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THE CLASS OF INVERSE M-MATRICES ASSOCIATED TO RANDOM WALKS*

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Abstract. Given $W = M^{-1}$, with M a tridiagonal M-matrix, we show that there are two diagonal matrices D, E and two nonsingular ultrametric matrices U, V such that DWE is the Hadamard product of U and V. If M is symmetric and row diagonally dominant, we can take $D = E = \mathbb{I}$. We relate this problem with potentials associated to random walks and study more closely the class of random walks that lose mass at one or two extremes.

Key words. M-matrix, potential matrix, tridiagonal matrix, substochastic matrix, random walk

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1. Introduction. A result of Gantmacher and Krein [9] shows that for a nonsingular symmetric matrix A, its inverse A^{-1} is an irreducible and tridiagonal matrix if and only if A is the Hadamard product of a weak type D-matrix and a flipped weak type D-matrix. In McDonald et al. [16], the authors extend this result when A is nonsymmetric and A^{-1} is a Z-matrix (for related results see Nabben [17]). In this situation, an extra diagonal matrix is needed. We connect this type of result to potentials of random walks. In what follows a random walk is a losing mass Markov chain on $\mathcal{I} = \{1, \dots, n\}$ with nearest neighbor transition probabilities (here n is the size of the matrices). In other words, it is a general birth and death chain on a finite set.

We shall prove that the inverse of a tridiagonal irreducible M-matrix can be written as the Hadamard product of two (or one in some extreme cases) special nonsingular ultrametric matrices, after a suitable change by two diagonal matrices, which can be taken as the identity if the M-matrix is symmetric and row diagonally dominant (see Corollary 2.8). We also study this type of decomposition using even simpler ultrametric matrices, but in this case we have to allow one, or both, of these matrices to be singular (see Theorems 2.1 and 2.3). Conversely, if $W = U \odot V$ is the Hadamard product of two nonsingular ultrametric matrices U, V associated to random walks, then W^{-1} is a tridiagonal M-matrix (see Theorem 2.3). We also give necessary and sufficient conditions in terms of U, V so that W^{-1} is row diagonally dominant. We also discuss the uniqueness of this decomposition and study the special case where the random walk associated to W loses mass at one or two ends.

In what follows we denote by $\mathcal{M}^{-1}$ the class of inverse M-matrices. Given $W = M^{-1} \in \mathcal{M}^{-1}$, we denote by $\mathbb{G} = \{(i, j) : M_{ij} < 0\}$ the incidence graph of M out of the diagonal. In most of this work we are interested in matrices for which $\mathbb{G}$ is linear

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(nearest neighbor); that is, $(i,j) \in \mathbb{G}$ if and only if $|i-j| = 1$. In particular, M is tridiagonal.

An interesting subclass of $\mathcal{M}^{-1}$ is the class of *potential matrices*, which are the matrices whose inverses are row diagonally dominant M-matrices. It is straightforward to show that a potential matrix W satisfies $W^{-1} = \theta(\mathbb{I} - P)$ for some constant θ and a substochastic matrix P, that is, $P \ge 0$, $P\mathbb{1} \le \mathbb{1}$ with strict inequality at some site i: $(P\mathbb{1})_i < 1$. This decomposition is not unique, and if there exists one for which $\theta = 1$, we say that W is a *sub-Markov* potential. Even if the decomposition of W^{-1} is not unique, the graph $\mathbb{G}$ associated to W is the graph of one-step transitions of P. Hence, $\mathbb{G}$ is linear if P is the transition kernel of a (nearest neighbor) random walk.

Recall that a substochastic matrix P is irreducible if for all i, j there exists m such that $P^m_{ij} > 0$. For an irreducible substochastic matrix P, the series $\sum_{m\ge 0} P^m$ is finite, and $W = (\mathbb{I} - P)^{-1} = \sum_{m\ge 0} P^m$ is the potential of P. Thus, W_{ij} is the expected number of visits to site j when the Markov chain whose transition kernel is P starts from site i. In applications potentials are interpreted in terms of electrical networks (see, for example, [12, Chapter 2], [11, Chapters 7 and 8]).

Note that for all $i \ne j$

$$W_{ij} = f^W_{ij} W_{jj}, \tag{1.1}$$

where $f^W_{ij} \in [0,1]$ is the probability that the chain ever visits j starting from i. In what follows we define $f^W_{ii} = 1$ so that (1.1) is satisfied for all i, j. Irreducibility of P is equivalent to $W > 0$; that is, all the entries of W are positive. For this reason we say that an M-matrix is irreducible if its inverse is a positive matrix.

In general the Hadamard product of two potentials is not an inverse M-matrix. For example, take the following ultrametric matrices (actually B is a permutation of A):

$$A = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 \\ 2 & 4 & 6 & 8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 8 & 6 & 4 \\ 2 & 6 & 6 & 4 \\ 2 & 4 & 4 & 4 \end{pmatrix}.$$

A difficult problem is to give conditions under which the Hadamard product of two inverse M-matrices is again an inverse M-matrix. There are a few results in this direction (see, for example, [19]). On the other hand, the square in the sense of Hadamard of an inverse M-matrix is an inverse M-matrix, which was conjectured in [18]. This is also true for every rth Hadamard power as long as $r \ge 1$ (see [1], [2], and [6]). The rth power, for $r \ge 1$, of a potential is again a potential (see again [6] and some results for fractional powers in [8]). This means that if a potential is realized by an electrical network, then it is possible to construct another electrical network whose potential is a power of the initial one. Nevertheless, to the best of our knowledge there is no probabilistic mechanism that relates one network with the other. This remains an important question in finite state Markov chain theory.

The decompositions studied in this paper, as well as those in [16], [17], allow us to study the Hadamard powers of inverses of irreducible and tridiagonal M-matrices. We show that the rth power of any of these matrices is again an inverse of an irreducible tridiagonal M-matrix when $r > 0$ (see Theorem 2.9). Thus, for $r \ge 1$ the rth power of a potential associated to a transient irreducible random walk is again a potential of another transient irreducible random walk. This provides a way to generate new random walks by using the Hadamard powers of random walk potentials. Furthermore, formula (2.4) below gives some probabilistic information on the random walk associated to the powers of such matrices.

Moreover, we show that if W, X are inverses of irreducible tridiagonal M-matrices, then their Hadamard product $W \odot X$ is an inverse of an irreducible and tridiagonal M-matrix (see Theorem 2.10).

To give an insight into the results of this work, let us recall that a transient Markov chain can be described uniquely by its potential. In many applications one measures directly the number of visits between sites (a measurement of its potential) instead of the transition frequencies of the underlying Markov chain. With this information one could estimate the potential matrix and then the transition probabilities. The main drawback of this approach is that structural restrictions for potentials are difficult to state. Nevertheless, as a consequence of [9] and our Theorems 2.3, 2.6, and 2.7, this approach is feasible for random walks.

For example, in the symmetric case, if W is a potential of a random walk, then W is determined by two monotone sequences of positive numbers $0 < x_1 \le x_2 \le \cdots \le x_n$, $0 < y_n \le y_{n-1} \le \cdots \le y_1$ such that

$$W_{ij} = x_{i\wedge j}\, y_{i\vee j} \quad \text{and} \quad x_i = \frac{W_{in}}{W_{1n}} x_1,\, y_j = \frac{W_{1j}}{W_{11}} y_1.$$

Thus, W must satisfy the structural equation and monotonicity

$$W_{ij} = \frac{W_{in} W_{1j}}{W_{1n}},\ 1 \le i \le j \le n, \quad \text{and} \quad W_{\bullet n} \uparrow,\ W_{1\bullet} \downarrow. \tag{1.2}$$

This condition is close to being sufficient for having a potential of a random walk (see Corollary 2.4 and formula (2.7)). Therefore, in applications if one wants to model a random walk by specifying its potential, a restriction like (1.2) must be imposed. Notice also that Theorem 2.6 discriminates between the ultrametric case and the non ultrametric one by the disposition of the sites where the chain will lose mass (roots). Thus, as a consequence of this result, if the model is not ultrametric, we expect to have at least two nonconsecutive roots.

Notably, restriction (1.2) is stable under Hadamard positive powers, which is in accordance with the fact that potentials are stable under Hadamard powers. On the other hand, (1.2) is also stable under Hadamard products, an indication that the product of two inverse tridiagonal M-matrices is again an inverse tridiagonal M-matrix. The probabilistic consequences of this fact and how the restriction that both graphs are linear intervenes in this property remain open questions.

Now we start with a concept that has revealed itself to be crucial in finite potential theory and that will play a key role in this work.

Definition 1.1. *Given a square matrix A, we say that μ is a* (right) signed equilibrium potential *if $A\mu = \mathbb{1}$, and a* (right) equilibrium potential *if in addition $\mu \ge 0$. When A is nonsingular such a right signed equilibrium potential exists and is unique, and we denote it by μ_A. In this situation, we say that i is a root for A if $(\mu_A)_i \ne 0$. The set of roots of A is denoted by $\mathscr{R}(A) = \{i : (\mu_A)_i \ne 0\}$, which is the support of the signed measure μ_A.*

The fact that W is a potential is equivalent to $W \in \mathcal{M}^{-1}$ and $\mu_W = W^{-1}\mathbb{1} \ge 0$. In this case $(\mu_W)_i = \theta(1 - \sum_j P_{ij})$, and therefore $(\mu_W)_i > 0$ if and only if $\sum_j P_{ij} < 1$. The support of μ_W is exactly the set of those indexes i for which P is losing mass, and we say that P is substochastic at those sites. We note that if there exists a path (of positive probability) connecting k with a root i, which does not contain j, then $f_{kj}^W < 1$ and so $W_{kj} < W_{jj}$. On the other hand, if every path (of positive probability) that connects k with any root also contains j, then $f_{kj}^W = 1$ and $W_{kj} = W_{jj}$. So, if W^{-1} is irreducible and tridiagonal, and, for example, $i \le j$, $\mathscr{R}(W) \cap [1, j) = \emptyset$, then $f_{ij}^W = 1$, but if $\mathscr{R}(W) \cap [1, j) \neq \emptyset$, then $f_{ij}^W < 1$.

Let us recall the definition of an ultrametric matrix (see [14]).

DEFINITION 1.2. *A nonnegative symmetric matrix U is said to be* ultrametric *if*

$$(\mathcal{UL}) \qquad \forall i, j, k \quad \textit{we have} \quad U_{ij} \ge \min\{U_{ik}, U_{kj}\}.$$

Remark 1.1. Notice that by taking $j = i$ in $(\mathcal{UL})$ one obtains for all i: $U_{ii} \ge \max\{U_{ij}, U_{ji} : j \neq i\}$. If this inequality is strict for all i, we shall say that U is *strictly ultrametric*.

The following special class of ultrametric matrices will play an important role.

DEFINITION 1.3. *The symmetric matrix U of size n will be called a* linear ultrametric matrix *if there exist $k \in \{1, \dots, n\}$ positive numbers $x_1, \dots, x_n$ such that*

$$x_1 \ge x_2 \ge \cdots \ge x_{k-1} \ge x_k \le x_{k+1} \le \cdots \le x_n \tag{1.3}$$

and U_{ij} is given by

$$U_{ij} = \min\{x_s : i \wedge j \le s \le i \vee j\} = \begin{cases} x_{i \vee j} & \textit{if } i, j \le k, \\ x_{i \wedge j} & \textit{if } i, j \ge k, \\ x_k & \textit{otherwise.} \end{cases}$$

We shall say that U is in class $\mathcal{L}(k)$ to emphasize the dependence on k. We call $x_1, \dots, x_n$ the characteristics of U.

Matrices in $\mathcal{L}(1)$ are a special case of D-matrices introduced by Markham [13]. In the same vein, $\mathcal{L}(n)$ are a special case of flipped D-matrices. Also, matrices in $\mathcal{L}(k)$ are a special case of cyclops with eye $k+$ in the notation of [16]. In particular, U can be described by blocks as

$$U = \begin{pmatrix} A & x_k E \\ x_k E' & B \end{pmatrix},$$

where A is an $\mathcal{L}(k)$ matrix of size k determined by $x_1 \ge \cdots \ge x_k > 0$, B is an $\mathcal{L}(1)$ matrix of size $n - k$ determined by $0 < x_{k+1} \le \cdots \le x_n$, and E is a matrix of ones of the appropriate size and $x_k \le x_{k+1}$.

If A is a matrix indexed by $\mathcal{I}$ and $\mathcal{J}, \mathcal{K}$ are nonempty subsets of $\mathcal{I}$, then, as is customary, $A_{\mathcal{J}\mathcal{K}}$ denotes the submatrix of A by selecting the rows in $\mathcal{J}$ and columns in $\mathcal{K}$. If there is no possible confusion, we use the notation $A_{\mathcal{J}}$ instead of $A_{\mathcal{J}\mathcal{J}}$.

Remark 1.2. If $\mathcal{J} \subseteq \{1, \dots, k\}$ has cardinal $p \ge 1$ and $U \in \mathcal{L}(k)$, then $U_{\mathcal{J}} \in \mathcal{L}(p)$. Similarly, if $\mathcal{J} \subseteq \{k, \dots, n\}$, then $U_{\mathcal{J}} \in \mathcal{L}(1)$.

Every linear ultrametric matrix is an ultrametric matrix. The following theorem collects known results about these matrices and shows the connection between them and symmetric random walks.

THEOREM 1.4.

(i) *Assume that $U \in \mathcal{L}(k)$. Then, U is nonsingular if and only if there are strict inequalities in* (1.3).

(ii) *Assume that $U \in \mathcal{L}(k)$ is nonsingular. Then, $U^{-1} = \theta(\mathbb{I} - P)$ for some constant θ and a symmetric irreducible substochastic and tridiagonal matrix P that loses mass only at k.*

(iii) *Assume that P is a symmetric irreducible substochastic and tridiagonal matrix. Then $U = (\mathbb{I} - P)^{-1}$ is an ultrametric matrix if and only if one of the following two cases occurs:*

(iii.1) *$\mathscr{R}(U) = \{k\}$, in which case $U \in \mathcal{L}(k)$.*

(iii.2) *$\mathscr{R}(U) = \{k, k+1\}$; that is, the roots are adjacent. In this case $U = V_1 \odot V_2$, where $V_1 \in \mathcal{L}(k), V_2 \in \mathcal{L}(k+1)$ are nonsingular.*

Part (iii.2) will be generalized in Theorem 2.6 to the case where the potential has more than two roots or has two roots in the general position. A formula for the tridiagonal matrix U^{-1} when $U \in \mathcal{L}(k)$, $2 \le k \le n-1$, is given by

$$
\begin{aligned}
&U^{-1}_{11} = -U^{-1}_{12} = -U^{-1}_{21} = \tfrac{1}{U_{11}-U_{22}},\\
&\forall\, i \le n-1, \quad U^{-1}_{i,i+1} = U^{-1}_{i+1,i} = \tfrac{-1}{|U_{i+1,i+1}-U_{ii}|},\\
&\forall\, 2 \le i \le n,\ i \ne k, \quad U^{-1}_{ii} = -U^{-1}_{i,i-1} - U^{-1}_{i,i+1},\\
&U^{-1}_{kk} = \tfrac{U_{k+1,k+1}}{U_{kk}U_{k+1,k+1}-U^2_{kk}} + \tfrac{1}{U_{k-1,k-1}-U_{kk}}.
\end{aligned} \tag{1.4}
$$

When $U \in \mathcal{L}(1)$, we have

$$
\begin{aligned}
&U^{-1}_{11} = \tfrac{1}{U_{11}} + \tfrac{1}{U_{22}-U_{11}},\\
&\forall i \le n-1, \quad U^{-1}_{i,i+1} = U^{-1}_{i+1,i} = \tfrac{-1}{U_{i+1,i+1}-U_{ii}},\\
&\forall\, 2 \le i \le n, \quad U^{-1}_{ii} = -U^{-1}_{i,i-1} - U^{-1}_{i,i+1}.
\end{aligned} \tag{1.5}
$$

In both cases we assume implicitly that $U^{-1}_{n,n+1} = 0$. A similar formula holds when $U \in \mathcal{L}(n)$.

2. Main results.

THEOREM 2.1. *Let M be a tridiagonal irreducible M-matrix, of size n, with inverse $W = M^{-1}$. Then, there exist two positive diagonal matrices D, E such that $(DWE)^{-1}$ is an irreducible symmetric tridiagonal and row diagonally dominant M-matrix, and, moreover,*

$$DWE = U \odot V \tag{2.1}$$

for some $U \in \mathcal{L}(1)$ and $V \in \mathcal{L}(n)$. If M is row diagonally dominant, we can take $D = \mathbb{I}$, and if M is symmetric, we can take $D = E$.

All diagonal matrices D, E for which $X = DWE$ is a symmetric potential are constructed in the following way. Take $\rho \in \mathbb{R}^n$ to be any nonnegative nonzero vector and define $D = D(\rho)$ as

$$\forall i, \quad D_{ii} = \frac{1}{(W\rho)_i}. \tag{2.2}$$

Next define $E = E(a, \rho)$ as the solution of the iteration: $E_{11} = a > 0$, and

$$\forall i \ge 2, \quad E_{ii} = \frac{D_{ii}E_{i-1,i-1}}{D_{i-1,i-1}} \frac{W_{i,i-1}}{W_{i-1,i}}. \tag{2.3}$$

The right equilibrium potential of X is $\mu_X = E^{-1}\rho$.

In particular, if we choose $\rho = e_1 = (1, 0, \dots, 0)'$ as the first vector in the canonical basis of $\mathbb{R}^n$, and we consider $\overline{D} = D(e_1)$, $\overline{E} = E(1, e_1)$ then $\overline{D}W\overline{E} = \overline{U}$ is a symmetric potential with an inverse that is an irreducible tridiagonal row diagonally dominant M-matrix, and its right equilibrium potential is $\mu_{\overline{U}} = e_1$. Hence, $\overline{U} \in \mathcal{L}(1)$.

In the last part of the previous theorem we have shown that for some special $\overline{D}$ and $\overline{E}$ the matrix $\overline{D}W\overline{E} \in \mathcal{L}(1)$. This is a special case of the general decomposition given in (2.1), where $V = \mathbb{1}\mathbb{1}'$ is the matrix full of ones. The importance of this special decomposition is that every potential of a random walk can be changed to a linear ultrametric matrix through two diagonal matrices.

Using this transformation and formula (1.5) we obtain a formula for W^{-1} in terms of W, where W^{-1} is an irreducible tridiagonal M-matrix. This formula is

$$\begin{aligned} &W^{-1}_{ij} = 0 \quad \text{if } |i-j| > 1,\\ &\forall i \le n-1, \quad \begin{cases} W^{-1}_{i,i+1} = \frac{-W_{i,i+1}W_{i1}W_{i+1,1}}{W_{i+1,i+1}W_{i+1,i}W_{i1}^2 - W_{ii}W_{i,i+1}W_{i+1,1}^2},\\ W^{-1}_{i+1,i} = W^{-1}_{i,i+1}\frac{W_{i+1,i}}{W_{i,i+1}}, \end{cases}\\ &\forall 2 \le i \le n-1, \quad W^{-1}_{ii} = -W^{-1}_{i,i-1}\frac{W_{i-1,1}}{W_{i1}} - W^{-1}_{i,i+1}\frac{W_{i+1,1}}{W_{i,1}},\\ &W^{-1}_{11} = -W^{-1}_{12}\frac{W_{2n}}{W_{1n}}, \quad W^{-1}_{nn} = -W^{-1}_{n,n-1}\frac{W_{n-1,1}}{W_{n,1}}. \end{aligned} \tag{2.4}$$

Remark 2.1. Recall that each diagonal entry of an ultrametric matrix dominates its corresponding column (and row) and this property is stable under Hadamard products. Thus, if $W = U \odot V$ is the Hadamard product of two ultrametric matrices, then its diagonal entries dominate the corresponding column.

In the next result we show uniqueness of the decomposition $W = U \odot V$ up to a multiplicative constant.

PROPOSITION 2.2. *Assume that* $W = U \odot V = \tilde{U} \odot \tilde{V}$ *for some* $U, \tilde{U} \in \mathcal{L}(1)$ *and* $V, \tilde{V} \in \mathcal{L}(n)$. *Then there exists* $a > 0$ *such that* $\tilde{U} = aU$ *and* $\tilde{V} = \frac{1}{a}V$.

Proof. Let $0 < x_1 \le \cdots \le x_n$, $0 < \tilde{x}_1 \le \cdots \le \tilde{x}_n$, $y_1 \ge \cdots \ge y_n > 0$, and $\tilde{y}_1 \ge \cdots \ge \tilde{y}_n > 0$ be the collection of numbers defining $U, \tilde{U}, V$, and $\tilde{V}$, respectively. We define $a = \tilde{x}_1/x_1$. We note that for all i we have $W_{i1} = x_1 y_i = \tilde{x}_1\tilde{y}_i$, and therefore for all i we obtain $y_i = a\tilde{y}_i$, which implies that $V = a\tilde{V}$. Similarly, we obtain that $\tilde{U} = aU$, and the result is shown. ∎

Example 2.1. *When M is not row diagonally dominant, a decomposition like $W = U \odot V$ may not exist, and the use of the diagonal matrices is necessary. Take, for example,*

$$W = \begin{pmatrix} 35 & 20 & 25 \\ 20 & 16 & 20 \\ 25 & 20 & 35 \end{pmatrix},$$

whose inverse is the irreducible tridiagonal M-matrix

$$M = \begin{pmatrix} 0.1 & -0.125 & 0 \\ -0.125 & 0.375 & -0.125 \\ 0 & -0.125 & 0.1 \end{pmatrix}.$$

We point out that M is row diagonally dominant only for the second row. A simple inspection shows that W is not the Hadamard product of linear ultrametric matrices (see Remark 2.1 *and Theorem* 2.3(ii)*).*

THEOREM 2.3. *Let* $U \in \mathcal{L}(1)$, *let* $V \in \mathcal{L}(n)$ *be of size* n, *and consider* $W = U \odot V$.

(i) *W is nonsingular if and only if for all* $i = 1, \ldots, n-1$

$$U_{i+1,i+1}V_{ii} > U_{ii}V_{i+1,i+1},$$

which is equivalent to

$$\frac{W_{n,i+1}W_{1,i}}{W_{1n}} > W_{i,i+1}.$$

In particular, W is nonsingular if U and V are nonsingular.

In what follows we assume that W is nonsingular with inverse $M = W^{-1}$.

(ii) *M is an irreducible tridiagonal M-matrix, which is row diagonally dominant at rows $1, n$.*

(iii) *M is strictly diagonally dominant at row 1 if and only if*

$$(U_{22} - U_{11})(U_{33}V_{22} - U_{22}V_{33}) > 0. \tag{2.5}$$

Since $V_{22} \geq V_{33} > 0$ a sufficient condition for this to happen is that $U_{33} > U_{22} > U_{11}$, which is the case when U is nonsingular. That is, if U is nonsingular, M is strictly row diagonally dominant at row 1.

Similarly, M is strictly diagonally dominant at row n if and only if

$$(V_{n-1,n-1} - V_{nn})(U_{n-1,n-1}V_{n-2,n-2} - U_{n-2,n-2}V_{n-1,n-1}) > 0. \tag{2.6}$$

Again, since $0 < U_{n-2,n-2} \leq U_{n-1,n-1}$, a sufficient condition is that $V_{n-2,n-2} > V_{n-1,n-1} > V_{nn}$ which is the case when V is nonsingular. That is, if V is nonsingular, M is strictly row diagonally dominant at row n.

(iv) *When $n \geq 3$, we define for any $i \in \{2, \dots, n-1\}$ the set $\mathcal{J} = \{i-1, i, i+1\}$. The matrix $W_{\mathcal{J}}$ is nonsingular, and its inverse $N(i)$ is a tridiagonal M-matrix, which is row diagonally dominant at rows $1, 3$ (strictly row diagonally dominant in the case $U_{\mathcal{J}}$ and $V_{\mathcal{J}}$ are nonsingular).*

(v) *Either of the following two conditions is necessary and sufficient for M to be row diagonally dominant at row $i \in \{2, \dots, n-1\}$:*

$$(U_{i+1,i+1} - U_{i-1,i-1})(V_{i-1,i-1} - V_{ii}) \geq (U_{ii} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i+1,i+1}), \tag{2.7}$$

$$N(i) \text{ is row diagonally dominant at row } 2. \tag{2.8}$$

Furthermore, M is strictly row diagonally dominant at row i if there is a strict inequality in (2.7), which is equivalent to saying that $N(i)$ is strictly row diagonally dominant at row 2.

When U and V are nonsingular, (2.7) and (2.8) are equivalent to

$$U^{-1}_{i,i-1}V^{-1}_{i,i+1} \geq U^{-1}_{i,i+1}V^{-1}_{i,i-1}. \tag{2.9}$$

(vi) *If U and V are sub-Markov potentials, that is, $U^{-1} = \mathbb{I} - P$, $V^{-1} = \mathbb{I} - Q$ with P, Q substochastic matrices, then the diagonal of M is bounded by one:*

$$\forall i, \quad M_{ii} \leq 1.$$

As consequence of these theorems, we obtain the following result.

COROLLARY 2.4. *Assume that W is a symmetric nonsingular matrix with inverse $M = W^{-1}$. The following two conditions are equivalent:*

(i) *M is a tridiagonal irreducible row diagonally dominant M-matrix.*

(ii) *$W = U \odot V$, where $U \in \mathcal{L}(1)$, $V \in \mathcal{L}(n)$, and (2.7) is satisfied for all $i \in \{2, \dots, n-1\}$.*

Under either of these two equivalent conditions we have

$$M_{ij} = \begin{cases} (W_{\{1,2,3\}})^{-1}_{ij} & \text{if } i = 1,\ j = 1, 2, \\ (W_{\{i-1,i,i+1\}})^{-1}_{2,j-i+2} & \text{if } 2 \le i \le n-1,\ |i-j| \le 1, \\ (W_{\{n-2,n-1,n\}})^{-1}_{3,j-n+3} & \text{if } i = n,\ j = n-1, n. \end{cases} \tag{2.10}$$

If, in addition, U and V are sub-Markov potentials, then W is also a sub-Markov potential.

Proof. (i)⇒(ii) follows from Theorem 2.1. (ii)⇒(i) follows from Theorem 2.3. That (2.10) holds under each one of these conditions follows from the proof we will do by induction of Theorem 2.3. In particular, we will show (see (3.8)) that for all $3 \le p \le n$

$$(W_{\{1,2,3\}})^{-1}_{11} = (W_{\{1,\dots,p\}})^{-1}_{11},$$

$$(W_{\{1,2,3\}})^{-1}_{12} = (W_{\{1,\dots,p\}})^{-1}_{12}.$$

The other cases in (2.10) follow similarly. Finally, if U and V are sub-Markov potentials, then by Theorem 2.3(vi) $M_{ii} \le 1$ for all i, which, together with the fact that M is a row diagonally dominant M-matrix, shows that $M = \mathbb{I} - N$ for some substochastic matrix N. Thus W is a sub-Markov potential. ◻

Remark 2.2. It may happen that $W = U \odot V$ is nonsingular and $U \in \mathcal{L}(1)$, $V \in \mathcal{L}(n)$ are singular. Indeed, consider the example

$$W = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \odot \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Moreover, if $W = A \odot B$ with $A \in \mathcal{L}(1)$, $B \in \mathcal{L}(n)$, then both A, B are singular. Notice that in this case $W \in \mathcal{L}(2)$.

The following result gives conditions to have U and V be nonsingular.

PROPOSITION 2.5. *Assume that $U \in \mathcal{L}(1)$, $V \in \mathcal{L}(n)$, and consider $W = U \odot V$, which is, of course, a symmetric matrix.*

(i) *A necessary and sufficient condition to have U and V be nonsingular is that for all $i = 1, \dots, n-1$*

$$W_{i,i+1} < \min\{W_{ii}, W_{i+1,i+1}\}. \tag{2.11}$$

Hence, using Theorem 2.3(i), *this condition implies that W is nonsingular.*

(ii) *Assume that W is nonsingular and that $M = W^{-1}$ is a tridiagonal irreducible row diagonally dominant M-matrix. Then, U is nonsingular if and only if M is strictly diagonally dominant at row* 1. *Similarly, V is nonsingular if and only if M is strictly diagonally dominant at row n.*

Proof. (i) Consider $0 < x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \ge y_2 \ge \cdots \ge y_n > 0$ the numbers defining U and V. Since $W_{i,i+1} = x_i y_{i+1}, W_{i+1,i+1} = x_{i+1} y_{i+1}, W_{ii} = x_i y_i$, condition (2.11) is equivalent to saying that x's are strictly increasing and y's are strictly decreasing, which is equivalent to U and V being nonsingular (see Theorem 1.4).

(ii) If U is nonsingular, then from Theorem 2.3(iii), we get that M is strictly diagonally dominant at row 1. To prove the converse we assume that M is strictly diagonally dominant at row 1. Without loss of generality we also assume that $M = \mathbb{I} - P$

with P symmetric, substochastic, irreducible, tridiagonal and strictly substochastic at row 1. The Markov chain associated to P loses mass at least at node 1, which, together with the fact that P is tridiagonal, implies that $f^W_{i,i+1} < 1$ for all $i = 1, \dots, n-1$.

We denote by $0 < x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \ge y_2 \ge \cdots \ge y_n > 0$ the numbers defining U and V. Since $x_i y_{i+1} = W_{i,i+1} = f^W_{i,i+1} W_{i+1,i+1} < W_{i+1,i+1} = x_{i+1} y_{i+1}$, we conclude that the x's are strictly increasing and therefore U is nonsingular. The conclusions for V follow similarly. $\square$

Remark 2.3. If $W = U \odot V$, then condition (2.11) is sufficient for W being nonsingular. On the other hand, if W is nonsingular, then the 2×2 matrix $W_{\{i,i+1\}}$ is positive definite, because W is an inverse M-matrix. In particular its determinant is positive, or, equivalently,

$$W_{i,i+1} < \sqrt{W_{ii} W_{i+1,i+1}}.$$

Thus, condition (2.11) is a strengthening of the necessary condition for W to be positive definite, namely, that each principal minor of size 2 must be positive.

The aim of the next result is to show that if W is the potential of a symmetric irreducible random walk, then it is either a linear ultrametric potential or the Hadamard product of two *nonsingular* linear ultrametric potentials.

THEOREM 2.6. *Assume that $M = W^{-1}$ is a symmetric irreducible tridiagonal and row diagonally dominant M-matrix. Then*

(i) *$\mathscr{R}(W) = \{k\}$ if and only if $W \in \mathcal{L}(k)$;*

(ii) *$|\mathscr{R}(W)| \ge 2$ if and only if $W = U \odot V$, where $U \in \mathcal{L}(k)$, $V \in \mathcal{L}(m)$ are nonsingular and $k = \min \mathscr{R}(W), m = \max \mathscr{R}(W)$.*

Remark 2.4. When $|\mathscr{R}(W)| \ge 2$, the decomposition given by this theorem is not unique. We shall see that there are

$$1 + (k-1) + (n-m)$$

degrees of freedom in such decomposition.

As a converse of the previous theorem, we have the following result.

THEOREM 2.7. *Assume that $U \in \mathcal{L}(k)$, $V \in \mathcal{L}(m)$ are nonsingular of size n (without loss of generality we assume $k \le m$). Then $W = U \odot V$ is nonsingular, and $M = W^{-1}$ is an irreducible tridiagonal M-matrix. The sum of row i is zero for $i \notin \{k \le j \le m\}$ and it is strictly positive for $i \in \{k, m\}$. When $k < i < m$, M is row diagonally dominant at row i if and only if* (2.7) *holds, that is,*

$$(U_{i+1,i+1} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i,i}) \ge (U_{i,i} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i+1,i+1}).$$

There is a strict inequality in this formula if and only if M is strictly diagonally dominant at row i.

We summarize some of these results in the next corollary.

COROLLARY 2.8. *Assume that W is a nonsingular positive matrix and denote by M its inverse. Then, M is an irreducible tridiagonal M-matrix if and only if there exist two positive diagonal matrices D, E such that $X = DWE$ is a symmetric potential and X is a linear ultrametric matrix or the Hadamard product of two nonsingular linear ultrametric matrices U, V. The first case occurs when X has one root. In the second case the roots of X are contained in the convex set determined by the roots of U and V. If M is row diagonally dominant we can take $D = \mathbb{I}$ and if M is symmetric we can take $D = E$.*

THEOREM 2.9. *Assume that W^{-1} is an irreducible tridiagonal M-matrix of size n. Then, for all $r > 0$ the matrix $W^{(r)}$, the rth Hadamard power of W, is nonsingular* *and its inverse is an irreducible tridiagonal M-matrix. If W is a potential (respectively, sub-Markov potential) and $r \geq 1$, then $W^{(r)}$ is also a potential (respectively, sub-Markov potential).*

For $r < 0$ the matrix $W^{(r)}$ is nonsingular, its inverse $C(r)$ is an irreducible tridiagonal matrix, and the following properties hold:

(i) $sign(\det(W^{(r)})) = (-1)^{n+1}$.

(ii) *If $n \geq 2$, then for all i, j we have*

$$C(r)_{ij} \text{ is } \begin{cases} < 0 & \text{if } i = j, \\ > 0 & \text{if } |i - j| = 1, \\ = 0 & \text{otherwise.} \end{cases}$$

(iii) *If W is symmetric, then the eigenvalues of $W^{(r)}$ are negative, except for the principal one, λ_1, which is positive and with maximal absolute value.*

In this theorem the *sign* function is given by

$$sign(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

THEOREM 2.10. *Assume that W and X are nonsingular and W^{-1}, X^{-1} are two irreducible tridiagonal M-matrices. Then the Hadamard product $W \odot X$ is again the inverse of an irreducible tridiagonal M-matrix.*

Proof. There exist diagonal matrices $D, \widetilde{D}, E, \widetilde{E}$ and nonsingular matrices $U, \widetilde{U} \in \mathcal{L}(1)$ such that

$$DWE = U, \quad \widetilde{D}X\widetilde{E} = \widetilde{U}.$$

Then

$$U \odot \widetilde{U} = (DWE) \odot (\widetilde{D}X\widetilde{E}) = D\widetilde{D}(W \odot X)E\widetilde{E}$$

is again a nonsingular matrix in $\mathcal{L}(1)$, and the result follows. ◻

3. Proof of main results. We start with a useful result about principal submatrices of inverse tridiagonal irreducible M-matrices.

LEMMA 3.1. *Assume $W = M^{-1}$ is the inverse of an irreducible tridiagonal M-matrix indexed by $\mathcal{I} = \{1, \ldots, n\}$. Let $\mathcal{J} = \{\ell_1 < \cdots < \ell_p\} \subset \mathcal{I}$, and let $X = W_{\mathcal{J}}$ be a principal submatrix of W. Then X^{-1} is an irreducible tridiagonal M-matrix. If W is a (sub-Markov) potential, then X is also a (sub-Markov) potential. Moreover, if $\mu = \mu_W$ is the right equilibrium potential of W, then*

$$\mu_X \geq \mu|_{\mathcal{J}} - M_{\mathcal{J}\mathcal{J}^c} M^{-1}_{\mathcal{J}^c\mathcal{J}^c} \mu|_{\mathcal{J}^c} \geq \mu|_{\mathcal{J}}, \tag{3.1}$$

and $\ell(\mathscr{R}(X)) := \{\ell_i : i \in \mathscr{R}(X)\} \supseteq \mathscr{R}(W) \cap \mathcal{J}$.

Consider $i = \ell_s$ for some $s = 1, \ldots, p$. If $\{i, i+1\} \subseteq \mathcal{J}$, then $X^{-1}_{s,s+1} = M_{i,i+1}$. Furthermore, if $\{i-1, i, i+1\} \subseteq \mathcal{J}$, then $X^{-1}_{s,s+t} = M_{i,i+t}$ for $t \in \{-1, 0, 1\}$. Finally, if $\{1, 2\} \subseteq \mathcal{J}$, we have $X^{-1}_{1k} = M_{1k}$ for $k \in \{1, 2\}$ (a similar relation holds when $\{n-1, n\} \subset \mathcal{J}$).

Proof. We assume that $\mathcal{J} = \{\ell_1 < \cdots < \ell_p\} \subsetneqq \mathcal{I}$. It is well known that X is an inverse M-matrix (see [10, p. 119]), and since $W > 0$, we know that X^{-1} is irreducible.

Using the inverse by block formula we obtain

$$X^{-1} = M_{\mathcal{J}\mathcal{J}} - M_{\mathcal{J}\mathcal{J}^c} M^{-1}_{\mathcal{J}^c\mathcal{J}^c} M_{\mathcal{J}^c\mathcal{J}}. \tag{3.2}$$

For the rest of the proof, we denote $Y = M^{-1}_{\mathcal{J}^c\mathcal{J}^c}$.

The set $\mathcal{J}$ induces a partition on $\mathcal{J}^c = \{l_1 < \cdots < l_{n-p}\}$, in at least one atom, given by the sets $[\ell_s, \ell_{s+1}] \cap \mathcal{J}^c$, where $\ell_s, \ell_{s+1} \in \mathcal{J}$ are consecutive in this subset, together with the sets $[1, \ell_1] \cap \mathcal{J}^c$ and $[\ell_p, n] \cap \mathcal{J}^c$. Denote the nonempty atoms by $\mathscr{A}_1, \ldots, \mathscr{A}_r$. The fact that M is tridiagonal implies that $M_{\mathscr{A}_a\mathscr{A}_b} = 0$ for $a \neq b$. This block structure of $M_{\mathcal{J}^c\mathcal{J}^c}$ is also present in Y. Therefore we have the formula, for $\ell_s = i < j = \ell_t$,

$$X^{-1}_{st} = M_{ij} - C_{i+1,j-1} M_{i,i+1} Y_{l_q,l_r} M_{j-1,j},$$

where $C_{i+1,j-1} = 1$ if $i+1$, $j-1$ belong to the same atom in $\mathcal{J}^c$ and $l_q = i+1$, $l_r = j-1$. When $i+1$ or $j-1$ does not belong to $\mathcal{J}^c$ or they belong to different atoms, we take $C_{i+1,j-1} = 0$.

Thus, if there exists $k \in \mathcal{J}$ such that $i < k < j$, we conclude that $X^{-1}_{st} = 0$ and therefore X^{-1} is a tridiagonal M-matrix.

Similarly, we obtain that

$$X^{-1}_{ss} = M_{ii} - C_{i+1,i+1} M_{i,i+1} Y_{l_q,l_q} M_{i+1,i} - C_{i-1,i-1} M_{i,i-1} Y_{l_{q'},l_{q'}} M_{i-1,i}, \tag{3.3}$$

from which the last part of the lemma follows (here $l_{q'} = i-1$ when this element belongs to $\mathcal{J}^c$).

Assume now that W is a potential, that is, $\mu = M\mathbb{1}_{\mathcal{I}} \geq 0$. Then, we get

$$\begin{aligned} M_{\mathcal{J}\mathcal{J}}\mathbb{1}_{\mathcal{J}} &= \mu|_{\mathcal{J}} - M_{\mathcal{J}\mathcal{J}^c}\mathbb{1}_{\mathcal{J}^c}, \\ M_{\mathcal{J}^c\mathcal{J}^c}\mathbb{1}_{\mathcal{J}^c} &= \mu|_{\mathcal{J}^c} - M_{\mathcal{J}^c\mathcal{J}}\mathbb{1}_{\mathcal{J}}. \end{aligned}$$

Since $Y = M^{-1}_{\mathcal{J}^c\mathcal{J}^c} \geq 0$ and $-M_{\mathcal{J}\mathcal{J}^c} \geq 0$, we get

$$M_{\mathcal{J}\mathcal{J}}\mathbb{1}_{\mathcal{J}} = \mu|_{\mathcal{J}} - M_{\mathcal{J}\mathcal{J}^c} Y \mu|_{\mathcal{J}^c} + M_{\mathcal{J}\mathcal{J}^c} Y M_{\mathcal{J}^c\mathcal{J}}\mathbb{1}_{\mathcal{J}},$$

which yields

$$\mu_X = X^{-1}\mathbb{1}_{\mathcal{J}} = M_{\mathcal{J}\mathcal{J}}\mathbb{1}_{\mathcal{J}} - M_{\mathcal{J}\mathcal{J}^c} Y M_{\mathcal{J}^c\mathcal{J}}\mathbb{1}_{\mathcal{J}} = \mu|_{\mathcal{J}} - M_{\mathcal{J}\mathcal{J}^c} Y \mu|_{\mathcal{J}^c} \geq \mu|_{\mathcal{J}}.$$

Thus X is a potential, and $\ell(\mathscr{R}(X)) \supseteq \mathscr{R}(W) \cap \mathcal{J}$.

Finally, assume that W is a sub-Markov potential. We have to prove that X is also a sub-Markov potential, which given that X is a potential, amounts to showing that the diagonal elements of X^{-1} are bounded above by 1. This follows from (3.3) because $X^{-1}_{ss} \leq M_{ii} \leq 1$. ∎

Proof of Theorem 1.4. (i) The property follows from the fact that a positive ultrametric matrix is nonsingular if and only if all rows are different (see [15] or [4]).

(ii) Every nonsingular ultrametric matrix is a potential. When $U \in \mathcal{L}(k)$, the kth column is constant and therefore $U^{-1}\mathbb{1} = \frac{1}{x_k} e_k$, where e_k is the kth vector of the canonical basis in $\mathbb{R}^n$. Thus, the only root of U is k. That U^{-1} is tridiagonal follows, for example, from Theorem 3 in [4], because the tree matrix extension of U is supported by a path or linear tree (see also Theorem 4.10 in [17]).

(iii) Assume that U^{-1} is tridiagonal, that is, its incidence graph is a path. For a nonsingular ultrametric matrix, all the roots are connected (see Theorem 4 in [4]), and so U can have one root or two adjacent roots. Conversely, assume that U has only one root at k. For $i \le j \le k$ we have $U_{ij} = f^U_{ij} U_{jj} = U_{jj}$. Similarly, if $k \le i \le j$, we have $U_{ij} = U_{ii}$. Finally, for the case $i \le k \le j$ one has $U_{ij} = U_{kj} = U_{jk} = U_{kk}$. Thus, $U \in \mathcal{L}(k)$ with characteristics $x_i = U_{ii}$.

Finally, assume that U has two consecutive roots at $k, k+1$; then according to Theorem 4.10 in [17] U is ultrametric. The fact that $U = V_1 \odot V_2$ with $V_1 \in \mathcal{L}(k), V_2 \in \mathcal{L}(k+1)$ is a particular case of what we will prove in Theorem 2.6. □

Now, we turn to the proof of the main results.

Proof of Theorem 2.1. Assume first that M is a symmetric tridiagonal irreducible row diagonally dominant M-matrix. Consider $W = M^{-1}$. The proof is done by induction on the order n of the matrix M. For $n = 1$ the result is obvious. So assume that the result holds for every symmetric row diagonally dominant matrix of order at most $n-1$. Without loss of generality we can assume that $M = \mathbb{I} - P$, where P is a substochastic tridiagonal matrix. We decompose M and W by blocks as follows:

$$M = \begin{pmatrix} 1 - P_{11} & -\zeta' \\ -\zeta & N \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} W_{11} & a' \\ a & T \end{pmatrix}.$$

Here $\zeta' = (P_{12}, 0, \ldots, 0) \ge 0$, $\zeta' \mathbb{1} \le 1 - P_{11}$, $a' = (W_{12}, \ldots, W_{1n})$, N is a tridiagonal, symmetric row diagonally dominant M-matrix, and, moreover, $N\mathbb{1} - \zeta \ge 0$. From Lemma 3.1 T is also a sub-Markov potential, and T^{-1} is tridiagonal. Then, by the induction hypothesis there exist two ultrametric matrices R, S, where $R \in \mathcal{L}(1)$ and $S \in \mathcal{L}(n-1)$ such that $T = R \odot S$. We denote by $0 < x_2 \le x_3 \le \cdots \le x_n$ and $y_2 \ge y_3 \ge \cdots \ge y_n > 0$ the numbers defining R and S, respectively.

The fact that P is tridiagonal and symmetric implies that for $i \ge 2$

$$\begin{aligned} W_{i1} &= f^W_{i2} W_{21} = f^W_{i2} W_{12} = f^W_{i2} f^W_{12} W_{22} \\ &= f^W_{12} W_{i2}. \end{aligned}$$

We take $x_1 = x_2 f^W_{12}$, which verifies $0 < x_1 \le x_2$. Then $W_{21} = x_1 y_2, W_{31} = x_1 y_3, \ldots, W_{n1} = x_1 y_n$. Finally, we take $y_1 = W_{11}/x_1 \ge W_{21}/x_1 = y_2$. We define $U \in \mathcal{L}(1)$ associated to $0 < x_1 \le x_2 \le x_3 \le \cdots \le x_n$, and $V \in \mathcal{L}(n)$ associated to $y_1 \ge y_2 \ge y_3 \ge \cdots \ge y_n > 0$ to get $W = U \odot V$, and the result is proved in this case.

Now, consider M a general irreducible tridiagonal M-matrix. Then, there exists a positive diagonal matrix F (see [10, Theorem 2.5.3]) such that $L = MF$ is a row diagonally dominant M-matrix. If M is a row diagonally dominant M-matrix, we can take $F = \mathbb{I}$. Clearly L is also tridiagonal and irreducible. Now, we look for a diagonal matrix G such that GL is also symmetric. The condition is that $G_{ii} L_{i,i-1} = G_{i-1,i-1} L_{i-1,i}$ for $i = 2, \ldots, n$. We take $G_{11} = F_{11}$, and we define inductively, for $i \ge 2$,

$$G_{ii} = G_{i-1,i-1} \frac{L_{i-1,i}}{L_{i,i-1}}.$$

The diagonal of G is positive by construction. Thus, $H = GMF$ is a symmetric tridiagonal irreducible M-matrix, and

$$H\mathbb{1} = G(MF\mathbb{1}) \ge 0.$$

Hence, H is also row diagonally dominant, and there exist two ultrametric matrices $U \in \mathcal{L}(1), V \in \mathcal{L}(n)$ such that $H^{-1} = U \odot V = F^{-1} M^{-1} G^{-1}$. Thus, we take $D = F^{-1}$ and $E = G^{-1}$.

If M is symmetric, we obtain for $i \geq 2$

$$G_{ii} = G_{i-1,i-1}\frac{L_{i-1,i}}{L_{i,i-1}} = G_{i-1,i-1}\frac{M_{i-1,i}F_{ii}}{M_{i,i-1}F_{i-1,i-1}} = G_{i-1,i-1}\frac{F_{ii}}{F_{i-1,i-1}}.$$

The solution is $G = F$, which that implies $H = FMF$ is a symmetric tridiagonal row diagonally dominant M-matrix, and we obtain $D = E = F^{-1}$.

Take D, E diagonal matrices with positive diagonal elements. Let us now assume that $X = DWE$ is a symmetric potential and consider μ its right equilibrium potential. Then $DWE\mu = \mathbb{1}$, and if we define $\rho = E\mu$, we obtain a nonzero, nonnegative vector. Obviously we get that $D = D(\rho)$, as defined in (2.2). Since X is symmetric, we get that E must satisfy (2.3). Also we obtain that $\mu = E^{-1}\rho$.

Conversely, assume that D, E are constructed as in (2.2) and (2.3). The matrix $X = DWE$ is a nonsingular matrix with an inverse $X^{-1} = E^{-1}MD^{-1}$. Thus, X^{-1} is an irreducible tridiagonal M-matrix. On the other hand, $XE^{-1}\rho = DW\rho = \mathbb{1}$, which means that X^{-1} is a row diagonally dominant matrix. The only thing left to be proved is that X is symmetric. This is equivalent to proving that $E^{-1}MD^{-1}$ is symmetric, which follows from the fact that for an irreducible tridiagonal M-matrix M with inverse W it holds that (see formula (2.4))

$$\frac{M_{i,i+1}}{M_{i+1,i}} = \frac{W_{i,i+1}}{W_{i+1,i}}.$$

Finally, if we take $\rho = e_1$, then $\overline{U} = \overline{D}W\overline{E}$ is a symmetric potential whose inverse is an irreducible symmetric tridiagonal row diagonally dominant M-matrix. Hence, $(\overline{U})^{-1} = \theta(\mathbb{I} - P)$ for some constant θ and an irreducible symmetric tridiagonal substochastic matrix P. Since the right equilibrium potential of $\overline{U}$ is $\mu_{\overline{U}} = e_1$, we get that P loses mass only at the first row. Then, from Theorem 1.4 we conclude that $\overline{U} \in \mathcal{L}(1)$. ∎

The proof of Theorem 2.3 requires the next lemma, which is essentially the result we want to show for dimension 3.

Lemma 3.2. *Consider the tridiagonal symmetric substochastic matrices*

$$P = \begin{pmatrix} 1-x-z & x & 0 \\ x & 1-x-y & y \\ 0 & y & 1-y \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1-p & p & 0 \\ p & 1-p-q & q \\ 0 & q & 1-q-s \end{pmatrix},$$

where $0 < x$, $0 < y < 1$, $0 < z$, $x + z < 1$, $x + y \leq 1$ and similarly $0 < p < 1$, $0 < q$, $0 < s$, $p + q \leq 1$, $q + s < 1$. Then $A = (\mathbb{I} - P)^{-1} \in \mathcal{L}(1)$, and $B = (\mathbb{I} - Q)^{-1} \in \mathcal{L}(n)$. The matrix $W = A \odot B$ is nonsingular, and $N = W^{-1}$ is the tridiagonal M-matrix given by

$$N = \begin{pmatrix} \frac{\beta}{\alpha} & -\frac{\gamma}{\alpha} & 0 \\ -\frac{\gamma}{\alpha} & \frac{\epsilon}{\alpha\delta} & -\frac{\eta}{\delta} \\ 0 & -\frac{\eta}{\delta} & \frac{\theta}{\delta} \end{pmatrix}, \quad \begin{cases} \alpha = pqz + psz + qsx + qsz; \\ \beta = (x+z)pqsz; \\ \gamma = xpqsz; \\ \epsilon = (pqzx + pqzy + psyx + pszx + pszy + qsyx + qszx + qszy)zqsx; \\ \delta = qxz + syx + sxz + syz; \\ \theta = (q+s)sxyz; \\ \eta = qsxyz. \end{cases}$$

The following inequalities hold:

$$0 < \gamma < \beta < \alpha, \quad 0 < \epsilon < \alpha\delta, \quad \text{and} \quad 0 < \eta < \theta < \delta.$$

Thus, N is strictly row diagonally dominant at rows 1, 3, *and the diagonal elements of N are smaller than* 1*. The sum of the second row is*

$$\frac{s^2z^2qx(qx-py)}{\alpha\delta}.$$

Hence, N is a tridiagonal row diagonally dominant M-matrix if and only if

$$qx - py \geq 0, \tag{3.4}$$

and in this case we have $N = \mathbb{I} - R$ for a substochastic matrix R, which is stochastic at the second row if and only if equality holds in (3.4)*.*

Proof. The proof is direct using MAPLE. ◻

Proof of Theorem 2.3. (i) The result follows from formula (2.10). When U and V are nonsingular they are inverse M-matrices and therefore they are positive definite matrices. Hence, W is positive definite and a fortiori nonsingular.

For proving parts (ii)–(vi) we shall first assume that U and V are nonsingular.

(ii)–(iv) We show by induction on n the size of W that $M = W^{-1}$ is a tridiagonal M-matrix, which is strictly row diagonally dominant at rows $1, n$. When the order is 1 or 2 the result is trivial. The case of order 3 is just Lemma 3.2.

Thus, we assume that the result is true up to order $n-1$, and we shall prove it for order $n \geq 4$. Without loss of generality we assume that $U^{-1} = \mathbb{I} - P$ and $V^{-1} = \mathbb{I} - Q$, where P, Q are symmetric substochastic kernels, which are also irreducible and tridiagonal. Moreover, P loses mass only at 1 and Q only at n. We decompose the matrices W and M in the following blocks:

$$W = \begin{pmatrix} X & V_{nn}u \\ V_{nn}u' & V_{nn}U_{nn} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \Omega & -\zeta \\ -\zeta' & \alpha \end{pmatrix},$$

where $X = W_{\{1,\dots,n-1\}}$, $u' = (U_{n1}, \dots, U_{n,n-1})$.

In what follows we denote by $e_i \in \mathbb{R}^{n-1}$ the ith vector of the canonical basis. The basic computation we need is, for $i = 1, \dots, n-1$,

$$(Xe_{n-1})_i = U_{n-1,i}V_{n-1,i} = V_{n-1,n-1}U_{i,n-1} = V_{n-1,n-1}f^U_{i,n-1}U_{n-1,n-1}.$$

On the other hand $U_{i,n} = U_{n,i} = f^U_{n,n-1}U_{n-1,i} = U_{n-1,i} = U_{i,n-1}$. This means that

$$Xe_{n-1} = V_{n-1,n-1}u. \tag{3.5}$$

Then, using the formulas for the inverse by blocks, we get that

$$\Omega = \left(X - \frac{V_{nn}}{U_{nn}}uu'\right)^{-1} = X^{-1} + \gamma e_{n-1}e'_{n-1},$$

with

$$\gamma = \frac{V_{nn}}{V_{n-1,n-1}(V_{n-1,n-1}U_{nn} - V_{nn}U_{n-1,n-1})} > 0.$$

Note that γ is well defined and positive because $U_{n-1,n} = f^U_{n-1,n}U_{nn} < U_{nn}$ and $V_{n-1,n} = f^V_{n-1,n}V_{nn} = V_{nn}$, but $V_{n,n-1} = f^V_{n,n-1}V_{n-1,n-1} < V_{n-1,n-1}$.

The induction hypothesis implies that X^{-1} is a tridiagonal M-matrix strictly row diagonally dominant at rows 1 and $n-1$. We conclude that Ω is a tridiagonal

M-matrix that is strictly row diagonally dominant at rows 1 and $n-1$, because the nonnegative term $\gamma e_{n-1}e'_{n-1}$ modifies just the diagonal element $\Omega_{n-1,n-1}$.

The equation for ζ is $V_{nn}\Omega u - V_{nn}U_{nn}\zeta = 0$, and therefore

$$U_{nn}\zeta = X^{-1}u + \gamma U_{n-1,n}e_{n-1} = \left(\frac{1}{V_{n-1,n-1}} + \gamma U_{n-1,n}\right)e_{n-1},$$

which gives

$$\zeta = \frac{1}{V_{n-1,n-1}U_{nn} - V_{nn}U_{n-1,n-1}}e_{n-1}. \tag{3.6}$$

Finally we compute $\alpha = M_{nn}$, which is given by

$$\alpha = \left(V_{nn}U_{nn} - V_{nn}^2 u'X^{-1}u\right)^{-1} = \frac{V_{n-1,n-1}}{V_{nn}(V_{n-1,n-1}U_{nn} - V_{nn}U_{n-1,n-1})}.$$

Since $V_{n-1,n-1} > V_{nn}$ we obtain that the sum of the row n is positive, and n is connected only to $n-1$ and n ($|M_{n,n-1}| > 0, M_{nn} > 0$). The connections of M on $\{1,\dots,n-1\}$ are the same as in X^{-1} (including the connection $(n-1,n-1)$, because $X^{-1}_{n-1,n-1} > 0$), and then the induction hypothesis shows that M is tridiagonal. With respect to the row sums, the only one that can change sign in $\{1,\dots,n-1\}$, with respect to the ones in X^{-1}, is that associated to row $n-1$ because the vector ζ is null out of the node $n-1$.

Thus, we have proved that M is a tridiagonal M-matrix, and the row sum of row n is positive (by symmetry the row sum of the row 1 is also positive). This proves in particular (iii). Since W is a positive matrix and its inverse is an M-matrix, we deduce that W^{-1} is irreducible. We also have proved (iv) because $W_{\mathcal{J}} = U_{\mathcal{J}} \odot V_{\mathcal{J}}$ and according to the extra hypothesis the three matrices are nonsingular (they are principal matrices of inverse M-matrices).

(v) To investigate the other row sums we have to look more closely at the previous induction and use the fact that the row sums of M in $\{1,\dots,n-2\}$ and those of X^{-1} are the same.

Taking $i \in \{2,\dots,n-1\}$, we shall prove that condition (2.9) is necessary and sufficient to have a nonnegative row sum at row i in M. Since $n \ge 4$ we choose $\mathcal{J} = \{i-1, i, i+1\}$. By Lemma 3.1 we have that

$$(U_{\mathcal{J}}^{-1})_{2,k-i+2} = U^{-1}_{ik}, \quad (V_{\mathcal{J}}^{-1})_{2,k-i+2} = V^{-1}_{ik}$$

holds for $k \in \mathcal{J}$. Lemma 3.2 gives a necessary and sufficient condition for $N = (U_{\mathcal{J}} \odot U_{\mathcal{J}})^{-1}$ to be a row diagonally dominant M-matrix, which written in terms of U, V is

$$U^{-1}_{i,i-1}V^{-1}_{i,i+1} \ge U^{-1}_{i,i+1}V^{-1}_{i,i-1}.$$

We conclude that (2.9) and (2.8) are equivalent (also the equivalence between their corresponding strict counterparts).

As we add states to this initial set $\mathcal{J}$, the only row sums that can be modified are those associated to nodes $i-1$ and $i+1$ (depending on which side we add nodes). Hence, the row sum associated to node i at the final stage on the matrix M is the same as the row sum of the second row in N, showing that (2.9) is necessary and sufficient for M to be row diagonally dominant at row i (again there is a correspondence between their strict counterparts).

Now, we show that conditions (2.9) and (2.7) are equivalent. For that purpose consider $\mu \in \mathbb{R}^3$ to be the unique solution of $(U_{\mathcal{J}} \odot V_{\mathcal{J}})\mu = \mathbb{1}$. This solution is

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix} = N\mathbb{1},$$

which implies that μ_2 is the sum of the second row of N. Thus, condition (2.9) is equivalent to $\mu_2 \geq 0$. According to Cramer's rule we get

$$\mu_2 \det(U_{\mathcal{J}} \odot V_{\mathcal{J}}) = \begin{vmatrix} U_{i-1,i-1}V_{i-1,i-1} & 1 & U_{i-1,i-1}V_{i+1,i+1} \\ U_{i-1,i-1}V_{i,i} & 1 & U_{i,i}V_{i+1,i+1} \\ U_{i-1,i-1}V_{i+1,i+1} & 1 & U_{i+1,i+1}V_{i+1,i+1} \end{vmatrix}. \tag{3.7}$$

Since $U_{\mathcal{J}} \odot V_{\mathcal{J}}$ is a positive definite matrix, the sign of μ_2 is the same as the sign of

$$(U_{i+1,i+1} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i,i}) - (U_{i,i} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i+1,i+1}),$$

and the equivalence is shown.

(vi) The result is straightforward when $n = 1, 2$. The case when $n = 3$ is done in Lemma 3.2. By Lemma 3.1, we obtain

$$\begin{aligned} M_{11} &= (W_{\{1,2,3\}})^{-1}_{11}, \\ M_{ii} &= (W_{\{i-1,i,i+1\}})^{-1}_{22} \text{ when } 2 \leq i \leq n\,, \\ M_{nn} &= (W_{\{n-2,n-1,n\}})^{-1}_{33}, \end{aligned}$$

and the theorem is proved under the extra hypothesis that U and V are nonsingular.

We now show (ii)–(v) without the assumption that U and V are nonsingular.

(ii), (iv) Denote by $0 < x_1 \leq \cdots \leq x_n$ and $y_1 \geq \cdots \geq y_n > 0$ the numbers associated to U and V, respectively. For $\epsilon > 0$ we consider $x_i^\epsilon = x_i + i\epsilon$ and $y_i^\epsilon = y_i + i\epsilon$. Then clearly $0 < x_1^\epsilon < \cdots < x_n^\epsilon$ and $y_1^\epsilon > \cdots > y_n^\epsilon$. Thus, the ultrametric matrices they induce, $U(\epsilon), V(\epsilon)$, are nonsingular. What we have proved, applied to the matrix $W(\epsilon) = U(\epsilon) \odot V(\epsilon)$, gives that $W^{-1}(\epsilon)$ is a tridiagonal M-matrix, which is strictly diagonally dominant at rows $1, n$. Since $W(\epsilon) \to W$ as $\epsilon \downarrow 0$ and W is assumed to be nonsingular, we conclude that $M = W^{-1}$ is a tridiagonal M-matrix that is row diagonally dominant at rows $1, n$. This shows (ii) and (iv).

(v) In order to prove that (2.7) is equivalent to W^{-1} being row diagonally dominant at row i, we note that the proof we have done in the restricted case relies on two facts. First, we have to prove that the condition is necessary and sufficient to have a row diagonally dominant at the second row of $W_{\mathcal{J}}^{-1}$. This was done in (3.7).

The second fact to be proved is that the row sum of node i does not change as we add more nodes until we arrive to $\mathcal{I}$. Assume that $i < n - 1$. As we have done before, we decompose W and W^{-1} into blocks as follows:

$$W = \begin{pmatrix} X & V_{nn}u \\ V_{nn}u' & V_{nn}U_{nn} \end{pmatrix} \quad \text{and} \quad W^{-1} = \begin{pmatrix} \Omega & -\zeta \\ -\zeta' & \alpha \end{pmatrix}.$$

The fact that W^{-1} is tridiagonal implies that $\zeta = ae_{n-1}$ for some $a > 0$. a has to be positive; otherwise node n cannot connect to any other node, and $W_{jn} = 0$ for $j < n$, which is not possible. Moreover,

$$a = \frac{1}{V_{n-1,n-1}U_{nn} - V_{nn}U_{n-1,n-1}},$$

where we note that $(V_{n-1,n-1}U_{nn} - V_{nn}U_{n-1,n-1})U_{n-1,n-1}V_{nn} = \det(W_{\{n-1,n\}}) > 0$. Also we have

$$\Omega = \left(X - \frac{V_{nn}}{U_{nn}}uu'\right)^{-1} = X^{-1} + \frac{V_{nn}}{V_{n-1,n-1}} a\, e_{n-1}e'_{n-1}. \tag{3.8}$$

The row sums of W^{-1} and X^{-1} are the same on $\{1,\dots,n-2\}$. The rest is done by induction. If $i = n-1$, we decompose the matrices in blocks indexed by $\{1\}$ and $\{2,\dots,n\}$ and proceed as before.

(iii) To show that (2.5) is equivalent to M being strictly row diagonally dominant at row 1, we proceed as in the proof of (iv). This condition is exactly that the inverse of $W_{\{1,2,3\}}$ is strictly row diagonally dominant at row 1. This row sum does not change as we add more states proving the desired equivalence. Similarly, (2.6) is equivalent to the property that the inverse of $W_{\{n-2,n-1,n\}}$ is strictly row diagonally dominant at row n. The rest of the argument is analogous. $\square$

Proof of Theorem 2.6. In what follows we assume that $M = \mathbb{I} - P$ for a substochastic matrix P.

(i) The proof follows immediately from Theorem 1.4.

(ii) Here we take $\mathcal{J} = \{k \le j \le m\}$ to be the smallest interval containing $\mathscr{R}(W)$, which by hypothesis has size at least 2. According to Lemma 3.1, $X = W_{\mathcal{J}}$ is a potential matrix. Its inverse is an irreducible tridiagonal row diagonally dominant M-matrix and $\mathscr{R}(X) + (k-1) \supseteq \mathscr{R}(W) \cap \mathcal{J} \supseteq \{k, m\}$.

The case $k = 1$ and $m = n$, that is, $X = W$, follows from Theorem 2.1 and Proposition 2.5. Thus, without loss of generality, for the rest of the proof we can assume that $k > 1$. Again Theorem 2.1 and Proposition 2.5 imply that $X = R \odot S$ for nonsingular $R \in \mathcal{L}(1)$, $S \in \mathcal{L}(m-k+1)$ of size $m-k+1$. The idea is now to extend these two matrices to a decomposition of W. We shall give an idea of how to extend this decomposition to $\mathcal{K} = \{k-1 \le j \le m\}$. Let us consider $W_{\mathcal{K}}$, the restriction of W to $\mathcal{K}$, that is,

$$W_{\mathcal{K}} = \begin{pmatrix} W_{k-1,k-1} & w' \\ w & X \end{pmatrix},$$

with $w = (W_{k-1,k},\dots,W_{k-1,m})' = (W_{kk},\dots,W_{km})'$ because $f^W_{k-1,k} = 1$. Using that $w' = X_{1\bullet}$ is the first row of X ($X_{\bullet 1}$ is the first column of X) we rewrite $W_{\mathcal{K}}$ as

$$W_{\mathcal{K}} = \begin{pmatrix} W_{k-1,k-1} & X_{1\bullet} \\ X_{\bullet 1} & X \end{pmatrix}.$$

Since $f^W_{k,k-1} < 1$ we have that $W_{k-1,k-1}$ strictly dominates the values in $X_{1\bullet}$. Indeed, for $j = k,\dots,m$ we have

$$W_{kj} = W_{k-1,j} = W_{j,k-1} = f^W_{j,k-1} W_{k-1,k-1} < W_{k-1,k-1}.$$

Let us now introduce the numbers associated to R and S:

$$0 < x_k < x_{k+1} < \cdots < x_m \quad \text{and} \quad y_k > y_{k+1} > \cdots > y_m > 0,$$

respectively. In particular $X_{1\bullet} = x_k(y_k,\dots,y_m)$. Hence, if we take $x_{k-1} > x_k$ and $y_{k-1} > y_k$ such that $x_{k-1}y_{k-1} = W_{k-1,k-1} > W_{kk} = x_k y_k$, we get

$$W_{\mathcal{K}} = \begin{pmatrix} x_{k-1} & x_k \mathbb{1}'_{m-k+1} \\ x_k \mathbb{1}_{m-k+1} & R \end{pmatrix} \odot \begin{pmatrix} y_{k-1} & (y_k,\dots,y_m) \\ (y_k,\dots,y_m)' & S \end{pmatrix}.$$

The rest of the proof is done by an argument based on induction. The matrix U constructed (from the x's) belongs to $\mathcal{L}(k)$, and V belongs to $\mathcal{L}(m)$. ∎

Proof of Theorem 2.7. W is a positive definite matrix, and therefore it is nonsingular. On the other hand, there exist $U1, U2 \in \mathcal{L}(1)$ and $V1, V2 \in \mathcal{L}(n)$ such that $U = U1 \odot V1$ and $V = U2 \odot V2$. Then $W = (U1 \odot U2) \odot (V1 \odot V2)$ is the Hadamard product of $U1 \odot U2 \in \mathcal{L}(1)$ and $V1 \odot V2 \in \mathcal{L}(n)$. Hence, M is an irreducible tridiagonal M-matrix (see Theorem 2.3).

Consider $\mathcal{J} = \{i : k \leq i \leq m\}$. Then, $W_{\mathcal{J}} = U_{\mathcal{J}} \odot V_{\mathcal{J}}$ is again an inverse M-matrix, and $U_{\mathcal{J}} \in \mathcal{L}(1), V_{\mathcal{J}} \in \mathcal{L}(m-k+1)$ are nonsingular of size $m-k+1$ (see Remark 1.2). In particular $R = W_{\mathcal{J}}^{-1}$ is an M-matrix, which is strictly row diagonally dominant at the first and last row. Now take the vector

$$\nu = R\mathbb{1}_{m-k+1} \in \mathbb{R}^{m-k+1},$$

the signed equilibrium potential of $W_{\mathcal{J}}$. We know that $\nu_1 > 0$ and $\nu_{m-k+1} > 0$, because of Theorem 2.3(ii). Consider $\mu \in \mathbb{R}^n$, the following extension of ν:

$$\mu_i = \begin{cases} 0 & \text{if } i \notin \mathcal{J}, \\ \nu_{i-k+1} & \text{if } i \in \mathcal{J}. \end{cases}$$

Let us prove that $W\mu = \mathbb{1}$. For that purpose we compute

$$(W\mu)_i = \sum_j U_{ij} V_{ij} \mu_j = \sum_{j \in \mathcal{J}} U_{ij} V_{ij} \nu_{j-k+1}.$$

There are three cases to analyze: $i < k$, $i \in \mathcal{J}$, and $i > m$. In the first case we use the fact that for $i < k$ and $j \in \mathcal{J}$

$$U_{ij} = U_{kj},\; V_{ij} = V_{kj},$$

and therefore this case is reduced to the second one. Similarly, the third case is reduced to the second one. Hence, we are left with $i \in \mathcal{J}$ in which case

$$(W\mu)_i = \sum_{j \in \mathcal{J}} U_{ij} V_{ij} \nu_{j-k+1} = \sum_{j \in \mathcal{J}} (W_{\mathcal{J}})_{i-k+1, j-k+1} \nu_{j-k+1} = (W_{\mathcal{J}}\, \nu)_{i-k+1} = 1.$$

Therefore, $\mu = W^{-1}\mathbb{1} = M\mathbb{1}$ is the right signed equilibrium potential of W. In particular, the row sums of M are 0 at rows $i \notin \mathcal{J}$. Also the row sums at rows k, m are strictly positive. Finally, M is row diagonally dominant at row $i : k < i < m$ if and only if $\nu_{i-k+1} \geq 0$ or, equivalently, R is diagonally dominant at row $i-k+1$. By Theorem 2.3(v) this is equivalent to

$$(U_{i+1,i+1} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i,i}) \geq (U_{i,i} - U_{i-1,i-1})(V_{i-1,i-1} - V_{i+1,i+1}).$$

Similarly, M is strictly diagonally dominant at row i if there is a strict inequality in the last formula. ∎

Proof of Theorem 2.9. If W^{-1} is an irreducible tridiagonal M-matrix, then W is an entrywise positive matrix, and there exist two positive diagonal matrices D, E and a nonsingular $U \in \mathcal{L}(1)$ such that

$$DWE = U.$$

Hence, for all $r \in \mathbb{R}$ we get

$$D^{(r)} W^{(r)} E^{(r)} = U^{(r)}.$$

For $r > 0$ we have that $U^{(r)} \in \mathcal{L}(1)$ is nonsingular and therefore $W^{(r)}$ is also nonsingular. Moreover, $(W^{(r)})^{-1} = E^{(r)}(U^{(r)})^{-1}D^{(r)}$. Since $U^{(r)} \in \mathcal{L}(1)$, its inverse is a symmetric tridiagonal row diagonally dominant M-matrix. Hence, $W^{(r)}$ is the inverse of an irreducible tridiagonal M-matrix.

When W is a potential and $r \geq 1$, the fact that $W^{(r)}$ is also a potential (respectively, sub-Markov potential) follows from Theorem 2.2 (respectively, Theorem 2.3) in [6].

Now, let us assume that W^{-1} is a tridiagonal irreducible M-matrix and $r < 0$. In order to prove (i), (ii) we can assume without loss of generality that $W = U$. We shall prove the desired properties by induction on n, the size of U. The cases $n = 1, 2$ are obtained immediately. So we assume the properties hold up to dimension $n - 1$, and we shall prove them for dimension $n \geq 3$. Also we shall assume that $r = -1$. The general case follows from the fact $W^{(r)} = (W^{(-r)})^{(-1)}$.

Take $0 < x_1 \leq x_2 \leq \cdots \leq x_n$, the numbers defining U. Then we have

$$T = W^{(-1)} = \begin{pmatrix} \frac{1}{x_1} & \frac{1}{x_1} & \cdots & \frac{1}{x_1} \\ \frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_1} & \frac{1}{x_2} & \cdots & \frac{1}{x_n} \end{pmatrix}.$$

Now, we partition T and T^{-1} (here for the moment we assume it exists) in blocks of sizes $n - 1$ and 1 as

$$T = \begin{pmatrix} A & a \\ a' & \frac{1}{x_n} \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \Omega & \zeta \\ \zeta' & z \end{pmatrix},$$

where $a' = (\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_{n-1}})$. The equations for Ω, z and ζ are

$$\begin{aligned} \Omega &= (A - x_n a a')^{-1}, \\ z &= (\tfrac{1}{x_n} - a' A^{-1} a)^{-1}, \\ \zeta &= -z A^{-1} a. \end{aligned}$$

The induction hypothesis implies that A is nonsingular. The important relation for solving these equations is $A e_{n-1} = a$, which implies that

$$\frac{1}{x_n} - a' A^{-1} a = \frac{1}{x_n} - a' e_{n-1} = \frac{1}{x_n} - \frac{1}{x_{n-1}} < 0$$

and $z = \frac{x_n x_{n-1}}{x_{n-1} - x_n} < 0$.

On the other hand, we have

$$\Omega = (A - x_n a a')^{-1} = A^{-1} + \alpha e_{n-1} e'_{n-1},$$

with $\alpha = \frac{x_n x_{n-1}}{x_{n-1} - x_n} = z < 0$. Finally, we get

$$\zeta = -z e_{n-1}.$$

The induction hypothesis implies that A^{-1} is an irreducible tridiagonal matrix, with negative diagonal elements and positive elements in the upper and lower next diagonals. The same is true for Ω and also for T^{-1} since $(T^{-1})_{nn} < 0$. Finally, from the well-known formula for the determinant of a matrix by blocks

$$\det(W^{(-1)}) = \det(A)\left(\frac{1}{x_n} - a'A^{-1}a\right) = \frac{\det(A)}{z},$$

we get that $sign(\det(W^{(-1)})) = -sign(\det(A))$, and the proof of (i), (ii) is complete.

(iii) The proof is done by induction on n, the size of W. When $n = 1, 2$, the proof is straightforward. So we assume that the property holds up to dimension $n-1$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $W^{(r)}$. We take $X = W_{\{1,\ldots,n-1\}}$ to be a principal submatrix of W, and we consider the ordered set of eigenvalues for $X^{(r)}$ given by $\mu_2 \geq \mu_3 \geq \cdots \geq \mu_n$. Since $X^{(r)}$ is a principal submatrix of $W^{(r)}$, we have by Cauchy's interlace theorem for eigenvalues

$$\lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_{k-1} \geq \mu_k \geq \lambda_k \geq \cdots \geq \mu_n \geq \lambda_n,$$

where $k = 2, \ldots, n$. The matrix X, of size $n-1$, satisfies the induction hypothesis, and therefore $\mu_2 > 0 > \mu_3$, which implies that $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_3 < 0 < \lambda_1$. We need to prove that $\lambda_2 < 0$. For that purpose we use that

$$\det(W^{(r)}) = \prod_{k=1}^{n} \lambda_k, \quad \det(X^{(r)}) = \prod_{k=2}^{n} \mu_k$$

and $sign(\det(W^{(r)})) = -sign(\det(X^{(r)}))$. This implies that necessarily $\lambda_2 < 0$. That λ_1 is maximal in absolute value follows from the Perron–Frobenius theorem. □

Remark 3.1. Note that $W^{(0)}$ is the constant matrix of ones, and so it is singular (unless the dimension of W is one). On the other hand, if W is a potential and $0 < r < 1$, the matrix $W^{(r)}$ is not in general a potential, as the following example shows. Take the matrices

$$U = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 4 & 4 \\ 1 & 4 & 9 & 9 \\ 1 & 4 & 9 & 16 \end{pmatrix} \in \mathcal{L}(1), \quad V = \begin{pmatrix} 25 & 16 & 9 & 1 \\ 16 & 16 & 9 & 1 \\ 9 & 9 & 9 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in \mathcal{L}(4).$$

The matrix $W = U \odot V$ is a potential (check numerically, or use Theorem 2.3), but $T = W^{(1/2)}$ is an inverse M-matrix that is not row diagonally dominant (the sum of the third row of T^{-1} is negative).

Remark 3.2. For $r < 0$, the inverse of $W^{(r)}$, which we denoted by $C(r)$, has the sign pattern opposite to an M-matrix, but clearly $-C(r)$ is not an M-matrix because its inverse is an entrywise negative matrix.

4. Potentials associated to random walks that lose mass only at the two ends. The aim of this section is to characterize the potentials associated to random walks on $\mathcal{I} = \{1, \cdots, n\}$ that lose mass exactly at $1, n$. As we know from Theorem 2.6 these potentials (at least the symmetric ones) are the Hadamard product of two nonsingular ultrametric matrices, one of them in $\mathcal{L}(1)$ and the other in $\mathcal{L}(n)$. We shall see that this decomposition has a very special form.

Consider the simplest random walk on $\{1,\dots,n\}$ losing mass at 1 and reflected at n; that is, $P^{01}=P^{01}(n)$ is the $n\times n$ matrix given by

$$\forall i,j,\quad P^{01}_{ij}=\begin{cases}1/2 & \text{if } |i-j|=1 \text{ or } i=j=n,\\ 0 & \text{otherwise.}\end{cases}\tag{4.1}$$

This matrix is stochastic except at $i=1$. According to Theorem 1.4, the matrix $U^{01}=(\mathbb{I}-P^{01})^{-1}$ is ultrametric and, moreover, is given by

$$\forall i,j,\quad U^{01}_{ij}=2(i\wedge j).\tag{4.2}$$

It can be checked directly that $U^{01}=\mathbb{I}+U^{01}P^{01}$, proving that $U^{01}=(\mathbb{I}-P^{01})^{-1}$. We also note that the matrix U^{10} obtained from U^{01} by

$$\forall i,j,\quad U^{10}_{ij}=U^{01}_{n+1-i,n+1-j}=2([n+1-i]\wedge[n+1-j])=2(n+1-(i\vee j))$$

is also ultrametric and is the potential of the random walk P^{10} which loses mass at n and is reflected at 1. Also we note that P^{01},P^{10} are tridiagonal.

The random walk on $\{1,\dots,n\}$ that loses mass at 1 and n has a kernel $P^{00}=P^{00}(n)$ given by

$$\forall i,j,\quad P^{00}_{ij}=\begin{cases}1/2 & \text{if } |i-j|=1,\\ 0 & \text{otherwise.}\end{cases}\tag{4.3}$$

This matrix is stochastic except at 1 and n, and it is tridiagonal. The matrix $W^{00}=(\mathbb{I}-P^{00})^{-1}$ is a potential and is given by

$$W^{00}_{ij}=\frac{2}{n+1}(i\wedge j)(n+1-(i\vee j))=\frac{2}{n+1}(i\wedge j)([n+1-i]\wedge[n+1-j])\,.\tag{4.4}$$

We note that W^{00} is not only symmetric but also is symmetric with respect to the change $(i,j)\to(n+1-i,n+1-j)$.

The main observation is that W^{00} is proportional to the Hadamard product $U^{01}\odot U^{10}$, where U^{01} is the ultrametric potential of the standard random walk that loses mass at node 1 (see (4.2)) and U^{10} is the one that loses mass at node n. Indeed we have

$$W^{00}=\frac{1}{2(n+1)}U^{01}\odot U^{10}.$$

This is a part of a general result that we now state.

Theorem 4.1. *Consider the matrix W indexed by $\mathcal{I}=\{1,\dots,n\}$ and given by*

$$W_{ij}=z_{i\wedge j}(\mathfrak{m}-z_{i\vee j})d_j,\tag{4.5}$$

where $0<z_1<z_2<\cdots<z_n<\mathfrak{m}$ and $d_j>0$ for all j. We note that W is symmetric if and only if (d_j) is constant. Then, W is a potential matrix with inverse $W^{-1}=\theta(\mathbb{I}-P)$, where θ is a constant and P is an irreducible tridiagonal substochastic matrix, which is stochastic except at 1 and n, and, moreover,

$$\begin{aligned}&W^{-1}_{ij}=0 \text{ if } |i-j|>1,\\ &W^{-1}_{11}=\tfrac{1}{d_1\mathfrak{m}}\left(\tfrac{1}{z_1}+\tfrac{1}{z_2-z_1}\right),\quad W^{-1}_{nn}=\tfrac{1}{d_n\mathfrak{m}}\left(\tfrac{1}{\mathfrak{m}-z_n}+\tfrac{1}{z_n-z_{n-1}}\right),\\ &\forall i\le n-1,\quad W^{-1}_{i,i+1}=-\tfrac{1}{d_i\mathfrak{m}(z_{i+1}-z_i)},\quad W^{-1}_{i+1,i}=-\tfrac{1}{d_{i+1}\mathfrak{m}(z_{i+1}-z_i)}\,,\\ &\forall 2\le i\le n-1,\quad W^{-1}_{ii}=-W^{-1}_{i,i-1}-W^{-1}_{i,i+1}\,.\end{aligned}\tag{4.6}$$

Conversely, assume that P is an irreducible tridiagonal substochastic matrix, which is stochastic except at 1 and n; then there exist $z, \mathfrak{m}, d$ as before such that $W = (\mathbb{I} - P)^{-1}$ has a representation like (4.5)*. In order to compute d, first calculate $a_1 = 1$ and $a_{i+1} = a_i \frac{P_{i,i+1}}{P_{i+1,i}}$ for $i = 1, \ldots, n-1$. Then, d is obtained from*

$$d_i = \frac{a_i}{\max\{a_j : j = 1, \ldots, n\}} \leq 1\,. \tag{4.7}$$

Clearly, if P is symmetric, we have $d_j = 1$ for all j. Moreover, $\mathfrak{m}$ and z can be calculated as

$$\begin{aligned} &\mathfrak{m}^2 = \tfrac{1}{d_1(1-P_{11}-P_{12})} + \sum_{j=1}^{n-1} \tfrac{1}{d_j P_{j,j+1}} + \tfrac{1}{d_n(1-P_{nn}-P_{n,n-1})}\,, \\ &\forall\, i \geq 1, \quad z_i = \tfrac{1}{\mathfrak{m} d_1(1-P_{11}-P_{12})} + \sum_{j=1}^{i-1} \tfrac{1}{\mathfrak{m} d_j P_{j,j+1}}\,. \end{aligned} \tag{4.8}$$

Remark 4.1. This result bears some similarities to the case of one-dimensional diffusions on the interval $[0,1]$ killed at both ends. In this context W plays the role of G, the Green potential of such diffusions. For example, in the case of Brownian motion killed at $0, 1$, the Green potential is given by

$$G(x,y) = 2(x \wedge y)(1 - (x \vee y)),$$

which is the analogue of (4.4).

Proof of Theorem 4.1. Assume that W has a representation like (4.5), where $d_j = 1$ for all j; that is,

$$W_{ij} = z_{i\wedge j}(\mathfrak{m} - z_{i \vee j}).$$

The first thing to notice is that $W = U \odot V$, where U, V are the ultrametric matrices given by

$$U_{ij} = z_{i\wedge j}, \quad V_{ij} = (\mathfrak{m} - z_i) \wedge (\mathfrak{m} - z_j).$$

It is straightforward to show that both $U \in \mathcal{L}(1), V \in \mathcal{L}(n)$ are nonsingular. Hence, U and V are symmetric inverse M-matrices and therefore positive definite. Thus, W is a positive definite matrix and a fortiori nonsingular. It is straightforward to see that U, V satisfy relations (2.7) with equality for all $i = 2, \ldots, n-1$. Hence, $W^{-1} = \theta(\mathbb{I} - P)$, where θ is a constant and P is an irreducible tridiagonal substochastic matrix, which is stochastic except at 1 and n (see Theorem 2.3(iii)).

Formula (4.6) for W^{-1} follows from formula (2.4), by using the symmetry of W and the fact that $W_{i1} = z_1(\mathfrak{m} - z_i)$. This shows the result when W is symmetric.

In the general case, $\widetilde{W} = WD$, where W is symmetric and D is a diagonal matrix, with strictly positive diagonal entries. Since $\widetilde{W}^{-1} = D^{-1}W^{-1}$, we conclude the proof of the first part.

Now we assume that P is an irreducible tridiagonal substochastic matrix such that P is stochastic except for the rows $1, n$. At the beginning we shall assume that P is symmetric. We shall prove that $W = (\mathbb{I} - P)^{-1}$ has a representation like (4.5). For that purpose we use representation (4.6) for $W^{-1} = \mathbb{I} - P$. First, consider

$\alpha(1) = \frac{1}{1-P_{11}-P_{12}}$, which is well defined because P is substochastic at 1. Notice that $\alpha(1)$ represents the unknown $\mathfrak{m}z_1$. In general, for $i = 2, \dots, n$ we define

$$\alpha(i) = \alpha(i-1) + \frac{1}{P_{i-1,i}}.$$

Also note here that $\alpha(i)$ represents $\mathfrak{m}z_i$. Then, the formula for W^{-1}_{nn} in (4.6) and the symmetry of P suggest that

$$\mathfrak{m}^2 = \alpha(n) + \frac{1}{1 - P_{nn} - P_{n,n-1}},$$

which is well defined because P is substochastic at n. Then, we obtain

$$\forall i \geq 1, \quad \alpha(i) = \tfrac{1}{1-P_{11}-P_{12}} + \sum_{j=1}^{i-1} \tfrac{1}{P_{j,j+1}}\,,$$

$$\mathfrak{m}^2 = \tfrac{1}{1-P_{11}-P_{12}} + \sum_{j=1}^{n-1} \tfrac{1}{P_{j,j+1}} + \tfrac{1}{1-P_{nn}-P_{n,n-1}}.$$

Given $\alpha(i)$, $i = 1, \dots, n$, and $\mathfrak{m}$, we can define $z_i = \alpha(i)/\mathfrak{m}$, which gives the formula (4.8).

It is clear that $0 < z_1 < z_2 < \cdots < z_n < \mathfrak{m}$. The matrix A defined by $A_{ij} = z_{i\wedge j}(\mathfrak{m} - z_{i\vee j})$ is, according to what we have proved, a potential matrix, and its inverse is given by formula (4.6). This shows that $A = W$, and the result is proved in this case.

Assume now that P is not symmetric. Take $a_1 = 1$, and define inductively for $i = 1, \dots, n-1$

$$a_{i+1} = a_i \frac{P_{i,i+1}}{P_{i+1,i}}.$$

We define $d_i = \frac{a_i}{\max\{a_j : j=1,\dots,n\}} \leq 1$ and the associated diagonal matrix D. It is straightforward to show that DP is symmetric. The symmetric matrix Q defined by

$$\mathbb{I} - Q = D(\mathbb{I} - P),$$

that is, $Q_{ij} = (1-d_i)\delta_{ij} + d_i P_{ij}$, is again irreducible tridiagonal and substochastic. Moreover, Q is symmetric and stochastic except at rows $1, n$. Hence, there exist $z, \mathfrak{m}$ such that $T = (\mathbb{I} - Q)^{-1}$ satisfies

$$T_{ij} = z_{i\wedge j}(\mathfrak{m} - z_{i\vee j}).$$

Let $W = (\mathbb{I} - P)^{-1}$; then $W = TD$. Hence, $W_{ij} = T_{ij} d_j$, and then W has a representation like (4.5), showing the result. $\square$

Remark 4.2. Assume that W is symmetric. The decomposition (4.5) is unique; that is, if for all i, j

$$W_{ij} = z_{i\wedge j}(\mathfrak{m} - z_{i\vee j}) = \tilde{z}_{i\wedge j}(\tilde{\mathfrak{m}} - \tilde{z}_{i\vee j}), \tag{4.9}$$

then $\tilde{\mathfrak{m}} = \mathfrak{m}$ and $z = \tilde{z}$ on $\mathcal{I}$. So, if $\mathcal{J} = \{\ell_1 < \cdots < \ell_k\} \subseteq \mathcal{I}$ and we define $V = W_{\mathcal{J}}$, then clearly for all $1 \leq s, t \leq k$ we have

$$V_{st} = W_{\ell_s \ell_t} = z_{\ell_s\wedge\ell_t}(\mathfrak{m} - z_{\ell_s\vee\ell_t}) = u_{s\wedge t}(\mathfrak{m} - u_{s\vee t}),$$

with $u_s = z_{\ell_s}$. In particular $\mathfrak{m}$ is the same quantity for all the principal submatrices of W. Thus, $\mathfrak{m}$ may be computed, for example, from $W_{\{1,2\}}$ and is given by

$$\mathfrak{m}^2 = \frac{(W_{11}W_{22} - W_{12}^2)^2}{(W_{11} - W_{12})(W_{22} - W_{12})W_{12}}.$$

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