$, where

$$N = \sum_{s=0}^{k} c_s \sigma_s l_s \mathbb{E}_s m_s + \sum_{s=0}^{k-1} \gamma_s \lambda_s p_s \mathbb{E}_s q_s \mu_s = \sum_{s=0}^{k} c_s \sigma_s l_s \mathbb{E}_s m_s + \gamma_s \lambda_s p_s \mathbb{E}_s q_s \mu_s. \tag{5.14}$$

Again $\lambda_{-1} = (\mathbb{I} - N)\mathbf{1} = \sigma_0 l_0$, and similarly $\mu_{-1} = \sigma_0 m_0$.

Let us introduce the following function:

$$\rho_s = \mathbb{E}_s(p_s)p_s + \mathbb{E}_s(q_s)q_s.$$

THEOREM 5.4. *Assume that the backward recursion* (5.13) *has a nonnegative solution starting with* $d_k = 1$. *Assume, moreover, that this solution verifies for* $s = k-1, \dots, 0$

$$\rho_s \gamma_s \le c_{s+1} + d_{s+1}. \tag{5.15}$$

Then $\lambda_s, l_s, \mu_s, m_s, \sigma_s$, *for* $s = k, \dots, 0$, *as well as* λ_{-1}, μ_{-1} *are well defined and nonnegative. Therefore,* $\mathbb{I} + U \in bi\mathcal{P}$, *and its inverse is* $\mathbb{I} - N$, *where* N *is given by* (5.14).

The proof of this result is based on the following lemma.

LEMMA 5.5. *Let* x, y *be nonnegative vectors, and* $\mathbb{E}$ *be a conditional expectation. If* $x\mathbb{E}(y) \le 1$, *then* $\mathbb{E}(xy) \le 1$.

Proof. We first assume that y is strictly positive. Since $x \le 1/\mathbb{E}(y)$ and $\mathbb{E}$ is an increasing operator, we have

$$\mathbb{E}(xy) \le \mathbb{E}\left(\frac{1}{\mathbb{E}(y)} y\right) = \frac{\mathbb{E}(y)}{\mathbb{E}(y)} = 1.$$

For the general case consider $(y + \epsilon\mathbf{1})/(1 + \epsilon|x|_\infty)$ instead of y and pass to the limit $\epsilon \to 0$. ∎

Proof of Theorem 5.4. We notice that condition (5.15) implies that

$$\frac{q_s \gamma_s}{c_{s+1} + d_{s+1}} \mathbb{E}_s(q_s) \le 1.$$

Since γ_s is $\mathbb{E}_s$-measurable and $q_s = q_s^2$, we obtain

$$\gamma_s \mathbb{E}_s\left(\frac{q_s}{c_{s+1} + d_{s+1}}\right) = \mathbb{E}_s\left(\frac{\gamma_s q_s^2}{c_{s+1} + d_{s+1}}\right).$$

This last quantity is bounded by one by Lemma 5.5. Similarly we have

$$\gamma_s \mathbb{E}_s\left(\frac{p_s}{c_{s+1} + d_{s+1}}\right) \le 1,$$

which implies that the algorithm is not stopped, and all the coefficients are nonnegative including λ_{-1}, μ_{-1}. ∎

COROLLARY 5.6. *Assume that for* $s = k-1, \dots, 0$ *we have*

$$\rho_s \gamma_s \le c_{s+1} + \gamma_{s+1}. \tag{5.16}$$

Then the recursion (5.13) *has a nonnegative solution that verifies* (5.15). *In particular,* $\mathbb{I} + tU$ *is in class* $bi\mathcal{P}$ *for all* $t \ge 0$, *and* U *is in* $bi\mathcal{P}$ *if it is nonsingular.*

Proof. Let us consider first the case $t = 1$. We prove by induction that $\gamma_s \le d_s$. For $s = k$ we have $0 = \gamma_k \le d_k = 1$. We point out that if we multiply in (5.13) by γ_s, we get

$$\frac{\gamma_s}{d_s} = \mathbb{E}_s\left(\frac{\gamma_s}{c_{s+1} + d_{s+1}}\right) - \mathbb{E}_s\left(\frac{\gamma_s p_s}{c_{s+1} + d_{s+1}}\right) \mathbb{E}_s\left(\frac{\gamma_s q_s}{c_{s+1} + d_{s+1}}\right),$$

which is of the form $x + y - xy$, where $x = \mathbb{E}_s\left(\frac{\gamma_s p_s}{c_{s+1} + d_{s+1}}\right)$. The inequality (5.16), the induction hypothesis $\gamma_{s+1} \le d_{s+1}$, and Lemma 5.5 imply $0 \le x \le 1,\ 0 \le y \le 1$. In particular,

$$0 \le \frac{\gamma_s}{d_s} \le 1,$$

and the induction is completed. Theorem 5.4 shows that $\mathbb{I} + U$ is in class $bi\mathcal{P}$. We notice that tU also verifies condition (5.16) because this condition is homogeneous, and the result follows. ∎

Remark 5.4. We notice that condition (5.16) can be expressed in terms of the original coefficients C, Γ in the dyadic case. In fact (see (5.7)),

$$p_s \mathbb{E}_s(p_s) = D_{p_s} D^{-1}_{F(\mathcal{R}_s)\mathbf{1}} F(\mathcal{R}_s) p_s = D_{p_s} D^{-1}_{F(\mathcal{R}_s)\mathbf{1}} F(\mathcal{R}_{s+1})\mathbf{1},$$

which implies that

$$\rho_s = (1/F(\mathcal{R}_s)\mathbf{1}) \odot (F(\mathcal{R}_{s+1})\mathbf{1}).$$

Then, inequality (5.16) is

$$\Gamma_s \le C_{s+1} + \Gamma_{s+1},$$

which is the condition for having a GUM (see (5.6)) . We mention here that condition (5.16) is more general than having a GUM, as the following example shows.

Remark 5.5. Consider the matrix U_β,

$$U_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & \beta & 1 & 0 \\ \beta & \beta & 0 & 1 \end{pmatrix} = D_{\Gamma_0} D_{p_0} F(\mathcal{R}_0) D_{q_0} + \mathbb{I},$$

where $\mathcal{R}_0 = \mathcal{N}$, $\Gamma_0 = \beta(1,1,1,1)' \le C_1 = (1,1,1,1)'$. We compute $c_0 = 0,\ \gamma_0 = 4\beta,\ c_1 = C_1,\ \gamma_1 = 0$ and also $\rho_0 = 1/2$.

It is direct to check that $U_\beta^{-1} = U_{-\beta}$. Then for all $\beta \ge 0$ the matrix $U_\beta \in \mathcal{M}^{-1}$. Also $U_\beta \in bi\mathcal{P}$ if and only if $0 \le \beta \le 1/2$. When $\beta \ge 0$ the condition (5.6), $\Gamma_0 \le C_1 + \Gamma_1$, is equivalent to $\beta \le 1$. Then, this condition does not ensure that $U \in bi\mathcal{P}$ (this happens because the filtration is not dyadic). Nevertheless, the analogous condition in terms of the normalized factors (5.16),

$$\rho_0 \gamma_0 \le c_1 + \gamma_1,$$

is equivalent to $\beta \le 1/2$, which is the correct condition.

COROLLARY 5.7. *Assume that*

$$\rho_s \gamma_s \le \sum_{r=s+1}^{k} c_r \tag{5.17}$$

hold for $s = k-1, \dots, 0$. *Then the recursion* (5.13) *has a nonnegative solution that verifies* (5.15). *In particular,* $\mathbb{I} + tU$ *is in class* $bi\mathcal{P}$ *for all* $t \ge 0$*, and* U *is in* $bi\mathcal{P}$ *if it is nonsingular.*

Proof. Consider the set of inequalities

$$\rho_s \gamma_s \vee \xi_s \le c_{s+1} + \xi_{s+1},$$

for $s = k-1, \dots, 0$. A nonnegative solution is given by

$$\xi_s = \sup\left\{0,\ \gamma_0\rho_0 - \sum_{r=1}^{s} c_r, \dots, \gamma_k\rho_k - \sum_{r=k+1}^{s} c_r, \dots, \gamma_{s-1}\rho_{s-1} - c_s\right\}.$$

The hypothesis of the corollary is that $\xi_k = 0$. We also notice that ξ_s is $\mathcal{R}_s$-measurable.

We show, using a backward recursion, that $\xi_s \le d_s$. Indeed, by construction, $1/\xi_s = \mathbb{E}_s(1/\xi_s) \ge (c_{s+1} + \xi_{s+1})^{-1}$ while $1/d_s \le \mathbb{E}_s((c_{s+1} + d_{s+1})^{-1})$. Then the inequality $\rho_s\gamma_s \le c_{s+1} + \xi_{s+1}$ implies $\rho_s\gamma_s \le c_{s+1} + d_{s+1}$, so the result holds (see Theorem 5.4). ∎

5.2. Conditions for class $\mathcal{T}$ and proof of Theorem 2.6.

THEOREM 5.8. *Assume that U has a decomposition*

$$U = \sum_{s=0}^{\ell} a_s \, \mathbb{E}_s \, b_s,$$

where a_s, b_s are nonnegative $\mathbb{E}_{s+1}$-measurable. Then U belongs to the class $\mathcal{T}$ and, moreover,

$$\tau(U) = \inf\{t > 0 : \; (\mathbb{I} + tU)^{-1}\mathbf{1} \not\geq 0 \text{ or } \mathbf{1}'(\mathbb{I} + tU)^{-1} \not\geq 0\}.$$

In particular, if $\tau(U) < \infty$, then $\mathbb{I} + \tau(U)\,U \in bi\mathcal{P}$.

Remark 5.6. In the case $\tau(U) < \infty$ we have that $\mathbb{I} + t\,U$ is nonsingular for $t > \tau(U)$ sufficiently close to $\tau(U)$. This follows from the fact that the set of nonsingular matrices is open.

Theorem 5.8 states that every filtered matrix with a nonnegative decomposition is in class $\mathcal{T}$, which proves Theorem 2.6.

Proof of Theorem 5.8. A warning about the use of vectors and functions. Here we consider vectors or functions on $\{1, \ldots, n\}$ indiscriminately. Thus for two vectors a, b the product ab makes sense as the product of two functions, which corresponds to the Hadamard product of the vectors. Also an expression as $(1 + ab)^{-1}$ is the vector whose components are the reciprocals of the components of $1 + ab$. We also recall that $(a)_i$ is the ith component of a.

Now, for $p = 0, \ldots, \ell$ consider the matrices

$$U(p) = \sum_{s=p}^{\ell} a_s \, \mathbb{E}_s \, b_s.$$

We notice that $U(0) = U$. We shall prove that $\tau_p = \tau(U(p))$ is increasing in p and $\tau_\ell = \infty$.

We rewrite the algorithm for $\mathbb{I} + tU$. This takes the form $\lambda_\ell(t) = \mu_\ell(t) = \kappa_\ell(t) = 1$, $\sigma_\ell(t) = (1 + t\,a_\ell b_\ell)^{-1}$, and for $p = \ell - 1, \ldots, 0$

$$\begin{aligned}
&\lambda_p(t) = \lambda_{p+1}(t)[1 - \sigma_{p+1}(t)\,t\,a_{p+1}\mathbb{E}_{p+1}(\kappa_{p+1}(t)b_{p+1})],\\
&\mu_p(t) = \mu_{p+1}(t)[1 - \sigma_{p+1}(t)\,t\,b_{p+1}\mathbb{E}_{p+1}(\kappa_{p+1}(t)a_{p+1})],\\
&\kappa_p(t) = \mathbb{E}_{p+1}(\lambda_p(t)) = \mathbb{E}_{p+1}(\mu_p(t)),\\
&\sigma_p(t) = (1 + \mathbb{E}_p(\kappa_p(t)ta_pb_p))^{-1}.
\end{aligned} \tag{5.18}$$

Also $\lambda_{-1}(t)$, $\mu_{-1}(t)$ are defined similarly. If $\lambda_s(t)$, $\mu_s(t)$, $\sigma_s(t)$, $s = \ell, \ldots, p$, are well defined, then

$$(\mathbb{I} + tU(p))^{-1} = \mathbb{I} - N(p,t),$$

where

$$N(p,t) = \sum_{s=p}^{\ell} \sigma_s(t)\lambda_s(t)\,t\,a_s\mathbb{E}_s b_s \mu_s(t). \tag{5.19}$$

If $\lambda_s(t),\ \mu_s(t),\ \sigma_s(t),\ s=\ell,\dots,p$, are nonnegative, then $N(p,t)\ge 0$ and $(\mathbb{I}+tU(p))\in\mathcal{M}^{-1}$. Moreover, $\lambda_{p-1}(t)$ and $\mu_{p-1}(t)$ are the right and left equilibrium potentials of $(\mathbb{I}+tU(p))$,

$$(\mathbb{I}+tU(p))\lambda_{p-1}(t)=\mathbf{1} \quad \text{and} \quad \mu'_{p-1}(t)(\mathbb{I}+tU(p))=\mathbf{1}'.$$

So, if they are nonnegative, we have $\mathbb{I}+tU(p)\in bi\mathcal{P}$. In particular, for $p=\ell$ we get

$$(\mathbb{I}+ta_\ell\,\mathbb{E}_\ell\,b_\ell)^{-1}=(\mathbb{I}+tU(\ell))^{-1}=\mathbb{I}-t(1+t\,a_\ell b_\ell)^{-1}a_\ell\,\mathbb{E}_\ell\,b_\ell.$$

Since $\mathbb{E}_\ell=\mathbb{I}$ we obtain that $\lambda_{\ell-1}=\mu_{\ell-1}=(1+t\,a_\ell b_\ell)^{-1}$. This means that $\mathbb{I}+tU(\ell)\in bi\mathcal{P}$ for all $t\ge 0$. Therefore $\tau_\ell=\infty$, and the result is true for $U(\ell)$. In particular, $\tau_{\ell-1}\le\tau_\ell$. We shall prove by induction that

- $\tau_{p+1}\le\cdots\le\tau_\ell$

and for $q=p+1,\dots,\ell$

- $\tau_q=\inf\{t>0:\lambda_{q-1}(t)\ngeqq 0 \text{ or } \mu_{q-1}(t)\ngeqq 0\}=\inf\{t>0:\lambda_{q-1}(t)\not> 0 \text{ or } \mu_{q-1}(t)\not>0\}$;
- $\lambda_s(t),\mu_s(t)$, for $s=\ell,\dots,q-1$, are strictly positive for $t\in[0,\tau_q)$;
- if $\tau_q<\infty$, we have $\mathbb{I}+\tau_qU(q)\in bi\mathcal{P}$.

The case $\tau_{p+1}=\infty$ is simple. Indeed, fix $t\ge 0$. From Lemma 3.1, $\mathbb{I}+tU(p+1)\in bi\mathcal{P}$ and its equilibrium potential are strictly positive; that is, $\lambda_p(t)>0,\ \mu_p(t)>0$. Thus, $\mathbb{I}+tU(p)$ is nonsingular; its inverse is $\mathbb{I}-N(p,t)$, where $N(p,t)\ge 0$ is given by (5.19). Hence, $\mathbb{I}+tU(p)\in\mathcal{M}^{-1}$. We conclude that

$$\tau_p=\inf\{t>0:\mathbb{I}+tU(p)\notin bi\mathcal{P}\}=\inf\{t>0:\lambda_{p-1}(t)\ngeqq 0 \text{ or } \mu_{p-1}(t)\ngeqq 0\}.$$

If $\tau_p=\infty$, Lemma 3.1 gives

$$\lambda_{p-1}(t)>0, \qquad \mu_{p-1}(t)>0,$$

and the induction step holds in this case.

Now if $\tau_p<\infty$, by continuity we have $\mathbb{I}+\tau_pU(p)\in bi\mathcal{P}$. We shall prove later on that $\lambda_{p-1}(t),\mu_{p-1}(t)$ are strictly positive in $[0,\tau_p)$.

We now analyze the case $\tau_{p+1}<\infty$. We first notice that in the algorithm the only possible problem could arise with the definition of $\sigma_p(t)$. Since $\sigma_p(\tau_{p+1})>0$, the algorithm is well defined, by continuity, for steps $\ell,\dots,p$ on an interval $[0,\tau_{p+1}+\epsilon]$ for $\epsilon>0$ small enough. This proves that the matrix $\mathbb{I}+tU(p)$ is nonsingular in that interval, and that $\lambda_{p-1},\ \mu_{p-1}$ exist in the same interval.

Now, for a sequence $t_n\downarrow\tau_{p+1}$, either $\lambda_p(t_n)$ or $\mu_p(t_n)$ has a negative component. Since there are a finite number of components, we can assume without loss of generality that for a fixed component i we have $(\lambda_p(t_n))_i<0$. Then, by continuity we get that $(\lambda_p(\tau_{p+1}))_i=0$, which implies (by the algorithm) that $(\lambda_{p-1}(\tau_{p+1}))_i=0$.

Assume now that for some $t>\tau_{p+1}$ the matrix $\mathbb{I}+tU(p)\in bi\mathcal{P}$. By Lemma 3.1 we will have that $\mathbb{I}+\tau_{p+1}U(p)\in bi\mathcal{P}$, but its equilibrium potential will satisfy $\lambda_{p-1}(\tau_{p+1})>0$, which is a contradiction. Therefore we conclude that $\tau_p\le\tau_{p+1}$.

The conclusion of this discussion is that the matrix $\mathbb{I}+tU(p)$, for $t\in[0,\tau_{p+1}]$, is nonsingular and its inverse is $\mathbb{I}-N(p,t)$, with $N(p,t)\ge 0$. That is, $\mathbb{I}+tU(p)\in\mathcal{M}^{-1}$ and therefore

$$\tau_p=\inf\{t>0:\mathbb{I}+tU(p)\notin bi\mathcal{P}\}=\inf\{t>0:\lambda_{p-1}(t)\ngeqq 0 \text{ or } \mu_{p-1}(t)\ngeqq 0\},$$

and by continuity $\mathbb{I}+\tau_pU(p)\in bi\mathcal{P}$.

To finish the proof we need to show that τ_p coincides with

$$S = \inf\{t > 0 : \lambda_{p-1}(t) \ngtr 0 \text{ or } \mu_{p-1}(t) \ngtr 0\}.$$

It is clear that $S \leq \tau_p$. If $S < \tau_p$, then, due to Lemma 3.1, we have that both $\lambda_{p-1}(S) > 0$ and $\mu_{p-1}(S) > 0$, which is a contradiction, and then $S = \tau_p$. This shows that $\lambda_{p-1}(t)$, $\mu_{p-1}(t)$ are strictly positive for $t \in [0, \tau_p)$, and the induction is proven. ∎

Remark 5.7. It is possible to prove that $\kappa_p(\tau_p) > 0$ when $\tau_p < \infty$, but this is not central to our discussion.

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