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On Liouville type theorems for fully nonlinear elliptic equations with gradient term

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ABSTRACT

In this paper, we prove a Hadamard property and Liouville type theorems for viscosity solutions of fully nonlinear elliptic partial differential equations with a gradient term, both in the whole space and in an exterior domain.

1. Introduction

In the study of nonlinear elliptic equations in bounded domains, non-existence results for entire solutions of related limiting equations appear as a crucial ingredient. In the search for positive solutions for semi-linear elliptic equations with nonlinearity behaving as a power at infinity, one is interested in the non-negative solutions of the equation

$$\Delta u + u^p = 0, \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

The question is for which value of p, typically $p > 1$, this equation has or has no solution. This has been one of the motivations that has pushed forward the study of Liouville type theorems for general equations in $\mathbb{R}^N$ and in unbounded domains like cones or exterior domains. On the other hand, the understanding of structural characteristics of general linear or nonlinear operators has been another

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motivation for advancing the study of Liouville type theorems that have attracted many researchers. See the work in [1,2,6,15,17]

If we consider the Pucci's operators instead of the Laplacian, the question set above becomes very interesting, since most of the techniques used in the case of the Laplacian are not available. The Liouville type theorem for the equation analogous to Eq. (1.1) has not been proved in full generality, but only in the radial case. On the other hand, the Liouville type theorem for non-negative solutions of

$$\mathcal{M}^- u + u^p \leqslant 0, \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

has been studied in full extent by Cutrì and Leoni [11] and generalized in various directions by Felmer and Quaas [12–14], Capuzzo-Dolcetta and Cutrì [10] and Armstrong and Sirakov in [3]. In all these cases the solutions of the inequality are considered in the viscosity sense.

In a recent paper, Armstrong and Sirakov in [4] made significant progress in the understanding of the structure of positive solutions of equations generalizing (1.2), shading light even for equations of the form

$$\Delta u + f(u) \leqslant 0, \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

They propose a general approach to non-existence and existence of solutions of the general inequality

$$Q(u) + f(x, u) \leqslant 0, \quad \text{in } \mathbb{R}^N, \tag{1.4}$$

where the second order differential operator Q satisfies certain scaling property, it possesses fundamental solutions behaving as power asymptotically and it satisfies some other properties, common to elliptic operators, like a weak comparison principle, a quantitative strong comparison principle and a very weak Harnack inequality, see hypotheses (H1)–(H5) in [4]. Regarding the nonlinearity f, the results in [4] unravel a very interesting property, that is, that the behavior of the function f only matters near $u = 0$ and for x large. These results are new even for the case of (1.3). Moreover, the authors in [4] are able to apply their approach to Eq. (1.4) in exterior domains without any boundary condition, providing another truly new result.

It is the purpose of this article to extend the results described above in order to include elliptic operators with first order term. The introduction of a first order term may brake the scaling property of the differential operator and it allows for the appearance of non-homogeneous fundamental solutions, not even asymptotically. Thus, the approach in [4] cannot be applied to this more general situation and we have to find different arguments. Interestingly, to prove our results we use the more elementary approach taken in the original work by Cutrì and Leoni, where the Hadamard property, obtained through the comparison principle, is combined with the appropriate choice of a function to test the equation. The underline principle is the asymptotic comparison between the solution of the inequality and the fundamental solution. This can be interpreted as the interaction between the elliptic operator, including first order term, and the nonlinearity (the zero order term).

We start the precise description of our results by recalling the definition of the Pucci's operators. In this paper we consider

$$\mathcal{M}^+\big(r, D^2u\big) = \Lambda(r) \sum_{e_i \geqslant 0} e_i + \lambda(r) \sum_{e_i < 0} e_i \tag{1.5}$$

and

$$\mathcal{M}^-\big(r, D^2u\big) = \lambda(r) \sum_{e_i \geqslant 0} e_i + \Lambda(r) \sum_{e_i < 0} e_i, \tag{1.6}$$

where $e_1,\dots,e_N$ are the eigenvalues of D^2u, $\lambda, \Lambda : [0,\infty) \to \mathbb{R}$ are continuous, λ_0 and Λ_0 are positive constants and

$$0 < \lambda_0 \leqslant \lambda(r) \leqslant \Lambda(r) \leqslant \Lambda_0 < \infty, \quad \forall r = |x|,\ x \in \mathbb{R}^N. \tag{1.7}$$

Our purpose is to study the non-negative solutions of

$$\mathcal{M}^-\big(r, D^2u\big) + \sigma(r)|Du| + f(x,u) \leqslant 0 \quad \text{in } \Omega, \tag{1.8}$$

with $\Omega = \mathbb{R}^N$ or an exterior domain and $\sigma : [0,\infty) \to \mathbb{R}$ and $f : \Omega \times (0,\infty) \to (0,\infty)$ are continuous. In this paper, by an exterior domain we mean a set $\Omega = \mathbb{R}^N \setminus K$ connected, where K is a nonempty compact subset of $\mathbb{R}^N$.

We consider the fundamental solutions for the second order differential operator $\varphi, \psi : (0,\infty) \to \mathbb{R}$ given in (3.5) and (3.6), which are non-trivial radially symmetric solutions of

$$\mathcal{M}^-\big(r, D^2u\big) + \sigma(r)|Du| = 0, \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{1.9}$$

satisfying

(i) ψ is increasing and either $\lim_{r\to\infty} \psi(r) = \infty$ or $\lim_{r\to\infty} \psi(r) = 0$ and
(ii) φ is decreasing and either $\lim_{r\to\infty} \varphi(r) = -\infty$ or $\lim_{r\to\infty} \varphi(r) = 0$.

Now we are in a position to make precise assumptions about the interaction between the differential operator and the nonlinearity. We assume:

(f_1) $f : \Omega \times (0,\infty) \to (0,\infty)$, $\lambda, \Lambda, \sigma : [0,\infty) \to \mathbb{R}$ are continuous.
(f_2) We have

$$\lim_{r=|x|\to\infty} \frac{r^2}{1+\sigma^-(r)r} f(x,s) = \infty,$$

uniformly on compact subsets of $(0,\infty)$. Here and in what follows $\sigma^- = \max\{-\sigma, 0\}$.

In order to state the next assumption we need a previous definition. Given $\mu > 0$, $a > 1$, $k > 0$ and $\tau > 0$ we define

$$\Psi_k(\tau) = \frac{\varphi(a\tau)}{\varphi(\tau)} \inf_{x\in B_{a\tau}\setminus B_\tau} \left\{ \frac{r^2}{\sigma^-(r)r+1} \inf_{k\varphi(ar)\leqslant s\leqslant\mu} \frac{f(x,s)}{s} \right\}. \tag{1.10}$$

We assume:

(f_3) If $\lim_{r\to\infty} \varphi(r) = 0$ we assume the existence of constants $\mu > 0$ and $a > 1$ such that, defining

$$h(k) = \limsup_{\tau\to\infty} \Psi_k(\tau),$$

one of the following holds:
(i) for all $k > 0$ we have $h(k) = \infty$ or
(ii) for all $k > 0$ we have

$$0 < \liminf_{\tau\to\infty} \Psi_k(\tau) \quad \text{and} \quad \lim_{k\to\infty} h(k) = \infty \tag{1.11}$$

and there is a constant $C \in \mathbb{R}$ such that

$$r\sigma(r) > C, \quad \text{for all } r > 0. \tag{1.12}$$

Now we state our first Liouville type theorem for inequality (1.8) in $\mathbb{R}^N$:

Theorem 1.1. *Assume f satisfies (f_1), (f_2) and (f_3). Then inequality (1.8) in $\mathbb{R}^N$ does not have a non-trivial viscosity solution $u \geqslant 0$.*

We observe that hypothesis (f_3) does restrict f when $\lim_{r\to\infty}\varphi(r) = -\infty$.

Regarding hypotheses (f_2) and (f_3) we would like to notice that they are natural extensions of hypotheses (f2)–(f3) in [4], when $\sigma \not\equiv 0$ and the fundamental solution φ is not necessarily power-like. Thus, we are generalizing the results in [4] in the case of a one-homogeneous differential operator in $\mathbb{R}^N$. It is also interesting to notice that hypotheses (f_2) and (f_3) appear explicitly and in a natural way in our proof of the theorem.

When the condition (i) is satisfied we say that inequality (1.8) is sub-critical and when condition (ii) holds, we say it is critical. In the case of

$$\Delta u + u^p \leqslant 0,$$

we say the inequality is sub-critical when $p < N/(N-2)$ and when $p = N/(N-2)$ it is critical. When $p > N/(N-2)$ we say the inequality is super-critical and here the existence of positive solution holds. Accordingly, we would like to define a notion of super-criticality the cases (i) and (ii) do not hold. However, in Theorem 2.3 we provide an example where there is no solution in a super-critical sub-region, showing that further study is required to understand the critical boundary.

In the case of an exterior domain, we need to consider also the interaction between the differential operator and the nonlinearity at ∞. We need a definition in order to state our assumptions. Given $\mu > 0$, $a > 1$, $k > 0$ and $\tau > 0$ we define

$$\tilde{\Psi}_k(\tau) = \frac{\psi(\tau)}{\psi(a\tau)} \inf_{x\in B_{a\tau}\setminus B_\tau} \left\{ \frac{r^2}{\sigma^-(r)r+1} \inf_{\mu\leqslant s\leqslant k\psi(ar)} \frac{f(x,s)}{s} \right\}.$$

Now we assume:

(f_4) If $\lim_{r\to\infty}\psi(r) = \infty$ then we assume the existence of $\mu > 0$ and $a > 1$, such that, defining

$$\tilde{h}(k) = \limsup_{\tau\to\infty} \tilde{\Psi}_k(\tau),$$

one of the following holds:
(i) for all $k > 0$ we have $\tilde{h}(k) = \infty$ or
(ii) for all $k > 0$ we have

$$0 < \liminf_{\tau\to\infty} \tilde{\Psi}_k(\tau) \quad \text{and} \quad \lim_{k\to 0^+} \tilde{h}(k) = \infty \tag{1.13}$$

and there is a constant $C \in \mathbb{R}$ such that (1.12) holds.

For an exterior domain we have the following non-existence result:

Theorem 1.2. *Assume Ω is an exterior domain and that f satisfies (f_1), (f_2), (f_3) and (f_4). Then inequality (1.8) in Ω does not have a non-trivial viscosity solution $u \geqslant 0$.*

We observe hypothesis (f_4) does not restrict f when $\lim_{r\to\infty}\psi(r)=0$.

As for (f_3), hypothesis (f_4) is the natural extension of (f4) in [4] to our case. Here we allow $\sigma\not\equiv 0$ and ψ not power-like, thus generalizing [4].

In case of (f_4) we may also define the notion of criticality for (1.8) in an analogous way as for (f_3). Since here the behavior of f is relevant at zero and infinity mixed cases appear, as for example, an inequality critical at 0 and sub-critical at ∞ or vice versa.

In the proofs of Theorems 1.1 and 1.2 we use some basic properties of the functions

$$m(r)=\inf_{x\in B_r}u(x),\qquad m_0(r)=\inf_{x\in B_r\setminus B_{r_0}}u(x)\quad\text{and}\quad M(r)=\inf_{x\notin B_r}u(x)\tag{1.14}$$

in connection with the fundamental solutions, as given by the Hadamard property provided in Theorem 4.3. Then we test the equation with an adequate function and we use the asymptotic assumptions on f and the fundamental solutions to obtain a contradiction with the existence of non-trivial non-negative solutions. In the proofs of our theorems we only consider $a=2$.

The interaction between the elliptic operator and the nonlinearity, that is expressed in hypotheses (f_3) and (f_4), is not easy to understand in full generality. However, beyond the cases studied in [4], there are many interesting examples that well illustrate the relevance of our results to understand the general structure of solutions for these equations. In particular, in Section 2 we discuss some examples for the inequality

$$\Delta u+\sigma(r)|Du|+f(u)\geqslant 0,\quad\text{in }\mathbb{R}^N,\tag{1.15}$$

which are not covered in the literature. In the first example we analyze the nonlinearity $f(u)=u^p$ with a function σ associated to a fundamental solution with oscillatory power, see (2.7). In this case, it is interesting to observe the way the introduction of σ affects the critical power of the nonlinearity. In the second example we analyze the case of $f(u)=u^p(1+\log|u|)^\nu$ and a function σ providing a fundamental solution matching the non-homogeneous nonlinearity, see (2.11). In this case we analyze the range of p and ν for non-existence of solutions to (1.15).

For the existence of positive solutions of (1.8), it is natural to consider the super-critical assumption, that is, the case when hypotheses (f_3) and (f_4) are not satisfied, which means

$$\liminf_{\tau\to\infty}\Psi_k(\tau)=0\quad\text{or}\quad\limsup_{k\to\infty}h(k)<\infty$$

and

$$\liminf_{\tau\to\infty}\tilde{\Psi}_k(\tau)=0\quad\text{or}\quad\limsup_{k\to 0^+}\tilde{h}(k)<\infty,$$

where h, $\tilde{h}$, Ψ_k and $\tilde{\Psi}_k$ were defined in (f_3) and (f_4). We observe that super-criticality holds when $h(k)=0$ or $\tilde{h}(k)=0$ for any $k>0$, but it is not true that under this notion of super-criticality a positive solution of (1.8) always exists, as we see in Section 2 through an example.

In the last part of this article we consider a Liouville type theorem in the case f is a linear function, that is, $f(x,s)=h(x)s$, that interestingly can be proved using the same techniques considered in the nonlinear case. This problem has been recently studied by Rossi [18] after some previous work by Berestycki, Hamel and Nadirashvili [7], Berestycki, Hamel and Roques [8] and Berestycki, Hamel and Rossi [9]. Rossi [18] proved a Liouville type theorem for general unbounded domains, assuming that

$$\liminf_{x\in\Omega,\,|x|\to\infty}\frac{u(x)+1}{dist(x,\partial\Omega)}=0.\tag{1.16}$$

It is clear that when Ω is an exterior domain then $dist(x,\partial\Omega)\sim|x|$, so that (1.16) implies a linear growth constraint on u. Thus, it is interesting to investigate the existence or non-existence of positive solutions of the corresponding equation when (1.16) does no longer hold. Here is our result:

Theorem 1.3. *Let u be a non-negative viscosity solution of*

$$\mathcal{M}^-\big(r,D^2u\big)+\sigma(r)|Du|+h(x)u\leqslant 0 \quad \text{in } \Omega, \tag{1.17}$$

where Ω is an exterior domain. Assume further that λ and Λ satisfy (1.7) and that

(h_1) *$h:\mathbb{R}^N\to\mathbb{R}$ and $\sigma:\mathbb{R}_+\to\mathbb{R}$ are continuous, h is positive and σ is negative.*
(h_2) *There exists a function $\kappa:\mathbb{R}_+\to\mathbb{R}_+$ of class C^1 such that*

$$\lim_{r\to\infty}\kappa'(r)=0 \tag{1.18}$$

and there is a constant $\mu\geqslant 1$ such that

$$1\leqslant\kappa(r)\max_{r-\kappa(r)\leqslant s\leqslant r}\big|\sigma(s)\big|\leqslant\mu, \quad \text{for all } r>0. \tag{1.19}$$

(h_3) *There exists a sequence $r_n\to\infty$ such that*

$$\lim_{n\to\infty}\inf_{r\in(r_n-\kappa(r_n),r_n)}\big\{h(r)-e^{\frac{\mu}{\lambda_0}}(2\Lambda_0+1)\sigma^2(r)\big\}>0. \tag{1.20}$$

Then $u\equiv 0$.

Besides avoiding assumption (1.16) we observe that we are not assuming that σ is bounded as in [18]. Examples of functions σ satisfying the hypotheses are given in Section 6.

This paper is organized as follows. In Section 2 we discuss the main theorems and provide examples. In Section 3 we find fundamental solutions and in Section 4 we prove the Hadamard property for the functions given in (1.14). In Section 5 we prove the non-existence of Theorems 1.1 and 1.2. Section 6 is devoted to the proof of the existence of Theorem 2.3 given in Section 2. Finally, in Section 7 we prove Theorem 1.3.

2. Discussion and examples

We devote this section to present various examples that illustrate the relevance of our results. We start discussing the relation between σ and the fundamental solution, then we present two examples for Theorem 1.1 and we conclude the section with a theorem related with the concept of super-criticality. We concentrate our discussion on Theorem 1.1 regarding inequality (1.8) in $\mathbb{R}^N$ in the case of the elliptic operator

$$Q(r,u)=\Delta u+\sigma(r)|Du|. \tag{2.1}$$

We may certainly construct examples for Theorem 1.2 regarding the inequality in an exterior domain and for general Pucci operators as in our theorems.

In Section 3 we study with details the fundamental solutions associated to the differential operator in Eq. (1.8). We see that in the case of Q, the decreasing fundamental solution is given by

$$\varphi(r) = -\int_1^r s^{1-N} e^{\int_1^s \sigma(\tau)\, d\tau}\, ds + L_\varphi,$$

where L_φ is a constant so that when $\lim_{r\to\infty} \varphi(r)$ exists, it becomes equal to 0, see Proposition 3.1 and its proof. With this formula we may construct many examples of fundamental solutions with a whole variety of asymptotic behavior. We start showing the effect of the first order term on the behavior of the fundamental solution. Our first example is for

$$\sigma(r) = \frac{d}{dr}(\sin r \log r), \quad r \geqslant 1,$$

properly extended to $[0, 1)$. Then we have, for a constant L_φ,

$$\varphi(r) = -\int_1^r s^{1-N+\sin(s)}\, ds + L_\varphi, \quad r \geqslant 1. \tag{2.2}$$

We observe that this fundamental solution does not behave like a power at infinity. The second example is given by

$$\sigma(r) = \frac{d}{dr}\big(\cos(\log\log r)\log r\big), \quad r \geqslant e,$$

properly extended to $[0, e)$. The associated fundamental solution does not behave like a power, not even asymptotically. Its behavior is oscillatory, with slower rate than (2.2). For a third example we consider

$$\sigma(r) = -\frac{d}{dr}\big((\alpha + 2 - N)\log r + \log\log r\big), \quad r \geqslant e, \tag{2.3}$$

extended to $[0, e)$ as a continuous function with fundamental solution

$$\varphi(r) = -e^{\alpha+1}\int_e^r \frac{s^{-1-\alpha}}{\log s}\, ds + L_\varphi, \quad r \geqslant e. \tag{2.4}$$

This example is different from earlier ones since it is not oscillatory, but with an asymptotic behavior which is not power-like because of its logarithmic term.

It is interesting to see that we may prescribe explicit fundamental solutions by providing functions q like

$$\varphi(r) = e^{-q(r)}, \quad r \geqslant 0, \tag{2.5}$$

assuming that q is increasing and $\lim_{r\to\infty} q(r) = \infty$. It is easy to check that this fundamental solution is obtained when the function σ is given by

$$\sigma(r) = \frac{N-1}{r} - q'(r) + \frac{q''(r)}{q'(r)} \quad \text{for } r \geqslant 0. \tag{2.6}$$

In view of our examples later, we will require q to be such that $r\sigma(r)$ is bounded. This condition is not necessary to use Theorem 1.1, but under this condition (f_2) and (f_3) greatly simplify. Assume that $N \geqslant 3$ and

$$q(r) = (N-2)\log r + \frac{1}{2}\sin\bigl(\log(\log r)\bigr)\log r, \quad r > e. \tag{2.7}$$

After some direct calculation, we see that $q'(r) > 0$ and, if σ is defined as in (2.6), $r\sigma(r)$ is bounded. In this case, the fundamental solution is

$$\varphi(r) = r^{-(N-2+\frac{1}{2}\sin(\log\log r))}, \quad r \geqslant e,$$

which is a power exhibiting an oscillatory exponent. In this situation we have

Theorem 2.1. *Assume $N \geqslant 3$ and*

$$1 < p < \frac{N-\frac{1}{2}}{N-\frac{5}{2}}, \tag{2.8}$$

then there is no positive solution to the nonlinear inequality

$$\Delta u + \sigma\bigl(|x|\bigr)|Du| + u^p \leqslant 0, \quad \text{in } \mathbb{R}^N. \tag{2.9}$$

This theorem shows the effect of the first order term on the critical exponent. It is interesting to notice that the critical exponent is enlarged because the 'dimension' is decreased by 1/2, the amplitude of the oscillatory power.

Proof of Theorem 2.1. The application of Theorem 1.1 requires to analyze the function Ψ_k in (f_3), since all other hypotheses are satisfied. Using the definition of Ψ_k, that $p > 1$ and that $r\sigma(r)$ is bounded, we find that for r large

$$\Psi_k(r) = k^{p-1}e^{-q(2r)p+q(r)+2\log r}. \tag{2.10}$$

Computing the exponent, from (2.7) we see that

$$\begin{aligned}-q(2r)p + q(r) = {} & -(N-2)p\log 2 - \frac{p}{2}\sin\bigl(\log\bigl(\log(2r)\bigr)\bigr)\log 2 \\ & + \left[-(N-2)(p-1) - \frac{p}{2}\sin\bigl(\log\bigl(\log(2r)\bigr)\bigr) + \frac{1}{2}\sin\bigl(\log\bigl(\log(r)\bigr)\bigr)\right]\log r.\end{aligned}$$

We claim that there exists a sequence $\{r_n\}$ such that $\lim_{n\to\infty} r_n = \infty$,

$$\lim_{n\to\infty}\sin\bigl(\log(\log 2r_n)\bigr) = -1 \quad \text{and} \quad \lim_{n\to\infty}\sin\bigl(\log(\log r_n)\bigr) = -1.$$

Assume the claim is true now, then we get $\lim_{n\to\infty}\Psi_k(r_n) = +\infty$ if we have $-(N-2)(p-1) + (p-1)/2 + 2 > 0$, which is exactly (2.8). To complete the proof we check the claim. We let r_n be the positive solution of the equation

$$\sin\left(\frac{\log(\log(2r_n)) + \log(\log(r_n))}{2}\right) = -1, \quad n \in \mathbb{N},$$

that satisfies $\lim_{n\to\infty} r_n = \infty$. Then we have

$$\sin\big(\log\big(\log(2r_n)\big)\big) + \sin\big(\log\big(\log(r_n)\big)\big) = -2\cos\left(\frac{\log(\log(2r_n)/\log(r_n))}{2}\right),$$

from where the claim follows, since

$$\lim_{n\to+\infty}\big[\sin\big(\log\big(\log(2r_n)\big)\big) + \sin\big(\log\big(\log(r_n)\big)\big)\big] = -2. \qquad \square$$

Now we consider another example for the function

$$q(r) = (N-2)\log r + \log(\log r), \quad r > e. \tag{2.11}$$

Its associated fundamental solution is a power with a logarithmic factor

$$\varphi(r) = \frac{1}{r^{N-2}\log r}, \quad r \geqslant 1,$$

and $r\sigma(r)$ is bounded, for σ as in (2.6). Next we apply Theorem 1.1 to the nonlinearity $f(u) = u^p(|\log u| + 1)^\nu$ with differential term Q with σ as above.

Theorem 2.2. *Assume that $N \geqslant 3$ and*

$$1 < p < \frac{N}{N-2} \quad \text{and} \quad \nu \in \mathbb{R}, \quad \text{or}$$
$$p = \frac{N}{N-2} \quad \text{and} \quad \nu \geqslant -\frac{2}{N-2},$$

then there is no positive solution to the nonlinear inequality

$$\Delta u + \sigma\big(|x|\big)|Du| + u^p\big(|\log u| + 1\big)^\nu \leqslant 0 \quad \text{in } \mathbb{R}^N. \tag{2.12}$$

This theorem provides an example of a non-existence result where the nonlinearity and the fundamental solution are not homogeneous and they match in such a way that the hypothesis (f_3) is satisfied.

Proof of Theorem 2.2. In this case, the function Ψ_k in (f_3) is given by

$$\Psi_k(r) = k^{p-1}e^{-q(2r)+q(r)+2\log r}\big[\big|\log k - q(2r)\big| + 1\big]^\nu. \tag{2.13}$$

From here and (2.11) we have

$$-q(2r)p + q(r) = (N-2)\big((1-p)\log r - p\log 2\big) - p\log\big(\log(2r)\big) + \log(\log r).$$

From here, there exists a constant $C > 0$ so that, for r large, we have

$$\Psi_k(r) \geqslant Ck^{p-1}r^{(N-2)(p-1)}(\log r)^{(p-1)}(\log r - \log k)^\nu.$$

If $p < \frac{N}{N-2}$ with $\nu \in \mathbb{R}$ or $p = \frac{N}{N-2}$ with $\nu > -\frac{2}{N-2}$, then $\lim_{r\to\infty} \Psi_k(r) = +\infty$. In the limit case, when $p = \frac{N}{N-2}$ with $\nu = -\frac{2}{N-2}$, we have $\lim_{r\to\infty} \Psi_k(r) \geqslant Ck^{p-1}$, from where we complete the proof using Theorem 1.1. $\square$

In the examples discussed above the fact that $f(s)/s$ is decreasing allowed to get the inner most infimum easily. Then, the monotonicity of the remaining term in r allowed to get the second infimum and thus Ψ_k was obtained explicitly. In what follows we give simplified versions of hypothesis (f_3).

Remark 2.1. In hypothesis (f_3), we may define the function h in a different way, namely we may consider

$$h_1(k) = \liminf_{\tau\to\infty} \Psi_k(\tau) \quad \text{or}$$

$$h_2(k) = \liminf_{|x|\to\infty} \frac{\varphi(ar)}{\varphi(r)} \frac{r^2}{\sigma^-(r)r+1} \inf_{k\varphi(ar)\leqslant s\leqslant\mu} \frac{f(x,s)}{s}.$$

These two definitions give rise to two stronger versions of hypothesis (f_3). We may use this condition to deal with the example given by (2.7).

Remark 2.2. If we assume that there exists $C \in \mathbb{R}$ such that

$$\int_R^{2R} \sigma(r)\, dr \geqslant C > -\infty \tag{2.14}$$

for each $R > 1$, for the function h_2 defined above, we have

$$h_2(k) = \liminf_{|x|\to\infty} \frac{r^2}{\sigma^-(r)r+1} \inf_{k\varphi(ar)\leqslant s\leqslant\mu} \frac{f(x,s)}{s}.$$

In case $f(x,u) = s^p$ and assuming $\lim_{r\to\infty} \varphi(r) = 0$, the function h_2 becomes

$$h_2(r) = k^{p-1} \lim_{r\to\infty} \frac{r^2\varphi(r)^{p-1}}{1+\sigma^-(r)r}.$$

We conclude this section discussing an example for the notion of super-criticality suggested by (f_3). For the power nonlinearity and the Laplacian

$$-\Delta u + u^p \leqslant 0, \quad x \in \mathbb{R}^N,$$

it is well known that a solution exists in the super-critical case, that is, when $p > \frac{N}{N-2}$. In our case, we defined super-critical inequality in the introduction, but our example below shows that this may not be fully appropriate.

We assume that α and $\nu > 0$ are positive numbers and $p > 1$. We let $\sigma : [0,\infty) \to \mathbb{R}$ as in (2.3) and $f : (0,\infty) \to \mathbb{R}$ be as in

$$f(s) = s^p\big(|\log s| + 1\big)^\nu, \quad s \in (0,\infty). \tag{2.15}$$

Considering the corresponding fundamental solution given in (2.2), we get

$$\lim_{r\to\infty}\varphi(r)=0 \quad \text{and} \quad \lim_{r\to\infty}\frac{\varphi(r)}{r^{-\alpha}(\log r)^{-1}}=e^{\alpha+2}\alpha^{-1}.$$

Next, given $k>0$, we find positive constants C and $\bar{R}$ such that for $r>\bar{R}$

$$\frac{Ck^{p-1}(\log r-\log k)^{\nu}}{r^{\alpha(p-1)-2}(\log r)^{(p-1)}}\leqslant \Psi_k(r)\leqslant \frac{k^{p-1}(\log r-\log k)^{\nu}}{Cr^{\alpha(p-1)-2}(\log r)^{(p-1)}},$$

where Ψ_k was defined in (1.10). Then we obtain the following three cases:

$$(\mathrm{C}_1)\ \text{sub-critical}\quad p<\frac{2}{\alpha}+1,\quad \text{or}\quad p=\frac{2}{\alpha}+1\quad\text{and}\quad \nu>\frac{2}{\alpha},$$
$$(\mathrm{C}_2)\ \text{critical}\quad p=\frac{2}{\alpha}+1\quad\text{and}\quad \nu=\frac{2}{\alpha},$$
$$(\mathrm{C}_3)\ \text{super-critical}\quad p>\frac{2}{\alpha}+1,\quad \text{or}\quad p=\frac{2}{\alpha}+1\quad\text{and}\quad \nu<\frac{2}{\alpha}.$$

And we obtain some non-existence and existence results as following:

Theorem 2.3. *Suppose σ and f are given as above and $\Omega=\mathbb{R}^N$, then:*

(i) *If (C_1) or (C_2) holds, then (1.8) does not have a positive solution.*
(ii) *If*

$$p=\frac{2}{\alpha}+1\quad \textit{and}\quad 0<\frac{2}{\alpha}-1<\nu<\frac{2}{\alpha}, \tag{2.16}$$

then (1.8) does not have a positive solution.
(iii) *If $p>\frac{2}{\alpha}+1$, then (1.8) has a positive solution.*

We see that the sub-region for (p,α) given in (2.16) is super-critical, however we can prove non-existence of a positive solution there. This fact shows that more analysis in needed to understand the critical boundary in general.

3. Fundamental solutions and basic properties

In this section we construct the fundamental solutions of the nonlinear second order operator with first order term given in (1.9). These special radial solutions are important tools for understanding the behavior of general viscosity solutions of (1.9).

We start defining the dimension like numbers, which are relevant in our construction. We let $n,N:(0,\infty)\to\mathbb{R}$ be the functions given by

$$n(r)=\begin{cases}\frac{\Lambda(r)}{\lambda(r)}(N-1)+1 & \text{if } r\sigma(r)\leqslant \Lambda(r)(N-1),\\ N & \text{if } r\sigma(r)>\Lambda(r)(N-1),\end{cases} \tag{3.1}$$

and

$$N(r)=\begin{cases}\frac{\lambda(r)}{\Lambda(r)}(N-1)+1 & \text{if } r\sigma(r)>-\lambda(r)(N-1),\\ N & \text{if } r\sigma(r)\leqslant -\lambda(r)(N-1).\end{cases} \tag{3.2}$$

We also need to consider the following functions

$$m_\lambda(r) = \begin{cases} \lambda(r) & \text{if } r\sigma(r) \leqslant \Lambda(r)(N-1), \\ \Lambda(r) & \text{if } r\sigma(r) > \Lambda(r)(N-1) \end{cases} \tag{3.3}$$

and

$$M_\lambda(r) = \begin{cases} \lambda(r) & \text{if } r\sigma(r) \leqslant -\lambda(r)(N-1), \\ \Lambda(r) & \text{if } r\sigma(r) > -\lambda(r)(N-1). \end{cases} \tag{3.4}$$

Given $r_1 > 0$ and constants L_φ and L_ψ we define the functions $\varphi, \psi : (0,\infty) \to \mathbb{R}$ as follows:

$$\varphi(r) = -\int_{r_1}^{r} s e^{\int_{r_1}^{s} (\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau})\, d\tau}\, ds + L_\varphi \tag{3.5}$$

and

$$\psi(r) = \int_{r_1}^{r} s e^{-\int_{r_1}^{s} (\frac{\sigma(\tau)}{M_\lambda(\tau)} + \frac{N(\tau)}{\tau})\, d\tau}\, ds + L_\psi. \tag{3.6}$$

Proposition 3.1.

(i) *The function φ defined in (3.5), is of class $C^{1,1}$ and it satisfies Eq. (1.9). Moreover, φ is a decreasing function and, by choosing the constant L_φ adequately, it satisfies*

$$\lim_{r\to\infty} \varphi(r) = -\infty \quad \textit{or} \quad \lim_{r\to\infty} \varphi(r) = 0. \tag{3.7}$$

(ii) *The function ψ defined in (3.6) is of class $C^{1,1}$ and it satisfies Eq. (1.9). Moreover, ψ is an increasing function and, by choosing the constant L_ψ adequately, it satisfies*

$$\lim_{r\to\infty} \psi(r) = \infty \quad \textit{or} \quad \lim_{r\to\infty} \psi(r) = 0. \tag{3.8}$$

The functions φ and ψ satisfying (3.7) and (3.8), respectively, are called fundamental solutions of the operator (1.9).

Proof of Proposition 3.1. We recall that, given a C^2 radially symmetric function $u(x) = v(|x|)$, the eigenvalues of $D^2u(x)$ are $v''(r)$ with multiplicity 1 and $v'(r)/r$ with multiplicity $N-1$.

(i) By the definition (3.5), we have

$$\varphi'(r) = -re^{\int_{r_1}^{r} (\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau})\, d\tau} \quad \text{and} \quad \varphi''(r) = \left[\frac{1-n(r)}{r} + \frac{\sigma(r)}{m_\lambda(r)}\right]\varphi'(r).$$

Then we readily see that $\varphi'(r) < 0$, so that φ is a decreasing function, and using (3.1) and (3.3) we find that

$$\begin{aligned} &\varphi''(r) \geqslant 0 \quad \text{if } r\sigma(r) \leqslant \Lambda(r)(N-1) \quad \text{and} \\ &\varphi''(r) < 0 \quad \text{if } r\sigma(r) > \Lambda(r)(N-1). \end{aligned}$$

Thus, whenever $r\sigma(r) \leqslant \Lambda(r)(N-1)$, we obtain

$$\begin{aligned}\mathcal{M}^-\big(r, D^2\varphi\big) + \sigma(r)|D\varphi| &= \lambda(r)\varphi''(r) + \Lambda(r)\frac{N-1}{r}\varphi'(r) - \sigma(r)\varphi'(r)\\ &= \lambda(r)\left[\varphi''(r) + \frac{n(r)-1}{r}\varphi'(r) - \frac{\sigma(r)}{\lambda(r)}\varphi'(r)\right] = 0\end{aligned}$$

and, whenever $r\sigma(r) > \Lambda(r)(N-1)$, we obtain

$$\begin{aligned}\mathcal{M}^-\big(r, D^2\varphi\big) + \sigma(r)|D\varphi| &= \Lambda(r)\varphi''(r) + \Lambda(r)\frac{N-1}{r}\varphi'(r) - \sigma(r)\varphi'(r)\\ &= \Lambda(r)\left[\varphi''(r) + \frac{N-1}{r}\varphi'(r) - \frac{\sigma(r)}{\Lambda(r)}\varphi'(r)\right] = 0.\end{aligned}$$

We conclude then, that φ is a solution of Eq. (1.9), it is of class $C^{1,1}$ and, since φ is decreasing, the limit in (3.7) exists. If it is bounded, we may find L_φ so that φ has limit equal to zero.

(ii) can be proved in a completely analogous way. □

Remark 3.1. We observe that the functions φ and ψ are not necessarily convex or concave and that they may change their concavity along r.

In what follows we derive various properties of the fundamental solutions that we need in the sequel. We start with properties for the function φ.

Lemma 3.1. *If $\lim_{r\to\infty}\varphi(r) = 0$, then there exists a sequence $\{r_n\}$ diverging to infinity such that*

$$\lim_{n\to\infty} r_n\varphi'(r_n) = 0. \tag{3.9}$$

Proof. This is equivalent to $\limsup_{r\to\infty} r\varphi'(r) < 0$ implies $\lim_{r\to\infty}\varphi(r) = -\infty$, which is obviously true. □

Proposition 3.2. *Suppose $\lim_{r\to\infty}\varphi(r) = 0$ and assume that (1.12) holds, then there is a constant $C_0 > 0$ such that*

$$-\frac{r\varphi'(r)}{\varphi(r)} \leqslant C_0, \quad \textit{for all } r \geqslant 1.$$

Proof. We first see that, from definition of φ and (1.12), we have

$$\frac{(r\varphi'(r))'}{\varphi'(r)} = \frac{r\varphi''(r) + \varphi'(r)}{\varphi'(r)} = -n(r) + 2 + \frac{r\sigma(r)}{m_\lambda(r)} \geqslant C,$$

for a certain negative constant C and all $r \geqslant 1$. Then, since φ is decreasing,

$$\big(r\varphi'(r)\big)' \leqslant C\varphi'(r), \quad \text{for all } r \geqslant 1.$$

Considering the sequence given in Lemma 3.1, we integrate to obtain

$$r_n\varphi'(r_n) - r\varphi'(r) \leqslant C\big(\varphi(r_n) - \varphi(r)\big), \quad \text{for all } n \in \mathbb{N}.$$

Then, taking limit as $n \to \infty$ and using the hypothesis, we find

$$-r\varphi'(r) \leqslant -C\varphi(r),$$

from where we conclude, taking $C_0 = -C$. $\square$

Proposition 3.3. *Assume* $\lim_{r\to\infty} \varphi(r) = 0$ *and that* σ *satisfies*

$$\int_r^{2r} \sigma(\tau)\,d\tau \geqslant C, \tag{3.10}$$

for some $C \in \mathbb{R}$ *and for all* $r \geqslant 1$*. Then, there exists* $C_0 > 0$ *such that*

$$\frac{\varphi(2r)}{\varphi(r)} \geqslant C_0, \quad \textit{for all } r \geqslant 1.$$

Proof. By definition of φ and hypothesis (3.10), we have

$$\frac{\varphi'(2r)}{\varphi'(r)} = 2e^{\int_r^{2r}(\frac{\sigma(\tau)}{m_\lambda(\tau)} - \frac{n(\tau)}{\tau})\,d\tau} \geqslant 2e^{c(\int_r^{2r}\sigma(\tau)\,d\tau) - C\log 2} \geqslant C_0,$$

for certain constants c, C and C_0. Then, since φ is decreasing, we have

$$\varphi'(2r) \leqslant C_0\varphi'(r), \quad \text{for all } r \geqslant 1.$$

Thus, integrating in $[r, R]$, taking limit as $R \to \infty$ and using the hypothesis we get the result. $\square$

Next we obtain two other propositions, but now regarding the function ψ.

Proposition 3.4. *Assume* $\lim_{r\to\infty} \psi(r) = \infty$ *and* σ *satisfies* (1.12)*, then there exist* $C_0 > 0$ *and* $r_1 > 0$ *such that*

$$\frac{r\psi'(r)}{\psi(r)} \leqslant C_0, \quad \textit{for all } r \geqslant r_1.$$

Proof. From (1.12) and definition of ψ we have

$$\frac{(r\psi'(r))'}{\psi'(r)} = \frac{r\psi''(r) + \psi'(r)}{\psi'(r)} = -N(r) + 2 - \frac{r\sigma(r)}{M_\lambda(r)} \leqslant C,$$

for some $C > 0$. Let r_1 be such that $\psi(r_1) > 0$ and consider that

$$\big(r\psi'(r)\big)' \leqslant C\psi'(r),$$

then we integrate in $[r_1, r]$ and get

$$\frac{r\psi'(r)}{\psi(r)} \leqslant C + \frac{r_1\psi'(r_1) - C\psi(r_1)}{\psi(r)} \leqslant C + \frac{r_1\psi'(r_1)}{\psi(r_1)} \equiv C_0,$$

for all $r \geqslant r_1$ completing the proof. $\square$

Proposition 3.5. *Assume* $\lim_{r\to\infty}\psi(r)=\infty$ *and* σ *satisfies* (3.10), *then there exist* $C_0>0$ *and* $r_1>0$ *such that*

$$\frac{\psi(r)}{\psi(2r)}\geqslant C_0,\quad \textit{for all } r\geqslant r_1.$$

Proof. By definition of ψ and from (3.10) we have

$$\frac{\psi'(r)}{\psi'(2r)}=2^{-1}e^{\int_r^{2r}(\frac{\sigma(\tau)}{M_\lambda(\tau)}+\frac{N(\tau)}{\tau})\,d\tau}\geqslant 4C_0,$$

for a certain positive constant C_0, and then

$$\psi'(r)\geqslant 4C_0\psi'(2r),\quad \text{for all } r\geqslant 1.$$

We let r_0 so that $\psi(2r_0)>0$ and we integrate from r_0 to r to obtain

$$\frac{\psi(r)}{\psi(2r)}\geqslant 2C_0+\frac{\psi(r_0)-C_0\psi(2r_0)}{\psi(2r)}.$$

From here we find r_1 such that the desired inequality holds for all $r\geqslant r_1$. $\square$

4. The Hadamard property

The Hadamard property and the Liouville type theorems are based on the Strong Maximum Principle and the Comparison Principle. Here we recall a version of these principles that are best suited for our purposes. We start with the Comparison Principle for viscosity solutions:

Theorem 4.1. *(See Ishii [16].) Let* $\Omega\subset\mathbb{R}^N$ *be a bounded open set. Let* λ, Λ *and* σ *satisfy hypothesis* (f_1) *and the functions* λ *and* Λ *satisfy* (1.7). *If* u *and* v *are respectively super- and sub-solutions in the viscosity sense of*

$$\mathcal{M}^-\big(r,D^2u\big)+\sigma\big(|x|\big)|Du|=0,\quad \textit{in } \Omega,$$

respectively and $u\geqslant v$ *on* $\partial\Omega$, *then* $u\geqslant v$ *in* Ω.

Next we have the Strong Minimum Principle:

Theorem 4.2. *(See Bardi and Da Lio [5].) Let* u *be a super-solution in the viscosity sense of*

$$\mathcal{M}^-\big(r,D^2u\big)+\sigma\big(|x|\big)|Du|=0,\quad \textit{in } \Omega.$$

If u *attains its minimum at an interior point of* Ω, *then* u *is a constant.*

Now we are in a position of proving the Hadamard property, a nonlinear Hadamard theorem. This theorem allows to obtain estimates for the behavior of super-solutions of (1.9) with respect to fundamental solutions. We have

Theorem 4.3. *Let* $\Omega=\mathbb{R}^N$ *or an exterior domain and suppose that* $u\in C(\Omega)$ *is a positive viscosity super-solution of* (1.9) *in* Ω. *We let* $r_0>0$ *be such that* $B^c_{r_0}\subset\Omega$ *and* $r_0<r_1<r_2$. *Then*

(i) *if* $\Omega = \mathbb{R}^N$*, for the functions* $m(r)$ *defined in* (1.14) *we have*

$$m(r) \geqslant \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} m(r_2) + \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} m(r_1), \quad r_1 < r < r_2, \tag{4.1}$$

(ii) *if* Ω *is an exterior domain,* $m_0(r)$ *is defined as in* (1.14) *and* r_1 *is large enough then for all* $r_1 < r < r_2$ *we have*

$$m_0(r) \geqslant \frac{\varphi(r) - \varphi(r_1)}{\varphi(r_2) - \varphi(r_1)} m_0(r_2) + \frac{\varphi(r_2) - \varphi(r)}{\varphi(r_2) - \varphi(r_1)} m_0(r_1), \tag{4.2}$$

(iii) *if* Ω *is an exterior domain, and the function* $M(r)$ *is defined as in* (1.14)*, for* $r_0 < r < r_1$ *we have*

$$\frac{M(r_1)}{\psi(r_1) - \psi(r_0)} \leqslant \frac{M(r)}{\psi(r) - \psi(r_0)}. \tag{4.3}$$

Proof. (i) It is clear that $m(r)$ is positive and non-increasing. By Proposition 3.1, we know that the function $\Phi(r) = C_1(\varphi(r) - \varphi(r_1)) + C_2$ with

$$C_1 = \frac{m(r_2) - m(r_1)}{\varphi(r_2) - \varphi(r_1)} > 0 \quad \text{and} \quad C_2 = m(r_1)$$

satisfies (1.9) for $0 < r_1 < r_2$ and $\Phi(r_1) = m(r_1)$ and $\Phi(r_2) = m(r_2)$. By the Comparison Principle (Theorem 4.1), we have

$$u(x) \geqslant \Phi(x), \quad x \in B_{r_2} \setminus B_{r_1}. \tag{4.4}$$

But, also by the Comparison Principle (Theorem 4.1), we have that $m(r) = \min\{u(x) \mid x \in \mathbb{R}^N, \ |x| = r\}$, so the conclusion follows from (4.4).

(ii) In the case of m_0 we observe that by the Strong Maximum Principle either $m_0(r)$ is constant for all $r \geqslant r_0$ or $m(r) = \min\{u(x) \mid x \in \mathbb{R}^N, \ |x| = r\}$, for all $r \geqslant r_1$ and r_1 large enough. Then the result is obtained in the same way as for m.

(iii) Let $r_1 > r_0$ and

$$\Phi(r) := M(r_1) \frac{\psi(r) - \psi(r_0)}{\psi(r_1) - \psi(r_0)}, \quad r \in (r_0, r_1),$$

which satisfies (1.9) and we see that $\Phi(r_1) = M(r_1) \leqslant u(x)$, for all $|x| = r_1$ and $0 = \Phi(r_0) \leqslant u(x)$ for all $|x| = r_0$. Then, by the Comparison Principle, we have

$$M(r_1) \frac{\psi(r) - \psi(r_0)}{\psi(r_1) - \psi(r_0)} \leqslant u(x),$$

for all $r_0 \leqslant r = |x| \leqslant r_1$. On the other hand, by the Strong Maximum Principle we see that either $M(r)$ is equal to a constant for all $r \geqslant r_0$ or

$$M(r) = \min\{u(x) \mid x \in \mathbb{R}^N, \ |x| = r\}, \quad \text{for all } r \geqslant r_0.$$

This completes the proof. □

From Theorem 4.3 we have

Corollary 4.1. *Assume that u is a non-negative viscosity solution of (1.8) in Ω, the whole space or an exterior domain, then we have*:

(i) *If* $\lim_{r\to\infty}\varphi(r)=0$, *then*

$$m(r)\geqslant \frac{m(r_1)}{\varphi(r_1)}\varphi(r) \quad \textit{and} \quad m_0(r)\geqslant \frac{m_0(r_1)}{\varphi(r_1)}\varphi(r), \quad \textit{for all } r\geqslant r_1\geqslant r_0.$$

(ii) *If* $\lim_{r\to\infty}\varphi(r)=-\infty$, *then*

$$m(r)\geqslant m(r_1) \quad \textit{and} \quad m_0(r)\geqslant m_0(r_1), \quad \textit{for all } r\geqslant r_1\geqslant r_0.$$

Proof. Since φ is decreasing, the result follows directly from Theorem 4.3 taking $r_2\to\infty$ in (4.1) and in (4.2). □

The next proposition provides additional properties of m, m_0 and M.

Proposition 4.1. *Suppose u is a positive viscosity solution of (1.8). Let*

$$g(r):=\min_{|x|=r}u(x).$$

Then there exists $\bar r$ such that g is either strictly increasing or strictly decreasing for $r>\bar r$. Either $m_0(r)$ is constant and $M(r)=g(r)$ strictly increasing or $m_0(r)=g(r)$ is strictly decreasing and $M(r)$ is constant for $r>\bar r$.

Proof. Let $r_1<r_2<r_3$ and $g(r_1)\geqslant g(r_2)$ and $g(r_3)\geqslant g(r_2)$, then u has a minimum point $x\in B_{r_3}\setminus B_{r_1}$, which contradicts with Minimum Principle. Then $g(t)$ may change monotonicity just once. So g is decreasing strictly or increasing strictly or first increasing and then decreasing. In the third case, let $\bar r$ be such that g is decreasing for $r\geqslant\bar r$. From here the result follows if we define $m_0(r)=\min_{\bar r\leqslant|x|\leqslant r}u(x)$. □

5. Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. The idea of the proof is to assume (1.8) has a solution and use an appropriate test function in order to get the behavior of u at infinity, which in view of our hypothesis is incompatible with the Hadamard property proved in the previous section.

Proof of Theorem 1.1. If the fundamental solution satisfies $\varphi(r)\to-\infty$, then by Corollary 4.1 we have

$$m(r)\geqslant m(r_1) \quad \text{for } r\geqslant r_1.$$

Since $m(r)$ is a non-increasing, we conclude that u attains an interior minimum, but then by the Strong Minimum Principle u is constant. From here $u\equiv 0$ since $f(x,s)>0$ if $s>0$ from our assumption (f_1).

If $\varphi(r)\to 0$, then we consider two cases: critical and sub-critical equations.

Sub-critical Case. We assume hypothesis (f_3) in case (i) holds. We may assume that $u>0$ by the Strong Minimum Principle. From Corollary 4.1 we have

$$m(r)\geqslant \frac{m(r_1)}{\varphi(r_1)}\varphi(r). \tag{5.1}$$

We also see that $m(r)$ is strictly decreasing. Considering $0<\tau<R$ as parameters, we define the test function

$$\zeta(x)=m(\tau)\left[1-\left\{\frac{(|x|-\tau)_+}{(R-\tau)}\right\}^3\right].$$

We observe that $\zeta(x)\leqslant 0<u(x)$ for $|x|\geqslant R$, $\zeta(x)\equiv m(\tau)<u(x)$ for $|x|<\tau$ and since m is strictly decreasing, $\zeta(\bar{x})=u(\bar{x})$ at some $\bar{x}$ with $|\bar{x}|=\tau$. Therefore, $u-\zeta$ attains a non-positive global minimum at some point x_R^τ such that $\tau\leqslant|x_R^\tau|<R$. By definition of viscosity solution we have

$$f\left(x_R^\tau,u\left(x_R^\tau\right)\right)\leqslant-\mathcal{M}^-\left(r,D^2\zeta\left(x_R^\tau\right)\right)-\sigma\left(\left|x_R^\tau\right|\right)\left|D\zeta\left(x_R^\tau\right)\right|. \tag{5.2}$$

Since ζ is radial we directly compute the right hand side and get

$$f\left(x_R^\tau,u\left(x_R^\tau\right)\right)\leqslant\frac{3\Lambda(|x_R^\tau|)m(\tau)}{(R-\tau)^3}\left\{2+\left(\frac{N-1}{|x_R^\tau|}-\frac{\sigma(|x_R^\tau|)}{\Lambda(|x_R^\tau|)}\right)\left(\left|x_R^\tau\right|-\tau\right)_+\right\}\left(\left|x_R^\tau\right|-\tau\right)_+.$$

If $|x_R^\tau|=\tau$, then $f(x_R^\tau,u(x_R^\tau))\leqslant 0$, contradicting (f_1). Thus, we may assume that $\tau<|x_R^\tau|<R$ and we have

$$f\left(x_R^\tau,u\left(x_R^\tau\right)\right)\leqslant Cm(\tau)\frac{1+\sigma^-(|x_R^\tau|)(R-\tau)}{(R-\tau)^2}, \tag{5.3}$$

for certain constant $C>0$. Now use the hypothesis (f_3)(i) to find a sequence $\{r_n\}$ diverging to infinity so that

$$\lim_{n\to\infty}\Psi_k(r_n)=h(k_1)=\infty, \tag{5.4}$$

with $k_1=m(r_1)/\varphi(r_1)$. We let $\tau=r_n$, $R=2r_n$ and $x_n=x_{2r_n}^{r_n}$, and recall that $r_n\leqslant|x_n|\leqslant 2r_n$. Next we see that $u(x_n)\to 0$ as $n\to\infty$, because (5.3) gives

$$\frac{|x_n|^2f(x_n,u(x_n))}{4(1+\sigma^-(|x_n|)|x_n|)}\leqslant Cm(r_n),$$

that contradicts (f_2) if $u(x_n)$, or a subsequence, is bounded away from zero. Then we use the monotonicity of $m(r)/\varphi(r)$ given by (5.1) and the fact that $u(x_n)\geqslant m(2r_n)$ to obtain

$$\frac{\varphi(2r_n)}{\varphi(r_n)}\frac{r_n^2}{1+\sigma^-(|x_n|)r_n}\frac{f(x_n,u(x_n))}{u(x_n)}\leqslant C. \tag{5.5}$$

But this contradicts (5.4), since by (5.1) $u(x_n)\geqslant m(2r_n)\geqslant k_1\varphi(2r_n)$, so that (5.5) gives that $\Psi_k(r_n)$ is bounded, completing the proof in this case.

Critical Case. If case (f_3)(ii) holds then there is no contradiction in case $h(k_1)<\infty$. In this case, arguing as above, we obtain $u(x_n)\to 0$ and, using hypothesis (1.12) and Proposition 3.3

$$\frac{r_n^2f(x_n,u(x_n))}{u(x_n)}\leqslant C, \tag{5.6}$$

for any sequence $\{r_n\}$ diverging to ∞. At this point we claim that

$$\lim_{r\to\infty}\frac{m(r)}{\varphi(r)}=\infty. \tag{5.7}$$

Assuming for a moment that (5.7) holds, we find M_k for every k so that $u(x)\geqslant k\varphi(x)$, for all $|x|\geqslant M_k$, consequently, from (5.6) we obtain that

$$\Psi_k(r_n)\leqslant\frac{r_n^2 f(x_n,u(x_n))}{u(x_n)}\leqslant C,$$

for n large. Since the sequence $\{r_n\}$ is arbitrary, we conclude that $h(k)\leqslant C$ for all k, which is a contradiction that completes the proof of the theorem.

Now we prove the claim (5.7). Let $\Omega_\tau=\{x\in\mathbb{R}^N\colon\ |x|>\tau,\ u(x)<\mu\}$, where $\tau>r_1$ and $\mu>0$ appears in (f_3). Ω_τ is open and nonempty. Next we consider the function

$$\Gamma(x):=-\varphi\big(|x|\big)\log\varphi\big(|x|\big)$$

and choose $\bar r\geqslant r_1$ such that $m(\tau)\leqslant\mu$ and $\Gamma(x)\leqslant\mu$, for all $|x|=\tau\geqslant\bar r$. Then we use (5.1) and the monotonicity of φ to find

$$\begin{aligned}\frac{|x|^2}{\varphi(|x|)}f\big(x,u(x)\big)&\geqslant k_1\frac{|x|^2}{1+\sigma^-(|x|)|x|}\frac{f(x,u(x))}{u(x)}\\&\geqslant k_1\frac{|x|^2}{1+\sigma^-(|x|)|x|}\inf_{k_1\varphi(|x|)\leqslant s\leqslant\mu}\frac{f(x,s)}{s}\\&\geqslant k_1\frac{\varphi(2\tau)}{\varphi(\tau)}\inf_{y\in B_{2\tau}\setminus B_\tau}\frac{|y|^2}{1+\sigma^-(|y|)|y|}\inf_{k_1\varphi(|y|)\leqslant s\leqslant\mu}\frac{f(y,s)}{s}\\&\geqslant k_1\Psi_{k_1}(\tau).\end{aligned}$$

From here, taking $\tau=|x|$ and using (1.11) we obtain

$$f\big(x,u(x)\big)\geqslant C\frac{\varphi(|x|)}{|x|^2}, \tag{5.8}$$

for certain constant C, for all $x\in\Omega_{\bar r}$. On the other hand, computing directly and using Proposition 3.2 we find $C_0>0$ such that

$$\mathcal{M}^-\big(r,D^2\Gamma\big)+\sigma(r)|D\Gamma|\geqslant -C_0\frac{\varphi(|x|)}{|x|^2},\quad |x|\geqslant\bar r. \tag{5.9}$$

Then we let $\tilde C:=\min\{\frac{C}{C_0},-k_1/\log\varphi(\bar r),1\}$ and from (1.8), (5.8) and (5.9) we obtain

$$\mathcal{M}^-\big(r,D^2(u+\varepsilon)\big)+\sigma(r)\big|D(u+\varepsilon)\big|\leqslant\tilde C\big(\mathcal{M}^-\big(D^2\Gamma\big)+\sigma(r)|D\Gamma|\big),$$

for all $x\in\Omega_{\bar r}$ and $\varepsilon>0$. By the choice of $\tilde C$ we have then

$$u(x)+\varepsilon\geqslant m(\bar r)\geqslant k_1\varphi(\bar r)=\frac{k_1}{-\log\varphi(\bar r)}\Gamma(\bar r)\geqslant\tilde C\Gamma(\bar r),\quad \text{for all } x\in\partial B_{\bar r}$$

and, since $\lim_{r\to\infty}\Gamma(r)=0$, there is R such that

$$u(x)+\varepsilon \geqslant \varepsilon \geqslant \Gamma(R) \geqslant \tilde{C}\Gamma(R), \quad \text{for all } x \in \partial B_R.$$

We also have $u(x)=\mu \geqslant \tilde{C}\Gamma(|x|)$ for $x\in (B_R\setminus \overline{B}_{\bar r})\cap \partial\Omega_{\bar r}$, thus

$$u(x)+\epsilon \geqslant \tilde{C}\Gamma\big(|x|\big), \quad x\in \partial(B_R\cap \Omega_{\bar r}).$$

Then we use Comparison Principle and then take $R\to\infty$ and $\epsilon\to 0^+$ to get

$$u(x)\geqslant \tilde{C}\Gamma\big(|x|\big), \quad x\in \Omega_{\bar r},$$

which implies (5.7). $\square$

Remark 5.1. If we have a non-negative super-solution u of (1.9) and the fundamental solution φ satisfies $\varphi(r)\to -\infty$ then u has to be constant. This result is usually known as Liouville property.

Now we prove Theorem 1.2 on the Liouville property in an exterior domain.

Proof of Theorem 1.2. According to Proposition 4.1 for some $\bar r\geqslant r_0$:

Case 1: $m_0(r)$ is strictly decreasing and $M(r)$ is constant for $r>\bar r$ or
Case 2: $M(r)$ is strictly increasing and $m_0(r)$ is constant for $r>\bar r$.

We recall the new definition of m_0 given in the proof of Proposition 4.1, for notational convenience, we just write m instead of m_0, from now on.

Proof in Case 1: If $\varphi(r)\to 0$ as $r\to\infty$, the proof follows step by step that of Theorem 1.1. A small change is needed in the complementary case: Given $\bar r<r_1<r_2$ we use inequality (4.2) and that $m(r_2)\geqslant 0$, to find

$$m(r)\geqslant m(r_1)\left(1-\frac{\varphi(r)}{\varphi(r_2)}\right) \quad \text{for } r\in [r_1,r_2]. \tag{5.10}$$

Then, we let $r_2\to\infty$ obtain $m(r)\geqslant m(r_1)$ for $r\geqslant r_1$, which is impossible since $m(r)$ is strictly decreasing.

Proof in Case 2 and sub-critical: We consider the test function

$$\zeta\big(|x|\big)=M(R)\left[1-\left\{\frac{(R-|x|)_+}{(R-\tau)}\right\}^3\right],$$

where $R>\tau\geqslant\bar r$ are parameters. As in the proof of Theorem 1.1, we see that $u-\zeta$ attains a non-positive global minimum at some point x_R^τ such that $\tau<|x_R^\tau|\leqslant R$ and $u(x_R^\tau)\leqslant M(R)$. Then, by the definition of viscosity solution and computing the differential operator we obtain

$$f\big(x_R^\tau, u\big(x_R^\tau\big)\big)\leqslant CM(R)\frac{1+\sigma^-(|x_R^\tau|)(R-\tau)}{(R-\tau)^2}. \tag{5.11}$$

Assume that $\lim_{r\to\infty}\psi(r)=0$ then, by Theorem 4.3 we have that $M(R)$ is bounded. Let us choose $\{r_n\}$ diverging to infinity and let $\tau=r_n$, $R=2r_n$ and write $x_n=x_{2r_n}^{r_n}$. We notice that $r_n\leqslant |x_n|\leqslant 2r_n$ and $u(x_n)\leqslant M(2r_n)$, so that $u(x_n)$ is bounded. But then, from (5.11), we find that

$$\frac{r_n^2}{1+\sigma(|x_n|)r_n} f\big(x_n, u(x_n)\big) \leqslant CM(2r_n), \tag{5.12}$$

contradicting (f_2).

Now we assume that $\lim_{r\to\infty}\psi(r)=\infty$ and we take, without loss of generality, that $\psi(r_0)=0$. From (4.3) and (5.11) we have

$$\frac{f(x_R^\tau, u(x_R^\tau))}{u(x_R^\tau)} \leqslant \frac{f(x_R^\tau, u(x_R^\tau))}{M(\tau)} \leqslant C\frac{\psi(R)}{\psi(\tau)}\frac{1+\sigma^-(|x_R^\tau|)(R-\tau)}{(R-\tau)^2}. \tag{5.13}$$

Next we use the hypothesis (f_4)(i) to find $\{r_n\}$ diverging to infinity so that

$$\lim_{n\to\infty}\tilde{\Psi}_{k_1}(r_n) = \tilde{h}(k_1) = \infty, \tag{5.14}$$

with $k_1 = M(\bar{r})/\psi(\bar{r})$. We let $\tau = r_n$, $R = 2r_n$ and we write $x_n = x_{2r_n}^{r_n}$. We notice that $r_n \leqslant |x_n| \leqslant 2r_n$ and $u(x_n) \leqslant M(2r_n)$, so that $u(x_n) \leqslant k_1\psi(2r_n)$, where this last inequality comes from (4.3). Again we have (5.12), but now we conclude that $M(2r_n)$ and consequently, $M(r_n)$ and $u(x_n)$ diverge to infinity. Now, from (5.13) we have the following inequality that contradicts (5.14)

$$\frac{\psi(r_n)}{\psi(2r_n)}\frac{r_n^2}{1+\sigma(|x_n|)r_n}\frac{f(x_n,u(x_n))}{u(x_n)} \leqslant C.$$

Proof in Case 2 and critical: Under hypothesis (f_4)(ii) then there is no contradiction in case $\tilde{h}(k_1) < \infty$. Arguing as above, using hypothesis (1.12) and Proposition 3.4 we obtain

$$\frac{r_n^2 f(x_n, u(x_n))}{u(x_n)} \leqslant C, \tag{5.15}$$

for any sequence $\{r_n\}$ diverging to ∞. At this point we claim that

$$\lim_{r\to\infty}\frac{M(r)}{\psi(r)} = 0. \tag{5.16}$$

Assuming that the claim is true, for every k there is M_k so that

$$M(r) \leqslant k\psi(r), \quad \text{for all } r \geqslant M_k,$$

consequently, from (5.15), we obtain that

$$\tilde{\Psi}_k(r_n) \leqslant \frac{r_n^2 f(x_n, u(x_n))}{u(x_n)} \leqslant C$$

for all n large and then

$$\limsup_{n\to\infty}\tilde{\Psi}_k(r_n) \leqslant C.$$

Since this inequality holds for all sequence $\{r_n\}$ diverging to infinity, we find that $\tilde{h}(k) \leqslant C$ for all k, contradicting (f_4)(ii).

Thus, we only need to prove (5.16) to complete the proof. We define the open set $\Omega_{\bar{r}} := \{x \in \Omega,\ |x| > \bar{r},\ u(x) < 3k_1\psi(|x|)\}$, which is nonempty since, given $r > \bar{r}$ we can find $\bar{x}$ with $|\bar{x}| = r$ and $u(\bar{x}) = M(r) \leqslant k_1\psi(r) < 3k_1\psi(r)$.

Assume our claim is not true, then there exists $\tilde{k} \in (0, k_1]$ such that

$$\lim_{r\to\infty} \frac{M(r)}{\psi(r)} = \tilde{k}. \tag{5.17}$$

Then we have

$$\tilde{k}\psi\big(|x|\big) \leqslant M\big(|x|\big) \leqslant k_1\psi\big(|x|\big), \quad |x| > \bar{r},$$

and $\tilde{k}\psi(|x|) \leqslant u(x)$ for all $x \in \Omega_{\bar{r}}$. From here and monotonicity of ψ we find

$$\begin{aligned}\frac{|x|^2}{\psi(|x|)} f\big(x, u(x)\big) &\geqslant \tilde{k}\frac{\psi(|x|)}{\psi(2|x|)} \inf_{y\in B_{2|x|}\setminus B_{|x|}} \frac{|y|^2}{1+\sigma^-(|y|)|y|} \inf_{\mu\leqslant s\leqslant \tilde{k}\psi(2|y|)} \frac{f(y,s)}{s} \\ &\geqslant \tilde{k}\tilde{\Psi}_{\tilde{k}}\big(|x|\big).\end{aligned}$$

Then, from (1.13), there exists $c > 0$ such that

$$f\big(x, u(x)\big) \geqslant c\frac{\psi(|x|)}{|x|^2}, \quad x \in \Omega_{\bar{r}}. \tag{5.18}$$

Next we define the auxiliary function

$$\tilde{\Gamma}(r) = \frac{\psi(r)}{\log\psi(r)}, \quad r = |x|.$$

Computing directly we obtain

$$\begin{aligned}\mathcal{M}^-\big(r, D^2\tilde{\Gamma}\big) + \sigma(r)|D\tilde{\Gamma}| \geqslant{}& \frac{\log\psi(r) - 1}{\log^2\psi(r)}\big(\mathcal{M}^-\big(r, D^2\psi\big) + \sigma(r)|D\psi|\big) \\ &- \Lambda\frac{\log\psi(r) - 2}{\log^3\psi(r)}\frac{(\psi'(r))^2}{\psi(r)}.\end{aligned}$$

Since ψ is the fundamental solution, by Proposition 3.4 we get

$$\mathcal{M}^-\big(r, D^2\tilde{\Gamma}\big) + \sigma(r)|D\tilde{\Gamma}| \geqslant -C\frac{\psi(r)}{r^2\log^2\psi(r)}.$$

On the other hand we can find $r_1 < r_2 < r_3$ such that

$$\log\big(\psi(r_1)\big) = n^2, \qquad \log\big(\psi(r_2)\big) = 2n^2 \quad\text{and}\quad \log\big(\psi(r_3)\big) = 3n^2,$$

with $n \in \mathbb{N}$ to be chosen later. We define

$$w(x) := \frac{M(r_3)}{\tilde{\Gamma}(r_3)}\big(\tilde{\Gamma}(r) - \tilde{\Gamma}(r_1)\big), \quad x \in B_{r_3} \setminus B_{r_1}.$$

There exists $n_0 > 0$ such that, for $n \geqslant n_0$ and $x \in (B_{r_3} \setminus B_{r_1}) \cap \Omega_{\bar{r}}$, we have

$$\mathcal{M}^-\big(r, D^2 w\big) + \sigma(r)|Dw| \geqslant -C \frac{\psi(r)}{r^2 \log^2(\psi(r))} \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}$$
$$\geqslant -f(x,u) \geqslant \mathcal{M}^-\big(D^2 u\big) + \sigma(r)|Du|,$$

where we used (5.18). Next we prove that

$$u(x) \geqslant w\big(|x|\big), \quad x \in \partial\big((B_{r_3} \setminus B_{r_1}) \cap \Omega_{\bar{r}}\big).$$

This is obvious for $|x| = r_3$ or $|x| = r_1$. For $x \in (B_{r_3} \setminus \overline{B}_{r_1}) \cap \partial\Omega_{\bar{r}}$ we have

$$w(x) = \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}\left(\frac{\psi(r)}{\log\psi(r)} - \frac{\psi(r_1)}{\log\psi(r_1)}\right)$$
$$\leqslant k_1\psi(r)\frac{\log\psi(r_3)}{\log\psi(r)} \leqslant k_1\psi(r)\frac{\log\psi(r_3)}{\log\psi(r_1)} = 3k_1\psi(r) = u(x).$$

Then we apply the Comparison Principle to obtain

$$u(x) \geqslant w(x) = \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}\left(\frac{\psi(r)}{\log\psi(r)} - \frac{\psi(r_1)}{\log\psi(r_1)}\right),$$

for $x \in (B_{r_3} \setminus \overline{B}_{r_1}) \cap \Omega_{\bar{r}}$. Then we take $x \in \partial B_{r_2} \cap \Omega_{\bar{r}}$, and we get

$$M(r_2) \geqslant \frac{M(r_3)\log\psi(r_3)}{\psi(r_3)}\left(\frac{\psi(r_2)}{\log\psi(r_2)} - \frac{\psi(r_1)}{\log\psi(r_1)}\right)$$

and then

$$\frac{M(r_2)}{\psi(r_2)} \geqslant \frac{M(r_3)}{\psi(r_3)}\left(\frac{3}{2} - \frac{3}{e^{n^2}}\right),$$

which is impossible if n is large enough, in view of (5.17). □

6. Proof of Theorem 2.3

In this section we prove Theorem 2.3. We observe that part (i) is a consequence of Theorem 1.1. In order to prove part (ii) we need a preliminary lemma. Given $\delta > 0$ we define

$$U_\delta(r) = \varphi(r)\big(-\log\varphi(r)\big)^\delta, \tag{6.1}$$

where $r > \bar{r} \geqslant e$ and $\bar{r}$ is such that $\varphi(\bar{r}) < 1$.

Lemma 6.1. *Assume the hypothesis of Theorem 2.3 and let $u > 0$ be a solution of (1.8). Then, for any $\delta > 0$, there exists $C_\delta \in (0,1)$ such that*

$$u(x) \geqslant C_\delta U_\delta\big(|x|\big), \quad |x| \geqslant \bar{r}.$$

Proof. By direct computation we find a constant $c > 0$ such that

$$\mathcal{M}^-\big(r, D^2U_\delta\big) + \sigma\big(|x|\big)|DU_\delta| \geqslant -c\frac{(\log|x|)^{-2+\delta}}{|x|^{2+\alpha}}. \tag{6.2}$$

On the other hand, by Hadamard theorem, there exists $c > 0$ such that $u(x) \geqslant c\varphi(|x|)$, for $r \geqslant \bar{r}$ and then there exists $\tilde{C} > 0$ such that

$$f(x,u) \geqslant f\big(x, c\varphi\big(|x|\big)\big) \geqslant \tilde{C}\frac{(\log|x|)^{\nu-p}}{|x|^{\alpha p}}, \quad \text{for all } |x| \geqslant \bar{r}. \tag{6.3}$$

If $0 < \delta \leqslant \delta_0 = 1 + \nu - \frac{2}{\alpha}$ and $\varepsilon > 0$ and using (6.2) and (6.3) we get

$$\mathcal{M}^-\big(r, D^2U_\delta\big) + \sigma\big(|x|\big)|DU_\delta| \geqslant \mathcal{M}^-\big(r, D^2(u+\epsilon)\big) + \sigma\big(|x|\big)\big|D(u+\epsilon)\big|, \quad |x| \geqslant \bar{r}.$$

By appropriately choosing C and R we find that

$$u(x) + \epsilon \geqslant CU_\delta\big(|x|\big), \quad x \in \partial(B_R \setminus B_{\bar{r}}),$$

thus, by the Comparison Principle and letting $R \to \infty$ and $\epsilon \to 0$, we obtain

$$u(x) \geqslant CU_\delta\big(|x|\big), \quad x \in B^c_{\bar{r}}. \tag{6.4}$$

For $\delta \in (\delta_0, (2 + \frac{2}{\alpha})\delta_0]$, we use (6.4) with $\delta = \delta_0$ to get, as in (6.3), that

$$f(x,u) \geqslant f\big(x, CU_{\delta_0}\big(|x|\big)\big) \geqslant \tilde{C}\frac{(\log|x|)^{\nu-p+\delta_0 p}}{|x|^{\alpha p}}. \tag{6.5}$$

Then, by making $\bar{C}$ smaller if necessary, we obtain

$$\mathcal{M}^-\big(r, D^2U_\delta\big) + \sigma\big(|x|\big)|DU_\delta| \geqslant -f(x,u), \tag{6.6}$$

for all $\delta \in (\delta_0, (2 + \frac{2}{\alpha})\delta_0]$. Then we use the Comparison Principle as before to prove that, for certain constant C, we have $u(x) \geqslant CU_\delta(|x|)$, for $x \in B^c_{\bar{r}}$. Repeating the argument we can prove similar result for every $\delta > 0$. $\square$

Proof of Theorem 2.3(ii). We assume that there exists a positive solution u of (1.8). By arguments as in the proof of Theorem 1.1, we find x^r_R such that $r < |x^r_R| < R$ and

$$u\big(x^r_R\big)^p\big(\big|\log u\big(x^r_R\big)\big| + 1\big)^\nu \leqslant 3m(r)\frac{\Lambda(x^r_R)(N+1) + \sigma^-(|x^r_R|)(R-r)}{(R-r)^2}.$$

From here and the monotonicity of $r \to \frac{m(r)}{\varphi(r)}$, we obtain

$$u\big(x^r_R\big)^{p-1}\big(\big|\log u\big(x^r_R\big)\big| + 1\big)^\nu \leqslant C\frac{\varphi(r)}{\varphi(R)}\frac{1 + \sigma^-(|x^r_R|)(R-r)}{(R-r)^2}.$$

At this point we choose $R = 2r$, we write $x_r = x^r_{2r}$ and we obtain

$$|x_r|^2 u(x_r)^{p-1}\big(\big|\log u(x_r)\big| + 1\big)^{\nu} \leqslant C, \tag{6.7}$$

for certain positive constant C. From here we easily conclude that $u(x_r) \to 0$, as $r \to \infty$. Now we choose $\delta > 0$ such that

$$\nu - \frac{2}{\alpha} + \frac{2\delta}{\alpha} > 0$$

and we use Lemma 6.1 to obtain

$$|x_r|^2 u(x_r)^{p-1}\big(\big|\log u(x_r)\big| + 1\big)^{\nu} \geqslant |x_r|^2\big(C_\delta U_\delta(x_r)\big)^{p-1}\big(\big|\log C_\delta U_\delta(x_r)\big| + 1\big)^{\nu}.$$

From the choice of δ and the definition of U_δ we see that the right hand side diverges to infinity, while from (6.7) the left hand side is bounded. This is a contradiction that completes the proof. □

We continue by proving the existence of a positive solution.

Proof of Theorem 2.3(iii). We consider the function $U(x) = \varphi(|x|)^{\theta}$, where $\theta \in (0,1)$ will be chosen later. By direct computation we find a constant $C > 0$ and $R > 0$ so that

$$\mathcal{M}^-\big(r, D^2U\big) + \sigma\big(|x|\big)|DU| \leqslant -C\frac{(\log(|x|))^{-\theta}}{|x|^{2+\alpha\theta}}$$

for $|x| > R$. On the other hand, we have

$$U^p(x)\big(\big|\log U(x)\big| + 1\big)^{\nu} \leqslant C|x|^{-\alpha\theta p}\big(\log\big(|x|\big)\big)^{\nu-\theta p}.$$

Now we choose

$$\theta = \frac{1}{2}\left(1 + \frac{2}{\alpha}\frac{1}{p-1}\right) < 1$$

and we use our assumption $p > \frac{2}{\alpha} + 1$ to obtain $\theta < 1$ and $(p-1)\theta > \frac{2}{\alpha}$. From here we find $\bar{R} > R$ such that for all $x \in B^c_{\bar{R}}$

$$\mathcal{M}^-\big(r, D^2U\big) + \sigma\big(|x|\big)|DU| + U^p(x)\big(\big|\log U(x)\big| + 1\big)^{\nu} \leqslant 0. \tag{6.8}$$

We notice that $U_\varepsilon(x) = \varepsilon U(x)$ also satisfies (6.8) if ε is small, since

$$\mathcal{M}^-\big(r, D^2U_\varepsilon(x)\big) + \sigma\big(|x|\big)\big|DU_\varepsilon(x)\big| \leqslant -C\varepsilon, \quad x \in B_{\bar{R}} \setminus B_{\frac{1}{2}}$$

and

$$U_\varepsilon{}^p(x)\big(\big|\log U_\varepsilon(x)\big| + 1\big)^{\nu} = o(\varepsilon), \quad x \in B_{\bar{R}} \setminus B_{\frac{1}{2}},$$

for $\varepsilon > 0$ small enough and $C > 0$. Thus (6.8) can be extended to $B^c_{\frac{1}{2}}$. Finally we let w be the unique radial solution of the problem

$$\begin{cases} \lambda\Delta(w) + \sigma\big(|x|\big)|Dw| = -1, & \text{in } B_1, \\ w(x) = 0, & \text{on } \partial B_1, \end{cases} \tag{6.9}$$

and let $w_\varepsilon = \varepsilon w$, with $\varepsilon > 0$. It is easy to see that there exists ε_0 small so that w_{ε_0} satisfies (1.8) in B_1. Since w is positive in B_1, there exists $\varepsilon_1 > 0$ such that $w_{\varepsilon_0}(x) > U_{\varepsilon_1}(x)$ for $|x| = 1/2$. On the other hand $U_{\varepsilon_1} \to \infty$ as $r \to 0$, so there exists $r \in (0, \frac{1}{2})$ such that $w(x) = U_{\varepsilon_1}(x)$ for all $|x| = r$. Now we define $V(x) = U_{\varepsilon_1}$ if $x \in B^c_r$ and $V(x) = w_{\epsilon_0}$ if $x \in B_r$, which is a solution of (1.8) in $\mathbb{R}^N$, completing the proof. $\square$

7. Liouville property for $f(x, u) = h(x)u$

In this section we study the Liouville type theorem for Eq. (1.17) in exterior domains, when the functions h and σ satisfy (h_1), (h_2) and (h_3). Before continuing we give two examples of functions satisfying (h_2):

Example 1. σ is a negative function such that $\liminf_{r\to\infty} \sigma(r) = c_0$, for some $c_0 < 0$. Then there is R_0 such that $c_0/2 \geqslant \sigma(r) \geqslant 2c_0$ for all $r \geqslant R_0$ and we can choose $\kappa(r) \equiv -\frac{1}{c_0}$.

We observe that if $\lim_{r\to 0} \sigma(r) = 0$, we may change σ by $\sigma - \varepsilon$, with $\varepsilon > 0$ and small enough so that inequality (1.17) and (h_3) are still satisfied.

Example 2. If σ is of class C^1 and satisfies

$$\lim_{r\to\infty} \sigma(r) = -\infty \quad \text{and} \quad \lim_{r\to\infty} \sigma'(r)/\sigma^2(r) = 0,$$

then we just let $\kappa = 1/\sigma$. If σ is not C^1, but the first limit still holds and $1/\sigma$ is convex, or if it does not differ too much from a convex function, then taking κ as an appropriate approximation of $1/\sigma$ will work.

Lemma 7.1. *Assume σ and κ satisfy hypotheses (h_1) and (h_2). Then $\varphi(r) \to 0$ and $\psi(r) \to \infty$ as $r \to \infty$, and for $\varepsilon > 0$, there exists $\bar{R} > R_0$ such that*

$$\frac{\varphi(r - \kappa(r))}{\varphi(r)} \leqslant (1+\varepsilon)e^{\frac{\mu}{\lambda_0}} \quad \textit{and} \quad \frac{\psi(r)}{\psi(r - \kappa(r))} \leqslant (1+\varepsilon)e^{\frac{\mu}{\lambda_0}}, \quad \forall r \geqslant \bar{R}.$$

Proof. As we have observed above, we may always assume that $|\sigma(r)| \geqslant \sigma_0 > 0$ for all r. We also see from (1.18) that for $r \geqslant \bar{R}$ we have $\kappa(r) \leqslant r/2$, for large. Next we see that

$$\int_{r-\kappa(r)}^{r} \frac{n(\tau)}{\tau}\, d\tau \leqslant C \int_{r-\kappa(r)}^{r} \frac{1}{\tau}\, d\tau \leqslant C\frac{\kappa(r)}{r - \kappa(r)} \leqslant 2C\frac{\kappa(r)}{r}. \tag{7.1}$$

By definition of φ and for $\varepsilon > 0$, we find $\bar{R} > 0$ large such that

$$\frac{(\varphi(r - \kappa(r)))'}{(\varphi(r))'} = \frac{\varphi'(r - \kappa(r))(1 - \kappa'(r))}{\varphi'(r)}$$

$$= \left(1-\kappa'(r)\right)\exp\left(\int_r^{r-\kappa(r)}\left(\frac{\sigma(\tau)}{m_\lambda(\tau)}-\frac{n(\tau)}{\tau}\right)d\tau\right)$$
$$\leqslant \left(1-\kappa'(r)\right)\exp\left(\frac{\kappa(r)}{\lambda_0}\left(\max_{r-\kappa(r)\leqslant s\leqslant r}\left|\sigma(s)\right|+\frac{2C\lambda_0}{r}\right)\right)$$
$$\leqslant (1+\varepsilon)e^{\frac{\mu}{\lambda_0}},$$

where we have used (7.1) and (1.19). Then we have

$$\left(\varphi\left(r-\kappa(r)\right)\right)' \geqslant (1+\varepsilon)e^{\frac{\mu}{\lambda_0}}\left(\varphi(r)\right)', \quad r>\bar{R}.$$

Integrating in $[r,R]$, letting R go to infinity and using the fact that $\varphi(r)\to 0$ we get the result. Proceeding as above, for $\varepsilon>0$ there exists $\bar{R}$ so that

$$\left(\psi(r)\right)' \leqslant \left(1+\frac{1}{2}\varepsilon\right)e^{\frac{\mu}{\lambda_0}}\left(\psi\left(r-\kappa(r)\right)\right)', \quad r>\bar{R}.$$

Then we integrate in $[\bar{R},r]$ and we divide by $\psi(r-\kappa(r))$ to get

$$\frac{\psi(r)}{\psi(r-\kappa(r))}-\frac{\psi(\bar{R})}{\psi(r-\kappa(r))} \leqslant \left(1+\frac{1}{2}\varepsilon\right)e^{\frac{\mu}{\lambda_0}}\left(1-\frac{\psi(r-\kappa(r))}{\psi(\bar{R}-\kappa(\bar{R}))}\right).$$

Using that $\psi(r)\to\infty$ as $r\to\infty$, we get the result. □

Proof of Theorem 1.3. If $u\geqslant 0$ is a non-trivial solution of (1.17), then

$$\mathcal{M}^-\left(r,D^2u\right)+\sigma\left(|x|\right)|Du| \leqslant 0, \quad x\in\Omega$$

and $u>0$ in Ω. Then we use Proposition 4.1 to consider two cases in the proof, depending on the behavior of $m_0(r)$ and $M(r)$, as defined in (1.14).

Case 1. $m_0(r)$ is strictly decreasing and $M(r)$ is constant for $r>\bar{r}$. We consider the test function

$$\zeta(x)=m_0(r)\left[1-\left\{\frac{(|x|-r)_+}{(R-r)}\right\}^3\right],$$

where r, R are parameters such that $R>r>\max\{\bar{r},r_0\}$. Proceeding as in the proof of Theorem 1.1, we obtain x_R^r such that $r<|x_R^r|<R$ and

$$h\left(|x_R^r|\right)u\left(x_R^r\right) \leqslant \frac{3\Lambda(x_R^r)m_0(r)}{(R-r)^3}\left\{2+\left(\frac{N-1}{|x_R^r|}-\frac{\sigma(|x_R^r|)}{\Lambda(x_R^r)}\right)\left(|x_R^r|-r\right)\right\}\left(|x_R^r|-r\right).$$

From here we obtain

$$h\left(|x_R^r|\right)u\left(x_R^r\right) \leqslant 3m_0(r)\left\{\frac{2\Lambda_0+|\sigma(x_R^r)|(R-r)+(N-1)(R-r)r^{-1}}{(R-r)^2}\right\}$$

and then, by the monotonicity of $r \to \frac{m_0(r)}{\varphi(r)}$,

$$h\big(\big|x_R^r\big|\big) \leqslant 3\frac{\varphi(r)}{\varphi(R)}\left\{\frac{2\Lambda_0 + |\sigma(x_R^r)|(R-r) + (N-1)(R-r)r^{-1}}{(R-r)^2}\right\}. \tag{7.2}$$

Next we choose $r = R - \kappa(R)$ with $R \geqslant \bar{R}$, and we use Lemma 7.1 to find

$$h\big(|x_R|\big) \leqslant (1+\varepsilon)e^{\frac{\mu}{\lambda_0}}\left\{\frac{2\Lambda_0 + |\sigma(x_R^r)|\kappa(R) + \frac{(N-1)\kappa(R)}{R-\kappa(R)}}{(\kappa(R))^2}\right\}. \tag{7.3}$$

From here, taking $R = r_n$ as in the hypothesis, we obtain

$$\lim_{n\to\infty}\inf_{r\in(r_n-\kappa(r_n),r_n)}\big[h(r) - (1+\varepsilon)e^{\frac{\mu}{\lambda_0}}(2\Lambda_0+1)\sigma^2(r)\big] \leqslant 0.$$

If $\varepsilon > 0$ is chosen properly, we obtain a contradiction with (1.20).

Case 2. $M(r)$ is strictly increasing and $m_0(r)$ is constant for $r > \bar{r}$. In this case we replace m_0 by $M(r)$ in the definition of the test function and we repeat step by step the proof, using Theorem 4.3 and the properties of ψ given in Lemma 7.1. □

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