Interaction of Defects in Two-Dimensional Systems

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We derive equations of motion for a diluted gas of spiral defects in the two-dimensional complex Ginzburg-Landau equation. The interaction of two defects is treated and our predictions agree with a recent numerical experiment.

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Defects in two-dimensional nonequilibrium extended systems play an important role.[1-3] In particular, it has been suggested that spiral defects could destroy long-range order in these systems,[1,2] and recent numerical experiments have provided clear evidence of the mechanism involved.[3] It has indeed been shown by Coullet, Gil, and Lega that if one starts in the Ginzburg-Landau equation with complex coefficients (which is taken as a representative example) with a homogeneous initial condition, strong phase gradients arise as soon as the relevant parameter crosses the point where the phase instability starts. At the same time spirals start to appear and their density tends to stabilize as one continues the variation of the parameter; in this state the equal-time correlation goes rapidly to zero with distance, thus showing that the system is now disorganized. It is this state that we describe here as a diluted gas of spiral defects interacting with a global phase. We consider then the Ginzburg-Landau equation with complex coefficients,

$$\partial_t A = \mu A + (1+i\alpha)\nabla^2 A - (1+i\beta)|A|^2 A\,, \qquad (1)$$

and we shall see that a solution of (1) exists in the form of a dominating term vanishing at N moving points $\{\mathbf{r}_1(t), \mathbf{r}_2(t), \ldots, \mathbf{r}_N(t)\}$ representing N spirals plus a small correction term.

We recall first what is known about single-spiral solutions.[2,4,5] Doing the transformation $A \to A\exp(-i\alpha\mu t)$ we obtain

$$\partial_t A = (1+i\alpha)\left[\mu A + \nabla^2 A - \frac{\gamma+i\nu}{1+\alpha^2}|A|^2 A\right], \qquad (2)$$

with $\gamma = 1+\alpha\beta$, $\nu = \beta-\alpha$. The defects here are spiral solutions[2,4,5] of (2) of the form

$$A(r,\varphi,t) = D(r)\exp\{i[\omega t + m\varphi - S(r)]\}\,, \qquad (3)$$

where $m \in Z$ and (r,φ) are polar coordinates in the plane. The functions $D(r)$ and $S_r(r) \equiv dS/dr$ satisfy the coupled ordinary differential equations

$$\left[\mu - \frac{\alpha\omega}{1+\alpha^2}\right]D(r) + \nabla_r^2 D - \left[\frac{m^2}{r^2} + S_r^2\right]D - \frac{\gamma}{1+\alpha^2}D^3 = 0\,, \qquad (4a)$$

$$2S_r D_r + \nabla_r^2 S D + \frac{\omega}{1+\alpha^2}D + \frac{\nu}{1+\alpha^2}D^3 = 0\,, \qquad (4b)$$

where

$$D_r \equiv \frac{dD}{dr}\,, \quad \nabla_r^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\,.$$

We consider from now on the case of one-armed spirals ($|m| = 1$). One has the asymptotic behavior

$$D(r) \to \begin{cases} \lambda r\,, & r \to 0\,, \\ D^\infty - \dfrac{k(1+\alpha^2)}{2\nu D^\infty}\dfrac{1}{r}\,, & r \to \infty\,, \end{cases} \qquad (5)$$

$$S_r(r) \to \begin{cases} \dfrac{\nu}{1+\alpha^2}\left[\dfrac{D^{\infty 2}}{4}r - \dfrac{\lambda^2}{6}r^3\right], & r \to 0\,, \\ k + \dfrac{\gamma}{2\nu}\dfrac{1}{r}\,, & r \to \infty\,, \end{cases} \qquad (6)$$

with $D^\infty = (\mu - k^2)^{1/2}$, $\omega = -\nu D^{\infty 2}$. Here λ and k are functions of (α,β), and k vanishes for $\nu = 0$. The region around the origin where $D(r)$ has not attained its asymptotic value D^∞ is called the core of the defect. We call ϵ its radius.

The solution (3) is topologically stable in the sense that it cannot disappear by continuous deformation. The amplitude of (3) vanishes at the origin and conversely a solution of (2) which vanishes at some point behaves locally as (3). Up to here, all of this is known. We remember now that, as stated before, numerical experiments show that for $\gamma < 0$ spirals appear spontaneously and a diluted gas of these defects establishes itself and dominates the behavior of the system. Consequently, the amplitude A vanishes at N points $\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N\}$,

$|\mathbf{r}_i-\mathbf{r}_j|\gg\epsilon$. We represent this situation by the *Ansatz* $A=(R^{(0)}+w)\exp(i\Theta^{(0)})$, where $R^{(0)}\exp(i\Theta^{(0)})$ represents the N spirals and $w\in\mathbb{C}$ is a small correction. We put

$$R^{(0)}=\sum_{i=1}^{N}[D(\rho_i)-D^\infty]+D^\infty \tag{7a}$$

and

$$\Theta^{(0)}=\omega t+\sum_{i=1}^{N}m_i\varphi_i-\sum_{i=1}^{N}\mathcal{S}(\rho_i) -k\min\{\rho_1,\rho_2,\ldots,\rho_N\}+\Psi(\mathbf{r},t)\,, \tag{7b}$$

where $\boldsymbol{\rho}_i\equiv\mathbf{r}-\mathbf{r}_i$, $\rho_i=|\boldsymbol{\rho}_i|$, $\mathcal{S}(\rho_i)\equiv S(\rho_i)-k\rho_i$, $\sin\varphi_i(\mathbf{r})=(y-y_i)/\rho_i$, $\cos\varphi_i(r)=(x-x_i)/\rho_i$, and $\Psi(\mathbf{r},t)$ is a slowly varying phase.

One can easily check that $A^{(0)}$ is not a solution of (2) and we account for this by allowing spirals to move [the $\{\mathbf{r}_i\}$ then become functions of time $\{\mathbf{r}_i(t)\}$] and considering that a small correction w arises. In order to obtain equations of motion for the new variables $\{\mathbf{r}_i(t),\Psi(\mathbf{r},t)\}$ we replace the *Ansatz* in (2) and, assuming that $\dot{\mathbf{r}}_i(t)$, $\dot\Psi(\mathbf{r},t)$, and w are small quantities of the same order [in fact we are assuming that the time dependence enters only through $\mathbf{r}_i(t)$ and $\Psi(\mathbf{r},t)$, and consequently w is a functional of them], we obtain in lowest order an equation of the form $\mathcal{L}w=I$. The solvability condition of this last equation is I orthogonal to kernel($\mathcal{L}^\dagger$) ($\mathcal{L}^\dagger\neq\mathcal{L}$ here since $\mathcal{L}$ is not self-adjoint) and gives equations for $\dot{\mathbf{r}}_i(t)$ and $\dot\Psi(\mathbf{r},t)$. Here we use $|\mathbf{r}_i-\mathbf{r}_j|\gg\epsilon$ and keep only the leading terms. We obtain

$$\frac{d\mathbf{r}_k}{dt}=2\left[\sum_{i\neq k}^{N}\left(m_im_k-\frac{\alpha\gamma}{2v}\right)\frac{\mathbf{r}_{ik}}{r_{ik}^2}+m_k\sum_{i\neq k}^{N}\left(\alpha m_im_k+\frac{\gamma}{2v}\right)\hat{\mathbf{z}}\frac{\mathbf{r}_{ik}}{r_{ik}^2}-m_k\hat{\mathbf{z}}\times\nabla\Psi|_{\mathbf{r}_k}+\alpha\nabla\Psi|_{\mathbf{r}_k}\right], \tag{8}$$

where $\mathbf{r}_{ik}=\mathbf{r}_k-\mathbf{r}_i$, $r_{ik}=|\mathbf{r}_{ik}|$, $\hat{\mathbf{r}}_{ik}\equiv\mathbf{r}_{ik}/r_{ik}$, and $\hat{\mathbf{z}}$ is a unitary vector orthogonal to the plane of the system. The first sum in (8) gives a force acting in the direction of the line $\mathbf{r}_{ik}$ joining the centers of two spirals and is attractive if $m_im_k-\alpha\gamma/2v<0$ and repulsive otherwise. The second gives rotational dynamics and, in particular, the term

$$m_k\sum_{i\neq k}\frac{m_im_k}{r_{ik}}\hat{\mathbf{z}}\times\hat{\mathbf{r}}_{ik}$$

is the usual vortex dynamics.[6-8] The phase Ψ satisfies a Kuramoto-like equation plus terms depending on the trajectories of the spirals.

We make some comments here concerning the reduction of the original Eq. (2) to Eq. (8) for the motion of the spirals and the phase equation for Ψ. If we had considered the Ginzburg-Landau equation with real coefficients, the spirals would be the usual vortices [$S(r)=0$, $\omega=0$], and proceeding in the same way the equation $\mathcal{L}w=I$ would now be $\mathcal{L}^\dagger=\mathcal{L}$, i.e., $\mathcal{L}$ is self-adjoint. As a result of this the elements of kernel($\mathcal{L}^\dagger$) =kernel($\mathcal{L}$) are known since they correspond to the Goldstone modes associated with translational invariance of (2) and the invariance $A\to A\exp(i\delta)$ with constant δ. The first set of Goldstone modes gives, through the solvability conditions, the equations for $\mathbf{r}_i(t)$, and the mode associated with $A\to A\exp(i\delta)$ gives the equation for Ψ. It is this well understood situation that has guided us to write our *Ansatz* (7) and it is the careful consideration of the known Goldstone modes of the real case which has guided us to find the elements of kernel($\mathcal{L}^\dagger$) in the complex case. Once these vectors in kernel($\mathcal{L}^\dagger$) are determined the solvability condition leads to (8).

We make the following remarks: (a) For $\gamma<0$ one has

$$\frac{\alpha\gamma}{2v}=\frac{[\gamma-(1+\alpha^2)]\gamma}{2v^2}>0$$

and the corresponding contribution to the force in (8) is attractive. (b) The origin of the terms depending on γ in (8) is the leading part of the asymptotic behavior $\mathcal{S}_r(r\to\infty)$ given by (6); in fact the form of the equations of motion is

$$\frac{d\mathbf{r}_k}{dt}=2\left\{\sum_{i\neq k}^{N}\left(\frac{m_im_k}{r_{ik}}-\alpha\mathcal{S}_r(r_{ik})\right)\hat{\mathbf{r}}_{ik}+\alpha\nabla\Psi +m_k\hat{\mathbf{z}}\times\left[\sum_{i\neq k}^{N}\left(\alpha\frac{m_im_k}{r_{ik}}+\mathcal{S}_r(r_{ik})\right)\hat{\mathbf{r}}_{ik}-\nabla\Psi\right]\right\}. \tag{9}$$

(c) Putting

$$H^{(1)}=-\frac{1}{2}\sum_{i\neq k}\left(m_im_k-\frac{\alpha\gamma}{2v}\right)\ln\left(\frac{r_{ik}}{\xi}\right),$$

$$H^{(2)}=-\frac{1}{2}\sum_{i\neq k}\left(\alpha m_im_k+\frac{\gamma}{2v}\right)\ln\left(\frac{r_{ik}}{\xi}\right),$$

where ξ is a constant with dimension of length, one can write (8) as

$$\frac{d\mathbf{r}_k}{dt}=-2\frac{\partial H^{(1)}}{\partial\mathbf{r}_k}-2m_k\hat{\mathbf{z}}\times\frac{\partial H^{(2)}}{\partial\mathbf{r}_k} +2\alpha\nabla\Psi|_{\mathbf{r}_k}-2m_k\hat{\mathbf{z}}\times\nabla\Psi|_{\mathbf{r}_k}\,, \tag{10}$$

where the dynamics of $H^{(2)}$ is of Hamiltonian type. (d) An alternative form of the equation of motion is obtained by defining

$$\Phi^{(k)}(\mathbf{r})=\sum_{i\neq k}m_i\varphi_i(\mathbf{r})-\sum_{i\neq k}\mathcal{S}(\rho_i)\,, \tag{11}$$

which gives

$$\frac{d\mathbf{r}_k}{dt}=-2m_k\hat{\mathbf{z}}\times\frac{\partial\tilde\Phi^{(k)}(\mathbf{r}_k)}{\partial\mathbf{r}_k}+2\alpha\frac{\partial\tilde\Phi^{(k)}(\mathbf{r}_k)}{\partial\mathbf{r}_k}\,, \tag{12}$$

where $\tilde{\Phi}^{(k)}(\mathbf{r}) = \Phi^{(k)}(\mathbf{r}) + \Psi(\mathbf{r},t)$ and $\Phi^{(k)}(\mathbf{r})$ is the total phase at $\mathbf{r}$ produced by all the spirals except the one in $\mathbf{r}_k$ (compare with Ref. 9 where this result is discussed in the variational case $\alpha=\beta=0$).

We consider now the case of two interacting defects leaving aside the contribution of the phase in (8). This is reasonable when one has phase stability, which must be checked from the phase equation for Ψ, which here is the equation for the phase of a perturbation to the single-spiral solution plus terms coupling the phase to the trajectories of the spirals. A complete discussion of the phase equation and its consequences (for example, the hysteresis loop observed in Ref. 3) will appear soon.[10] Defining the "center-of-mass" coordinate $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and the relative motion coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, we obtain from (8) ($r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$)

$$\frac{d\mathbf{R}}{dt} = (m_1 - m_2)\left[\alpha m_1 m_2 + \frac{\gamma}{2v}\right]\frac{1}{r}\hat{\mathbf{z}}\times\hat{\mathbf{r}}, \tag{13a}$$

$$\frac{d\mathbf{r}}{dt} = 4\left[m_1 m_2 - \frac{\alpha\gamma}{2v}\right]\frac{\hat{\mathbf{r}}}{r} + 2(m_1+m_2)\left[\alpha m_1 m_2 + \frac{\gamma}{2v}\right]\frac{1}{r}\hat{\mathbf{z}}\times\hat{\mathbf{r}}. \tag{13b}$$

In the spiral-spiral case ($m_1 = m_2 = m$, $m^2 = 1$) ($\hat{\varphi} = \hat{\mathbf{z}} \times \hat{\mathbf{r}}$),

$$\frac{d\mathbf{R}}{dt} = 0, \quad \frac{d\mathbf{r}}{dt} = 4\left[1 - \frac{\alpha\gamma}{2v}\right]\frac{\hat{\mathbf{r}}}{r} + 4m\left[\alpha + \frac{\gamma}{2v}\right]\frac{1}{r}\hat{\varphi}. \tag{14}$$

We see then that the center of mass does not move and the spirals rotate and move away if $\sigma = 1 - \alpha\gamma/2v > 0$, in which case the integration of (13b) gives

$$r(\varphi) = r_0 \exp\left[\frac{\sigma}{m(\alpha + \gamma/2v)}\varphi\right], \quad r(t)^2 = r_0^2 + 8\sigma t. \tag{15}$$

This behavior has been observed in a recent numerical experiment[11] involving a direct simulation of the Ginzburg-Landau equation [see Figs. 2(b) and 2(c) of this reference] done for $\alpha=0$ ($\sigma=1$), in which case one has phase stability. In the spiral-antispiral case ($m_1 = -m_2 = m$) one has

$$\frac{d\mathbf{R}}{dt} = 2m\left[-\alpha + \frac{\gamma}{2v}\right]\frac{1}{r}\hat{\varphi}, \tag{16a}$$

$$\frac{d\mathbf{r}}{dt} = -4\left[1 + \frac{\alpha\gamma}{2v}\right]\frac{1}{r}\hat{\mathbf{r}}. \tag{16b}$$

The spirals do not rotate now and are attracted if $\tau = 1 + \alpha\gamma/2v > 0$. The center of mass moves in the direction $\hat{\varphi} = \hat{\mathbf{z}} \times \hat{\mathbf{r}}$ which is perpendicular to the line joining the two spirals which has direction $\hat{\mathbf{r}}$=const. This behavior has again been observed in Ref. 11 [Fig. 1(b) of this reference] for $\alpha=0$ ($\tau=1$).

The only point in which we find no agreement with the direct simulation in Ref. 11 is in their Fig. 1(c) where it is observed that the spiral and the antispiral move away if their initial distance is sufficiently big. Our equations cannot explain this behavior.

We consider some special cases of (2). If $\alpha=\beta$, one has $v=0$ and Eq. (2) is of the form

$$\partial_t A = -(1+i\alpha)\delta\phi/\delta A,$$

with

$$\phi = -\int d\mathbf{r}(\mu|A|^2 - |\nabla A|^2 - \tfrac{1}{2}|A|^4),$$

and $\phi[A]$ is a Lyapunov functional for (2).[12] In fact, $\phi[A]$ is a nonequilibrium potential;[13] i.e., if one adds δ-correlated white noise $\sqrt{\eta}f(\mathbf{r},t)$ to (2), the stationary probability is $p_{\rm st}[A] \approx \exp(-1/\eta\phi[A])$ (a nonpolynomial potential in one space dimension has been found in Ref. 14 for $\alpha\neq\beta$). The function $S(r)\equiv 0$ for $v=0$, the defects are vortices, and the equations of motion reduce to

$$\frac{d\mathbf{r}_k}{dt} = 2\left[\sum_{i\neq k}\frac{m_i m_k}{r_{ik}}\hat{\mathbf{r}}_{ik} + \alpha m_k \sum_{i\neq k}\frac{m_i m_k}{r_{ik}}\hat{\mathbf{z}}\times\hat{\mathbf{r}}_{ik} - m_k\hat{\mathbf{z}}\times\nabla\Psi|_{\mathbf{r}_k} + \alpha\nabla\Psi|_{\mathbf{r}_k}\right], \tag{17}$$

which can be written in the form (10) with $H^{(2)} = \alpha H^{(1)}$, $H^{(1)} = -\frac{1}{2}\sum m_i m_k \ln(r_{ik}/\xi)$. For the real Ginzburg-Landau equation ($\alpha=\beta=0$) Eq. (17) reduces to

$$\frac{d\mathbf{r}_k}{dt} = 2\left[\sum_{i\neq k}\frac{m_i m_k}{r_{ik}}\hat{\mathbf{r}}_{ik} - m_k\hat{\mathbf{z}}\times\nabla\Psi|_{\mathbf{r}_k}\right]. \tag{18}$$

One then obtains for the vortex-antivortex interaction in the presence of thermal noise the equation $\dot{r} = -4/r + \sqrt{T}f(t)$, where $\langle f(t)\rangle = 0$, $\langle f(t)f(t')\rangle = \delta(t-t')$, r is the relative coordinate [see Eq. (16b)], and T is proportional to the temperature. The stationary probability is then $p_{\rm st}(r) \approx (r/\epsilon)^{-8/T}$, $r > \epsilon$, and $p_{\rm st}(r) = 0$, $r < \epsilon$ (vortices annihilate when they touch). Then

$$\langle r\rangle = \frac{1-8/T}{2-8/T}\epsilon$$

and for $T < T_c = 4$ one has a finite $\langle r\rangle$ which increases with the temperature and diverges at $T = T_c$ (Kosterlitz-Thouless mechanism).

The generalization to Λ-Ω systems[2,4] is elementary. Here (2) is changed to

$$\partial_t A = (a+ib)[\{\Lambda(|A|) + i\Omega(|A|)\}A + \nabla^2 A], \tag{19}$$

and the equations of motion for the spirals are [see Eq. (9)]

$$\frac{d\mathbf{r}_k}{dt} = 2\left[a\sum_{i\neq k}\frac{m_i m_k}{r_{ik}}\hat{\mathbf{r}}_{ik} - b\sum_{i\neq k}\mathcal{S}_r(r_{ik})\hat{\mathbf{r}}_{ik} + b\nabla\Psi\Big|_{\mathbf{r}_k}\right] + 2m_k\hat{\mathbf{z}}\times\left[b\sum_{i\neq k}\frac{m_i m_k}{r_{ik}}\hat{\mathbf{r}}_{ik} + a\sum_{i\neq k}\mathcal{S}_r(r_{ik})\hat{\mathbf{r}}_{ik} - a\nabla\Psi\Big|_{\mathbf{r}_k}\right]. \tag{20}$$

The asymptotic behavior of S_r for $r \to \infty$ is now

$$S_r \to k + \frac{\Lambda'(D^\infty)}{2\Omega'(D^\infty)}\frac{1}{r},$$

where $D^\infty = D(r \to \infty)$ with $D(r)$ the amplitude of the spiral solution of (19) [see Eq. (3)].

If we choose a $a = \Omega(|A|) = 0$, Eq. (19) is a nonlinear Schrödinger-type equation, $S(r) = 0$, the defects are vortices, and from (20) the dynamics is the usual vortex dynamics,[6-8]

$$\frac{d\mathbf{r}_k}{dt} = 2b \sum_{i \neq k} \frac{m_i}{r_{ik}} \hat{\mathbf{z}} \times \hat{\mathbf{r}}_{ik}. \tag{21}$$

If, furthermore, $b = \hbar/2m$ and

$$\Lambda(|A|) = (2m/\hbar^2)(\mu - g|A|^2),$$

then (19) becomes the well-known equation for the macroscopic condensate wave function on which the analysis of superfluidity is based[15,16] and (21) is the equation for superfluid vortices.

A last remark: We can write (20) in the form (12), and with the same notation one has

$$\frac{d\mathbf{r}_k}{dt} = -2am_k \hat{\mathbf{z}} \times \frac{\partial \tilde{\Phi}^{(k)}(\mathbf{r}_k)}{\partial \mathbf{r}_k} + 2b \frac{\partial \tilde{\Phi}^{(k)}(\mathbf{r}_k)}{\partial \mathbf{r}_k}. \tag{22}$$

Here $b = 0$ corresponds to the real variational case and the right-hand side (RHS) of (22) reduces to the term $\hat{\mathbf{z}} \times \nabla \tilde{\Phi}^{(k)}$, a result obtained by Kawasaki[9] [see after Eq. (12)]. The case $a = 0$ corresponds to a nonlinear Schrödinger-type equation and the RHS of (22) reduces to the term $\nabla \tilde{\Phi}^{(k)}$, a result obtained in Ref. 16 for the equation satisfied by the condensate wave function.

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