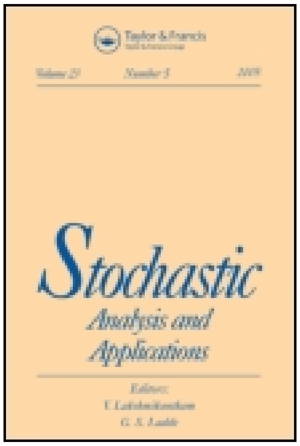

Stochastic Analysis and Applications
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lsaa20

Numerical Method for Reflected Backward Stochastic Differential Equations

Miguel Martínez [a] , Jaime San Martín [b] & Soledad Torres [c]

[a] Laboratoire d'Analyse et de Mathématiques Appliqueés, Université de Marne-La-Vallée, Champs-sur-Marne, France

[b] Facultad de Ciencias Físicas y Matemáticas, Departamento de Ingeniería Matemática and CMM-CNRS UMI 2807, Universidad de Chile, Santiago, Chile

[c] DEUV and CIMFAV, Universidad de Valparaíso, Valparaíso, Chile

Version of record first published: 18 Oct 2011.

To cite this article: Miguel Martínez , Jaime San Martín & Soledad Torres (2011): Numerical Method for Reflected Backward Stochastic Differential Equations, Stochastic Analysis and Applications, 29:6, 1008-1032

To link to this article: http://dx.doi.org/10.1080/07362994.2011.610162

Stochastic Analysis and Applications, 29: 1008–1032, 2011

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362994.2011.610162

Numerical Method for Reflected Backward Stochastic Differential Equations

MIGUEL MARTÍNEZ,[1] JAIME SAN MARTÍN,[2] AND SOLEDAD TORRES[3]

[1]Laboratoire d'Analyse et de Mathématiques Appliqueés, Université de Marne-La-Vallée, Champs-sur-Marne, France
[2]Facultad de Ciencias Físicas y Matemáticas, Departamento de Ingeniería Matemática and CMM-CNRS UMI 2807, Universidad de Chile, Santiago, Chile
[3]DEUV and CIMFAV, Universidad de Valparaíso, Valparaíso, Chile

In this article we propose a numerical method for reflected backward stochastic differential equations (RBSDE). This method is based on the simple random walk, and the convergence is related to the Skorohod topology.

Keywords Backward SDEs with reflections; Skorohod topology.

Mathematics Subject Classification Primary 60H10; Secondary 34F05, 93E03.

1. Introduction

In this article, we are interested in the following backward stochastic differential equation with reflection (in short RBSDE).

$$\begin{aligned} &Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \le t \le T, \\ &Y_t \ge S_t, \quad 0 \le t \le T, \quad \text{and} \quad \int_0^T (Y_t - S_t)\, dK_t = 0, \end{aligned} \tag{1.1}$$

Received July 22, 2008; Accepted May 3, 2011

The authors are thankful for the suggestions made by an anonymous referee. M. M. was partially support by PostDoc Fondecyt grant #3060118. S. T. was supported in part by Fondecyt grant #1070919, Dipuv grant 26/2009, ECOS/CONICYT C10E03 2010, MATHAMSUD/SAMP Stochastic Analysis and Mathematical Physics Research Network, and Anillo PBCT-ACT 13, Laboratorio de Análisis estocástico. J. S. M. and S. T. were partially supported by the research projects Nucleus Millenium P04-069-F and Basal project CMM.

Address correspondence to Soledad Torres, DEUV and CIMFAV, Universidad de Valparaíso, Casilla 5030, Valparaíso; E-mail: soledad.torres@uv.cl

where f is called the *coefficient*, ξ the *terminal condition*, and S_t is the process representing the reflecting barrier. It is assumed that $\xi \geq S_T$.

It is well known in the case without reflection, that the RBSDE (actually a BSDE) has a unique solution under the usual assumptions on the generator f and the final condition ξ, see e.g., the work of Pardoux and Peng [30, 31]. For RBSDE and under the Lipschitz assumption of f, El Karoui, Kapoudjian, Pardoux and Quenez in [10] stated the first existence and uniqueness result, for which the solution is constrained to stay above an *obstacle* $\{S_t, 0 \leq t \leq T\}$.

The solution of a RBSDE with obstacle S, terminal condition ξ and coefficient f, consists of a triple of a progressive measurable process (Y, Z, K), taking values in $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}_+$, respectively, where K is an increasing process introduced in order to force the solution Y within a boundary area, satisfying:

i) $\mathbb{E} \int_0^T |Z_t|^2 dt < \infty$;
ii) $\{Y_t \geq S_t, 0 \leq t \leq T\}$;
iii) $\{K_t\}$ is a continuous increasing process, $K_0 = 0$ and $\int_0^T (Y_t - S_t) dK_t = 0$.

Several works have been made to prove existence and uniqueness under different assumptions on f. For instance, in [20], Lepeltier and San Martín proved the existence of a solution for BSDE's with a coefficient which is only continuous with linear growth. In [17], Kobylanski studied the case of BSDE's without reflection and proved an existence result in the case when the coefficient is only linear growth in y, and quadratic in z. In [21], Lepeltier and San Martín generalized the result to a superlinear case in y. Matoussi [28] established the existence of a solution for RBSDE's with continuous and linear growth coefficient. Cvitanic and Karatzas [7] generalized their results for two reflecting barriers, in which the solution process of BSDE has to stay within two pre-specified limits U and L. In this situation see also Xu [36], Lepeltier and Xu [22], among others. In [16], Kobylanski, Lepeltier, Quenez and Torres proved existence of a reflected solution of the one-dimensional BSDE when the coefficient is continuous, has a superlinear growth in y and quadratic growth in z. In [19], Lepeltier, Matoussi, and Xu gave results on existence under monotonicity and general increasing growth conditions. For reflected backward doubly stochastic differential equations, see [1, 13], and the references therein.

In [32], Ren proved existence and uniqueness of a solution for RBSDE driven by Teugels martingales associated with a Lévy process. For RBSDE driven by a Lévy process see [33–35], and references therein. Finally, to better understand the penalization and apriori method in BSDE, see [9, 23].

RBSDEs are a useful tool for the pricing of American options. In a complete market, the process Y of an American option associated with payoff $\{S_t, 0 \leq t \leq T\}$ is a solution of a RBSDE such as (1.1). On the other hand, in [18], El Karoui and Rouge studied the problem of pricing European options via exponential utility. In the case of an incomplete market, they stated that the price of such option is a solution of a quadratic BSDE. Thus, if we are concerned with American options instead of European options, we are naturally led to the study of reflected quadratic BSDE's. See [11] for imperfect market and BSDEs. For a survey on the developed theory of forward-backward stochastic differential equations (FBSDEs) see [26].

A long-standing problem in the theory of BSDEs is to find an implementable numerical method. For example, in the Markovian case, Douglas, Ma, and Protter [8] established a numerical method for a class of forward–backward SDEs, based on a four step scheme developed by Ma, Protter, and Yong [25]. On the other hand, Chevance [5] proposed a numerical method for BSDEs. In [37], Zhang proposed a numerical scheme for a class of backward stochastic differential equations (BSDEs) with possible path-dependent terminal values. For a class of decoupled forward–backward stochastic differential equations Bouchard and Touzi [2], proposed a discrete-time approximation, and the convergence is related with the L^p norm. In [12], a regression method to solve BSDE was developed by Gobet, Lemor, and Warin. In the case of a random walk instead of a Brownian motion see [3, 24], in the case of BSDEs and [14, 15, 29] for the RBSDE case.

This article is organized as follows. In Section 2, we present some preliminaries. In Section 3, we introduce the numerical method for solutions of RBSDE. Section 4 is devoted to the proof of the convergence of the numerical scheme proposed in Section 3. Finally, in Section 5 we give an alternative procedure to compute the solution of a RBSDE, using the ideas of Ma and Zhang given in [27].

2. Preliminaries

In this section we will use the following space: A hilbert space $\mathbb{H}$

$$\mathbb{H}^2 = \left\{ \{\phi_t : 0 \le t \le T\} \text{ is a predictable process s.t. } \mathbb{E}\left[\int_0^T |\phi_t|^2 dt\right] < +\infty \right\}.$$

A Banach space $\mathcal{S}$,

$$\mathcal{S}^2 = \left\{ \{\phi_t : 0 \le t \le T\} \text{ is a predictable process s.t. } \mathbb{E}\left[\sup_{0\le t\le T} |\phi_t|^2\right] < +\infty \right\}.$$

Let W be a Brownian motion. Throughout this article we denote by $c_n(t) = [nt]/n$. Then there exists a family of independent random variables ζ_k^n such that

$$\mathbb{P}(\zeta_k^n = 1) = \mathbb{P}(\zeta_k^n = -1) = 1/2 \quad \text{and} \quad W_t^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{c_n(t)} \zeta_i^n$$

converges uniformly in probability to W.

We define $\mathcal{G}_k^n := \sigma(\zeta_1^n, \ldots, \zeta_k^n)$. We will consider the following standing assumptions:

- **(A1)** the function f is continuous and bounded.
- **(A2)** the function f is uniformly Lipschitz with respect to variables (y, z) with Lipschitz constant K.
- **(A3)** the Barrier S is assumed to be almost surely constant.
- **(H)** $\lim_{n\to+\infty} \mathbb{E}\left[\sup_{s\in[0,T]} \left|\mathbb{E}[\xi|\mathcal{F}_s] - \mathbb{E}[\xi^n|\mathcal{G}^n_{c_n(s)}]\right|\right] = 0.$

Remark 1. For simplicity we assume that $T \equiv 1$.

Remark 2. The assumption (H) is related with the so called weak convergence of filtrations defined by Coquet, Mémin and Slominski in [6]. On the other hand, the strongest assumption is (A3). For the general case, we consider $S_t = S_0 + \int_0^t J_s ds + \int_0^t H_s dB_s$, then $R_t = Y_t - S_t$ satisfies the following RBSDE:

$$
\begin{aligned}
&R_t = \hat{\xi} + \int_t^T \hat{f}(s, R_s, \Gamma_s)ds - \int_t^T \Gamma_s dB_s + K_T - K_t \quad 0 \le t \le T, \\
&R_t \ge 0, \quad 0 \le t \le T, \quad \text{and} \quad \int_0^T R_t dK_t = 0,
\end{aligned} \tag{2.2}
$$

where $\hat{\xi} = \xi - S_T$, $\Gamma_t = Z_t + H_t$, and $\hat{f}(s, r, \gamma) = f(s, r + S_s, \gamma - H_s) + J_s$.

We notice that $\hat{f}$ is also uniformly Lipschitz on (r, γ), with the same constant as f. Nevertheless $\hat{f}$ is in general a random function. We present in this paper the case where S is constant, but the results hold as soon as H, J are smooth functionals of B, to ensure that a method based on Euler scheme jointly with a random walk approximation of B converges to them.

3. Numerical Method for RBSDE

In this section, we define the numerical scheme for RBSDE. The method is based in two steps:

- **Step I:** The penalization term and Picard's ieration procedure in the continuous case. In this case we follow with the main ideas given in [10].
- **Step II:** The penalization term and Picard's iteration procedure in the discrete case. In this step we will follow the ideas given in [3] and/or [4].

3.1. *Step I: The Penalization Term and Picard's Iteration Procedure in the Continuous Case*

3.1.1. Penalization Term. In this step we are interested in the following penalization scheme associated to the unique solution of the RBSDE given in (1.1). For each $\varepsilon > 0$, let $\{(Y_t^\varepsilon, Z_t^\varepsilon); 0 \le t \le 1\}$ denote the unique pair of progressively measurable $\mathcal{F}_t$ processes with values in $\mathbb{R} \times \mathbb{R}$ satisfying the following BSDE:

$$
Y_t^\varepsilon = \xi + \int_t^1 f(s, Y_s^\varepsilon, Z_s^\varepsilon)ds - \int_t^1 Z_s^\varepsilon dB_s + \frac{1}{\varepsilon}\int_t^1 (S - Y_s^\varepsilon)^+ ds, \tag{3.3}
$$

where ξ and f satisfy the above assumptions (A1), (A2). In this famework, we define

$$
K_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t (S - Y_s^\varepsilon)^+ ds, \quad 0 \le t \le 1. \tag{3.4}
$$

Proposition 1. *Assume the standard conditions* (A1), (A2), (A3), *and* (H). *Then, we have that*

$$
\mathbb{E}\left[\int_0^1 |Y_t^\varepsilon - Y_t|^2 dt + \int_0^1 |Z_t^\varepsilon - Z_t|^2 dt + \sup_{0 \le t \le 1} |K_t^\varepsilon - K_t|^2\right] \to 0 \quad as\ \varepsilon \to 0. \tag{3.5}
$$

Proof. We follow the proof given in [10]. □

Remark 3. For the penalization method we can obtain the following result. There exists a constant c such that for every $\varepsilon \in]0, 1]$,

$$\mathbb{E}\left[\sup_{0\le t\le 1}|Y_t^\varepsilon|^2 + \int_0^1 |Z_t^\varepsilon|^2 dt + \sup_{0\le t\le 1}|K_t^\varepsilon|^2\right] \le c.$$

Remark 4. Let us notice that for each fixed t, the sequence (Y_t^ε) is decreasing in ε.

Lemma 1. *Under the assumption* (A1), (A2), (A3), Y^ε *the unique solution of equation* (3.3) *satisfies the following property*:

$$\mathbb{E}\left[\sup_{0\le t\le 1}|(S - Y_t^\varepsilon)^+|^2\right] \to 0 \quad as\ \varepsilon \to 0. \tag{3.6}$$

Proof. We follow again the proof given in [10]. □

3.1.2. Picard's Iteration Procedure. Remember that $\mathbb{H}^2$ is a Hilbert space. For each ε fixed in $]0, 1]$, we define a mapping Φ^ε of $\mathbb{H}^2 \times \mathbb{H}^2$ into itself as follows: Given $(U, V) \in \mathbb{H}^2 \times \mathbb{H}^2$, we define $(Y^\varepsilon, Z^\varepsilon) := \Phi^\varepsilon(U, V)$ to be the unique element of $\mathbb{H}^2 \times \mathbb{H}^2$, which is such that the couple $(Y^\varepsilon, Z^\varepsilon)$ solves the following BSDE:

$$Y_t^\varepsilon = \xi + \int_t^1 f(s, U_s, V_s)ds + \frac{1}{\varepsilon}\int_t^1 (S - U_s)^+ ds - \int_t^1 Z_s^\varepsilon dB_s. \tag{3.7}$$

This equation corresponds to the Picard's iteration associated to the BSDE (3.3). In [10], the authors show that the mapping Φ^ε is a strict contraction on $\mathbb{H}^2 \times \mathbb{H}^2$ equipped with the norm $\|\cdot\|_\beta$ defined by

$$\|(U, V)\|_\beta := \left(\mathbb{E}\left[\int_0^1 e^{\beta s}\left(|U_s|^2 + |V_s|^2\right) ds\right]\right)^{1/2}.$$

Thus, the mapping Φ^ε has a unique fixed point, which is the solution $(Y^\varepsilon, Z^\varepsilon)$ of the penalized BSDE (3.3). This fixed point can be reached via Picard's iteration procedure. More precisely, we define the sequence $(Y^{\varepsilon,p}, Z^{\varepsilon,p})_{p\in\mathbb{N}}$ as follows:

$$\begin{cases} (Y^{\varepsilon,0}, Z^{\varepsilon,0}) := (0, 0); \\ (Y^{\varepsilon,p+1}, Z^{\varepsilon,p+1}) = \Phi^{\varepsilon,p+1}(Y^{\varepsilon,0}, Z^{\varepsilon,0}) := \Phi^\varepsilon(Y^{\varepsilon,p}, Z^{\varepsilon,p}). \end{cases} \tag{3.8}$$

In particular, for each $p \in \mathbb{N}$,

$$\|(Y^\varepsilon - Y^{\varepsilon,p}, Z^\varepsilon - Z^{\varepsilon,p})\|_\beta \le \frac{1}{2^p}\|(Y^\varepsilon, Z^\varepsilon)\|_\beta, \tag{3.9}$$

and this inequality permits to understand the influence of the initial condition used for Picard's iteration method. In conclusion,

$$\|(Y^\varepsilon - Y^{\varepsilon,p}, Z^\varepsilon - Z^{\varepsilon,p})\|_\beta \to 0, \quad \text{as } p \text{ tends to } \infty. \tag{3.10}$$

3.2. *Step II: The Penalization Term and Picard's Iteration Procedure in the Discrete Case*

In this step we are interested in the following discrete RBSDE. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ a discretization step for the interval $[0, 1]$ such that $\forall 1 \leq i \leq n$, $t_i - t_{i-1} = 1/n$.

$$\begin{aligned} Y_t^{\infty,n} &= \xi^n + \int_t^1 f(s, Y_s^{\infty,n}, Z_s^{\infty,n})ds - \int_t^1 Z_s^{\infty,n} dW_s^n + K_{t_i}^{\infty,n} - K_t^{\infty,n}, \quad 0 \leq t \leq 1, \\ Y_t^{\infty,n} &\geq S, \quad 0 \leq t \leq 1, \\ Y_1^{\infty,n} &= \xi^n. \end{aligned} \tag{3.11}$$

3.2.1. Penalization Term. For $t \in [t_{i-1}, t_i[$, and for each $\varepsilon > 0$, let $\{(Y_t^{\varepsilon,\infty,n}, Z_t^{\varepsilon,\infty,n}, K_t^{\infty,n}); 0 \leq t \leq 1\}$ denote the unique pair of progressively measurable $\mathcal{F}_t$ processes with values in $\mathbb{R} \times \mathbb{R}$ satisfying the following discrete BSDE:

$$\begin{aligned} Y_t^{\varepsilon,\infty,n} &= Y_{t_i}^{\varepsilon,\infty,n} + \int_t^{t_i} f(s, Y_s^{\varepsilon,\infty,n}, Z_s^{\varepsilon,\infty,n})ds - \int_t^{t_i} Z_s^{\varepsilon,\infty,n} dW_s^n + K_{t_i}^{\varepsilon,\infty,n} - K_t^{\varepsilon,\infty,n}; \\ Y_1^{\varepsilon,\infty,n} &= \xi^n. \end{aligned} \tag{3.12}$$

where $K_0^{\varepsilon,\infty,n} = 0$ and for $t \in]t_{i-1}, t_i[$ and we define

$$K_t^{\varepsilon,\infty,n} := \frac{1}{n\varepsilon} \sum_{j=1}^{i} \left(S - Y_{t_{j-1}}^{\varepsilon,\infty,n} \right)^+. \tag{3.13}$$

Proposition 2. *Assume the standard conditions* (A1), (A2). *Then, we have that*

$$\mathbb{E}\left[\int_0^1 |Y_t^{\varepsilon,\infty,n} - Y_t^{\infty,n}|^2 dt + \int_0^1 |Z_t^{\varepsilon,\infty,n} - Z_t^{\infty,n}|^2 dt + \sup_{0 \leq t \leq 1} |K_t^{\varepsilon,\infty,n} - K_t^{\infty,n}|^2\right] \to 0. \tag{3.14}$$

as $\varepsilon \to 0$, *where* $(Y^{\infty,n}, Z^{\infty,n}, K_t^{\infty,n})$ *is the unique solution of the equation* (3.11).

Proof. See the proof in [29], for RBSDE and [3] in the BSDE case. □

In this section we follow the ideas given in [3], in order to get the exact solution of the discrete equation (3.12). For each fixed $\varepsilon > 0$, we introduce the following implicit discrete-time scheme BSDE:

$$Y_{t_i}^{\varepsilon,\infty,n} = Y_{t_{i+1}}^{\varepsilon,\infty,n} + \frac{1}{n} f\left(t_i, Y_{t_i}^{\varepsilon,\infty,n}, Z_{t_i}^{\varepsilon,\infty,n}\right) + \frac{1}{n\varepsilon}\left(S - Y_{t_i}^{\varepsilon,\infty,n}\right)^+ - \frac{1}{\sqrt{n}} Z_{t_i}^{\varepsilon,\infty,n} \zeta_{i+1}, \tag{3.15}$$

for $i \in \{n-1, \ldots, 0\}$, with $Y_1^{\varepsilon,\infty,n} = \xi^n$. Remember that, $\{\zeta_k\}_{1 \leq k \leq n}$ is an i.i.d. Bernouilli symmetric sequence, and ξ^n is a square integrable random variable, measurable w.r.t. $\mathcal{G}_n$ with $\mathcal{G}_k := \sigma(\zeta_1, \ldots, \zeta_k)$.

3.2.2. Picard's Iteration Procedure. An explicit solution of (3.15) can be found using a discrete Picard's iteration method. Let us set $Y^{\varepsilon,0,n} \equiv 0$, $Z^{\varepsilon,0,n} \equiv 0$, we define $(Y^{\varepsilon,p+1,n}, Z^{\varepsilon,p+1,n})$ by induction as the solution of the iterated discrete-time scheme

BSDE:

$$Y_{t_i}^{\varepsilon,p+1,n} = Y_{t_{i+1}}^{\varepsilon,p+1,n} + \frac{1}{n} f\left((t_i, Y_{t_i}^{\varepsilon,p,n}, Z_{t_i}^{\varepsilon,p,n}\right) + \frac{1}{n\varepsilon}\left(S - Y_{t_i}^{\varepsilon,p,n}\right)^+ - \frac{1}{\sqrt{n}} Z_{t_i}^{\varepsilon,p+1,n} \zeta_{i+1} \tag{3.16}$$

for $i \in \{n-1, \ldots, 0\}$. Remember the definition of Picard's iteration procedure (3.16) and the definition of our scheme as a solution of (3.15). Just like in the continuous setting, we begin to show the following lemma:

Lemma 2. *There exists $\alpha_\varepsilon > 1$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, for all $p \in \mathbb{N}^*$,*

$$\left\| \left(Y^{\varepsilon,p+1,n} - Y^{\varepsilon,p,n}, Z^{\varepsilon,p+1,n} - Z^{\varepsilon,p,n}\right) \right\|_{\alpha_\varepsilon}^2 \leq \frac{1}{4} \left\| \left(Y^{\varepsilon,p,n} - Y^{\varepsilon,p-1,n}, Z^{\varepsilon,p,n} - Z^{\varepsilon,p-1,n}\right) \right\|_{\alpha_\varepsilon}^2,$$

where, for $p \in \mathbb{N}$,

$$\begin{aligned} &\left\| \left(Y^{\varepsilon,p+1,n} - Y^{\varepsilon,p,n}, Z^{\varepsilon,p+1,n} - Z^{\varepsilon,p,n}\right) \right\|_{\alpha_\varepsilon}^2 \\ &\quad := \mathbb{E}\left[\sup_{0\leq k\leq n} {\alpha_\varepsilon}^{k/n} \left|Y^{\varepsilon,p+1,n} - Y^{\varepsilon,p,n}\right|^2\right] + \frac{1}{n}\mathbb{E}\left[\sum_{k=0}^{n-1} {\alpha_\varepsilon}^{k/n} \left|Z^{\varepsilon,p+1,n} - Z^{\varepsilon,p,n}\right|^2\right]. \end{aligned}$$

Proof. We follow again [3]. □

Just like in the continuous case, we can use the Cauchy criterion and the preceding lemma to get the following result:

Proposition 3. *Following the notations of* (3.15) *and* (3.16), *for each fixed ε in*]0, 1], *we have that*

$$\sup_n \mathbb{E}\left[\sup_{0\leq t\leq 1} \left|Y_t^{\varepsilon,p,n} - Y_t^{\varepsilon,\infty,n}\right|^2 + \int_0^1 \left|Z_t^{\varepsilon,p,n} - Z_t^{\varepsilon,\infty,n}\right|^2 dt\right] \to 0, \quad \text{as } p \to \infty. \tag{3.17}$$

4. Main Result

Theorem 1. *Under the assumptions* (A1), (A2), (A3), *and* (H) *we have*

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} (\xi^n, Y^{\varepsilon,\infty,n}, Z^{\varepsilon,\infty,n}, K^{\varepsilon,\infty,n}) = (\xi, Y, Z, K),$$

in the Skorohod topology, where (ξ, Y, Z, K) is the solution of the RBSDE (1.1).

Proof. We shall prove the convergence of $Y^{\varepsilon,\infty,n}$ toward Y. The convergence of the other terms follows along the lines of the proof given in [10]. □

The main idea of the proof is the following decomposition for the error:

$$Y_t - Y_t^{\varepsilon,\infty,n} = (Y_t - Y_t^\varepsilon) + (Y_t^\varepsilon - Y_t^{\varepsilon,p}) + (Y_t^{\varepsilon,p} - Y_t^{\varepsilon,p,n}) + (Y_t^{\varepsilon,p,n} - Y_t^{\varepsilon,\infty,n}),$$

where the first term corresponds to the penalization term in the continuous setting, the second one is the Picard's iteration procedure for the continuous BSDE, the third term correspond to the discretization of a BSDE by using a random walk instead of the Brownian motion, and the last term is related to a Picard's iteration procedure in the discrete case. The proof follows from the next result in which the main technical point is to control the limit in n of

$$\sup_{0\le t\le 1}\left|\frac{1}{\varepsilon}\int_0^t (S-Y^{\varepsilon,p,n}_{s-})^+ dc_n(s) - \frac{1}{\varepsilon}\int_0^t (S-Y^{\varepsilon,p}_{s})^+ ds\right|$$

uniformly in ε, p.

In order to established convergence in probability we consider that all the processes are defined in the same probability space.

Lemma 3. *Let the assumptions* (A1), (A2), (A3), *and hypothesis* (H). *Let us consider the scaled random walks W^n. We have that for each fixed $\varepsilon \in]0, 1]$,*

$$\sup_{0\le t\le 1}\left|Y^{\varepsilon,p,n}_{t-} - Y^{\varepsilon,p}_{t}\right| + \int_0^1 \left|Z^{\varepsilon,p,n}_{s-} - Z^{\varepsilon,p}_{s}\right|^2 ds \to 0 \quad \textit{as } n\to+\infty \textit{ in probability}. \tag{4.18}$$

Proof. The proof will be done by induction on p. For the sake of clarity, we express everything in continuous time, so that equation (3.16) becomes for $t\in[0,1]$;

$$\begin{aligned}Y^{\varepsilon,p+1,n}_t &= \xi^n + \int_t^1 f(s-, Y^{\varepsilon,p,n}_{s-}, Z^{\varepsilon,p,n}_{s-})dc_n(s)\\ &\quad + \frac{1}{\varepsilon}\int_t^1 f(S-Y^{\varepsilon,p,n}_{s-})^+ dc_n(s) - \int_t^1 Z^{\varepsilon,p+1,n} dW^n_s,\end{aligned}$$

where $c_n(s)=\frac{[sn]}{n}$, and $Y^{\varepsilon,\cdot,n}$ denotes the càglàd process associated to $Y^{\varepsilon,\cdot,n}$. The assumption is that $(Y^{\varepsilon,p,n}, Z^{\varepsilon,p,n})$ converges to $(Y^{\varepsilon,p}, Z^{\varepsilon,p})$ in the sense of (4.18) so that our aim is to prove that $(Y^{\varepsilon,p+1,n}, Z^{\varepsilon,p+1,n})$ converges to $(Y^{\varepsilon,p+1}, Z^{\varepsilon,p+1})$ in the same sense.

Taking conditional expectations w.r.t. $\mathcal{G}_k$ in (3.16) and using the fact that $Y^{\varepsilon,p+1,n}_{t_k}$ is $\mathcal{G}_k$-measurable, we find that for $t_k \le \frac{[tn]}{n} < t_{k+1}$:

$$\begin{aligned}Y^{\varepsilon,p+1,n}_t &= \mathbb{E}\left[\xi^n + \int_t^1 f(s-, Y^{\varepsilon,p,n}_{s-}, Z^{\varepsilon,p,n}_{s-})dc_n(s)\middle|\mathcal{G}_k\right]\\ &\quad + \mathbb{E}\left[\frac{1}{\varepsilon}\int_t^1 (S-Y^{\varepsilon,p,n}_{s-})^+ dc_n(s)\middle|\mathcal{G}_k\right].\end{aligned}$$

So that

$$\begin{aligned}M^{\varepsilon,p+1,n}_t &:= Y^{\varepsilon,p+1,n}_t + \int_0^t f(s-, Y^{\varepsilon,p,n}_{s-}, Z^{\varepsilon,p,n}_{s-})dc_n(s) + \frac{1}{\varepsilon}\int_0^t (S-Y^{\varepsilon,p,n}_{s-})^+ dc_n(s)\\ &= \mathbb{E}\left[\xi^n + \int_0^1 f(s-, Y^{\varepsilon,p,n}_{s-}, Z^{\varepsilon,p,n}_{s-})dc_n(s)\middle|\mathcal{G}_k\right]\\ &\quad + \mathbb{E}\left[\frac{1}{\varepsilon}\int_0^1 (S-Y^{\varepsilon,p,n}_{s-})^+ dc_n(s)\middle|\mathcal{G}_k\right]\\ &= \mathbb{E}\left[M^{\varepsilon,p+1,n}_1 \mid \mathcal{G}_k\right] \quad \text{is a } \mathcal{G} \text{ martingale.}\end{aligned}$$

Moreover, we have the representation

$$\begin{aligned} M_t^{\varepsilon,p+1,n} &= \mathbb{E}\left[\int_0^1 Z_{s-}^{\varepsilon,p+1,n} dW_s^n, \bigg| \mathcal{G}_k\right] \\ &= \int_0^{t_k} Z_{s-}^{\varepsilon,p+1,n} dW_s^n \\ &= \int_0^{t} Z_{s-}^{\varepsilon,p+1,n} dW_s^n. \end{aligned}$$

Now, the idea is to apply Corollary 3.2 in [3] to the sequence of martingales $\{(M_t^{\varepsilon,p+1,n})_{0\le t\le 1}; n\in\mathbb{N}\}$. For this, we have to prove the L^1 ($\mathbb{P}$) convergence of $M_1^{\varepsilon,p+1,n}$.

Using the fact that $Y^{\varepsilon,p,n}$ and $Z^{\varepsilon,p,n}$ are piecewise constant, we have that

$$\begin{aligned} &\left| M_1^{\varepsilon,p+1,n} - \xi - \int_0^1 f(s, Y_s^{\varepsilon,p}, Z_s^{\varepsilon,p}) ds - \frac{1}{\varepsilon}\int_0^1 (S - Y_s^{\varepsilon,p})^+ ds \right| \\ &\le |\xi^n - \xi| + \int_0^1 \left| f(s, Y_s^{\varepsilon,p}, Z_s^{\varepsilon,p}) - f(s, Y_s^{\varepsilon,p,n}, Z_s^{\varepsilon,p,n}) \right| ds + \frac{1}{\varepsilon}\int_0^1 \left\{ \left| Y_s^{\varepsilon,p} - Y_s^{\varepsilon,p,n} \right| \right\} ds \end{aligned}$$

(where we have used triangular inequalities together with $|u^+ - v^+| \le |u - v|$).

Now using the fact that f is Lipschitz with constant K,

$$\begin{aligned} &\left| M_1^{\varepsilon,p+1,n} - \xi - \int_0^1 f(s, Y_s^{\varepsilon,p}, Z_s^{\varepsilon,p}) ds - \frac{1}{\varepsilon}\int_0^1 (S - Y_s^{\varepsilon,p})^+ ds \right| \\ &\le (1 + K + \frac{1}{\varepsilon}) \sup_{0\le s\le 1} \left| Y_s^{\varepsilon,p,n} - Y_s^{\varepsilon,n} \right| + K\int_0^1 \left| Z_s^{\varepsilon,p} - Z_s^{\varepsilon,p,n} \right| ds, \end{aligned}$$

and using the recurrence assumption, this last term tends to zero in probability and then in L^1 ($\mathbb{P}$) (using the L^2 ($\mathbb{P}$)-boundedness).

Applying Corollary 3.2 in [3], we see that $M^{\varepsilon,p+1,n}$ converges to

$$\begin{aligned} M_t^{\varepsilon,p+1} &:= \mathbb{E}\left[\xi + \int_0^1 f(s, Y_s^{\varepsilon,p}, Z_s^{\varepsilon,p}) ds \bigg| \mathcal{F}_t\right] + \mathbb{E}\left[\frac{1}{\varepsilon}\int_0^1 (S - Y_s^{\varepsilon,p})^+ ds \bigg| \mathcal{F}_t\right] \\ &= Y_t^{\varepsilon,p+1} + \int_0^t f(s, Y_s^{\varepsilon,p}, Z_s^{\varepsilon,p}) ds + \frac{1}{\varepsilon}\int_0^t (S - Y_s^{\varepsilon,p})^+ ds, \end{aligned}$$

in the sense that

$$\sup_{0\le t\le 1} \left| M_t^{\varepsilon,p+1,n} - M_t^{\varepsilon,p+1} \right| + \int_0^1 \left| Z_s^{\varepsilon,p+1,n} - Z_s^{\varepsilon,p+1} \right| \to 0 \quad \text{in probability.}$$

Since we want to prove

$$\sup_{0\le t\le 1} \left| Y_t^{\varepsilon,p+1,n} - Y_t^{\varepsilon,p+1} \right| + \int_0^1 \left| Z_s^{\varepsilon,p+1,n} - Z_s^{\varepsilon,p+1} \right| \to 0 \quad \text{in probability.}$$

The theorem will be demonstrated if we prove that

$$\sup_{0\le t\le 1}\left|\int_0^t f\left(s-, Y_{s-}^{\varepsilon,p,n}, Z_{s-}^{\varepsilon,p,n}\right) dc_n(s) - \int_0^t f\left(s, Y_s^{\varepsilon,p,n}, Z_s^{\varepsilon,p,n}\right) ds\right| \to 0 \quad \text{in probability} \tag{4.19}$$

and that

$$\sup_{0\le t\le 1}\left|\frac{1}{\varepsilon}\int_0^t (S - Y_{s-}^{\varepsilon,p,n})^+ dc_n(s) - \frac{1}{\varepsilon}\int_0^t (S - Y_s^{\varepsilon,p})^+ ds\right| \to 0 \quad \text{in probability.} \tag{4.20}$$

We begin to check (4.20). We have that

$$\begin{aligned}\sup_{0\le t\le 1}&\left|\frac{1}{\varepsilon}\int_0^t (S - Y_{s-}^{\varepsilon,p,n})^+ dc_n(s) - \frac{1}{\varepsilon}\int_0^t (S - Y_s^{\varepsilon,p})^+ ds\right| \\ &\le \frac{1}{\varepsilon}\left\{\int_0^1 \left|Y_s^{\varepsilon,p} - Y_s^{\varepsilon,p,n}\right| ds\right\} + \frac{1}{n\varepsilon}\left\{|S| + \sup_{0\le t\le 1}\left|Y_{c_n(t)}^{\varepsilon,p,n}\right|\right\}\end{aligned} \tag{4.21}$$

The first term in (4.21) tends to 0 as n tends to ∞ in probability because of (H) and the recurrence assumption.

For the second term in (4.21), we have:

$$\begin{aligned}\frac{1}{n\varepsilon}\left\{|S| + \sup_{0\le t\le 1}\left|Y_{c_n(t)}^{\varepsilon,p,n}\right|\right\} &\le \frac{1}{n\varepsilon}\left\{|S| + \sup_{0\le t\le 1}\left|Y_{c_n(t)}^{\varepsilon,p,n} - Y_{c_n(t)}^{\varepsilon,p}\right|\right\} \\ &\quad + \frac{1}{n\varepsilon}\left\{\sup_{0\le t\le 1}\left|Y_{c_n(t)}^{\varepsilon,p}\right|\right\}\end{aligned} \tag{4.22}$$

The first term on the right side of (4.22) tends to 0 with n in probability because of (H) and the recurrence assumption. The second term tends to zero $\mathbb{P}$-a.s. because $|S| + \sup_{0\le t\le 1}\left|Y_{c_n(t)}^{\varepsilon,p}\right|$ is an almost surely finite random variable.

We can apply the same method to get (4.19): thanks to Corollary 3.2 in [3], we may use the fact that

$$\sup_{0\le t\le 1}\left|\int_0^t Z_s^{\varepsilon,p,n} dc_n(s) - \int_0^t Z_s^{\varepsilon,p} ds\right| \to 0, \quad \text{as } n \to \infty \text{ in probability.} \tag{4.23}$$

□

5. A New Procedure: Ma and Zhang's Method

Let us introduce the following 2-step scheme given in [27] page 562 in the discrete case:

- $Y_1^n := \xi^n$.
- for $i = n, n-1, \dots 1$, and $t \in [t_{i-1}, t_i[$, let $\left(\widetilde{Y}^n, Z^n\right)$ be the solution of the BSDE:

$$\widetilde{Y}_t^n = Y_{t_i}^n + \int_t^{t_i} f(s, \widetilde{Y}_s^n, Z_s^n) ds - \int_t^{t_i} Z_s^n dW_s^n. \tag{5.24}$$

- for each i and $t \in]t_{i-1}, t_i]$, define $Y_t^n = \widetilde{Y}_t^n \vee S$
- let $K_0^n = 0$ and for $t \in]t_{i-1}, t_i]$, define $K_t^n \equiv K_{t_i}^n := \sum_{j=1}^i \left(Y_{t_{j-1}}^n - \widetilde{Y}_{t_{j-1}}^n\right)$.

Clearly, K^n is predictable. And observe also that we have

$$Y^n_{t_{i-1}} = Y^n_{t_i} + \int_{t_{i-1}}^{t_i} f\left(s, \widetilde{Y}^n_s, Z^n_s\right) ds - \int_{t_{i-1}}^{t_i} Z^n_s dW^n_s + K^n_{t_i} - K^n_{t_{i-1}}. \tag{5.25}$$

In the appendix we prove that $(\widetilde{Y}^n, Z^n)$ is contraction and then it converges towards (Y, Z) the unique solution of (1.1).

Now we define a modified Picard's iteration procedure for a penalization discrete BSDE. This is the numerical algorithm that we propose to approximate (Y, Z).

We define $\ddot{Y}^{\varepsilon,p+1,n}$ for $i = 0, \dots, n-1$, by

$$\begin{aligned} \ddot{Y}^{\varepsilon,p+1,n}_{t_i} &= \ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}} + \frac{1}{n} f\left(t_i, \ddot{Y}^{\varepsilon,p,n}_{t_i}, \ddot{Z}^{\varepsilon,p,n}_{t_i}\right) - \frac{1}{\sqrt{n}} \ddot{Z}^{\varepsilon,p+1,n}_{t_i} \zeta^n_{i+1} + \left(\ddot{K}^{\varepsilon,p+1,n}_{t_{i+1}} - \ddot{K}^{\varepsilon,p+1,n}_{t_i}\right) \\ \ddot{Y}^{\varepsilon,p,n}_1 &:= \xi^n \end{aligned} \tag{5.26}$$

where we have set for $0 \le i \le n$,

$$\ddot{K}^{\varepsilon,p+1,n}_{t_{i+1}} - \ddot{K}^{\varepsilon,p+1,n}_{t_i} := \frac{1}{n\varepsilon}\left(S - \ddot{Y}^{\varepsilon,p+1,n}_{t_i}\right)^+. \tag{5.27}$$

Remark 5. The main difference between this approximation and the Picard's iteration procedure is that instead of p we use $p+1$ in the last term of (5.26). This makes the method explicit. In fact, the equation to be solved has the form $y = a + b(S-y)^+$, where $a, b > 0$ are known. The solution is $y = a$ if $a \ge S$. When $a < S$ the solution is $y = \frac{a+bS}{1+b} < S$. We expect this method to have a better performance, because in the first method the Lipschitz constant, of the implicit equation to be solved in y, is large for small ϵ.

In this section $(\widetilde{Y}^{p,n}_{t_i}, Z^{p,n}_{t_i})$ denotes the the Picard iteration procedure for the couple of processes $(\widetilde{Y}^n_{t_i}, Z^n_{t_i})$ defined as the solution of the BSDE equation (5.24).

Theorem 2. *Assume* (A1)–(A3) *and* (H). *Then*

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\lim_{p\to+\infty} \mathbb{E}\left[\sup_{0\le i\le n}\left|\widetilde{Y}^n_{t_i} - \ddot{Y}^{\varepsilon,p,n}_{t_i}\right|^2 + \int_0^1 \left|Z^n_t - \ddot{Z}^{\varepsilon,p,n}_t\right|^2 dt\right] = 0. \tag{5.28}$$

The proof of this theorem follows from the convergence of $(\widetilde{Y}^{p,n}, Z^{p,n})$, on p, to the solution of the discrete solution (Y^n, Z^n) (given in the Appendix) and the next result.

Proposition 4. *Assume* (A1)–(A3) *and* (H). *Then, for all* $p \in \mathbb{N}$,

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty} \mathbb{E}\left[\sup_{0\le i\le n}\left\{\left|\widetilde{Y}^{p,n}_{t_i} - \ddot{Y}^{\varepsilon,p,n}_{t_i}\right|^2 + \frac{1}{n}\sum_{i=0}^n \left|Z^{p,n}_{t_i} - \ddot{Z}^{\varepsilon,p,n}_{t_i}\right|^2\right\}\right] = 0. \tag{5.29}$$

The proof of Proposition 4 relies on the following two lemmas that control the distance between S and $\widetilde{Y}^{p+1,n}$ or $\ddot{Y}^{\varepsilon,p+1,n}$.

Lemma 4. *For all $n \in \mathbb{N}$, $p \in \mathbb{N}$ and $0 \leq i \leq n$,*

$$|S - \widetilde{Y}_{t_i}^{p+1,n}| \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} \leq \frac{1}{n} |f(t_i, \widetilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n})| \leq \frac{1}{n} \|f\|_\infty. \tag{5.30}$$

Proof. We have that

$$\widetilde{Y}_{t_i}^{p+1,n} \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} = \left\{ Y_{t_{i+1}}^{p+1,n} + \frac{1}{n} f(t_i, \widetilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n}) - \frac{1}{\sqrt{n}} Z_{t_i}^{p+1,n} \zeta_{i+1}^n \right\} \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S}.$$

Conditioning w.r.t. $\mathscr{G}_i^n$, from this equality we get that:

$$(\widetilde{Y}_{t_i}^{p+1,n} - S) \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} = \left\{ \mathbb{E}\left[Y_{t_{i+1}}^{p+1,n} - S | \mathscr{G}_i^n \right] + \frac{1}{n} f(t_i, \widetilde{Y}_{t_i}^{p,n}, \tilde{Z}_{t_i}^{p,n}) \right\} \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S}.$$

Now, recall that by construction, we know that $(Y_{t_{i+1}}^{p+1,n} - S) \geq 0$ (and in particular $\mathbb{E}\left[Y_{t_{i+1}}^{p+1,n} - S | \mathscr{G}_i^n \right] \geq 0$).

We deduce that

$$\left\{ (\widetilde{Y}_{t_i}^{p+1,n} - S) - \frac{1}{n} f(t_i, \widetilde{Y}_{t_i}^{p,n}, \tilde{Z}_{t_i}^{p,n}) \right\} \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} \geq 0.$$

Obviously,

$$(\widetilde{Y}_{t_i}^{p+1,n} - S) \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} \leq 0,$$

In conclusion, we have that

$$\frac{1}{n} f(t_i, \widetilde{Y}_{t_i}^{p,n}, \tilde{Z}_{t_i}^{p,n}) \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} \leq (\widetilde{Y}_{t_i}^{p+1,n} - S) \mathbf{1}_{\widetilde{Y}_{t_i}^{p+1,n} \leq S} \leq 0,$$

which implies inequality (5.30). □

Lemma 5. *Suppose that* (H) *is satisfied. Then*

$$\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \mathbb{E}\left[\left(\sup_i (S - \ddot{Y}_{t_i}^{\varepsilon,p+1,n})^+ \right)^2 \right]^{1/2} = 0.$$

Proof. Consider the auxiliary process $(X^{\varepsilon,n}, \Gamma^{\varepsilon,n})$ solution of the discrete backward equation

$$X_{t_i}^{\varepsilon,n} = X_{t_{i+1}}^{\varepsilon,n} - \frac{1}{n} \|f\|_\infty - \frac{1}{\sqrt{n}} \Gamma_{t_i}^{\varepsilon,n} \zeta_{i+1}^n + \frac{1}{n\varepsilon} (S - X_{t_i}^{\varepsilon,n})^+$$

with $X_1^{\varepsilon,n} = \xi^n$. A backward induction on i shows that for all $\varepsilon > 0, p \geq 1, n \geq 1, 0 \leq i \leq n$ we have

$$X_{t_i}^{\varepsilon,n} \leq \ddot{Y}_{t_i}^{\varepsilon,p+1,n}.$$

In fact, this property holds for $i = n$. Assume that it holds for $i \geq k+1$ and we want to prove it for $i = k$. Recall that if $y = \mathbb{E}\big(\ddot{Y}^{\varepsilon,p+1,n}_{t_{k+1}} + \frac{1}{n} f\left(t_k, \ddot{Y}^{\varepsilon,p,n}_{t_k}, \ddot{Z}^{\varepsilon,p,n}_{t_k}\right) \big| \mathcal{G}_k\big)$ and $b = \frac{1}{n\varepsilon}$ then (see Remark 5)

$$\ddot{Y}^{\varepsilon,p+1,n}_{t_k} = \begin{cases} y & \text{if } y \geq S \\ \dfrac{y+bS}{1+b} & \text{if } y < S. \end{cases}$$

On the other hand, the same result gives for $x = \mathbb{E}\left(X^{\varepsilon,n}_{t_{k+1}} - \frac{1}{n}\|f\|_\infty \big| \mathcal{G}_k\right)$ that

$$X^{\varepsilon,n}_{t_k} = \begin{cases} x & \text{if } x \geq S \\ \dfrac{x+bS}{1+b} & \text{if } x < S. \end{cases}$$

The induction hypothesis gives $x \leq y$. Now, if $x \geq S$ clearly also $y \geq S$ and in this case $X^{\varepsilon,n}_{t_k} \leq \ddot{Y}^{\varepsilon,p+1,n}_{t_k}$. On the other hand, if $x < S$ and $y \geq S$ we have $X^{\varepsilon,n}_{t_k} \leq S \leq y = \ddot{Y}^{\varepsilon,p+1,n}_{t_k}$. Finally, if $x < S, y < S$ we get $X^{\varepsilon,n}_{t_k} = \frac{x+bS}{1+b} \leq \frac{y+bS}{1+b} = \ddot{Y}^{\varepsilon,p+1,n}_{t_k}$ and the claim is proven.

In particular, we have for all $\varepsilon > 0, p \geq 1, n \geq 1, 0 \leq i \leq n$

$$(S - \ddot{Y}^{\varepsilon,p,n}_{t_i})^+ \leq (S - X^{\varepsilon,n}_{t_i})^+$$

It is straightforward to prove, using induction on i, the inequality

$$(S - X^{\varepsilon,n}_{t_i})^+ \leq (1-t_i)\|f\|_\infty \leq \|f\|_\infty.$$

As n tends to infinity $(X^{\varepsilon,n})$ converges in the Skorohod topology to the unique solution of

$$X^\varepsilon_t = \xi + \int_t^1 \left(-\|f\|_\infty + \frac{1}{\varepsilon}(S - X^\varepsilon_s)^+\right) ds - \int_t^1 \Gamma^\varepsilon_s dW_s.$$

Again it is direct to prove that $(S - X^\varepsilon_t)^+ \leq (1-t)\|f\|_\infty$.

In particular, if ψ is any continuous bounded function such that $\psi(x) = x^2$ for $0 \leq x \leq \|f\|_\infty$ we obtain

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\left[\left(\sup_i (S - X^{\varepsilon,n}_{t_i})^+\right)^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[\psi\left(\sup_i (S - X^{\varepsilon,n}_{t_i})^+\right)\right] \\ &= \mathbb{E}\left[\psi\left(\sup_t (S - X^\varepsilon_t)^+\right)\right] \\ &= \mathbb{E}\left[\left(\sup_t (S - X^\varepsilon_t)^+\right)^2\right]. \end{aligned}$$

From Lemma 1 (which is Lemma 6.1, p. 723 in [10]) applied to $(X^\varepsilon)_\varepsilon$, we obtain the result. □

Proof of Proposition 4. The proof is based on an induction argument on the variable p. For any $\ell > 1$ consider $\beta = \ell^{1/n}$ and $p \in \mathbb{N}$, we introduce the following induction hypothesis:

$$\mathscr{H}_p^\ell : \left\{ \overline{\lim_{\epsilon\to 0}}\ \overline{\lim_{n\to+\infty}}\ \mathbb{E}\left[\sup_{0\le i\le n} \left\{ \beta^i \left| \widetilde{Y}_{t_i}^{p,n} - \ddot{Y}_{t_i}^{\varepsilon,p,n} \right|^2 + \frac{1}{n}\sum_{i=0}^{n} \beta^i \left| Z_{t_i}^{p,n} - \ddot{Z}_{t_i}^{\varepsilon,p,n} \right|^2 \right\} \right] = 0 \right\} \quad (5.31)$$

Note that $\mathscr{H}_0^\ell$ is satisfied for any choice of ℓ. Our choice of ℓ (independent of ε, p, n) will be done latter. For all ε, p, n we define

$$\begin{aligned} \Delta v_i^{\varepsilon,p+1,n} &:= \ddot{Y}_{t_i}^{\varepsilon,p+1,n} - \widetilde{Y}_{t_i}^{p+1,n} \\ &= U_{t_{i+1}}^{\varepsilon,p+1,n} + \frac{1}{n}\left\{ f\left(t_i, \ddot{Y}_{t_i}^{\varepsilon,p,n}, \ddot{Z}_{t_i}^{\varepsilon,p,n}\right) - f\left(t_i, \widetilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n}\right) \right\} \\ &\quad - \frac{1}{\sqrt{n}}\left(\ddot{Z}_{t_i}^{\varepsilon,p+1,n} - Z_{t_i}^{p+1,n}\right)\zeta_{i+1}^n, \end{aligned} \quad (5.32)$$

where: $U_{t_{i+1}}^{\varepsilon,p+1,n} = \ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - Y_{t_{i+1}}^{p+1,n} + \ddot{K}_{t_{i+1}}^{\varepsilon,p+1,n} - \ddot{K}_{t_i}^{\varepsilon,p+1,n}$. We also define

$$\Delta Z_i^{\varepsilon,p,n} := \ddot{Z}_{t_i}^{\varepsilon,p+1,n} - Z_{t_i}^{p+1,n}.$$

Since $\Delta v_n^{\varepsilon,p+1,n} = 0$, we have, for $k = 0, \ldots, n-1$, that

$$\begin{aligned} \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2 &= \sum_{i=k}^{n-1} \beta^i|\Delta v_i^{\varepsilon,p+1,n}|^2 - \beta^{i+1}|\Delta v_{i+1}^{\varepsilon,p+1,n}|^2 \\ &= (1-\beta)\sum_{i=k}^{n-1}\beta^i|\Delta v_i^{\varepsilon,p+1,n}|^2 + \beta\sum_{i=k}^{n-1}\beta^i\left(|\Delta v_i^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right), \end{aligned}$$

which implies

$$\begin{aligned} \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2 &= (1-\beta)\sum_{i=k}^{n-1}\beta^i|\Delta v_i^{\varepsilon,p+1,n}|^2 + \beta\sum_{i=k}^{n-1}\beta^i\left(|\Delta v_i^{\varepsilon,p+1,n}|^2 - |U_{t_{i+1}}^{\varepsilon,p+1,n}|^2\right) \\ &\quad + \beta\sum_{i=k}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right). \end{aligned}$$

We know that

$$(\Delta v_i^{\varepsilon,p+1,n})^2 - (U_{t_{i+1}}^{\varepsilon,p+1,n})^2 = 2\Delta v_i^{\varepsilon,p+1,n}\left(\Delta v_i^{\varepsilon,p+1,n} - U_{t_{i+1}}^{\varepsilon,p+1,n}\right) - \left(\Delta v_i^{\varepsilon,p+1,n} - U_{t_{i+1}}^{\varepsilon,p+1,n}\right)^2$$

Since f is Lipschitz in (y, z) with constant K, we have for each $\alpha > 0$,

$$\begin{aligned} &\frac{2}{n}\Delta v_i^{\varepsilon,p+1,n}\left\{ f\left(t_i, \ddot{Y}_{t_i}^{\varepsilon,p,n}, \ddot{Z}_{t_i}^{\varepsilon,p,n}\right) - f\left(t_i, \widetilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n}\right) \right\} \\ &\le \frac{2K}{n}|\Delta v_i^{\varepsilon,p+1,n}|\left(|\Delta v_i^{\varepsilon,p,n}| + |\Delta Z_i^{\varepsilon,p,n}|\right) \\ &\le \frac{4K^2}{n}|\Delta v_i^{\varepsilon,p+1,n}|^2 + \frac{1}{2n}\left\{|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right\}, \end{aligned}$$

and from (5.32)

$$\frac{1}{n}|\Delta Z_i^{\varepsilon,p+1,n}|^2 \leq 2\left(\Delta v_i^{\varepsilon,p+1,n} - U_{t_{i+1}}^{\varepsilon,p+1,n}\right)^2 + \frac{2K^2}{n^2}\left(|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right) \quad (5.33)$$

As a by product of this discussion, we deduce that

$$\begin{aligned} &2\sum_{i=k}^{n-1}\beta^i \Delta v_i^{\varepsilon,p+1,n}\left(\Delta v_i^{\varepsilon,p+1,n} - U_{t_{i+1}}^{\varepsilon,p+1,n}\right) \\ &\quad \leq \frac{4K^2}{n}\sum_{i=k}^{n-1}\beta^i|\Delta v_i^{\varepsilon,p+1,n}|^2 + \frac{1}{2n}\sum_{i=k}^{n-1}\beta^i\left\{|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right\} \\ &\qquad - \frac{2}{\sqrt{n}}\sum_{i=k}^{n-1}\beta^i \Delta v_i^{\varepsilon,p+1,n}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n, \end{aligned}$$

and

$$\begin{aligned} &-\sum_{i=k}^{n-1}\beta^i\left(\Delta v_i^{\varepsilon,p+1,n} - U_{t_{i+1}}^{\varepsilon,p+1,n}\right)^2 \\ &\quad \leq -\frac{1}{2n}\sum_{i=k}^{n-1}\beta^i|\Delta Z_i^{\varepsilon,p+1,n}|^2 + \frac{2K^2}{n^2}\sum_{i=k}^{n-1}\beta^i\left(|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right). \end{aligned}$$

Set $\rho := \left(\frac{2K^2}{n} + \frac{1}{2}\right)\frac{\beta}{n}$ and $A = 1 - \beta + \beta\frac{4K^2}{n}$, which is negative if ℓ is large enough. We have

$$\begin{aligned} &\beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2 + \frac{\beta}{2n}\sum_{i=k}^{n-1}\beta^i|\Delta Z_i^{\varepsilon,p+1,n}|^2 \\ &\quad \leq A\sum_{i=k}^{n-1}\beta^i|\Delta v_i^{\varepsilon,p+1,n}|^2 + \rho\sum_{i=k}^{n-1}\beta^i\left\{|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right\} \\ &\qquad + \beta\sum_{i=k}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right) - \frac{2\beta}{\sqrt{n}}\sum_{i=k}^{n-1}\beta^i\Delta v_i^{\varepsilon,p+1,n}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n. \quad (5.34) \end{aligned}$$

Our aim is now to study the term $\beta\sup_k\sum_{i=k}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right)$. We have the following technical lemma:

Lemma 6. *Suppose $\beta = \ell^{\frac{1}{n}}$, with $\ell > 1$, then*

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{n\to+\infty}}\ \mathbb{E}\left[\beta\sup_k\sum_{i=k}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right)\right] \leq 0. \quad (5.35)$$

Proof. We will distinguish four cases, where we denote $\Delta\ddot{K}_i^{\varepsilon,p+1,n} = \ddot{K}_{t_{i+1}}^{\varepsilon,p+1,n} - \ddot{K}_{t_i}^{\varepsilon,p+1,n} = \frac{1}{n\varepsilon}\left(S - \ddot{Y}_{t_i}^{\varepsilon,p+1,n}\right)^+$, which makes $\ddot{K}^{\varepsilon,p+1,n}$ predictable.

$$\begin{cases} \text{I} : \widetilde{Y}_{t_{i+1}}^{p+1,n} \geq S \text{ and } \ddot{Y}_{t_i}^{\varepsilon,p+1,n} \geq S; & U_{t_{i+1}}^{\varepsilon,p+1,n} = \ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - \widetilde{Y}_{t_{i+1}}^{p+1,n} = \Delta v_{i+1}^{\varepsilon,p+1,n} \\ \text{II} : \widetilde{Y}_{t_{i+1}}^{p+1,n} \geq S \text{ and } \ddot{Y}_{t_i}^{\varepsilon,p+1,n} < S; & U_{t_{i+1}}^{\varepsilon,p+1,n} = \Delta v_{i+1}^{\varepsilon,p+1,n} + \Delta\ddot{K}_i^{\varepsilon,p+1,n} \\ \text{III} : \widetilde{Y}_{t_{i+1}}^{p+1,n} < S \text{ and } \ddot{Y}_{t_i}^{\varepsilon,p+1,n} \geq S; & U_{t_{i+1}}^{\varepsilon,p+1,n} = \ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - S \leq \Delta v_{i+1}^{\varepsilon,p+1,n} \\ \text{IV} : \widetilde{Y}_{t_{i+1}}^{p+1,n} < S \text{ and } \ddot{Y}_{t_i}^{\varepsilon,p+1,n} < S; & U_{t_{i+1}}^{\varepsilon,p+1,n} = \ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - S + \Delta\ddot{K}_i^{\varepsilon,p+1,n} \end{cases}$$

Let us set

$$M := \sup_{\varepsilon\in]0,1[}\sup_{p\in\mathbb{N}}\sup_{n\in\mathbb{N}}\left[\mathbb{E}\left(\sup_{0\leq i\leq n}\left(|S|^2 + |\ddot{Y}_{t_i}^{\varepsilon,p+1,n}|^2 + |\widetilde{Y}_{t_i}^{p+1,n}|^2\right)\right)\right]^{1/2} < \infty.$$

Recalling the definition of $U_{t_{i+1}}^{\varepsilon,p+1,n}$, we see that

$$\begin{aligned} &\left\{|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right\} \\ &\leq |\Delta\ddot{K}_i^{\varepsilon,p+1,n}|^2\mathbf{1}_{\text{IV}\cup\text{II}} + \left\{\left(\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - S\right)^2 - \left(\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - \widetilde{Y}_{t_{i+1}}^{p+1,n}\right)^2\right\}\mathbf{1}_{\text{IV}\cup\text{III}} \\ &\quad + 2\left(\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - \widetilde{Y}_{t_{i+1}}^{p+1,n}\right)\Delta\ddot{K}_i^{\varepsilon,p+1,n}\mathbf{1}_{\text{IV}\cup\text{II}} \\ &= |\Delta\ddot{K}_i^{\varepsilon,p+1,n}|^2\mathbf{1}_{\text{IV}\cup\text{II}} + \left(S - \widetilde{Y}_{t_{i+1}}^{p+1,n}\right)\left(S + \widetilde{Y}_{t_{i+1}}^{p+1,n} - 2\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n}\right)\mathbf{1}_{\text{IV}\cup\text{III}} \\ &\quad + 2\left(\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} - \widetilde{Y}_{t_{i+1}}^{p+1,n}\right)\Delta\ddot{K}_i^{\varepsilon,p+1,n}\mathbf{1}_{\text{IV}\cup\text{II}} \end{aligned}$$

Convergence of the first and second terms: For the first term, by definition of $\Delta\ddot{K}_i^{\varepsilon,p+1,n}$, it is easily seen that

$$\mathbb{E}\left[\sup_k\sum_{i=k}^{n-1}\beta^i|\Delta\ddot{K}_i^{\varepsilon,p+1,n}|^2\right] = \mathbb{E}\left[\sum_{i=0}^{n-1}\beta^i|\Delta\ddot{K}_i^{\varepsilon,p+1,n}|^2\right] \leq \frac{2M^2}{n\varepsilon^2}\frac{1}{n}\frac{\beta^n - 1}{\beta - 1} \tag{5.36}$$

Since $\beta = l^{1/n}$ we obtain for every fixed $\varepsilon > 0$

$$\lim_{n\to+\infty}\mathbb{E}\left[\sup_k\sum_{i=k}^{n-1}\beta^i|\Delta\ddot{K}_i^{\varepsilon,p+1,n}|^2\right] = 0.$$

For the second term, note that if $\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} \geq S$, then

$$\left(S - \widetilde{Y}^{p+1,n}\right)\left(S + \widetilde{Y}_{t_{i+1}}^{p+1,n} - 2\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n}\right)\mathbf{1}_{\text{IV}\cup\text{III}} \leq 0,$$

so we just need to consider the case $\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n} \leq S$, getting

$$\left(S - \widetilde{Y}_{t_{i+1}}^{p+1,n}\right)\left(S + \widetilde{Y}_{t_{i+1}}^{p+1,n} - 2\ddot{Y}_{t_{i+1}}^{\varepsilon,p+1,n}\right)\mathbf{1}_{\text{IV}\cup\text{III}}$$

$$\leq 2\left(S-\widetilde{Y}^{p+1,n}_{t_{i+1}}\right)\left(S-\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}}\right)^{+}\mathbf{1}_{\mathrm{IV}\cup\mathrm{III}},$$

which includes both cases. We get using Lemma 4

$$\begin{aligned}
&\mathbb{E}\left[\sup_k\sum_{i=k}^{n-1}\beta^i\left(S-\widetilde{Y}^{p+1,n}_{t_{i+1}}\right)\left(S-\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}}\right)^{+}\mathbf{1}_{\mathrm{IV}\cup\mathrm{III}}\right]\\
&\quad=\mathbb{E}\left[\sum_{i=0}^{n-1}\beta^i\left(S-\widetilde{Y}^{p+1,n}_{t_{i+1}}\right)\left(S-\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}}\right)^{+}\mathbf{1}_{\mathrm{IV}\cup\mathrm{III}}\right]\\
&\quad\leq\|f\|_\infty\frac{\beta^n-1}{n(\beta-1)}\left[\mathbb{E}\left(\sup_{0\leq i\leq n-1}(S-\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}})^{+}\right)^2\right]^{1/2},
\end{aligned}$$

which converges to zero by Lemma 5 by letting $n\to\infty$, and then $\varepsilon\to 0$.

Convergence of last term: First, notice that $\mathrm{IV}\cup\mathrm{II}=S>\ddot{Y}^{\varepsilon,p+1,n}_{t_i}$. From the equation satisfied by $\ddot{Y}^{\varepsilon,p+1,n}$ we have that

$$\begin{aligned}
\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}}-\widetilde{Y}^{p+1,n}_{t_{i+1}}&=\ddot{Y}^{\varepsilon,p+1,n}_{t_i}-\widetilde{Y}^{p+1,n}_{t_{i+1}}\\
&\quad-\frac{1}{n}f(t_i,\ddot{Y}^{\varepsilon,p,n}_{t_i},\ddot{Z}^{\varepsilon,p,n}_{t_i})+\frac{1}{\sqrt{n}}\ddot{Z}^{\varepsilon,p+1,n}_{t_i}\zeta^n_{i+1}-\Delta\ddot{K}^{\varepsilon,p+1,n}_i,
\end{aligned}$$

which implies

$$\begin{aligned}
&(\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}}-\widetilde{Y}^{p+1,n}_{t_{i+1}})\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}}\\
&\quad\leq(S-\widetilde{Y}^{p+1,n}_{t_{i+1}})\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}}-\frac{1}{n}\|f\|_\infty+\frac{1}{\sqrt{n}}\ddot{Z}^{\varepsilon,p+1,n}_{t_i}\zeta^n_{i+1}\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}}\\
&\quad\leq(S-\widetilde{Y}^{p+1,n}_{t_{i+1}})\mathbf{1}_{\widetilde{Y}^{p+1,n}_{t_{i+1}}<S}+\frac{1}{n}\|f\|_\infty+\frac{1}{\sqrt{n}}\ddot{Z}^{\varepsilon,p+1,n}_{t_i}\zeta^n_{i+1}\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}}\\
&\quad\leq|S-\widetilde{Y}^{p+1,n}_{t_{i+1}}|\mathbf{1}_{\widetilde{Y}^{p+1,n}_{t_{i+1}}<S}+\frac{1}{n}\|f\|_\infty+\frac{1}{\sqrt{n}}\ddot{Z}^{\varepsilon,p+1,n}_{t_i}\zeta^n_{i+1}\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}}\\
&\quad\leq\frac{2}{n}\|f\|_\infty+\frac{1}{\sqrt{n}}\ddot{Z}^{\varepsilon,p+1,n}_{t_i}\zeta^n_{i+1}\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}},
\end{aligned}$$

where the last inequality follows from Lemma 4. Then

$$\begin{aligned}
&\sum_{i=k}^{n}\beta^i\left(\ddot{Y}^{\varepsilon,p+1,n}_{t_{i+1}}-\widetilde{Y}^{p+1,n}_{t_{i+1}}\right)\Delta\ddot{K}^{\varepsilon,p+1,n}_i\mathbf{1}_{\mathrm{IV}\cup\mathrm{II}}\\
&\quad\leq\frac{2\|f\|_\infty}{n}\sum_{i=k}^{n}\beta^i\Delta\ddot{K}^{\varepsilon,p+1,n}_i+\sum_{i=k}^{n}\beta^i\frac{1}{\sqrt{n}}\ddot{Z}^{\varepsilon,p+1,n}_{t_i}\Delta\ddot{K}^{\varepsilon,p+1,n}_i\mathbf{1}_{S>\ddot{Y}^{\varepsilon,p+1,n}_{t_i}}\zeta^n_{i+1}.
\end{aligned}$$

Using (5.36) we control the first term in this inequality. So we need to control

$$\mathbb{E}\left[\sup_k|\Theta_n-\Theta_k|\right],$$

where $\Theta_k = \sum_{i=0}^{k} \beta^i \frac{1}{\sqrt{n}} \ddot{Z}_{t_i}^{\varepsilon,p+1,n} \Delta\ddot{K}_i^{\varepsilon,p+1,n} \mathbf{1}_{S>\ddot{Y}_{t_i}^{\varepsilon,p+1,n}} \zeta_{i+1}^n$. Using BDG inequalities we obtain that

$$\begin{aligned}
\mathbb{E}\left[\sup_k |\Theta_n - \Theta_k|\right] &\leq \mathbb{E}\left[\sup_k |\Theta_n| + |\Theta_k|\right] \leq 2\mathbb{E}\left[\sup_k |\Theta_k|\right] \\
&\leq C\mathbb{E}\left[\left(\sum_{i=0}^{n} \frac{\beta^{2i}}{n}\left[\ddot{Z}_{t_i}^{\varepsilon,p+1,n}\Delta\ddot{K}_i^{\varepsilon,p+1,n}\right]^2\right)^{1/2}\right] \\
&\leq C\ell\left(\mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n}[\ddot{Z}_{t_i}^{\varepsilon,p+1,n}]^2\right]\right)^{1/2}\left(\mathbb{E}\left[\sup_k \frac{1}{n\varepsilon}(S - \ddot{Y}_{t_k}^{\varepsilon,p+1,n})^+\right]^2\right)^{1/2}
\end{aligned}$$

In the last inequality, the first term is uniformly bounded and the second converges to 0 according to Lemma 5. □

We now prove the induction step. Let $k = 0$ in (5.34), and taking expectation we obtain (recall that $A < 0$)

$$\begin{aligned}
\mathbb{E}\left(\frac{\beta}{2n}\sum_{i=0}^{n-1}\beta^i|\Delta Z_i^{\varepsilon,p+1,n}|^2\right) &\leq \left(\frac{2K^2}{n}+\frac{1}{2}\right)\frac{\beta}{n}\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left\{|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right\}\right) \\
&\quad + \beta\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right)\right) \\
&\leq \left(\frac{2K^2}{n}+\frac{1}{2}\right)\beta\left(\mathbb{E}\left(\sup_k \beta^k|\Delta v_k^{\varepsilon,p,n}|^2 + \frac{1}{n}\sum_i|\Delta Z_i^{\varepsilon,p,n}|^2\right)\right) \\
&\quad + \beta\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right)\right)
\end{aligned}$$

This inequality, Lemma 6 and the induction hypothesis $\mathscr{H}_p^\ell$ show that

$$\overline{\lim_{\varepsilon\to 0}}\ \overline{\lim_{n\to\infty}}\ \frac{1}{n}\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i|\Delta Z_i^{\varepsilon,p+1,n}|^2\right) = 0. \tag{5.37}$$

In the inequality (5.34), we take supremum and expectation to get

$$\begin{aligned}
&\mathbb{E}\left(\sup_k \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2 + \frac{\beta}{2n}\sum_{i=0}^{n-1}\beta^i|\Delta Z_i^{\varepsilon,p+1,n}|^2\right) \\
&\leq \left(\frac{2K^2}{n}+\frac{1}{2}\right)\frac{\beta}{n}\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left\{|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right\}\right) \\
&\quad + \beta\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right)\right) \\
&\quad + \frac{2\beta}{\sqrt{n}}\mathbb{E}\left(\sup_k \sum_{i=k}^{n-1}\beta^i\Delta v_i^{\varepsilon,p+1,n}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n\right).
\end{aligned} \tag{5.38}$$

Let us study the last term in the previous inequality. We use again BDG inequality to get the upper bound

$$\begin{aligned}
&\frac{2\beta}{\sqrt{n}}\mathbb{E}\left(\sup_k \sum_{i=k}^{n-1}\beta^i \Delta v_i^{\varepsilon,p+1,n}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n\right)\\
&\le C\frac{2\beta}{\sqrt{n}}\mathbb{E}\left(\left(\sum_{i=0}^{n-1}\beta^{2i}[\Delta v_i^{\varepsilon,p+1,n}\Delta Z_i^{\varepsilon,p+1,n}]^2\right)^{\frac{1}{2}}\right)\\
&\le 2C\beta\mathbb{E}\left(\sup_k \beta^{k/2}|\Delta v_k^{\varepsilon,p+1,n}|\left(\frac{1}{n}\sum_{i=0}^{n-1}\beta^i[\Delta Z_i^{\varepsilon,p+1,n}]^2\right)^{\frac{1}{2}}\right)\\
&\le 2C\beta\left(\mathbb{E}\left(\sup_k \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2\right)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\frac{1}{n}\sum_{i=0}^{n-1}\beta^i[\Delta Z_i^{\varepsilon,p+1,n}]^2\right)\right)^{\frac{1}{2}}
\end{aligned}$$

We use the inequality $2xy \le \mu x^2 + \frac{1}{\mu}y^2$, for any $\mu > 0$, which gives

$$\begin{aligned}
&\frac{2\beta}{\sqrt{n}}\mathbb{E}\left(\sup_k \sum_{i=k}^{n-1}\beta^i \Delta v_i^{\varepsilon,p+1,n}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n\right)\\
&\quad\le C\beta\mu\mathbb{E}\left(\sup_k \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2\right) + \frac{C\beta}{\mu}\mathbb{E}\left(\frac{\beta}{2n}\sum_{i=0}^{n-1}\beta^i[\Delta Z_i^{\varepsilon,p+1,n}]^2\right)
\end{aligned}$$

Choose, now, a small $\mu > 0$ such that $C\beta\mu \le \frac{1}{2}$, which gives in 5.38

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left(\sup_k \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2\right) \le{}& \left(\frac{2K^2}{n}+\frac{1}{2}\right)\frac{\beta}{n}\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left\{|\Delta v_i^{\varepsilon,p,n}|^2 + |\Delta Z_i^{\varepsilon,p,n}|^2\right\}\right)\\
&+\beta\mathbb{E}\left(\sum_{i=0}^{n-1}\beta^i\left(|U_{t_{i+1}}^{\varepsilon,p+1,n}|^2 - |\Delta v_{i+1}^{\varepsilon,p+1,n}|^2\right)\right)\\
&+\frac{C\beta}{\mu}\mathbb{E}\left(\frac{\beta}{2n}\sum_{i=0}^{n-1}\beta^i[\Delta Z_i^{\varepsilon,p+1,n}]^2\right)
\end{aligned}$$

Again Lemma 6, the induction hypothesis $\mathscr{H}_p^\ell$ and 5.37 show that

$$\overline{\lim_{\varepsilon\to 0}}\,\overline{\lim_{n\to\infty}}\,\mathbb{E}\left(\sup_k \beta^k|\Delta v_k^{\varepsilon,p+1,n}|^2\right) = 0.$$

which finally shows $\mathscr{H}_{p+1}^\ell$.

The only thing left to show is how to choose ℓ. This is done by imposing that

$$A = \left(1 - \ell^{\frac{1}{n}} + \frac{4\ell^{\frac{1}{n}}K^2}{n}\right) \le 0$$

Set $M^* := 4K^2$ and let us take n large enough such that $\frac{4K^2}{n} < 1$. With such a choice we need

$$\ell \geq \exp\left(-n \ln\left(1 - \frac{M^*}{n}\right)\right)$$

Since $\exp\left(-n \ln\left(1 - \frac{M^*}{n}\right)\right)$ tends to $\exp(M^*)$ as n tends to ∞, we may choose $\ell := \exp(1 + M^*)$. Now note that our choice of ℓ does not depend on ε, p, n and the theorem is proved. □

6. Notations

1. (Y, Z, K) denotes the unique solution of a RBSDE with coefficients (ξ, f, W). Equation (1.1).
2. $(Y^\varepsilon, Z^\varepsilon)$ denotes the penalization of a RBSDE, the unique solution of a BSDE with coefficients (ξ, f, W). Equation (3.3).
3. $(Y^{\varepsilon,p}, Z^{\varepsilon,p})$ Picard iteration for a Penalization BSDE $(Y^\varepsilon, Z^\varepsilon)$. Equation (3.8).
4. $(Y^{\infty,n}, Z^{\infty,n}, K^{\infty,n})$ denotes the unique solution of a discrete RBSDE with coefficients (ξ^n, f, W^n). Equation (3.11).
5. $(Y^{\varepsilon,\infty,n}, Z^{\varepsilon,\infty,n})$ denotes the penalization of the unique solution of a discrete BSDE with coeffcients (ξ^n, f, W^n) Equation (3.12).
6. $(Y^{\varepsilon,p,n}, Z^{\varepsilon,p,n})$ denotes the Picard iteration procedure for a penalization discrete BSDE $(Y^{\varepsilon,\infty,n}, Z^{\varepsilon,\infty,n})$. Equation (3.16).
7. $(Y^n, \widetilde{Y}^n, Z^n, K^n)$ denotes the two step scheme given by Ma and Zhang in [27], in the contex of the random walk. See Equations (5.24)–(5.25).
8. $(\widetilde{Y}^{p,n}, Z^{p,n})$ is the Picard iteration for the process $(\widetilde{Y}^n, Z^n)$ given in formula (5.24).
9. $(\ddot{Y}^{\varepsilon,p+1,n}_{t_i}, \ddot{Z}^{\varepsilon,p+1,n}_{t_i}, \ddot{K}^{\varepsilon,p+1,n}_{t_i})$ denotes the Modified Picard's iteration procedure for a penalization discrete BSDE $(Y^{\varepsilon,\infty,n}, Z^{\varepsilon,\infty,n})$, Equation (3.16). See (5.26).

7. Appendix

In this Appendix, we prove the convergence for the Picard method associated to the Ma and Zhang procedure, in the discrete setting. This is the aim on the next lemma.

Lemma 7. *Let K the lipschitz constant of f and $0 < \gamma < 1$ any fixed number. Take A large enough such that $32K^2/A \leq \gamma < 1$ and $n_0 = n_0(A) > 4A$ such that for all $t \in [0, 1]$ and all $n \geq n_0$*

$$\frac{1}{2} e^{-4At} \leq \left(1 - \frac{4A}{n}\right)^{nt}.$$

Then, for all $n \geq n_0$ the Picard iteration satisfies for all $p \geq 2$ the contraction property

$$|||(\widetilde{Y}^{p+1,n}, Z^{p+1,n}) - (\widetilde{Y}^{p,n}, Z^{p,n})|||^2 \leq \gamma |||(\widetilde{Y}^{p,n}, Z^{p,n}) - (\widetilde{Y}^{p-1,n}, Z^{p-1,n})|||^2,$$

where the norm $|||.|||$ is defined by

$$|||\,(Y, Z)\,||| := \left\{\frac{1}{n} \sum_{0 \leq t_i \leq 1} e^{4At_i} \mathbb{E}\left[|Y_{t_i}|^2 + |Z_{t_i}|^2\right]\right\}^{1/2}, \tag{7.39}$$

Proof. Let us recall that for $i \leq n-1$ we have

$$\widetilde{Y}_{t_i}^{p+1,n} = \widetilde{Y}_{t_{i+1}}^{p+1,n} \vee S + \frac{1}{n} f\left(t_i, \widetilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n}\right) - \frac{1}{\sqrt{n}} Z_{t_i}^{p+1,n} \zeta_{i+1}^n, \tag{7.40}$$

where $\widetilde{Y}_1^{p,n} = \xi^n$.

We denote $\Delta y_i^{p+1} := \widetilde{Y}_{t_i}^{p+1,n} - \widetilde{Y}_{t_i}^{p,n}$, $\Delta z_i^{p+1} := Z_{t_i}^{p+1,n} - Z_{t_i}^{p,n}$, $\Delta s_{i+1}^{p+1} := \widetilde{Y}_{t_{i+1}}^{p+1,n} \vee S - \widetilde{Y}_{t_{i+1}}^{p,n} \vee S$ and $\Delta f_i^p := f(t_i, \widetilde{Y}_{t_i}^{p,n}, Z_{t_i}^{p,n}) - f(t_i, \widetilde{Y}_{t_i}^{p-1,n}, Z_{t_i}^{p-1,n})$.

From Equation (7.40), we have that:

$$\Delta s_{i+1}^{p+1} = \Delta y_i^{p+1} - \frac{1}{n}\Delta f_i^p + \frac{1}{\sqrt{n}}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n.$$

from which we deduce that

$$\begin{aligned}|\Delta s_{i+1}^{p+1}|^2 &= |\Delta y_i^{p+1}|^2 + \frac{1}{n^2}|\Delta f_i^p|^2 + \frac{1}{n}|\Delta Z_i^{\varepsilon,p+1,n}|^2 \\ &\quad - \frac{2}{n}\Delta y_i^{p+1}\Delta f_i^p + \frac{2}{\sqrt{n}}\Delta y_i^{p+1}\Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n - \frac{2}{n^{3/2}}\Delta f_i^p \Delta Z_i^{\varepsilon,p+1,n}\zeta_{i+1}^n.\end{aligned}$$

Taking expectations in the previous equality and using the inequality $|y_1 \vee S - y_2 \vee S|^2 \leq |y_1 - y_2|^2$, yields

$$\begin{aligned}\mathbb{E}\left[|\Delta y_i^{p+1}|^2 + \frac{1}{n}|\Delta Z_i^{\varepsilon,p+1,n}|^2\right] &= \mathbb{E}\left[|\Delta s_{i+1}^{p+1}|^2\right] - \frac{1}{n^2}\mathbb{E}\left[|\Delta f_i^p|^2\right] + \frac{2}{n}\mathbb{E}\left[\Delta y_i^{p+1}\Delta f_i^p\right] \\ &\leq \mathbb{E}\left[|\Delta y_{i+1}^{p+1}|^2\right] + \frac{2}{n}\mathbb{E}\left[\Delta y_i^{p+1}\Delta f_i^p\right] \\ &\leq \mathbb{E}\left[|\Delta y_{i+1}^{p+1}|^2\right] + \frac{4A}{n}\mathbb{E}\left[|\Delta y_i^{p+1}|^2\right] + \frac{4}{An}\mathbb{E}\left[|\Delta f_i^p|^2\right].\end{aligned} \tag{7.41}$$

Here, we have used the inequality $ab \leq 2(Aa + b/A)$. In particular, we obtain

$$\theta\mathbb{E}\left[|\Delta y_i^{p+1}|^2\right] \leq \left\{\mathbb{E}\left[|\Delta y_{i+1}^{p+1}|^2\right] + \frac{4}{An}\mathbb{E}\left[|\Delta f_i^p|^2\right]\right\},$$

where $\theta = \left(1 - \frac{4A}{n}\right)$. Iterating this inequality, we get for all i

$$\theta^{n-i}\mathbb{E}\left[|\Delta y_i^{p+1}|^2\right] \leq \frac{4}{An}\sum_{j=i}^{n-1}\theta^{n-j-1}\mathbb{E}\left[|\Delta f_j^p|^2\right]. \tag{7.42}$$

Summing up this inequalities we obtain

$$\begin{aligned}\sum_{i=0}^{n}\theta^{-i}\mathbb{E}\left[|\Delta y_i^{p+1}|^2\right] &\leq \frac{4}{An}\sum_{i=0}^{n}\sum_{j=i}^{n-1}\theta^{-j-1}\mathbb{E}\left[|\Delta f_j^p|^2\right] \\ &= \frac{4}{A\theta n}\sum_{j=0}^{n-1}\theta^{-j}\mathbb{E}\left[|\Delta f_j^p|^2\right]\sum_{i\leq j}1\end{aligned}$$

$$\leq \frac{4}{A\theta}\sum_{j=0}^{n-1}\theta^{-j}\mathbb{E}\left[|\Delta f_j^p|^2\right]$$

$$\leq \frac{8k^2}{A\theta}\sum_{j=0}^{n-1}\theta^{-j}\mathbb{E}\left[|\Delta y_j^p|^2+|\Delta z_j^p|^2\right].$$

Again from (7.41), we deduce that

$$\frac{\theta^{-i}}{n}\mathbb{E}\left[|\Delta Z_i^{\varepsilon,p+1,n}|^2\right]\leq \theta^{-i}\mathbb{E}\left[|\Delta y_{i+1}^{p+1}|^2\right]+\frac{4}{An}\theta^{-i}\mathbb{E}\left[|\Delta f_i^p|^2\right].$$

If we use 7.42, for $i+1$, we have

$$\theta^{-i}\mathbb{E}\left[|\Delta y_{i+1}^{p+1}|^2\right]\leq \frac{4}{An}\sum_{j=i+1}^{n-1}\theta^{-j}\mathbb{E}\left[|\Delta f_j^p|^2\right],$$

and then

$$\theta^{-i}\mathbb{E}\left[|\Delta Z_i^{\varepsilon,p+1,n}|^2\right]\leq \frac{4}{An}\sum_{j=i}^{n-1}\theta^{-j}\mathbb{E}\left[|\Delta f_j^p|^2\right].$$

Once more summing up we conclude

$$\sum_{i=0}^{n}\theta^{-i}\mathbb{E}\left[|\Delta Z_i^{\varepsilon,p+1,n}|^2\right]\leq \frac{8K^2}{A}\sum_{i=0}^{n}\theta^{-i}\mathbb{E}\left[|\Delta y_i^p|^2+|\Delta z_i^p|^2\right],$$

and therefore

$$\sum_{i=0}^{n}\theta^{-i}\mathbb{E}\left[|\Delta y_i^{p+1}|^2+|\Delta Z_i^{\varepsilon,p+1,n}|^2\right]\leq \frac{16K^2}{A}\sum_{i=0}^{n}\theta^{-i}\mathbb{E}\left[|\Delta y_i^p|^2+|\Delta z_i^p|^2\right],$$

Notice that $\theta^i=\left(1-\frac{4A}{n}\right)^i=\left(1-\frac{4A}{n}\right)^{nt_i}$. Since the sequence of functions $f_n(t)=\left(1-\frac{4A}{n}\right)^{nt}$ converges increasingly to e^{-4At} we obtain that the convergence is uniform. This implies that for all $n\geq n_0$, which depends only on A and all $t\in[0,1]$ we have

$$\frac{1}{2}e^{-4At}\leq\left(1-\frac{4A}{n}\right)^{nt}\leq e^{-4At}.$$

This implies that

$$\sum_{i=0}^{n}e^{(4At_i)}\mathbb{E}\left[|\Delta y_i^{p+1}|^2+|\Delta Z_i^{\varepsilon,p+1,n}|^2\right]\leq \frac{32K^2}{A}\sum_{i=0}^{n}e^{(4At_i)}\mathbb{E}\left[|\Delta y_i^p|^2+|\Delta z_i^p|^2\right]$$

and the result is proved. □

Corollary 1. *Given $\gamma < 1$, there exists $n_0 = n_0(\gamma)$ and a finite constant $D = D(\gamma) < \infty$ such that for all $p \geq 1$ and $n \geq n_0$*

$$\mathbb{E}\left(\frac{1}{n}\sum_{t_i\geq 0}|\widetilde{Y}^{p,n}_{t_i} - \widetilde{Y}^{n}_{t_i}|^2 + \frac{1}{n}\sum_{t_i\geq 0}|Z^{p,n}_{t_i} - Z^{n}_{t_i}|^2\right) \leq D\gamma^p.$$

Proof. From the previous Lemma it follows that

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{n}\sum_{t_i\geq 0}|\widetilde{Y}^{p,n}_{t_i} - \widetilde{Y}^{n}_{t_i}|^2 + \frac{1}{n}\sum_{t_i\geq 0}|Z^{p,n}_{t_i} - Z^{n}_{t_i}|^2\right)\\ &\quad\leq \frac{\gamma^{p-1}}{(1-\gamma^{1/2})^2}|||(\widetilde{Y}^{2,n}, Z^{2,n}) - (\widetilde{Y}^{1,n}, Z^{1,n})|||^2.\end{aligned}$$

On the other hand

$$\begin{aligned}&|||(\widetilde{Y}^{2,n}, Z^{2,n}) - (\widetilde{Y}^{1,n}, Z^{1,n})|||^2\\ &\quad\leq e^{4A}\mathbb{E}\left[\frac{1}{n}\sum_{0\leq t_i\leq 1}|\widetilde{Y}^{2,n}_{t_i} - \widetilde{Y}^{1,n}_{t_i}|^2 + |Z^{2,n}_{t_i} - Z^{1,n}_{t_i}|^2\right],\end{aligned}$$

which is uniformly bounded in n using standard a priori estimates. □

Corollary 2. *Given $\gamma < 1$, there exists $n_0 = n_0(\gamma)$ and a finite constant $D = D(\gamma) < \infty$ such that for all $p \geq 1$ and $n \geq n_0$*

$$\mathbb{E}\left(\sup_{t_i\geq 0}|\widetilde{Y}^{p,n}_{t_i} - \widetilde{Y}^{n}_{t_i}|^2 + \frac{1}{n}\sum_{t_i\geq 0}|Z^{p,n}_{t_i} - Z^{n}_{t_i}|^2\right) \leq D\gamma^p.$$

Proof. Using the recurrence (7.40), we get

$$\begin{aligned}\widetilde{Y}^{p+1,n}_{t_i} - \widetilde{Y}^{p,n}_{t_i} &= \widetilde{Y}^{p+1,n}_{t_{i+1}} \vee S - \widetilde{Y}^{p,n}_{t_{i+1}} \vee S + \frac{1}{n}[f(t_i, \widetilde{Y}^{p,n}_{t_i}, Z^{p,n}_{t_i})\\ &\quad - f(t_i, \widetilde{Y}^{p-1,n}_{t_i}, Z^{p-1,n}_{t_i})] + \frac{1}{\sqrt{n}}(Z^{p+1,n}_{t_i} - Z^{p,n}_{t_i})\zeta^n_{i+1}.\end{aligned}$$

Moreover, it holds

$$|\Delta y^{p+1}_i| = \left|\mathbb{E}\left(\Delta s^{p+1}_{i+1} + \frac{1}{n}\Delta f^p_i \middle| \mathcal{G}^n_i\right)\right| \leq \mathbb{E}\left(|\Delta y^{p+1}_{i+1}| + \frac{1}{n}|\Delta f^p_i| \middle| \mathcal{G}^n_i\right).$$

Iterating this inequality, we get

$$|\Delta y^{p+1}_i| \leq \mathbb{E}\left(\frac{1}{n}\sum_{j=0}^{n}|\Delta f^p_j| \middle| \mathcal{G}^n_i\right),$$

and from Doob's inequality, we get finally

$$\mathbb{E}(\sup_i |\Delta y_i^{p+1}|^2) \leq 4\mathbb{E}\left(\frac{1}{n}\sum_{j=0}^{n}|\Delta f_j^p|^2\right).$$

The result follows from the previous corollary. □

References

1. Bahlali, K., Hassani, M., Mansouri, B., and Mrhardy, N. 2009. One barrier rejected backward doubly stochastic differential equations with continuous generator. *C.R. Math. Acad. Sci. Paris* 347:1201–1206.
2. Bouchard, B., and Touzi, N. 2004. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Process. Appl.* 111:175–206.
3. Briand, P., Delyon, B., and Mémin, J. 2001. Donsker–Type Theorem for BSDEs. *Electronic Communications in Probability* 6:1–14.
4. Briand, P., Delyon, B., and Mémin, J. 2002. On the Robustness of backward stochastic differ ential equations. *Stochastic Process. Appl.* 97:229–253.
5. Chevance, D. 1997. *Numerical Methods for Backward Stochastic Differential Equations*. Numerical Methods in Finance, 232–244. Cambridge University Press, Cambridge, UK.
6. Coquet, F., Mémin, J., and Slomiński, L. 2001. *On Weak Convergence of Filtrations. Lecture Notes in Mathematics 1755*. Springer Verlag, New York.
7. Cvitanic, J., and Karatzas, I. 1996. Backward stochastic differential equations with rejections and dynkin games. *Ann. Prob.* 24:2024–2056.
8. Douglas,, J., Ma, J., and Protter, P. 1996. *Numerical methods for forward-backward stochastic differential equations. Ann. Appl. Probab.* 6:940–968.
9. Fan, L., and Ma, N. 2008. Analysis of rejected backward stochastic differential equation by an apriori method. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* 15(Suppl. 1):64–68.
10. El Karoui, N., Kapoudjian, C., Pardoux, E., and Quenez, M.C. 1997. Rejected solutions of backward sdes, and related obstacle problems for pdes. *Ann. Probab.* 25:702–737.
11. El Karoui, N., and Quenez, M. C. 1997. *Imperfect Markets and Backward Stochastic Differential Equation. Numerical Methods in Finance. 181214.* Cam bridge University Press, Cambridge, UK.
12. Gobet, E., Lemor, J.-P., and Warin, X. 2005. A regression-based Monte Carlo method to solve Backward Stochastic Differential equations. *Ann. Appl. Probab.* 15:2172–2202.
13. Huang, Z., Lepeltier, J.-P., and Wu, Z. 2010. Rejected forward-backward stochastic differential equations with continuous monotone coeffients. *Statist. Probab. Lett.* 80:1569–1576.
14. Jańczak, K. 2009. Generalized rejected backward stochastic differential equations. *Stochastics* 81:147–170.
15. Jańczak, K. 2008. Discrete approximations of rejected backward stochastic differential equations with random terminal time. *Probab. Math. Statist.* 28:41–74.
16. Kobylanski, M., Lepeltier, J.-P., Quenez, M. C., and Torres S. 2002. Rejected BSDE with Superlinear quadratic coeffcient. *Probability and Mathematical Statistics* 22:51–83.
17. Kobylanski, M. 2000. Backward Stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.* 28:558–602.
18. El Karoui, N., and Rouge, R. 2000. Contingent claim pricing via utility maximization. *Mathematical Finance* 10:259–276.
19. Lepeltier, J.-P., Matoussi, A., and Xu, M. 2005. Rejected backward stochastic differential equations under monotonicity and general increasing growth conditions. *Adv. Appl. Probab.* 37:134–159.

20. Lepeltier, J.-P., and San Martín, J. 1998. *Existence for BSDE with superlinear-quadratic coeffcients. Stochastics Stochastics Reports* 63:227–240.
21. Lepeltier, J.-P., and San Martín, J. 1997. Backward Stochastic Differential Equations with continuous coefficient. *Stat. Probab. Lett.* 32:425–430.
22. Lepeltier, J.-P., and Xu, M. 2007. Rejected backward stochastic differential equations with two RCLL barriers. *ESAIM Probab. Stat.* 11:3–22.
23. Lepeltier, J.-P., and Xu, M. Penalization Method for Rejected backward stochastic differential equations with one R.C.L.L. barrier. *Statist. Prob. Lett.* 75:55–66.
24. Ma, J., Protter, P., San Martín, J., and Torres, S. 2002. Numerical method for Backward Stochastic Differential Equations. *Ann. Appl. Probab.* 12:302–316.
25. Ma, J., Protter, P., and Yong, J. 1994. Solving forward–backward stochastic differential equa tions explicitly – A four step scheme. *Probability Theory Related Fields* 93:339–359.
26. Ma, J., and Yong, J. 1999. *Forward-Backward Stochastic Differential Equations and Their Applications. Lecture Notes in Mathematics, 1702.* Springer Verlag, Berlin.
27. Ma J., and Zhang L. Representations and regularities for solutions to bsdes with rejections. *Stochastic Proc. Appl.* 115:539–569.
28. Matoussi, A. 1197. Rejected solutions of backward stochastic differential equations with continuous coeffcient. *Statist. Probab. Lett.* 34:347–354.
29. Mémin, J., Peng S., and Xu, M. 2008. Convergence of Solutions of discrete Rejected Back ward SDE's and simulations. *Acta Math. Appl. Sin. Engl. Ser.* 24:1–18.
30. Pardoux, P., and Peng, S. 1990. Adapted solution of backward stochastic differential equation. *Systems Control Lett.* 14:55–61.
31. Pardoux, P., and Peng, S. 1992. Backward stochastic differential equation and quasilinear parabolic partial differential equations. *Lecture Notes in Control and Information Science* 176:200–217.
32. Ren, Y. 2007. Rejected backward doubly stochastic differential equations driven by Levy process. *C.R. Math. Acad. Sci. Paris* 348:439–444.
33. Ren Y., and Hu, L. Rejected backward stochastic differential equations driven by Levy processes. *Statist. Probab. Lett.* 77:1559–1566.
34. Ren, Y., Lin, A., and Hu, L. 2009. Stochastic PDIEs and backward doubly stochastic differential equations driven by Levy processes. *J. Comput. Appl. Math.* 223:901–907.
35. Ren, Y., and Fan, X. 2009. Rejected Backward stochastic differential equations driven by a Levyprocess. *ANZIAM J.* 50:486–500.
36. Xu, M. 2007. Rejected backward SDEs with two barriers under monotonicity and general increasing conditions. *J. Theoret. Probab.* 20:1005–1039.
37. Zhang, J. 2004. A numerical scheme for BSDEs. *Ann. Appl. Probab.* 14:459–488.