SIAM J. DISCRETE MATH.
Vol. 27, No. 2, pp. 697–707

FORCING LARGE COMPLETE (TOPOLOGICAL) MINORS IN INFINITE GRAPHS*

MAYA STEIN† AND JOSÉ ZAMORA‡

Abstract. It is well known that in finite graphs, large complete minors/topological minors can be forced by assuming a large average degree. Our aim is to extend this fact to infinite graphs. For this, we generalize the notion of the relative end degree, which had been previously introduced by the first author for locally finite graphs, and show that large minimum relative degree at the ends and large minimum degree at the vertices imply the existence of large complete (topological) minors in infinite graphs with countably many ends.

Key words. infinite graph, minor, topological minor, degree

AMS subject classifications. 05C35, 05C63, 05C83

DOI. 10.1137/100819722

1. Introduction.

A recurrent question in finite graph theory is how certain substructures, such as specific minors or subgraphs, can be forced by density assumptions. The density assumptions are often expressed via a lower bound on the average or minimum degree of the graph.

The classical example for this type of question is Turán's theorem, or, in the same vein, the Erdős–Stone theorem. Infinite analogues of these results are not difficult: it is easy to see that if the upper density of an infinite graph G is at least the Turán density for k, i.e., $(k-2)/(k-1)$, then $K^k \subseteq G$; that is, G contains the complete graph of order k as a subgraph (see Bollobás [2] and also [13]). At the other extreme, a result of Mader affirms that an average degree of at least $4k$ ensures the existence of a $(k+1)$-connected subgraph. This result has been extended to infinite graphs by the first author [12].

Halfway between the two types of results just discussed lies the question of how to force complete (topological) minors with large average degree. First steps in this direction were taken by Mader [10]. The results we present in the following theorem are due to Kostochka [9] for minors, and to Komlós and Szemerédi [7, 8] and Bollobás and Thomason [3] for topological minors.

THEOREM 1. *There are constants c_1, c_2 so that for each $k \in \mathbb{N}$ and each graph G the following holds. If G has average degree at least $c_1 k\sqrt{\log k}$, then $K^k \preceq G$, and if G has average degree at least $c_2 k^2$, then $K^k \preceq_{top} G$.*

Our aim is to extend this result to infinite graphs (qualitatively, that is, without necessarily using the same functions $c_1 k\sqrt{\log k}$ and $c_2 k^2$ from Theorem 1). We call a finite graph H a minor/topological minor of a graph G if it is a minor/topological minor of some finite subgraph of G.[1]

*Received by the editors December 30, 2010; accepted for publication (in revised form) November 14, 2012; published electronically April 16, 2013.
http://www.siam.org/journals/sidma/27-2/81972.html

†Centro de Modelamiento Matemático, Universidad de Chile, Santiago 8370459, Chile (mstein@dim.uchile.cl). This author's research was supported by Fondecyt grant 1104099.

‡Departamento de Matemáticas, Universidad Andres Bello, Santiago 8370134, Chile (josezamora@unab.cl). This author's work was partially supported by Nucleo Milenio Información y Coordinación en Redes ICM/FIC P10-024F.

[1] Note that this means all branch-sets of our minor are finite, but as long as H is finite, it clearly makes no difference whether we allow infinite branch-sets or not.

Avoiding the difficulty of defining an average degree for infinite graphs (the upper density mentioned above is too strong for our purposes; cf. [13]), we shall stick to the minimum degree for our extension of Theorem 1 to infinite graphs. This works fine for rayless graphs: the first author showed in [14] that every rayless graph of minimum degree $\geq m \in \mathbb{N}$ has a finite subgraph of minimum degree $\geq m$. Thus, Theorem 1 can be extended to rayless graphs if we replace the average with the minimum degree.

In general, however, large minimum degree at the vertices alone is not strong enough to force large complete minors. This is so because of the infinite trees, which may attain any minimum degree condition without containing any interesting substructure. So we need some additional condition that prevents the density from "escaping to infinity."

The most natural way to impose such an additional condition is to impose it on the ends[2] of the graph. This approach has also proved successful in other recent work [4, 14, 12]. In this way, that is, defining the degree of an end in an appropriate way, the minimum degree, now taken over vertices and ends, can continue to serve as our condition for forcing large complete minors.

In [14], the *relative degree* of an end was introduced for locally finite graphs. In order to explain it, a few notions come in handy.

Let G be a locally finite graph. The *edge-boundary* of a subgraph H of G, denoted by $\partial_e^G H$, or $\partial_e H$ where no confusion is likely, is the set $E(H, G - H)$. The *vertex-boundary* $\partial_v^G H$, or $\partial_v H$, of a subgraph H of G is the set of all vertices in H that have neighbors in $G - H$. Now, the idea of the relative degree is to calculate the ratios of $|\partial_e H_i|/|\partial_v H_i|$ of certain subgraphs H_i of G, and then take the relative degree to be the limit of these ratios as the H_i in some sense converge to ω.

For this, define an ω-region of an end ω of a graph G as an induced connected subgraph which contains some ray of ω and whose vertex-boundary is finite. For $V' \subseteq V(G)$, $\Omega' \subseteq \Omega(G)$, a V'–Ω' separator is a set $S \subseteq V(G)$ such that $V' \not\subseteq S$ and such that no component of $G \setminus S$ contains both a vertex of V' and a ray from ω. We write $\Omega^G(H)$ for the set of all ends of G that have a ray in $H \subseteq G$.

Write $(H_i)_{i\in\mathbb{N}} \to \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of distinct ω-regions such that $H_{i+1} \subseteq H_i - \partial_v H_i$ and $\partial_v H_{i+1}$ is a minimal $\partial_v H_i$–$\Omega^G(H_{i+1})$ separator in G for each $i \in \mathbb{N}$. Now, define

$$d_{e/v}(\omega) := \inf_{(H_i)_{i\in\mathbb{N}}\to\omega} \liminf_{i\to\infty} \frac{|\partial_e H_i|}{|\partial_v H_i|}.$$

Note that it does not matter whether we consider the lim inf or the lim sup, because if $(H_i)_{i\in\mathbb{N}} \to \omega$, all subsequences of $(H_i)_{i\in\mathbb{N}}$ also converge to ω. (For the same reason we could restrict our attention only to sequences (H_i) for which $\lim_{i\to\infty} \frac{|\partial_e H_i|}{|\partial_v H_i|}$ exists.)

For more discussion of this notion see section 2. From now on we write $\delta^{V,\Omega}(G)$ for the minimum degree/relative degree, taken over all vertices and ends of G. The first author showed the following.

THEOREM 2 (see [14]). *Let $m \in \mathbb{N}$, and let G be a locally finite graph. If $\delta^{V,\Omega}(G) \geq m$, then G has a finite subgraph H of average degree at least m.*

[2]The *ends* of a graph are the equivalence classes of the rays (i.e., one-way infinite paths) under the following equivalence relation. Two rays are equivalent if no finite set of vertices separates them. (A set S of vertices is said to *separate* two rays R_1, R_2 if $V(R_1) \setminus S$ and $V(R_2) \setminus S$ lie in different components of $G - S$.) The set of ends of a graph G is denoted by $\Omega(G)$. See Chapter 8 of [6] for more on ends.

We remark that the finite subgraph H from Theorem 2 may be required to have any given minimum order.[3]

Theorem 2 readily serves as a black box for extensions of extremal results using degree conditions from finite to infinite locally finite graphs. In particular,[4] we deduce that if $\delta^{V,\Omega}(G) \geq c_1 k\sqrt{\log k}$ for a locally finite graph G, then $K^k \preceq G$, and if $\delta^{V,\Omega}(G) \geq c_2 k^2$, then $K^k \preceq_{top} G$. (The c_i are the constants from Theorem 1.)

In this paper, we extend the notion of the relative degree to arbitrary infinite graphs. For a given end ω of some infinite graph G, let $Dom(\omega)$ denote the set of all vertices that dominate[5] ω. If $G_\omega := G - Dom(\omega)$ does not contain any rays from ω, or if $|Dom(\omega)| \geq \aleph_0$, then we set $d_{e/v}(\omega) := |Dom(\omega)|$. Otherwise, writing $\hat{\omega}$ for the unique[6] end of G_ω that contains rays from ω, we define

$$d_{e/v}(\omega) := |Dom(\omega)| + \inf_{(\hat{H}_i)_{i\in\mathbb{N}}\to\hat{\omega}} \liminf_{i\to\infty} \frac{|\partial_e \hat{H}_i|}{|\partial_v \hat{H}_i|}.$$

Note that the $\hat{H}_i$ are $\hat{\omega}$-regions of G_ω. Also note that for locally finite graphs, our definition coincides with the one given earlier. For further discussion of our notion of the relative degree, for examples, and for alternative definitions that do (not) work, see section 2.

Our main result is the following version of Theorem 2 for graphs with countably many ends.

THEOREM 3. *Let $k \in \mathbb{N}$, let $m \in \mathbb{Q}$, and let G be a graph such that $|\Omega(G)| \leq \aleph_0$ and $\delta^{V,\Omega}(G) > m$. Then $K^k \preceq_{top} G$ or G has a finite subgraph H of average degree $> m - k + 1$.*

Again, the graph H may be required to have any minimum order (see the remark after the proof of Theorem 3 at the end of section 4).

Combining Theorem 3 with Theorem 1, we obtain the desired extension of Theorem 1 to graphs with countably many ends.

THEOREM 4. *Let $k \in \mathbb{N}$, and let G be a graph with $|\Omega(G)| \leq \aleph_0$.*

(a) *If $\delta^{V,\Omega}(G) \geq c_1 k\sqrt{\log k} + k$, then $K^k \preceq G$.*

(b) *If $\delta^{V,\Omega}(G) \geq c_2 k^2 + k$, then $K^k \preceq_{top} G$.*

We remark that in Theorem 4, the assumption that G has countably many ends is necessary. To see this, consider the graph we obtain by the following procedure. Let G_1 be a double-ray, and let $E_1 := E(G_1)$. Now, for $i \geq 2$, replace each edge $e \in E_{i-1}$ with $\aleph_0$ many paths of length two whose middle points are connected by a (fresh) double-ray D_e. Let E_i be the edges on the D_e. After $\aleph_0$ many steps, we obtain a planar graph G with $\delta^{V,\Omega}(G) = \aleph_0$. Because of its planarity, G has no complete minors of order greater than 4.

PROBLEM 5. *Is there a "degree condition" which forces large complete (topological) minors in arbitrary infinite graphs?*

---

[3] This is very easy to check in the proof of Theorem 2 of [14], which uses a compactness argument. In fact, we may require H to contain any previously fixed finite subgraph of the graph G.

[4] Other applications include forcing a minor isomorphic to $K_{r,s}$ or to the union of s disjoint copies of K_r, using the respective result for finite graphs [1]. Also, as for finite graphs [5], we can use large (but linear) degree/relative degree to force (strong) complete graph immersions. The latter also works for arbitrary infinite graphs with countably many ends: this will follow from Theorem 3.

[5] We say a vertex v *dominates* an end ω if v dominates some ray R of ω, that is, if there are infinitely many v–$V(R)$-paths that are disjoint except in v. (It is not difficult to see that then v dominates all rays of ω.)

[6] The uniqueness of $\hat{\omega}$ follows at once from the fact that $|Dom(\omega)| < \aleph_0$.

2. Discussion of the relative degree. In this section, we will discuss our definition as well as possible alternative definitions of the relative degree of an end. This motivation is not necessary for understanding the rest of the paper and may be skipped during a first reading.

2.1. Large vertex-degree is not enough. As we have already seen in the introduction, large minimum degree at the vertices alone is not sufficient for forcing large complete (topological) minors in infinite graphs, because of the trees. A similar example discards the following alternative. The *vertex*- and the *edge-degree* of an end ω were introduced in [12] (see also [6]) as the supremum of the cardinalities of sets of vertex- or edge-disjoint rays in ω. Clearly, the vertex-degree of an end is always at most its edge-degree, so for our purposes we may restrict our attention to the vertex-degree. It is not difficult to show [11] that an end has vertex-degree $\geq k$ if and only if there is a finite set $S \subseteq V(G)$ so that every S–$\{\omega\}$ separator has order at least k.

Large vertex-degree at the ends together with large degree at the vertices ensures the existence of grid minors [14], and of highly connected subgraphs [12], but it is not strong enough for forcing (topological) minors. This can be seen by inserting the edge set of a spanning path at each level of a large-degree tree in such a way that the obtained graph is still planar.

The reason that the vertex-degree fails to force large complete minors is that it gives information only about the sizes of vertex-separators S_i "converging" to the end in question. But imagine that we wish to "cut off" an end. Then the information we need is not the size of the S_i but the average number of edges that the vertices in S_i send "into" the graph. This idea is made precise in the definition of the relative degree for locally finite graphs as given in the introduction.

We defined the relative degree for locally finite graphs in the introduction as $\inf_{(H_i)} \liminf_i(|\partial_e H_i|/|\partial_v H_i|)$. Let us remark that instead, we might have defined the relative degree as $\inf_{(H_i)} \liminf_i((|\partial_e H_i| + |E(G[\partial_v H_i])|/|\partial_v H_i|)$. This change, and the corresponding change for nonlocally finite graphs, would alter the relative degree of the ends (it would make more ends have large relative degree). Thus, using such an altered definition, our result would cover a larger class of graphs. In fact it is very easy to verify that all our proofs go through in the same way for the altered definition. However, we believe that our definition is more natural.

Let us quickly evaluate a possible alternative definition, which at first sight might seem as plausible as ours but less complicated. Consider the ratio $d_e(\omega)/d_v(\omega)$ of the edge- and the vertex-degree of some end ω. If we defined this ratio as the degree of ω, then large degrees at vertices and ends do not force large complete minors, not even in locally finite graphs. For an example, see [14].

Finally, it is quite clear that because we do not wish ends of trees to have large degree, we cannot replace the $\inf_{(H_i)}$ with a $\sup_{(H_i)}$. But even adding some additional property, such as requiring connectivity of the H_i–∂H_i, does not work, by the example from [14] mentioned in the previous paragraph.

2.2. The role of the separation property. Let us now explain the reason for requiring the vertex-boundaries $\partial_v H_i$ of the graphs H_i to be minimal $\partial_v H_{i-1}$–$\Omega(H_i)$ separators (we shall call this property the *separation property*). First, let us show that a very similar condition, namely, requiring the $\partial_v H_i$ to be minimal $\partial_v H_{i-1}$–$\{\omega\}$ separators, is too weak. Even locally finite graphs can satisfy a degree condition thus modified and still not contain any large complete minors. In order to see this, consider the following example.

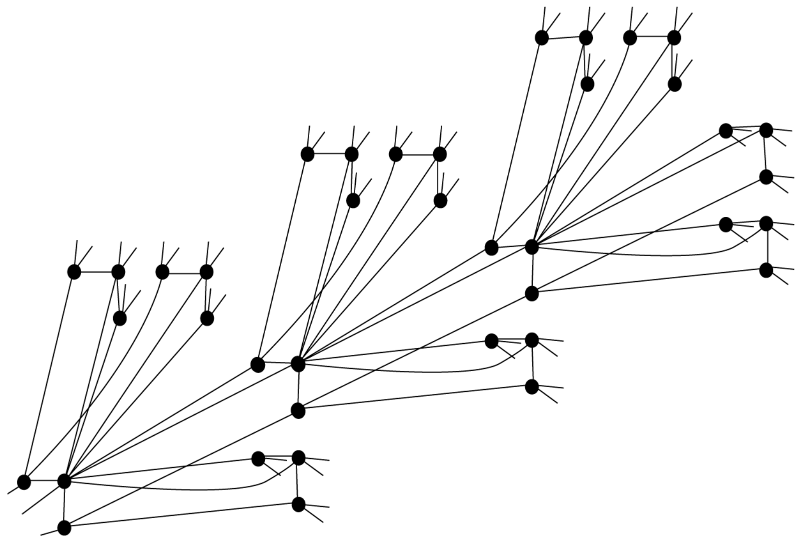

FIG. 1. *The graph from Example* 6*, with* $r = 5$.

Example 6. We start out with the r-regular (infinite) tree T_r with levels L_i. For each of its vertices, we color half of the edges going to the next level blue and the other half red. Then we replace each vertex $v \in V(T_r)$ with a path $x_v y_v z_v$. For each $vw \in E(T_r)$, with v from the ith and w from the $(i+1)$th level of T_r, say, we do the following. If vw is blue, then add the edges $x_v x_w$, $y_v y_w$, and $y_v z_w$, and if vw is red, then add the edges $y_v x_w$, $y_v y_w$, and $z_v z_w$. See Figure 1.

The resulting graph Γ_r is locally finite, and it is easy to see that Γ_r does not contain any complete minor of order greater than 4 using the fact that the vertices of any such minor cannot be separated by any 2 separator of Γ_r.

Note that each end of Γ_r has relative degree 1. In fact, let $\omega \in \Omega(\Gamma_r)$, and let $\omega' \in \Omega(T_r)$ be the corresponding end of T_r. Suppose $v_1 v_2 v_3 \ldots \in \omega'$ with v_1 being the root of T_r. For $v \in V(T_r)$, let H_v be the subgraph of Γ_r that is induced by all vertices x_w, y_w, z_w such that w lies in the upper closure $\lceil v \rceil$ of v in the tree order of T_r. Then $(H_{v_i})_{i\in\mathbb{N}} \to \omega$, and thus $d_{e/v}(\omega) = 1$.

However, if we required the $\partial_v H_i$ to be minimal $\partial_v H_{i-1}$–$\{\omega\}$ separators instead of minimal $\partial_v H_{i-1}$–$\Omega^G(H_i)$ separators, then the ends of Γ_r would have relative degree $(r+2)/2$. In fact, as the $\partial_v H_{v_i}$ are not minimal $\partial_v H_{i-1}$–$\{\omega\}$ separators, the sequence $(H_{v_i})_{i\in\mathbb{N}}$ is no longer taken into account when we calculate the relative degree of ω. It is not difficult to see that the relative degree would then be determined by the sequence $(K_{v_i})_{i\in\mathbb{N}}$, which we define now. If $v_i v_{i+1}$ is blue, then K_{v_i} is the subgraph of Γ_r that is induced by $\{x_{v_i}, y_{v_i}\} \cup \{x_w, y_w, z_w : w \in \lceil v \rceil, v \in L_{i+1}, v_i v \text{ is a blue edge}\}$. Otherwise, let K_{v_i} be induced by $\{x_{v_i}, y_{v_i}\} \cup \{x_w, y_w, z_w : w \in \lceil v \rceil, v \in L_{i+1}, v_i v \text{ is a red edge}\}$. As $|\partial_e K_i|/|\partial_v K_i| = (r+2)/2$ for all $i > 1$, the end ω would thus have relative degree $(r+2)/2$.

On the other hand, if we totally gave up the requirement that the $\partial_v H_i$ be minimal separators, then basically every end of every graph would have relative degree 1. To see this, write $(H_i)_{i\in\mathbb{N}} \rightsquigarrow \omega$ if $(H_i)_{i\in\mathbb{N}}$ is an infinite sequence of distinct regions of G with $H_{i+1} \subseteq H_i - \partial_v H_i$ such that ω has a ray in H_i for each $i \in \mathbb{N}$. Let $(H_i)_{i\in\mathbb{N}}$ with $(H_i)_{i\in\mathbb{N}} \rightsquigarrow \omega$, and let $v_i \in \partial_v H_{3i}$ for $i \in \mathbb{N}$. Then the v_i do not have common neighbors. We construct a sequence $(H'_j)_{j\in\mathbb{N}}$ with $H'_0 := H_0$, and, for $j > 0$, we let $H'_j := H_{i_j} - V_j$, where i_j is such[7] that $H_{i_j} \subseteq H'_j - \partial_v H'_j$, and V_j consists of $j|\partial_e H_{i_j}|$ vertices v_i with $i \geq i_j$. Then $(H'_j)_{j\in\mathbb{N}} \rightsquigarrow \omega$, and

$$\liminf_{j\to\infty} \frac{|\partial_e H'_j|}{|\partial_v H'_j|} = \liminf_{j\to\infty} \frac{|\partial_e H_{i_j}| + \sum_{v\in V_j} d(v)}{|\partial_v H_{i_j}| + \sum_{v\in V_j} d(v)} = 1.$$

This shows that the additional condition that $\partial_v H_{i+1}$ be an $\subseteq$-minimal $\partial_v H_i$–$\Omega^G(H_{i+1})$ separator is indeed necessary for our definition of the relative degree to make sense.

For our notion, every integer, and also $\aleph_0$, appears as the relative degree of an end in some locally finite graph (larger cardinals appear only in nonlocally finite graphs). Indeed, let $k \in \mathbb{N}$, and let G be obtained from the disjoint union of $\aleph_0$ many copies K_i of K^k by adding all edges between K_i and K_{i+1} for all i. Suppose $(H_i)_{i\in\mathbb{N}} \to \omega$ for the unique end ω of G. Then by the separation property, we can conclude inductively that each $\partial_v H_i$ is contained in some K_{i_j}. Thus $d_{e/v}(\omega) = k$. A similar example can be constructed to find an end of relative degree $\aleph_0$.

2.3. From locally finite to arbitrary graphs. In arbitrary infinite graphs, we have to face the problem that there might be vertices dominating our end ω, and hence no sequence of subgraphs H_i can satisfy $H_{i+1} \subseteq H_i - \partial_v H_i$. Our way out of this dilemma was to delete the dominating vertices temporarily, find the sequences as above and calculate the corresponding infimum, and then add $|Dom(\omega)|$ to the relative degree.

One might think that, alternatively, we might have weakened our requirements on the sequences of ω-regions H_i. For instance, we might be satisfied with them obeying $H_{i+1} \subseteq H_i - (\partial_v H_i - Dom(\omega))$. Then we should also require that $H_i - Dom(\omega)$ be connected, as otherwise our sequence may "converge" to more than one end. When no such sequence existed, we would set $\tilde{d}_{e/v}(\omega) := \infty$. This alternative definition would have the advantage that the contribution of a dominating vertex, in terms of outgoing edges, to the edge-boundary of an ω-region is counted.

However, the approach does not allow for an extension of Theorem 1. The problem is vertices that dominate more than one end. Consider an infinite r-regular tree to which we add one vertex that is adjacent to all other vertices. The ends of this graph have infinite degree in the sense just discussed (and relative degree 2) but, of course, no K^k-minor for large k, no matter how large r is with respect to k. We can give a similar example for a graph with only two ends.[8]

3. Dominating vertices and topological K^k-minors. This section provides some results about dominating vertices that will be needed later. First, we give a useful characterization of dominating vertices.

LEMMA 7. *In any graph, a vertex v dominates an end ω if and only if there is no finite v–ω separator.*

Proof. For the forward direction, note that every finite set of vertices intersects only a finite number of the infinitely many v–$V(R)$ paths, where $R \in \omega$ is dominated by v. Hence v and ω cannot be finitely separated.

For the backward direction we inductively construct a set of v–$V(R)$ paths, where R is any ray in ω. At each step we use the fact that all paths constructed so far form

[7]For instance, set $i_j := \max\{dist(v, w) | v \in \partial_v H_0, w \in \partial_v H'_j\} + 1$.

[8]Take two copies of the r-regular tree and add a spanning path in each level of each of the two trees and a new vertex adjacent to all other vertices.

a finite set which (without v) does not separate v from ω. Hence we can always add a new v–$V(R)$ path, which is disjoint from all the others (except in v). ∎

We now show that for an end ω which is dominated by only finitely many vertices, the graph G_ω contains sequences of subgraphs converging to $\hat{\omega}$.

LEMMA 8. *Let G be a graph, and let $\omega \in \Omega(G)$ with $|Dom(\omega)| < \aleph_0$. Then G_ω contains a sequence $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$.*

Proof. We define a sequence of disjoint finite sets $S_i \subseteq V(G_\omega)$, starting with any finite nonempty set S_1. For $i > 1$ and for each $v \in S_{i-1}$, let S_v^{i-1} be a finite v–$\hat{\omega}$ separator in G_ω (note that such a separator exists by Lemma 7). Set

$$\tilde{S}_i := \bigcup_{v\in S_{i-1}} S_v^{i-1} \setminus S_{i-1}.$$

Then $\tilde{S}_i$ separates S_{i-1} from $\hat{\omega}$. In fact, otherwise there would be a ray $R \in \hat{\omega}$ with only its first vertex v in S_{i-1}, and disjoint from $\tilde{S}_i$. But R must meet S_v^{i-1}, a contradiction.

Choose $S_i \subseteq \tilde{S}_i$ minimal such that it separates S_{i-1} from $\hat{\omega}$. Now, for $i \in \mathbb{N}$, let K_i be the component of $G_\omega - S_i$ that contains a ray of $\hat{\omega}$ (since $S_i \cup Dom(\omega)$ is finite, there is a unique such component K_i). Let $\hat{H}_i$ be the subgraph of G_ω that is induced by S_i and K_i. Then, by the choice of S_i, we find that S_i is a minimal S_{i-1}–$\Omega^G(\hat{H}_i)$ separator. Hence, $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$, as desired. ∎

Next, we will see that our desired minor is easy to find whenever there are enough vertices dominating the same end.

LEMMA 9. *Let $k \in \mathbb{N}$, let G be a graph, and let $\omega \in \Omega(G)$. If $|Dom(\omega)| \geq k$, then $K^k \preceq_{top} G$.*

Lemma 9 follows at once from Lemma 10 below. The *branching vertices* of a subdivision are those vertices that did not arise from subdividing edges.

LEMMA 10. *Let $k \in \mathbb{N}$, let G be a graph, let $\omega \in \Omega(G)$, and let $S \subseteq Dom(\omega)$ with $|S| = k$. Then G contains a subdivision TK^k of K^k whose branching vertices are in S.*

Proof. We use induction on k; the base case $k = 0$ is trivial. So suppose $k \geq 1$. Then, let $S \subseteq Dom(\omega)$ be a set of size k, and let $s \in S$. By the induction hypothesis, $G - s$ contains a subdivision TK^{k-1} of K^{k-1} with branching vertices in $S' := S \setminus \{s\}$.

Successively we define sets $\mathcal{P}_i$ of s–S' paths in G which are disjoint except in s. We start with $\mathcal{P}_0 := \emptyset$. For $i > 0$, suppose there is a vertex $v \in S'$ which is not the endpoint of a path in $\mathcal{P}_{i-1}$. Then, $S_i := S \cup \bigcup_{P\in\mathcal{P}_{i-1}} V(P)$ is finite, and $G - S_i$ has a unique component C_i which contains rays of ω. Since Lemma 7 implies that neither s nor v can be separated from ω by a finite set of vertices, both s and v have neighbors in C_i. Hence there is an s–v path P_i that is internally disjoint from S_i. Set $\mathcal{P}_i := \mathcal{P}_{i-1} \cup \{P_i\}$. The procedure stops after step $k - 1$, when all vertices of S' are connected to s by a path in $\mathcal{P}_i$. This gives the desired subdivision TK^k. ∎

With a very similar proof,[9] we also get the following statement (which will not be needed in what follows).

LEMMA 11. *Let G be a graph, and let $\omega \in \Omega(G)$. If $|Dom(\omega)| \geq \aleph_0$, then $K^{\aleph_0} \preceq_{top} G$.*

[9] We construct the $TK^{\aleph_0}$ step by step, adding one branching vertex plus the corresponding paths at a time. In each step, the finiteness of the already constructed part ensures the existence of an unused dominating vertex s of ω and enough paths to connect s to the already defined branching vertices.

We finish this section with one more basic lemma. This lemma implies that removing a finite part of a graph, or even an infinite part with a finite vertex-boundary, will not alter the relative degree of the remaining ends.

LEMMA 12. *Let G be a graph, let $\omega \in \Omega(G)$, and let G' be an induced subgraph of G such that $\partial_v^G G'$ is finite, and such that G' has an end ω' that contains rays of ω. Then $d_{e/v}(\omega') = d_{e/v}(\omega)$.*

Proof. First of all, observe that since $\partial_v^G G'$ is finite, it follows that $Dom(\omega) \subseteq V(G')$. Hence, we need only show that $\inf_{(\hat{H}_i)_{i\in\mathbb{N}}} \liminf_i(|\partial_e \hat{H}_i|/|\partial_v \hat{H}_i|)$ is the same for sequences $(\hat{H}_i)_{i\in\mathbb{N}}$ in G_ω and for sequences $(\hat{H}_i)_{i\in\mathbb{N}}$ in $G'_{\omega'}$.

For this, note that every sequence $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$ in G_ω has a subsequence $(\hat{H}_i)_{i\geq i_0} \to \hat{\omega}'$ in $G'_{\omega'}$ (it suffices to take $i_0 := \max\{dist(v,w) : v \in \partial_v \hat{H}_0, w \in \partial_v G'\} + 1$). Moreover, for every sequence $(\hat{H}'_j)_{j\in\mathbb{N}} \to \hat{\omega}'$ in $G'_{\omega'}$, there is an index j_0 such that $\partial_v \hat{H}'_j \cup N(\hat{H}'_j)$ is the same set in G_ω and in G'_ω (for instance, take $j_0 := \max\{dist(v,w) : v \in \partial_v \hat{H}'_0, w \in \partial_v G'\} + 1$). Hence, the relative degrees of ω and ω' are the same. ∎

4. Proof of Theorem 3. In this section we prove our main result, Theorem 3. We start by showing how to find, for a fixed end ω of some graph G, an $\hat{\omega}$-region $\hat{H}$ of G_ω that has an acceptable average degree into $G - \hat{H}$.

LEMMA 13. *Let G be a graph, let $\omega \in \Omega(G)$ with $d_{e/v}(\omega) > m$ for some $m \in \mathbb{Q}$, and let $S \subseteq V(G)$ be finite. If $|Dom(\omega)| < \aleph_0$, then G_ω has an $\hat{\omega}$-region $\hat{H}$ such that*

(a) *$S \cap V(\hat{H}) = \emptyset$ and*

(b) *$\frac{|\partial_e \hat{H}|}{|\partial_v \hat{H}|} > m - |Dom(\omega)|$.*

Proof. By Lemma 8, there is a sequence of regions $\hat{H}_i$ such that $(\hat{H}_i)_{i\in\mathbb{N}} \to \hat{\omega}$ in G_ω. Let $i_0 = \max\{dist_{G_\omega}(v,w) : v \in S \setminus Dom(\omega), w \in \partial_v \hat{H}_0\}+1$. Then $S \cap V(\hat{H}_i) = \emptyset$ for all $i \geq i_0$. As $d_{e/v}(\omega) > m$, there is a $j_0 \geq i_0$ such that $(|\partial_e \hat{H}_j|/|\partial_v \hat{H}_j|) > m - |Dom(\omega)|$ for all $j \geq j_0$. ∎

We now apply Lemma 13 repeatedly to the ends of any suitable fixed countable subset of $\Omega(G)$. Lemma 12 will ensure that the relative degree of the ends is not disturbed by what has been cut off earlier.

LEMMA 14. *Let $m \in \mathbb{Q}$, let G be a graph, let $X \subseteq V(G)$ be finite, and, for $i \in \mathbb{N}$, let $\omega_i \in \Omega(G)$ with $|Dom(\omega_i)| < \aleph_0$. If $\delta^{V,\Omega}(G) > m$, then G contains induced subgraphs G_i, finite sets $X \subseteq X_i \subseteq V(G_i)$, and finite sets $\mathcal{F}_i$ of pairwise disjoint finite subsets of $V(G_i)$ such that, for each $i \in \mathbb{N}$,*

(A) *$G_{i+1} \subseteq G_i$, $X_i \subseteq X_{i+1}$, and $\mathcal{F}_{i+1} \supseteq \mathcal{F}_i$,*

(B) *$X_i \cup \bigcup \mathcal{F}_i \subseteq V(G_i)$,*

(C) *there is a family $\mathcal{H}_i = \{H_F : F \in \mathcal{F}_i\}$ of disjoint connected subgraphs such that $V(H_F) \cap V(G_i) = F$ for each $F \in \mathcal{F}_i$ and such that $G = G_i \cup \bigcup \mathcal{H}_i$,*

(D) *$\partial_v^G G_i$ is finite,*

(E) *the average degree of F into $F \cup X_i$ is $> m - |Dom(\omega_i)|$ for each $F \in \mathcal{F}_i$,*

(F) *$d_{X_i}(v) > m$ for all $v \in \partial_v^G G_i \setminus \bigcup \mathcal{F}_i$, and*

(G) *ω_i has no rays in G_i.*

Proof. Set $G_0 := G$, $X_0 := X$, and $\mathcal{F}_0 := \emptyset$. Now, for $i \geq 1$ we do the following. If ω_i has no ray in G_{i-1}, then we set $G_i := G_{i-1}$, $X_i := X_{i-1}$, and $\mathcal{F}_i := \mathcal{F}_{i-1}$, which ensures all the desired properties (as they hold for $i - 1$).

So suppose ω_i does have a ray in G_{i-1}. Then let D_i be a finite set of vertices of G_{i-1} so that each dominating vertex of ω_i in G_{i-1} has degree $> m$ into D_i. (This is possible since by assumption there are only finitely many vertices dominating ω_i. See Figure 2.)

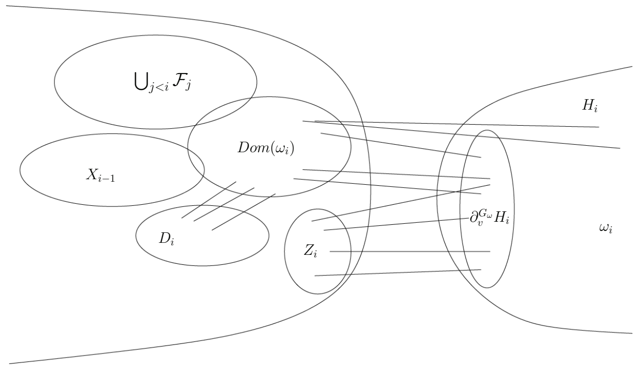

FIG. 2. *Construction of the graph G_i in Lemma* 14.

Observe that because of (D) we may apply Lemma 12 to obtain that all ends of G_{i-1} have relative degree $> m$. Hence Lemma 13 applied to G_{i-1} and the finite set

$$S := X_{i-1} \cup D_i \cup \bigcup \mathcal{F}_{i-1}$$

yields an $\hat{\omega}_i$-region $\hat{H}_i$ of G_ω. We set $\mathcal{F}_i := \mathcal{F}_{i-1} \cup \partial_v \hat{H}_i$ and $G_i := G_{i-1} - (\hat{H}_i - \partial_v \hat{H}_i)$. Choose a finite subset Z_i of $V(G_i)$ such that $\partial_v \hat{H}_i$ has average degree $> m - |Dom(\omega_i)|$ into $Z_i \cup \partial_v \hat{H}_i$, and set $X_i := X_{i-1} \cup D_i \cup Z_i$.

Then, conditions (A), (D), and (G) are clearly satisfied for step i, as they hold for step $i-1$. Conditions (B) and (C) for i follow from Lemma 13(a), and from (B) and (C) for $i-1$. Condition (E) follows from Lemma 13(b) and (E) for $i-1$.

Finally, for (F) suppose that $v \in \partial_v^G G_i \setminus \bigcup \mathcal{F}_i$. If $v \in \partial_v^G G_{i-1}$, then (F) for i follows from (F) for $i-1$. Otherwise, v dominates ω_i. Then by construction, v has sufficiently many neighbors in $D_i \subseteq X_i$. ∎

If G has only countably many ends, then the procedure just described can be used to cut off all ends.

LEMMA 15. *Let $k \in \mathbb{N}$, let $m \in \mathbb{Q}$, let G be a graph with $|\Omega(G)| \leq \aleph_0$, and let $X \subseteq V(G)$ be finite. Suppose $|Dom(\omega)| < k$ for all ends $\omega \in \Omega(G)$. If $\delta^{V,\Omega}(G) > m$, then G has an induced subgraph G' and a set $\mathcal{F}$ of finite pairwise disjoint vertex sets such that*

(i) *$X \cup \bigcup \mathcal{F} \subseteq V(G')$,*

(ii) *there is a family $\{H_F : F \in \mathcal{F}\}$ of disjoint connected subgraphs of G such that $V(H_F) \cap V(G') = F$ for each $F \in \mathcal{F}$,*

(iii) *for each vertex $v \in V(G')$ of degree $\leq m$, there is an $F \in \mathcal{F}$ so that $v \in F$ and the average degree of F in G' is $> m - k + 1$, and*

(iv) *every ray of G has only finitely many vertices in $V(G')$.*

Proof. Let $\omega_1, \omega_2, \omega_3, \ldots$ be a (possibly repetitive) enumeration of $\Omega(G)$. Apply Lemma 14, and then set $G' := \bigcap_{i\in\mathbb{N}} G_i$ and $\mathcal{F} := \bigcup_{i\in\mathbb{N}} \mathcal{F}_i$. We claim that G' and $\mathcal{F}$ are as desired. Indeed, properties (A) and (B) imply property (i), and property (C) together with (A) implies property (ii).

For property (iii) observe that (A) and (B) imply that $X_i \subseteq V(G')$ for all $i \in \mathbb{N}$. Now (iii) follows from (E) and (F) together with the assumption that $\delta^{V,\Omega}(G) > m$.

In order to see (iv), suppose that R is a ray of G that has infinitely many vertices in G'. Say $R \in \omega_j$. Then by (A) for j, the ray R has infinitely many vertices in G_j. So, as $\partial_v^G G_j$ is finite by (D), R has a subray in G_j, a contradiction to (G) for j. ∎

We are now almost ready to prove our main theorem. We will make use of a standard tool from infinite graph theory, König's infinity lemma.

LEMMA 16 (see [6]). *Let $V_1, V_2, V_3, \ldots$ be disjoint finite nonempty sets, and let G be a graph on their union. Suppose that for all $i \in \mathbb{N}$, each vertex of V_{i+1} has a neighbor in V_i. Then G has a ray $v_1v_2v_3\ldots$, with $v_i \in V_i$ for each $i \in \mathbb{N}$.*

Let us now prove Theorem 3.

Proof of Theorem 3. Suppose that K^k is not a topological minor of G. Then by Lemma 9, $|Dom(\omega)| < k$ for all $\omega \in \Omega(G)$. Let $u \in V(G)$ and let G' be the subgraph we obtain from Lemma 15 applied to G and $X := \{u\}$, and let $\mathcal{F}$ be the corresponding set of disjoint finite vertex sets.

For $i \in \mathbb{N}$, we shall successively define finite sets S_i with $S_i \subseteq S_{i+1}$. We start by setting $S_0 := \emptyset$ and $S_1 := \{u\}$ if $u \notin \bigcup \mathcal{F}$, or $S_1 := F_u$ if there is an $F_u \in \mathcal{F}$ with $u \in F_u$ (by the disjointness of the sets in $\mathcal{F}$, there is at most one such F_u). Note that $S_0 \subseteq S_1 \subseteq V(G')$ because of Lemma 15(i).

Our sets S_i will have the following properties for $i \geq 1$:

(I) the average degree of the vertices of S_{i-1} in $G'[S_i]$ is $> m - k + 1$,

(II) the average degree of the vertices of $S_i \setminus S_{i-1}$ in G' is $> m - k + 1$, and

(III) for each $F \in \mathcal{F}$ with $F \cap S_i \neq \emptyset$ there is a $j \leq i$ such that $F \subseteq S_j \setminus S_{j-1}$.

For $i = 1$, property (I) holds trivially, and (II) is satisfied because of Lemma 15(iii). By the choice of S_1 and since the $F \in \mathcal{F}$ are disjoint, (III) also holds.

Now, for $i \geq 2$ we choose a finite subset X_i of the neighborhood of $S_{i-1} \setminus S_{i-2}$ in $G' - S_{i-1}$ so that the average degree of the vertices in S_{i-1} in the graph $G'[S_{i-1} \cup X_i]$ is at least $m - k + 1$. Such a choice is possible by (I) and (II) for $i - 1$.

Let Y_i denote the union of all $F \in \mathcal{F}$ that contain some $v \in X_i$. Note that Y_i is finite since the F are all disjoint and because X_i is finite. Then set $S_i := S_{i-1} \cup X_i \cup Y_i$. Our choice of the S_i clearly satisfies conditions (I), (II), and (III). This finishes our definition of the sets S_i.

First suppose that $S_i \neq S_{i-1}$ for all $i \in \mathbb{N}$. Then, for each $i \in \mathbb{N}$, let V_i be obtained from $S_i \setminus S_{i-1}$ by collapsing each $F \in \mathcal{F}$ with $F \subseteq S_i \setminus S_{i-1}$ to one vertex v_F (which will be adjacent to all neighbors of F outside F). So, for all $i \in \mathbb{N}$, each vertex of V_{i+1} has a neighbor in V_i, and therefore, we may apply König's infinity lemma (Lemma 16) to the sets V_i in order to find a ray R in $\bigcup_{i \in \mathbb{N}} V_i$. We use (III) and Lemma 15(ii) to expand R to a ray R' in G. As R' has infinitely many vertices in G', this establishes a contradiction to Lemma 15(iv).

So we may assume that there is an $i \in \mathbb{N}$ such that $S_i = S_{i-1}$. Then, by (I), $H := G'[S_i]$ is a finite graph of average degree $> m - k + 1$, which is as desired. ∎

We remark that the finite subgraph H from Theorem 3 may be required to have any desired minimum order, or, even more generally, to contain any fixed finite subgraph X of G. For this, it suffices to choose the set S_1 in the proof of Theorem 3 as $V(X)$ together with all elements of $\mathcal{F}$ that touch X. Then all the rest of the proof will proceed in exactly the same way.

REFERENCES

[1] T. BÖHME AND A. KOSTOCHKA, *Disjoint K_r-minors in large graphs with given average degree*, European J. Combin., 26 (2005), pp. 289–292.

[2] B. BOLLOBÁS, *Modern Graph Theory*, Springer-Verlag, New York, 1998.

[3] B. BOLLOBÁS AND A. THOMASON, *Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs*, European J. Combin., 19 (1998), pp. 883–887.

[4] H. BRUHN AND M. STEIN, *On end degrees and infinite circuits in locally finite graphs*, Combinatorica, 27 (2007), pp. 269–291.

[5] M. DeVos, Z. Dvořák, J. Fox, J. McDonald, B. Mohar, and D. Scheide, *Minimum Degree Condition Forcing Complete Graph Immersion*, preprint, arXiv:1101.2630v1 [math.CO], 2011.

[6] R. Diestel, *Graph Theory*, 4th ed., Springer-Verlag, Heidelberg, 2010.

[7] J. Komlós and E. Szemerédi, *Topological cliques in graphs*, Combin. Probab. Comput., 3 (1994), pp. 247–256.

[8] J. Komlós and E. Szemerédi, *Topological cliques in graphs.* II, Combin. Probab. Comput., 5 (1996), pp. 79–90.

[9] A. Kostochka, *Lower bound of the Hadwiger number of graphs by their average degree*, Combinatorica, 4 (1984), pp. 307–316.

[10] W. Mader, *Homomorphieeigenschaften und mittlere Kantendichte von Graphen*, Math. Ann., 174 (1967), pp. 265–268.

[11] M. Stein, *Arboricity and tree-packing in locally finite graphs*, J. Combin. Theory Ser. B, 96 (2006), pp. 302–312.

[12] M. Stein, *Forcing highly connected subgraphs*, J. Graph Theory, 54 (2007), pp. 331–349.

[13] M. Stein, *Degree and Substructure in Infinite Graphs*, KAM series preprint (MCW 2010), Charles University, Prague, 2010.

[14] M. Stein, *Extremal Infinite Graph Theory*, Discrete Math., 311 (2011), pp. 1472–1496.