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Equilibrium analysis of dynamic models of imperfect competition

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ABSTRACT

Motivated by recent developments in applied dynamic analysis, this paper presents new sufficient conditions for the existence of a Markov perfect equilibrium in dynamic stochastic games. The main results imply the existence of a Markov perfect equilibrium provided the sets of actions are compact, the set of states is countable, the period payoff functions are upper semi-continuous in action profiles and lower semi-continuous in actions taken by rival firms, and the transition function depends continuously on actions. Moreover, if for each firm a static best-reply set is convex, the equilibrium can be taken in pure strategies. We present and discuss sufficient conditions for the convexity of the best replies. In particular, we introduce new sufficient conditions that ensure the dynamic programming problem each firm faces has a convex solution set, and deduce the existence of a Markov perfect equilibrium for this class of games. Our results expand and unify the available modeling alternatives and apply to several models of interest in industrial organization, including models of industry dynamics.

1. Introduction

This paper considers infinite horizon games in which at each period, after observing a payoff-relevant state variable, players choose actions simultaneously. The state of the game evolves stochastically parameterized by past history in a stationary Markov fashion. The setting includes a broad class of models, including Ericson and Pakes' (1995) model, as well as more general dynamic models of imperfect competition.

We present a general existence theorem for dynamic stochastic games and offer several applications to industrial organization. A strict implication from our main result, Theorem 1, is the following. A dynamic stochastic game possesses a behavior strategy Markov perfect equilibrium if the sets of actions are compact, the set of states is countable, the period payoff functions are upper semi-continuous in action profiles and lower semi-continuous in rivals' actions, and the probability distribution of the next state depends continuously on the actions chosen. Moreover, if for each player a static best-reply set is convex, the equilibrium can be taken in pure strategies.

As in previous work (Doraszelski and Satterthwaite, 2010; Horst, 2005), to obtain existence in pure strategies, we need to impose convexity restrictions on the dynamic game. Our result requires the game to have convex best replies, meaning that for all rivals' actions and all (bounded) continuation functions, each firm's static best-reply set is convex. This condition resembles (and indeed reduces to) the standard convexity restriction imposed on the payoff functions in strategic-form games to ensure the existence of Nash equilibrium. We state independent, sufficient conditions that ensure the convexity of the best replies. Our first sufficient condition is the uniqueness of the set of best replies, a condition requiring best-reply sets to be single-valued. This condition reduces to the convexity condition introduced by Doraszelski and Satterthwaite (2010) in an industry dynamics model. The second sufficient condition, satisfied by the so-called games with concave reduced payoffs, ensures each player's maximization problem is concave and so best replies are convex-valued. Although these two conditions do not cover all the games that have convex best replies, they significantly broaden the modeling alternatives that existing results offer.

Our main results have several applications; Section 4 provides a few. We analyze an industry dynamics model similar to that introduced by Ericson and Pakes (1995). Doraszelski and Satterthwaite (2010) have recently studied a version of the Ericson–Pakes model and introduced a condition, the *unique investment choice* (UIC) condition, to guarantee equilibrium existence. Under the UIC condition, best replies are single-valued and thus our convexity restrictions are met. Moreover, we provide a new alternative condition for the existence in the Ericson–Pakes model and discuss how this new condition permits modeling alternatives uncovered by Doraszelski and Satterthwaite's (2010) analysis. In particular, our results allow for multidimensional investment decisions and complementarities among firms' investments.

We also study a Markov Cournot game — in which firms compete in quantities, and at each round, a decision-controlled demand shock is realized. We provide sufficient conditions ensuring equilibrium existence. We show how restrictions on how rivals' actions affect payoffs and on how the transition function depends on action profiles make current results unsatisfactory (Amir, 1996; Horst, 2005; Nowak, 2007). Notably, to ensure equilibrium existence, we do not need to restrict the number

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of firms nor do we need to assume the transition function is linear in action profiles. We also consider a version of the Markov Cournot game in which firms have fixed costs, and show results ensuring the existence of behavior strategy equilibrium.

Finally, we also apply our results to an incomplete information dynamic model extensively studied and applied recently (e.g. Bajari et al., 2007; Doraszelski and Escobar, 2010).

Dating back to Shapley (1953), several authors have studied the problem of equilibrium existence in dynamic stochastic games. Among these, Mertens and Parthasarathy (1987), Nowak and Raghavan (1992), and Duffie et al. (1994) constitute important contributions that neither generalize nor are generalized by our results. Two strands of the literature are more closely related to this work. First, Horst (2005), Doraszelski and Satterthwaite (2010), and Nowak (2007) deal with the pure-strategy equilibrium existence problem. Some of these results cover state spaces uncovered by our results and prove not only existence but also uniqueness. Although our main result is formally unrelated to these authors', this paper identifies convexity conditions that expand and unify available modeling alternatives. Indeed, a game satisfying any of the convexity conditions those authors impose has convex best replies as required by our main result. Moreover, games such as those that Horst (2005) and Nowak (2007) consider are games with concave reduced payoffs and so, according to Proposition 3, have convex best replies.[1] This work contributes to this literature by identifying convexity restrictions that are significantly weaker than the conditions so far available.[2]

These results also contribute to the literature on dynamic games with countable state spaces. Federgruen (1978) and Whitt (1980) provide existence results that are corollaries to our main behavior strategy result, Corollary 2, in that they do not permit payoffs to be discontinuous. In particular, they do not deal with the problem of pure strategy existence, nor do they answer whether a nontrivial class of models could satisfy a convexity condition as the one we impose.

The paper is organized as follows. Section 2 presents the model. Section 3 presents and discusses the main theorem. Section 4 provides a number of applications of our results. Section 5 concludes. All proofs are in the appendix, except where the proof provides important intuition.

2. Setup

In this section we introduce our dynamic game model and define our equilibrium notion. Similar to many studies in industrial organization, we consider a dynamic stochastic game played by a finite set of firms. In each round of play, there is a payoff-relevant state variable (e.g., the identity of the incumbent firms). The state variable evolves stochastically, and firms can influence its evolution through actions (e.g., by entering or exiting the market). The goal of each firm is to maximize the expected present value of its stream of payoffs.

2.1. Model

There is a finite set of firms denoted by I. At the outset of period $t=1$, firms are informed about the initial state of the game, s_1. Then they simultaneously pick their actions $a_1=(a_1^i)_{i\in I}$. At the outset of period $t=2$, firms are informed of the new state of the game s_2 and then simultaneously pick their actions $a_2=(a_2^i)_{i\in I}$. And so on for $t\geq 3$.

The state space is S. For each firm i, the set of actions is A^i. In most applications, we will assume A^i is contained in $\mathbb{R}^{L^i}$, where L^i is a natural number, but allowing some more generality will be useful when studying models of imperfect competition in which firms have private information (see Section 4.3).

When firms make decisions at round t, they know the whole sequence of realized states $s_1,\ldots,s_t$, and past actions $a_1,\ldots,a_{t-1}$. The evolution of the state variable is Markovian in the sense that (a_t,s_t) fully determines the distribution over the state in the next round s^{t+1}. The Markovian transition function takes the form $\mathbb{P}[s_{t+1}\in B\,|(a_t,s_t)]=Q(B;a_t,s_t)$, where $B\subseteq S$. Given realized sequences of actions $(a_t)_{t\geq 1}$ and states $(s_t)_{t\geq 1}$, the total payoff for firm i is the discounted sum of period payoffs

$$\sum_{t=1}^{\infty}\left(\delta^i\right)^{t-1}\pi^i(a_t,s_t),$$

where $\delta^i\in[0,1[$ is the discount factor, and $\pi^i(a,s)$ is the per period payoff function.

This dynamic stochastic game model is flexible and, indeed, several models widely used in the literature fit into this framework. We will discuss applications and examples in detail in Section 4.

Throughout the paper, we will maintain the following assumptions.

- A1 S is a countable set.
- A2 For all i, A^i is compact and contained in a linear metric space.[3]
- A3 For all i, π^i is a bounded function.
- A4 The transition function Q is *setwise continuous* in $a\in A$ (Royden, 1968, Chapter 11.4): for every $B\subseteq S$ and $s\in S$, $Q(B;a,s)$ is continuous in $a\in A$.

In applications, Assumption (A1) is perhaps the most demanding one. While this assumption is usually made in industry dynamics models (Doraszelski and Satterthwaite, 2010; Ericson and Pakes, 1995), it rules out dynamic stochastic games in which the state variable is continuous. From Assumption (A3), we can define π_l^i and π_u^i as, respectively, the lower and upper bounds for the function π^i, and denote $\|\pi^i\|_\infty=\sup_{a\in A,s\in S}|\pi^i(a,s)|$.

2.2. Markov perfect equilibria

We now present the equilibrium notion with which we work. One may study subgame perfect equilibria of our dynamic model, but recent research has focused on Markov perfect equilibria. Markov perfect equilibria are a class of subgame perfect equilibrium strategies in which players condition their play only on payoff-relevant information.[4] The idea is that, in a given round, firms choose actions depending on the current state, with the purpose of maximizing the sum of current and future expected discounted payoffs.

A *Markov strategy* for firm i is a function $\bar{a}^i:S\to A^i$ mapping current states to actions. Thus, a Markov strategy defines a dynamic game strategy in which in each round t, firm i chooses action $\bar{a}^i(s_t)$, where s_t is the state realized in round t. A tuple of Markov strategies $\left(\bar{a}^i\right)_{i\in I}$ is a *Markov perfect equilibrium* if it is a subgame perfect equilibrium of the dynamic game. In a Markov perfect equilibrium, although firms condition their play only on the current state, they may deviate to arbitrary strategies conditioning on the whole transpired history. We will also consider *behavior Markov perfect equilibria*, defined as subgame perfect equilibria in which each firm i uses a strategy $\bar{a}^i:S\to\Delta(A^i)$ that maps current states to a distribution over actions.

[1] Bernheim and Ray (1989) and Dutta and Sundaram (1992) derive pure strategy results formally unrelated to ours. For a class of dynamic models, they restrict the strategy sets so that best replies are single valued and the games therefore satisfy the convexity restrictions required by our analysis.

[2] While Amir (1996) and Curtat (1996) restrict their attention to supermodular stochastic games, they do need to impose convexity conditions that, as we explain in Section 3.2, cannot be deemed as less stringent than ours.

[3] A linear metric space is a vector space endowed with a metric. For example, A^i could be a compact subset of $\mathbb{R}^{L_i}$ for some L_i.

[4] Several arguments in favor of this restriction can be given; see Maskin and Tirole (2001) for a particularly insightful discussion.

3. The main result

In this section, we present our main existence result, Theorem 1. We then derive several sufficient conditions for Theorem 1 to be applicable.

3.1. Statement

As in many dynamic models, dynamic programming tools will be useful for analyzing our setup. We thus define

$$\Pi^i\left(a,s;v^i\right)=\pi^i(a,s)+\delta^i\sum_{s'\in S}v^i(s')Q(s';a,s),$$

where $a\in A$, $s\in S$, and $v^i:S\to\mathbb{R}$ are bounded functions. The number $\Pi^i(a,s;v^i)$ is the total expected payoff for player i, given that the current state is $s\in S$, the current action profile is $a\in A$, and the continuation payoff, as a function of the next state $s'\in S$, is $v^i(s')$. Intuitively, fixing a state s and continuation value functions $\left(v^i\right)_{i\in I}$, the functions $\Pi^i(\cdot,s;v^i)$, $i\in I$, define a static game in which firms' action profiles are $a\in A$. The Markov perfect equilibrium requirement, on the one hand, restricts continuation value functions and, on the other hand, induces Nash equilibrium behavior in the corresponding family of static games.

To guarantee the existence of a Markov perfect equilibrium, we will impose convexity and regularity restrictions on our dynamic game. The dynamic stochastic game is said to have *convex best replies* if for all i, all $s\in S$, all a^{-i} in A^{-i}, and all bounded function $v^i:S\to\left[\frac{\pi_l^i}{1-\delta^i},\frac{\pi_u^i}{1-\delta^i}\right]$ the best-reply set

$$\arg\max_{x^i\in A^i}\Pi^i\left(\left(x^i,,a^{-i}\right),s;v^i\right)\tag{3.1}$$

is convex. This condition says the static optimization problem in which each firm i chooses an action $x^i\in A^i$ with the purpose of maximizing its total expected payoffs has a convex solution set, given the profile played by its rivals a_{-i}, the current state s, and the continuation value function v^i.

Imposing some continuity restrictions on payoffs will also be useful. Recall that a function $f:X\to\mathbb{R}$, where X is a metric space, is said to be upper semi-continuous if for all sequence $x_n\to x$ in X, $\lim\sup_{no\infty}f(x_n)\leq f(x)$, and is said to be lower semi-continuous if for all $x_n\to x$ in X, $\lim\inf_{n\to\infty}f(x_n)\geq f(x)$.

The following is our main existence result.

Theorem 1. *The dynamic stochastic game possesses a Markov perfect equilibrium if it has convex best replies and for all i, $\pi^i(a,s)$ is upper semi-continuous in $a\in A$ and lower semi-continuous in $a^{-i}\in A^{-i}$.*

We provide a proof of this result in Appendix A. We employ a fixed point argument on the space of best replies and continuation values to find a solution to the conditions imposed by Markov equilibria. We use upper and lower semi-continuity[5] to ensure that best replies and continuation values exist and are sufficiently continuous in rivals' strategies.[6] Together with the convexity of the best replies, these continuity restrictions ensure equilibrium existence using Kakutani's fixed point theorem.

Section 3.2 presents some easy-to-check conditions to apply Theorem 1. Before discussing those conditions, we observe that when the game does not have convex best replies, our main result can still be applied to deduce the existence of behavior Markov perfect equilibria. Appendix B presents a proof.

Corollary 2. *The dynamic stochastic game possesses a behavior Markov perfect equilibrium if for all i, $\pi^i(a,s)$ is upper semi-continuous in $a\in A$ and lower semi-continuous in $a^{-i}\in A^{-i}$.*

Although computing behavior strategy equilibrium is difficult (if not impossible) when the set A is not finite, this corollary can be applied to analytically study dynamic oligopoly games in which firms incur fixed costs; see Section 4.4.

3.2. Discussion and sufficient conditions

The game will have convex best replies whenever the set of maximizers (3.1) is a singleton; in this case, we say the game has *single-valued best replies*. Games studied by Horst (2005) and Doraszelski and Satterthwaite (2010), among others, have single-valued best replies. Section 4 presents the models extensively employed in the applied IO literature where this kind of uniqueness restriction can be exploited.

We will now introduce a new class of dynamic stochastic games exhibiting convex best replies. To present these games (and in the rest of this section), we will restrict our main setting by assuming that the set of actions of each player i, A^i, is a convex set contained in $\mathbb{R}^{L^i}$, where L^i is a natural number. For a given real-valued symmetric square matrix M, we denote

$$\mathbf{mev}(M)=\max\{\lambda\,|\,\lambda\text{ is an eigenvalue of }M\}.$$

We also assume that $\pi^i(a,s)$ and $Q(s';a,s)$ are twice continuously differentiable with respect to $a^i\in A^i$, and denote the Hessian matrices with respect to $a^i\in A^i$ by $\pi^i_{ii}(a,s)$ and $Q_{ii}(s';a,s)$ respectively.[7] We say that the game has *concave reduced payoffs* if for all i, the function $\pi^i(a,s)$ is concave in a^i, and for all $(a,s)\in A\times S$ either

$$\left(\pi_u^i-\pi_l^i\right)\sum_{s'\in S}\max\{0,\mathbf{mev}(Q_{ii}(s';a,s))\}-\mathbf{mev}\left(\pi^i_{ii}(a,s)\right)=0$$

or this expression is strictly positive and

$$\delta^i\leq\frac{-\mathbf{mev}\left(\pi^i_{ii}(a,s)\right)}{(\pi_u^i-\pi_l^i)\sum_{s'\in S}\max\{0,\mathbf{mev}(Q_{ii}(s';a,s))\}-\mathbf{mev}(\pi^i_{ii}(a,s))}.\tag{3.2}$$

The following result provides a sufficient condition for a game to have convex best replies.

Proposition 3. *Suppose that the game has concave reduced payoffs. Then, for all $i\in I$, all $a\in A$, all $s\in S$, and all $v^i:S\to\left[\frac{\pi_l^i}{1-\delta^i},\frac{\pi_u^i}{1-\delta^i}\right]$, $\Pi^i(a,s;v^i)$ is concave in a^i. In particular, the game has convex best replies.*

Because the proof of this result is simple and intuitive, we present the argument in the text. Assume first $\pi_l^i=0$. Observe that $\Pi^i_{ii}(a,s;v^i)=\pi^i_{ii}(a,s)+\delta^i\sum_{s'\in S}v^i(s')Q_{ii}(s';a,s)$. Therefore, Π^i is concave in a^i if for all $x\in\mathbb{R}^{L^i}$,

$$x'\pi^i_{ii}(a,s)x+\delta^i\sum_{s'\in S}v^i(s')x'q_{ii}(s';a,s)x\leq 0.$$

To prove this property, observe that for any symmetric matrix (Aleskerov et al., 2011, section 9.4),

$$x'Mx\leq\mathbf{mev}(M)\|x\|^2.$$

[5] A previous version of the paper relaxed the lower semi-continuity assumption in Theorem 1.

[6] Observe that if $\pi^i(a,s)$ is upper semi-continuous in a and lower semi-continuous in a^{-i}, then it is continuous in a^{-i}. This observation implies that our result only allows π^i to be discontinuous in a^i. We note this observation does not imply that existence is obtained when $\pi^i(a,s)$ is upper semi-continuous in a^i and continuous in a^{-i}, because to ensure best replies are continuous in other strategies some degree of joint continuity in own and rivals' actions is needed.

[7] Observe that when Q is twice continuously differentiable in a^i, then our assumption (A4) is satisfied.

Therefore, for all $v^i:S\to[0,\pi_i^u]$

$$
\begin{aligned}
&x'\pi_{ii}^i(a,s)x+\delta^i\sum_{s'\in S}v^i\left(s'\right)x'Q_{ii}\left(s';a,s\right)x\\
&\le\mathbf{mev}\left(\pi_{ii}^i(a,s)\right)\|x\|^2+\delta^i\frac{\pi_u^i}{1-\delta^i}\sum_{s'\in S}\max\left\{0,x'Q_{ii}\left(s';a,s\right)x\right\}\\
&\le\mathbf{mev}\left(\pi_{ii}^i(a,s)\right)\|x\|^2+\delta^i\frac{\pi_u^i}{1-\delta^i}\sum_{s'\in S}\max\left\{0,\mathbf{mev}\left(Q_{ii}\left(s';a,s\right)\right)\|x\|^2\right\}\\
&=\left(\mathbf{mev}\left(\pi_{ii}^i(a,s)\right)+\delta^i\frac{\pi_u^i}{1-\delta^i}\sum_{s'\in S}\max\left\{0,\mathbf{mev}\left(Q_{ii}\left(s';a,s\right)\right)\right\}\right)\|x\|^2.
\end{aligned}
$$

Under the conditions of the proposition, this expression is less than or equal to 0. When $\pi_l^i\neq0$, consider $\bar\pi^i(a,s)=\pi^i(a,s)-\pi_l^i$ and apply the result above to the modified payoffs. This step completes the proof.

Proposition 3 provides a condition under which $\Pi^i(a,s;v^i)$ is a concave function of a^i for all $a^{-i}\in A^{-i}$, all $s\in S$, and all v^i. To gain intuition, suppose first that $\pi^i(a,s)$ is strictly concave in a^i. Then, even if for some v^i the term $\sum_{s'\in S}v^i(s')Q(s';a,s)$ is highly non-linear, the sum of $\pi^i(a,s)$ and $\delta^i\sum_{s'\in S}v^i(s')Q(s';a,s)$ can still be a concave function if δ^i is small enough. More generally, Eq. (3.2) can be seen as making explicit a tension between the discount factor δ^i and the second derivative with respect to a^i of $\pi^i(a,s)$. Note that, as the following result shows, in some models, restricting attention to games with concave reduced payoffs imposes no restriction on δ^i.

Corollary 4. *Suppose that the transition function can be written as*

$$
Q\left(s';a,s\right)=\sum_{k=1}^{K}\alpha_k(a)F_k\left(s';s\right)
$$

where for all s, $F_k(\cdot;s)$ is a probability distribution over S, and for all $a\in A$, $\sum_{k=1}^{K}\alpha_k(a)=1$ with $\alpha_k(a)\ge0$ for all k. Assume that for all k, α_k is twice continuously differentiable as a function of $a^i\in A^i$ $\mathbb{R}^{L^i}$ with a Hessian matrix that equals 0. Then, the game has concave reduced payoffs and best replies are convex.

This result simply follows by noting that under the assumptions on the transition in the corollary, $\mathbf{mev}(Q_{ii}(s';a,s))$ equals 0 for all i,s', a, and s, and therefore the game has concave reduced payoffs and, from Proposition 3, best replies are convex.

The importance of this corollary is that it provides easy-to-check sufficient conditions ensuring the convexity of the best replies. In particular, when the transition function Q is a multilinear function of $(a^1,\ldots,a^{|I|})$, the game has concave reduced payoffs provided each $\pi^i(a,s)$ is a concave function of $a^i\in A^i$. Although this restriction on Q may seem demanding in some applications, it provides as much flexibility in period payoffs as one can hope for (i.e., concavity of payoffs as functions of own actions) and imposes no restriction on the discount factors.

One of the attractive features of games with concave reduced payoffs is their tractability. Indeed, in games with concave reduced payoffs, first-order conditions, being necessary and sufficient for optimality, can be used to characterize equilibrium strategies. This observation is important not only when analytically deriving properties of the equilibrium strategies, but also when numerically solving for those strategies.

The restriction to games having concave reduced payoffs relates to similar conditions imposed in previous work. Horst's (2005) Weak Interaction Condition (2005) is strictly more demanding than our sufficient condition; this can be seen by noting that any stochastic game satisfying condition (7) in Horst's (2005) paper also satisfies condition (3.2).[8] Indeed, Horst's (2005) assumption additionally makes reaction functions virtually flat functions of others' strategies. More recently, Nowak (2007) works under the assumption that $\pi^i(a,s)$ is concave in a^i, $Q(s';a,s)$ is affine in a^i (as in Corollary 4), and a strict diagonal dominance assumption holds.[9] It is not hard to see that under Nowak's (2007) concavity condition, $Q_{ii}=0$ and so, from Corollary 4, the game has concave reduced payoffs and convex best replies for all $\delta^i<1$.

Amir (1996) and (Curtat (1996) have studied supermodular stochastic games. These authors work under the assumption that the payoffs and the transition are supermodular and satisfy a positive spillovers property.[10] Moreover, these works still need to impose some convexity restrictions. Consider, for example, Curtat's (1996) strict diagonal dominance (SDD) assumption. To simplify the exposition, assume first that for all i, $A^i\subset\mathbb{R}$. Then, the SDD condition can be expressed as follows: For all i and all $(a,s)\in A\times S$, $\sum_{i\in I}\frac{\partial^2\pi^i(a,s)}{\partial a^i\partial a^j}<0$. Since π^i is supermodular, SDD implies that π^i is strictly concave in a^i. More generally, if $A^i\subset\mathbb{R}^{L^i}$, $L^i\neq1$, SDD is related to the concavity of $\pi^i(a,s)$ in a, but neither condition implies the other. Yet, the SDD condition on the transition restricts the model dynamics substantially. Indeed, in all the examples studied by Curtat (1996), the transition is a linear function of the action profile $a\in A$.

4. Applications

This section provides a number of applications of our main results. Section 4.1 studies a model similar to Ericson and Pakes (1995) and relates our sufficient conditions to those recently derived by Doraszelski and Satterthwaite (2010). Section 4.2 shows a dynamic version of the textbook Cournot game with stochastic demand. Section 4.3 ensures equilibrium existence in a dynamic model of incomplete information (Bajari et al., 2007; Doraszelski and Escobar, 2010). Finally, Section 4.4 ensures existence in behavior strategies in a Markov Cournot game with fixed costs.

4.1. Ericson–Pakes industry dynamics model

We now study an industry dynamics game in the spirit of Ericson and Pakes's (1995) seminal model. Consider a finite set I of firms. At each t, some of the firms are incumbent; the others are entrant. The state of firm i is $s^i=\left(\bar s^i,,\eta^i\right)\in\bar S^i\times\{0,1\}$, where $\bar s^i$ reflects a demand or technology shock; $\eta^i=1$ if firm i is an incumbent, and $\eta^i=0$ if firm i is an entrant. The state of the industry is $s=\left(s^i\right)_{i\in I}$.

The action of firm i is $(1^i,x^i)\in\{0,1\}\times X$, with $X\subset\mathbb{R}_+^L$, where $1^i=1$ (resp. $1^i=0$) if firm i changes (resp. does not change) its incumbency/entrance status and x^i is a vector of investment projects firm i undertakes. In other words, if firm i is an entrant (resp. incumbent) and $1^i=1$, then i becomes an incumbent (resp. entrant). Since the set $\{0,1\}$ is not convex, we allow firms to randomize. Let $p^i\in[0,1]$ be the probability with which i changes its statues. Firm i's action is therefore a vector $a^i=(p^i,x^i)\in[0,1]\times X$; we assume that X is convex and compact.

[8] To see this, assuming that $\pi_l^i=0$, $L^i=1$, and $\delta^i=\delta$ for all i, our condition (3.2) can be equivalently written as $\frac{\delta}{1-\delta}\|\pi^i\|_\infty\sup_{a\in A}\sum_{s'\in S}\frac{\max\{0,Q_{ii}(s';a,s)\}}{|\pi_{ii}^i(a,s)|}\le1$ for all i and all s. Denoting by $\pi_{ij}^i(a,s)=\frac{\partial\pi^i(a,s)}{\partial a^i\partial a^j}$, condition 7 on assumption 2.2 in Horst (2005) can be written as $\sum_{j\neq i}\sup_{a\in A}\frac{|\pi_{ij}^i(a,s)|}{|\pi_{ii}^i(a,s)|}+\frac{\delta}{1-\delta}\|\pi^i\|_\infty\sum_{j\neq i}\sup_{a\in A}\frac{\sum_{s'\in S}|Q_{ij}(s';a,s)|}{|\pi_{ii}^i(a,s)|}+\frac{\delta}{1-a}\|\pi^i\|_\infty\sup_{a\in A}\sum_{s'\in S}\frac{|Q_{ii}(s';a,s)|}{|\pi_{ii}^i(a,s)|}<1$ for all i and all s. It follows that the left-hand side of my restriction is strictly less than the left-hand side of the restriction above.

[9] This assumption makes reaction functions in the static one-shot game a contraction.

[10] A game has positive spillovers if payoff functions are nondecreasing in rivals' actions.

Given a state s and an action profile $a=(a^i)_{i\in I}$, the per-period payoff for firm i is given by

$$\pi^i(a,s)=\eta^i\left(g^i(s)+\psi^i(s)p^i\right)+\left(1-\eta^i\right)\left(-\bar{\psi}^i(s)p^i\right)-c^i\left(x^i,s\right).$$

The first term is the profit that firm i obtains when competing in a spot market, $g^i(s)$, plus the scrap value at which firm i may be sold, $\psi^i(s)$, times the probability of exit p^i when firm i is incumbent, $\eta^i=1$. The second term is the set-up price firm i must pay to enter the market, $-\bar{\psi}^i(s)$, times the probability of entry, p^i, when firm i is an entrant, $\eta^i=0$. The third term is the cost of investment x^i when the state is s. In applied work, one would restrict $g^i(s)$ to depend not on the whole vector s, but only on those s^j for which firm j is an incumbent $\eta^j=1$. Analogously, the scrap and set-up values will typically depend only on the firm's own state s^i. Firm i's discount factor is δ^i.

For a given vector $1=(1_i)_{i\in I}\in\{0,1\}^{|I|}$ of decisions on status changes and a profile $x=(x^i)_{i\in I}$ of investment decisions, the state of the system in the following period is distributed according to $Q^r(\cdot;1,x,s)$. It is relatively easy to see that given the vector of actions $a=(p,x)$, the next period state is distributed according to

$$Q(s';a,s)=\sum_{1\in\{0,1\}^{|I|}}Q^r\left(s';1,x,s\right)\left(\prod_{j=1}^{|I|}\left(p^j\right)^{1^j}\left(1-p^j\right)^{1-1^j}\right),$$

where we define $0^0=0$. We assume that $c^i(x^i,s)$ and $Q^r(s';1,x,s)$ are twice continuously differentiable functions of x^i.

Doraszelski and Satterthwaite (2010) study a similar model. They introduce the unique investment choice (UIC) condition, a condition implying that the best-reply set (3.1) is unique. It is therefore evident that after introducing a UIC condition in our model, the stochastic game has convex best replies and so the existence of equilibrium is a consequence of Theorem 1.[11] Although the UIC condition may be applied to many variations of the Ericson–Pakes model, we provide a new condition that applies to important situations that Doraszelski and Satterthwaite's (2010) result does not.

First note that

$$\pi^i_{ii}(a,s)=-c^i_{ii}\left(x^i,s\right),$$

where $c^i_{ii}(x^i,s)$ denotes the matrix of second derivatives with respect to x^i. Now, the Hessian matrix of the transition function, Q, can be expressed as

$$Q_{ii}(s';p,x,s)=\sum_{1\in\{0,1\}^{|I|}}\left(\prod_{j\neq i}\left(p^j\right)^{1^j}\left(1-p^j\right)^{1-1^j}\right)\times\begin{pmatrix}0 & \left[Q^r_{x^i}\left(s';1,x,s\right)\right]'\left(21^i-1\right)\\ Q^r_{x^i}\left(s';1,x,s\right)\left(21^i-1\right) & Q^r_{x^ix^i}\left(s';1,x,s\right)\left(p^i\right)^{1^i}\left(1-p^i\right)^{1-1^i}\end{pmatrix},$$

where $Q_{x^i}{}^r(s';1,x,s)$ (resp. $Q^r_{x^ix^i}(s';1,x,s)$) denotes the column vector of derivatives (resp. matrix of second derivatives) of $Q^r(s';1,x,s)$ with respect to the variable x^i. Denoting $\lambda^i(s';p,x,s)=\mathbf{mev}(Q_{ii}(s';p,x,s))$, it follows that the Ericson–Pakes industry dynamics model has an equilibrium provided

$$\delta^i\leq\frac{-\mathbf{mev}\left(-c^i_{ii}\left(x^i,s\right)\right)}{\mu^i\sum_{s'\in S}\max\{0,\lambda^i(s';p,x,s)\}-\mathbf{mev}(-c^i_{ii}(x^i,s))}\tag{4.1}$$

for all i,p,x, and s, where $\mu^i=\pi^i_u-\pi^i_l$ equals

$$\mu^i=\max_{s''\in S\text{ with }\eta^i=1,y^i\in X}\left(g^i\left(s''\right)+\psi^i\left(s''\right)-c^i\left(y^i,,s''\right)\right)+\max_{s''\in S\text{ with }\eta^i=0,y^i\in X}\left(\bar{\psi}^i\left(s''\right)+c^i\left(y^i,,s''\right)\right).$$

Although Eq. (4.1) is not more general than the UIC condition (a condition already shown to fit into our general framework), this new condition allows modeling alternatives uncovered by Doraszelski and Satterthwaite (2010). Doraszelski and Satterthwaite's (2010) analysis hinges on the unidimensionality of the investment decisions – ruling out, for example, investment plans that can affect the demand and the cost structure independently –, and the separable form of the transition — ruling out several transitions exhibiting non-trivial complementarities among the investment decisions. These and other modeling alternatives can be analyzed with this new alternative condition.

Condition (4.1) involves the maximum eigenvalue of the matrix of second derivatives of minus the cost function. Intuitively, the condition says that $-c^i_{ii}$ must be sufficiently concave, given the discount factor δ^i and the transition function (as captured by the non-linear term $\sum_{s'\in S}\max\left\{0,\lambda^i(s';p,x,s)\right\}$), so that all of its eigenvalues are negative enough. Alternatively, the firm must be sufficiently impatient given the technology c^i_{ii} and the transition function Q. Condition (4.1) resonates well with other existence results in equilibrium theory, which emphasize the importance of ruling out increasing returns of production to ensure equilibrium existence (Mas-Colell et al., 1995, Proposition 17.BB.2). The novel aspect of condition (4.1) is that, because of the dynamics, convexity of the cost functions must be strengthened so that even when firms maximize total payoffs, best replies are convex-valued. We therefore restrict attention to models in which returns to scale are "sufficiently decreasing".

Also of interest is the observation that when Eq. (4.1) holds, each firm's payoff is a concave function of its decision variables. Thus, first-order conditions are necessary and sufficient for optimality. The problem of numerically finding equilibrium strategies is therefore effectively reduced to the problem of solving a (potentially huge) system of first-order conditions (equalities or variational inequalities). Remarkably, under Eq. (4.1), we can be confident that a converging method solving first-order conditions will yield an equilibrium of the model.

In applications, checking condition (4.1) amounts to solving $|I|\times|S|$ nonlinear minimization problems on $([0,1]\times X)^{|I|}$. In complicated models, one can solve such problems numerically before running the routines to solve for the equilibria. This first step is relatively easy to implement numerically because the $|I|\times|S|$ minimization problems are unrelated. If this initial step is successful, our model is well behaved in that it possesses an equilibrium and all the dynamic programming problems involved will be concave maximization problems.

The following example presents a simple model in which investment decisions are multidimensional and returns to scale are decreasing; we observe that Doraszelski and Satterthwaite (2010) results do not apply.

Example 5. Suppose that firms invest jointly in a project and the total investment determines the common state of the industry $\bar{s}$. In other words, we now assume that $\bar{s}^i=\bar{s}^j$ for all $i\neq j$, and that this state is stochastically determined by $\sum_{i=1}^{|I|}x^i$, where $x^i\in[0,1]^2$. The state $\bar{s}$ only determines spot market profits so that the profit of a firm competing in the spot market is $g(\bar{s},\eta)$, whereas scrap values and set-up prices are $\psi^i(s)=\psi\in\mathbb{R}_+$ and $\bar{\psi}^i(s)=\bar{\psi}\in\mathbb{R}_+$ for all i and all s. Each firm may carry out two types of investment projects so that $x^i\in[0,1]^2$, and the cost functions take the form $c^i(x^i,s)=\frac{1}{2}\left(\left(x^i_1\right)^2+\left(x^i_2\right)^2\right)$. We assume that $\delta^i=\delta$ for all i. This model is a symmetric one in which firms jointly

[11] For completeness, let me simplify the model to offer a UIC condition in the spirit of Doraszelski and Satterthwaite (2010). Suppose that the status change decision is (payoff) irrelevant, that is, the only choice variable is x^i. Also suppose that $X=[0,1]$ and $c^i(x^i,s)=x^i$. Then the UIC condition holds provided for all i, $Q(s';x,s)=a^i(s';x^{-i},s)\eta^i(s,x^i)+b^i(s';x^{-i},s)$, where η^i is twice differentiable, strictly increasing, and strictly concave in x^i. Under this condition, Eq. (3.1) is single valued.

invest in improving spot market profits for example, by advertising or by making the (non-proprietary) production technology more efficient. We refer to the state $\bar{s}$ as the spot market conditions.

The set of spot market conditions $\bar{S}$ is ordered and can be written $\bar{S} = \{1, \dots, |\bar{S}|\}$. Higher states result in higher profits so that $g(\bar{s}, \eta)$ is increasing in $\bar{s}$. The evolution of the spot market conditions $\bar{s}$ takes the form

$$Q^r\left(\bar{s}^{'};x,\bar{s}\right) = \alpha\left(\sum_{i=1}^{|I|} x^i\right) F_1\left(\bar{s}^{'};\bar{s}\right) + \left(1-\alpha\left(\sum_{i=1}^{|I|} x^i\right)\right) F_2\left(\bar{s}^{'};\bar{s}\right),$$

where $\alpha(y) \in [0,1]$ for all $y \in [0,|I|]^2$, and $F_k(\cdot|\bar{s})$ is a probability distribution over $\bar{S}$ (as in Corollary 4). The transitions are such that F_1 moves the subsequent state up deterministically in one step (or stays in the same state if the current step is $|\bar{S}|$), whereas F_2 moves the state down in one step (or stays in the same state if the current step is 1). Intuitively, the higher the joint effort $y = \sum_{i \in I} x^i$, the more likely the next spot market conditions will be favorable. We assume that $\alpha(\cdot)$ is a linear function and that the first dimension of the investments is more effective: $\alpha(y) = \frac{1}{2} + \alpha_1 y_1 + \alpha_2 y_2$ with $\alpha_2 = \alpha_1/2$. If no firm invests, the subsequent state is equally likely to go up or down.

Whether firms enter or exit the market is determined by

$$q\left(\eta^{'};p,\eta\right) = \left(\prod_{i:\eta^i = \eta^{'i}} p^i\right)\left(\left(\prod_{i:\eta^i \neq \eta^{'i}} \left(1-p^i\right)\right)\right).$$

Therefore the transition takes the form

$$Q\left(s^{'};p,x,s\right) = \left(\prod_{j:\eta^j = \eta^{'j}} p^j\right)\left(\prod_{j:\eta^j \neq \eta^{'j}} \left(1-p^j\right)\right) \cdot Q^r\left(\bar{s}^{'};x,s\right).$$

Once we have an expression for the transition function, it is relatively easy to show that[12]

$$\sum_{s^{'} \in S} \max\left\{0, \lambda^i\left(s^{'};p,x,s\right)\right\} = 2\sqrt{\alpha_1^2 + \alpha_2^2} = \sqrt{5}\alpha_1$$

and that $\mathbf{mev}(-c^i_{ii}(x^i,s)) = -1$. We also assume that $\bar{g} + \psi + \bar{\psi} = 1$, where $\bar{g} = \max_s g(s)$, meaning that the sum of spot market profits, scrap values, and set-up costs is at most 1 (which is the maximum investment cost a firm can incur). We derive the following sufficient condition for equilibrium existence:

$$\delta \leq \frac{1}{2\sqrt{5}\alpha_1 + 1}.$$

For example, if $\alpha_1 = 1/100$ (so that a firm investing $\Delta > 0$ units can increase the probability of the high state in Δ per cent), then the condition above amounts to $\delta \leq .96$.[13]

When firms make entry and exit decisions before investment decisions, and if when making investment decisions firms observe the identity of the market participants, then in the model above the existence of Markov perfect equilibrium can be guaranteed using Corollary 4 and Theorem 1, regardless of the discount factor. In such a model, because the transition Q^r is linear in the investment decisions and entry and exit decisions are randomized strategies, the game has concave reduced payoffs for all discount factors. In some industries, advertising decisions are likely to take place after the identity of the market participants is publicly known, and therefore a model of sequential decisions would seem more appropriate.

4.2. Markov Cournot oligopoly

We now consider a simple dynamic version of the textbook Cournot game. A finite set I of oligopolists exists. At each t, oligopolists set quantities $a^i_t \in [0,1]$, $i \in I$, simultaneously and independently. The (inverse) demand function takes the form $P(\sum_{i \in I} a^i_t, s_t)$, where the state, s_t, belongs to a finite set. No costs of production exist. Thus, the period payoff to firm i is

$$\pi^i(a,s) = a_i \cdot P\left(\sum_{i \in I} a^i, s\right).$$

The demand function assumes the functional form $P(\sum_{i \in I} a^i, s) = s \cdot (1 - \sum_{i \in I} a^i)$. Players discount future payoffs at a constant rate $\delta \in]0,1[$.

The set of states S is a countable subset of $]0,\infty[$. The evolution of the state $(s_t)_{t \geq 0}$ is given by the transition

$$Q\left(s^{'};a,s\right) = \sum_{k=1}^{K} \alpha_k\left(\sum_{i \in I} a^i\right) F_k\left(s^{'};s\right),$$

where K is a finite number, α_k is a quadratic function of $\sum_{i \in I} a^i$, and $F_k(\cdot;s)$ is a probability distribution on S. As previously discussed, we can interpret this transition as being drawn in two steps: first, we draw a lottery over the set $\{1,\dots,K\}$ (where the weights are determined by the total production $\sum_{i \in I} a^i$), then, given the result k of the lottery, a draw from the distribution $F_k(\cdot;s)$ is realized and determines the subsequent state. The assumption that α_k is a quadratic function of $\sum_{i \in I} a^i$ implies that its second derivative is constant; let $\bar{\alpha}_k$ be the second derivative of α_k. We also assume that $F_k(\cdot;s)$ puts weight 1 on some state $s' \in S$ (for example, it may put weight 1 on the state immediately below or above s).

It is relatively easy to see that[14]

$$\sum_{s^{'} \in S} \max\left\{0, Q_{ii}\left(s^{'};a,s\right)\right\} \leq \sum_{k=1}^{K} |\bar{\alpha}_k|.$$

for all $a \in [0,1]^{|I|}$ and all $s \in S$. The existence of a Markov perfect equilibrium is guaranteed provided

$$\delta \leq \frac{1}{\frac{H}{4L}\sum_{k=1}^{K} |\bar{\alpha}_k| + 1},$$

where $H = \max\{s \in S\}$ and $L = \min(s \in S)(>0)$. Note that the results by Curtat (1996) do not apply to this Cournot setting because he considers supermodular games satisfying strong monotonicity restrictions. To ensure existence we do not need to impose conditions on the number of players, nor do we need to assume that $\bar{\alpha}_k = 0$ for all k. To apply Horst's (2005) and Nowak's (2007) results, regardless of the transition function, we would need to impose that $|I| \leq 2$.

[12] Note that Q_{ii} takes the form $P[\eta_{-i}|\eta_{-i},p](F_1(\bar{s}';\bar{s}) - F_2(\bar{s}';\bar{s}))\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & 0 \\ \alpha_2 & 0 & 0 \end{pmatrix}$ and therefore $\mathbf{mev}(Q_{ii}(s';p,x,s)) = P\left[\eta_{-i}|\eta_{-i},p\right]|F_1(\bar{s}';\bar{s}) - F_2(\bar{s}';\bar{s})|\sqrt{\alpha_1^2 + \alpha_2^2}$. Noting that each $F_k(\cdot;\bar{s})$ puts positive weight on 2 states and summing up over subsequent states $s' \in S$, the result follows.

[13] Observe that because $\alpha(y)$ must belong to [0,1], α_1 is bounded above by $\frac{1}{3|I|}$.

[14] To see this, note that $Q_{ii}(s';a,s) = \sum_{k=1}^{K} \bar{\alpha}_k F_k(s';s)$. Fix s, and let $s'(k)$ be the state with weight 1 given $F_k(\cdot;s)$. Then $\sum_{s' \in S} \max\{0, Q_{ii}(s';a,s)\} \leq \sum_{s' \in S}\sum_{k=1}^{K} |\bar{\alpha}_k| F_k(s';s) = \sum_{k=1}^{K} |\bar{\alpha}_k| F_k(s(k);s) = \sum_{k=1}^{K} |\bar{\alpha}_k|$.

Moreover, results by Nowak (2007) can be applied only if we considered a linear transition.

For example, consider a model of habit formation, in which $K=2$, $\alpha_1(\sum_{i\in I}a^i)=\bar{\alpha}_1(\sum_{i\in I}a^i)^2$, with $\bar{\alpha}_1>0$, and given $s\in S$, $F_1(\cdot;s)$ puts weight 1 on a point strictly greater than the point in which $F_2(\cdot;s)$ puts weight on. The idea is that the higher the volume of sales $\sum_{i\in I}a^i$, the higher the probability the next demand state s' will be high. Because $\alpha_1(\sum_{i\in I}a^i)\in[0,1]$ and $\alpha_1(\sum_{i\in I}a^i)+\alpha_2(\sum_{i\in I}a^i)=1$, it follows that $|\bar{\alpha}_k|\leq\frac{1}{|I|^2}$ for $k=1,2$. Assuming that $H/L=2$, the sufficient condition for equilibrium existence is

$$\delta\leq\frac{|I|^2}{|I|^2+1.}$$

When $|I|=2$, an equilibrium exists provided $\delta\leq\frac{4}{5}$, whereas when $|I|=6$, equilibrium existence is guaranteed when $\delta\leq\frac{36}{37}$.

4.3. Dynamic model with incomplete information

We now consider a model of dynamic interaction with private information. The applied literature extensively employs similar models; consult Bajari et al. (2007) for a recent application.

Consider a model similar to that introduced in Section 2, but now suppose that at the beginning of each period, each firm not only observes the public state s_t but also receives a private shock $\nu_t^i\in\mathbb{R}^{N_i}$. Then each firm picks its action a_t^i and obtains a period profit $\pi^i(a,s,\nu^i)$. Private shocks are drawn independently according to a distribution function $G^i(\cdot)$, $i\in I$, and the transition function takes the form $Q(s';a,s)$.

A pure strategy for a firm is a function $\bar{a}^i: S\times\mathbb{R}^{N_i}\to A^i$. However, to apply our general framework, we interpret a strategy as a function $\bar{a}^i: S\to\mathbb{A}^i$, where

$$\mathbb{A}^i=\left\{a^i:\mathbb{R}^{N_i}\to A^i\middle|a^i\text{ is measurable}\right\}.$$

Functions in $\mathbb{A}^i$ are deemed identical if they are equal G^i — almost sure. Given functions $a^i\in\mathcal{A}^i$ and a public state $s\in S$, define

$$\tilde{\pi}^i(a,s)=\int\pi^i\left(\left(a^1\left(\nu^1\right),\ldots,a^I\left(\nu^I\right)\right),s,\nu^i\right)G^1\left(d\nu^1\right)\ldots G^I\left(d\nu^I\right),$$

and

$$\tilde{Q}\left(s';a,s\right)=\int Q\left(s';\left(a^1\left(\nu^1\right),\ldots,a^I\left(\nu^I\right)\right),s\right)G^1\left(d\nu^1\right)\ldots G^I\left(d\nu^I\right).$$

We thus have defined a dynamic model that fits into our dynamic stochastic game framework.

To see the importance of the private shocks when applying our results, assume that A^i is finite, G^i is absolutely continuous with respect to the Lebesgue measure, $N^i=|A^i|$, and the period payoff function takes the form

$$\pi^i\left(a,s,\nu^i\right)=g^i(a,s)+\sum_{k\in A^i}1_{a^i=k}\nu^{i,k}.$$

Now, endow A^i with the discrete topology and $\mathbb{A}^i$ with the convergence in measure metric. That is, given measurable functions $a^i,b^i:\mathbb{R}^{N^i}\to A^i$, define

$$d^{\mathbb{A}^i}\left(a^i,,b^i\right)=\int\frac{d^i\left(a^i\left(\nu^i\right),b^i\left(\nu^i\right)\right)}{1+d^i(a^i(\nu^i),b^i(\nu^i))G^i(d\nu^i).}$$

where d^i is the discrete metric over A^i. Under $d^{\mathbb{A}^i}$, $\mathbb{A}^i$ is compact. The transition $\tilde{Q}\left(ds';a,s\right)$ is continuous and for all i, the payoff $\tilde{\pi}^i(a,s)$ is continuous in $a\in\mathbb{A}$. Private signals come crucially into play when verifying the convexity of the best replies. Indeed, it is not hard to see that the best-reply set of each firm is (essentially) unique for any continuation value function.[15] So, the game has single-valued best replies and the existence of an equilibrium follows from Theorem 1.

4.4. Markov Cournot oligopoly with fixed costs

We finally apply our results to a Markov Cournot game with fixed costs. Fixed costs introduce discontinuities that make all previous results in the literature unapplicable.

A finite set I of oligopolists exists that, at each t, set quantities $a^i\in[0,1]$, $i\in I$, simultaneously and independently. The (inverse) demand function takes the form $P(\sum_{i\in I}a^i,s)$, where the state belongs to a countable set S. $P(\sum_{i\in I}a^i,s)$ is a continuous function of $\sum_{i\in I}a^i$. Firm i's cost function, $c^i(a^i,s)$, is lower semi-continuous in a^i. For example, suppose that each firm must incur a fixed cost $\kappa>0$ to produce any (strictly) positive quantity, and that marginal costs equal 0. Then, firm i's cost function can be written as

$$c^i\left(a^i,s\right)=\begin{cases}0 & \text{if } a^i=0,\\ \kappa & \text{if } a^i>0.\end{cases}$$

This cost function is lower semi-continuous at $a^i=0$. More generally, the presence of fixed costs (that, by definition, can be avoided if production is suspended in a given round) makes cost functions naturally lower semi-continuous, but not continuous, at $a^i=0$. Hence, the period payoff to firm i is

$$\pi^i(a,s)=a_iP\left(\sum_{j\in I}a^j,s\right)-c^i\left(a^i,s\right).$$

The transition Q is assumed setwise continuous in $a\in A$. For example, the state variable s_{t+1} may represent a demand shock that is continuously modified by current sales a_t. Because π^i is upper semi-continuous in a and lower semi-continuous in a^{-i}, the existence of behavior strategy Markov perfect equilibrium follows from Corollary 2.[16]

5. Concluding comments

This paper offers results that guarantee the existence of Markov perfect equilibria in a class of dynamic stochastic games. Dynamic models that can be solved analytically are exceptional; therefore researchers often need to resort to computational routines to analyze their models. Yet unless an equilibrium is guaranteed to exist, a non-converging algorithm designed to compute an equilibrium may fail either because an equilibrium exists and the algorithm is not suitable for its computation or, more dramatically, because an equilibrium does not exist. The results in this paper provide guidance on the nature of dynamic models that possess Markov perfect equilibria (in pure and behavior strategies). In doing so, we expand and unify several modeling alternatives (Doraszelski and Satterthwaite, 2010; Horst, 2005) and apply our results to several dynamic models of imperfect competition. We impose restrictions on the fundamentals of the model, ensuring that each firm's optimization problem has a concave objective function. This property not only constitutes a sufficient condition for equilibrium existence, but also makes available numerical algorithms more reliable.

The continuity restrictions we impose on the payoff functions limit the applicability of our results. As a consequence of these

[15] Indeed, a firm is indifferent between two actions with probability zero.

[16] As in static models, it is hard to ensure the existence of (pure) Markov perfect equilibria in models with fixed costs.

assumptions, our setting does not permit applications to auction and pricing games. In fact, in those games, the possibility of equilibrium ties makes payoff functions not upper semi-continuous. Unless one rules out equilibrium ties, our results are not applicable to auction and pricing games. This observation opens an important research avenue.

Appendix A

A.1. Proof of Theorem 1

In this appendix, we prove Theorem 1. We begin by stating the following key result (Stokey and Lucas, 1989, Theorem 9.2).

Lemma A.1. *For each $i\in I$, consider a function $\bar{a}^i:S\to A^i$. Suppose that there is a tuple $(v^i)_{i\in I}$, where $v^i: S\to\mathbb{R}$ is bounded, such that for all i and for all $s\in S$*

$$v^i(s)=\max\left\{\pi^i\left(\left(x^i,\bar{a}^{-i}(s)\right),s\right)+\delta^i\sum_{s'\in S}v^i\left(s'\right)Q\left(s';\left(x^i,\bar{a}^{-i}(s)\right),s\right)|x^i\in A^i\right\} \tag{A1}$$

and

$$\bar{a}^i(s)\in\arg\max\left\{\pi^i\left(\left(x^i,\bar{a}^{-i}(s)\right),s\right)+\delta^i\sum_{s'\in S}v^i\left(s'\right)Q\left(s';\left(x^i,\bar{a}^{-i}(s)\right),s\right)|x^i\in A^i\right\}. \tag{A2}$$

Then, $\left(\bar{a}^i\right)_{i\in I}$ is a Markov perfect equilibrium.

This result allows us to reduce the problem of finding an equilibrium to the problem of solving a system of functional equations. We will therefore verify the existence of solutions to this system of functional equations using Kakutani's fixed point theorem.

Before presenting the details of the proof of Theorem 1, we present some dynamic programming results.

A.2. Dynamic programming results

Consider the functional equation

$$V(s)=\sup_{x\in X}\left\{\pi(x,s)+\delta\sum_{s'\in S}V\left(s'\right)\nu\left(s';x,s\right)\right\},s\in S, \tag{A3}$$

where X is a compact subset of a metric space, $\pi(x,s)$ is the per-period profit function, $\delta\in[0,1[$, and $\nu(\cdot;x,s)$ is a probability distribution over S.

In this subsection, we study the solutions to the functional Eq. (A3) and provide results concerning the existence and continuity of those solutions. Consider the following assumptions.

(D1) $\pi(x,s)$ is upper semi-continuous in $x\in X$.

(D2) $\nu(s';x,s)$ is setwise continuous in $x\in X$: for all $E\subseteq S$, $\nu(s';x,s)$ is continuous in $x\in X$.

The following result guarantees the existence of a bounded solution for Eq. (A3).

Theorem A.2. *Assume (D1)–(D2). Then, there exists a single solution $\bar{V}$ to Eq. (A3). Moreover, $\|\bar{V}\|_\infty\leq\|\pi\|_\infty/(1-\delta)$. Further, there exists a policy function $\bar{x}: S\to X$.*

The proof of this result is standard (see Theorem 9.6 in Stokey and Lucas, 1989).

Now, let us study continuity properties for the only solution to Eq. (A3), viewing this solution as a function of the transition function ν and the per-period payoff π. We can define

$$TV(s)=\sup_{x\in X}\left\{\pi(x,s)+\delta\sum_{s'\in S}V\left(s'\right)\nu\left(s';x,s\right)\right\}. \tag{A4}$$

For each $n\in\mathbb{N}$, consider a transition function ν_n and a per-period payoff function π_n. For each n, consider the operator T_n defined as we did in Eq. (A4), but replacing π and ν with π_n and ν_n respectively.

Let $\bar{V}_n$ and $\bar{V}$ be the only bounded functions such that $T_n\bar{V}_n=\bar{V}_n$ and $T\bar{V}=\bar{V}$. Additionally, let the set-valued maps $\bar{X}_n$ and $\bar{X}$ be defined by

$$\bar{X}_n(s;V)=\arg\max_{x\in X}\left\{\pi_n(x,s)+\delta\sum_{s'\in S}V\left(s'\right)\nu_n\left(s';x,s\right)\right\}$$

and

$$\bar{X}(s;V)=\arg\max_{x\in X}\left\{\pi(x,s)+\delta\sum_{s'\in S}V\left(s'\right)\nu\left(s';x,s\right)\right\}.$$

The following result shows that the only solution to Eq. (A3) and the related policy map depends continuously on ν and π.

Proposition A.3. *For all $n\in\mathbb{N}$, assume (D1)–(D2) for the problems defined by T_n and T.*

Suppose that for all sequence $x_n\to x$ in X, E S, and $s\in S$, the sequence of real numbers $(\nu_n(E;x_n,s))_{n\in\mathbb{N}}$ converges to $\nu(E;x,s)$ Further suppose that for all $s\in S$

1. *For all sequence $x_n\to x$ in X, $\lim\sup_{n\to\infty}\pi_n(x_n,s)\leq\pi(x,s)$;*
2. *For all $x\in X$, there exists $y_n\to x$ in X, such that $\lim\inf_{n\to\infty}\pi_n(y_n,s)\geq\pi(x,s)$.*

Then, the following statements hold.

(α) *For all subsequence $V_{n_j}\to V$ (pointwise) and given any selection $\bar{x}_{n_j}(\cdot)\in\bar{X}_{n_j}\left(\cdot;V_{n_j}\right)$ converging to $\bar{x}: S\to X, \bar{x}(s)\in\bar{X}(s;V)$ for all $s\in S$.*

(b) *For all $s\in S$, $\bar{V}_n(s)\to\bar{V}(s)$.*

(c) *The policy sets are closed maps of the per period payoff and transition functions.*

Before proving this proposition, we establish a preliminary lemma.

Lemma A.4. *Let $(P_n)_{n\in\mathbb{N}}$ and P be probability measures on S. Suppose that for all $E\subseteq S$, $P_n(E)$ converges to $P(E)$. Fix $\alpha>0$ and let $V_n:S\to[-\alpha,\alpha]$ be a sequence of functions pointwise converging to $V:S\to[-\alpha,\alpha]$. Then, $\sum_{s'\in S}V_n(s')P_n(s)\to\sum_{s'\in S}V(s')P(s')$.*

Proof. Observe that $|V_n|$ is bounded by α and that $\sum_{s'\in S}\alpha P_n(s')=\alpha$ converges to $\sum_{s'\in S}\alpha P(s')=\alpha$. The result follows from Theorem 18 in Royden (1968, Chapter 11). □

Proof of Proposition A.3. Let us begin proving (b). Consider any converging subsequence $(\bar{V}_{n_k})_{k\in\mathbb{N}}$ to $\bar{V}_0$ (such a subsequence always exists). Fix $s\in S$. Consider any sequence $x_k\to x$ in X. Since $\bar{V}_{n_k},\bar{V}_0$ are uniformly bounded by the same constant, we can apply Lemma A.4 above to deduce that for any $x_k\to x$, $\sum_{s'\in S}\bar{V}_{n_k}(s')\nu_{n_k}(s';x_k,s)\to\sum_{s'\in S}\bar{V}_0(s')\nu(s';x,s)$. So, defining $\psi_n(x,s)=\sum_{s'\in S}\bar{V}_n(s')\nu_n(s';x,s)$ and $\psi_0(x,s)=\sum_{s'\in S}\bar{V}_0(s')\nu(s';x,s)$, we deduce that for all s, $\psi_{n_k}(\cdot,s)$ converges to $\psi_0(\cdot,s)$ uniformly on the compact set X.

Fix now $x\in X$. Condition 1 in Proposition A.3 permits us to deduce that for all $x_k\to x$

$$\limsup_{k\to\infty}\pi_{n_k}(x_k,s)+\delta\psi_{n_k}(x_k,s)\leq\pi(x,s)+\delta\psi(x,s). \tag{A5}$$

Additionally, there exists a sequence y_k such that

$$\liminf_{k\to\infty} \pi_{n_k}(y_k,s)+\delta\psi_{n_k}(y_k,s)\geq\pi(x,s)+\delta\psi(x,s). \tag{A6}$$

To prove this result, define $\varphi_{n_k}(x,s)=\pi_{n_k}(x,s)+\delta\sum_{s'\in S}\bar{V}_{n_k}(s')\nu_{n_k}(s';x,s)$ and $\varphi_0(x,s)=\pi(x,s)+\delta\sum_{s'\in S}\bar{V}_0(s')\nu(s';x,s)$. Fix $\eta>0$. From condition 2 in the proposition, there exists $\hat{x}_k\to x$ in X such that $\pi_{n_k}(\hat{x}_k,s)\geq\pi(x,s)-\frac{\eta}{3}$. We further know that the function $x\in X\mapsto\delta\sum_{s'\in S}\bar{V}_{n_k}(s')\nu_{n_k}(s';x,s)$ converges continuously to $x\in X\mapsto\delta\sum_{s'\in S}\bar{V}_0(s')\nu(s';x,s)$. Consequently, for k big enough $\delta\sum_{s'\in S}\bar{V}_0(s')\nu(s';x,s)-\delta\sum_{s'\in S}\bar{V}_{n_k}(s')\nu(s';\hat{x}_k,s)\leq\frac{\eta}{3}$. Therefore,

$$\begin{aligned}\max_{x\in X}\varphi_{n_k}(x,s)&\geq\varphi_{n_k}(\hat{x}_k,s)\\&\geq\pi(x,s)-\frac{\eta}{3}+\delta\sum_{s'\in S}\bar{V}_{n_k}\left(s'\right)\nu\left(s';\hat{x}_k,s\right)\\&\geq\pi(x,s)-\frac{\eta}{3}+\delta\sum_{s'\in S}\bar{V}_0\left(s'\right)\nu\left(s';x,s\right)-\frac{\eta}{3}.\end{aligned}$$

Taking y_k to be a $\eta/3$-maximizer of the maximization problem above, we deduce that for all k big enough $\varphi_{n_k}(y_k,s)\geq\varphi(x,s)-\eta$. Taking liminf, inequality (A6) follows.

With these preliminary results, we are in position to prove (b). We will prove that for any subsequence $\bar{V}_{n_k}\to\bar{V}_0$, where $\bar{V}_0$ is some function, $T\bar{V}_0=\bar{V}_0$. The result then follows from the uniqueness property stated in Theorem A.2. Let $\bar{x}_{n_k}\in\bar{X}_{n_k}(s;\bar{V}_{n_k})$, $s\in S$. Since X is compact, we assume without loss of generality that $\bar{x}_{n_k}\to\bar{x}$ (eventually through a subsequence). Let $x\in\bar{X}_0(s;\bar{V}_0)$ and consider y_k as in Eq. (A6). Then

$$\varphi(x,s)\leq\liminf_{k\to\infty}\varphi_{n_k}(y_k,s)\leq\liminf_{k\to\infty}\varphi_{n_k}\left(\bar{x}_{n_k},s\right)\leq\limsup_{k\to\infty}\varphi_{n_k}\left(\bar{x}_{n_k},s\right)\leq\varphi(\bar{x}).$$

The first inequality is by construction of the sequence y_k. The second inequality follows since $\bar{x}_{n_k}\in\bar{X}_{n_k}(s;\bar{V}_{n_k})$. The third inequality follows by definition. The fourth inequality holds by virtue of Eq. (A5).

It follows that $\bar{x}\in\bar{X}_0(s;V_0)$ and that the sequence of inequalities above is actually equalities. Therefore, $\bar{V}_0(s)=\lim_{k\to\infty}\bar{V}_{n_k}(s)=\lim_{k\to\infty}T_{n_k}\bar{V}_{n_k}(s)=\lim_{k\to\infty}\varphi_{n_k}(\bar{x}_{n_k},s)=\varphi(\bar{x},s)=T\bar{V}_0(s)$, proving the first part of the proposition. Finally, to see (a), just apply the argument above to $V_{n_j}\to V$. Finally, (c) follows from (a) and (b). □

A.3. Proving Theorem 1

For each i, define $\mathcal{A}^i$ as the set of functions $a^i:S\to A^i$ and $\mathcal{V}^i$ as the set of functions $v^i:S\to\left[\frac{\pi_l^i}{1-\delta^i},\frac{\pi_u^i}{1-\delta^i}\right]$. Each of these sets is contained in a vectorial space that, when endowed with the product topology, is a Hausdorff topological vector space. Since A^i and $\left[\frac{\pi_l^i}{1-\delta^i},\frac{\pi_u^i}{1-\delta^i}\right]$ are compact, Tychonoff's theorem (Royden, 1968, Chapter 9, Theorem 14) implies that $\mathcal{A}^i$ and $\mathcal{V}^i$ are compact. Additionally, since S is countable, $\mathcal{A}^i$ and $\mathcal{V}^i$ are metric spaces (Royden, 1968, Chapter 8, Exercise 45).[17] We define $\mathcal{A}=\prod_{i\in I}\mathcal{A}^i$, and $\mathcal{V}=\prod_{i\in I}\mathcal{V}^i$. We have therefore shown the following result.

Lemma A.5. *$\mathcal{A}\times\mathcal{V}$ is a convex compact subset of a linear metric space.*

Now, for $i\in I$, consider the map Φ^i defined on $\mathcal{A}\times\mathcal{V}$ by

$$\begin{aligned}&(a,v)\in\mathcal{A}\times\mathcal{V}\mapsto\Phi^i(a,v)\\&=\left\{\bar{a}^i\in\mathcal{A}|\bar{a}^i(s)\in\arg\max_{x^i\in A^i}\Pi^i\left(\left(x^i,a^{-i}(s)\right),s;v^i\right)\ \text{forall}\ s\in S\right\}.\end{aligned}$$

This map yields the best Markov strategies $\bar{a}^i$ for firm i, given arbitrary strategies for i's rivals and given continuation values v^i. Observe that the arguments $(v^j)_{j\neq i}$ and a^i in the definition of Φ^i appear only for consistency (as will be shown soon).

It will also be useful to nail down continuation values. To do that, define the map T^i by

$$\begin{aligned}&(a,v)\in\mathcal{A}\times\mathcal{V}\mapsto T^i(a,v)\\&=\left\{\bar{v}^i\in\mathcal{V}^i|\bar{v}^i(s)=\max_{x^i\in A^i}\Pi^i\left(\left(x^i,a^{-i}(s)\right),s;\bar{v}^i\right)\ \text{forall}\ s\in S\right\}.\end{aligned}$$

This map yields the solutions to the dynamic programming problem faced by firm i, given the Markov strategies followed by i's rivals. Again, the dependance of this map on $(v^j)_{j\in I}$ and a^i is just for consistency.

Finally define Φ by $\Phi(a,v)=(\prod_{i\in I}\Phi^i(a,v))\times(\prod_{i\in I}T^i(a,v))$. We state three preliminary results.

Lemma A.6. *$\Phi(\mathcal{A}\times\mathcal{V})$ $\mathcal{A}\times\mathcal{V}$.*

To see this lemma, fix i and note that for any $(a,v)\in\mathcal{A}\times\mathcal{V}$, $\Phi^i(a,v)$ $\mathcal{A}^i$ and $T^i(a,v)$ $\mathcal{V}^i$. This implies that $\Phi(\alpha,v)$ $\mathcal{A}\times\mathcal{V}$.

Lemma A.7. *Φ is nonempty- and convex-valued.*

The proof of this lemma is as follows. Since the product of sets which are nonempty- and convex-valued inherits these properties, it is enough to prove that for each i, Φ^i and T^i are nonempty- and convex-valued. Fix i. Given $a^{-i}:S\to A^{-i}$ and a continuation value $v^i:S\to\mathbb{R}_+$, for each s the existence of solution for firm i's static problem (A2) is evident; let $a_i(s)$ be such a solution. By definition, $a_i\in\Phi_i$ so that Φ_i is nonempty-valued. Moreover, Φ^i is convex-valued. Indeed, fix $\lambda\in[0,1]$ and consider $\phi^i,\phi^i\in\Phi^i(a,v)$, where $a,v\in\mathcal{A}\times\mathcal{V}$. Then, for all s, $\phi^i(s),\phi^i(s)\in\text{argmax}_{x^i\in A^i}\Pi^i((x^i,a^{-i}(s)),s;v^i)$. Since the game has convex best replies, for all s, $\lambda\phi^i(s)+(1-\lambda)\phi^i(s)\in\text{argmax}_{x^i\in A^i}\Pi^i((x^i,a^{-i}(s)),s;v^i)$. This observation implies that $\lambda\phi^i+(1-da)\phi^i\in\Phi^i(a,v)$ and proves that Φ^i is convex-valued. Now, let us analyze T^i. Given $a^{-i}:S\to A^{-i}$, Theorem A.2 in the Appendix implies the existence of a single function $v^i\in\mathcal{V}^i$ satisfying the dynamic programming condition (A1) for firm i. Consequently, T^i, being the set of all such solutions to Eq. (A1), is nonempty- and single-valued. The proof is complete.

Lemma A.8. *Φ has closed graph.*

Because $\mathcal{A}\times\mathcal{V}$ is a metric space, to prove this lemma it is enough to prove that for any sequence $\phi_n\in\Phi(a_n,v_n)$ with $\phi_n\to\phi$ and $(a_n,v_n)\to(a,v)$, we have $\phi\in\Phi(a,v)$. This result follows as an immediate consequence of Proposition A.3 in the Appendix.

We are now in a position to provide a proof of Theorem 1. Lemma A.5 implies that $\mathcal{A}\times\mathcal{V}$ is a compact convex set contained in a metric linear space. From Lemmas A.6, A.7 and A.8, $\Phi:\mathcal{A}\times\mathcal{V}\to\mathcal{A}\times\mathcal{V}$ is nonempty- and convex-valued, and its graph is closed. From Glicksberg's (1952) generalization of Kakutani's fixed point theorem, we deduce the existence of a fixed point $(\bar{a},v)\in\mathcal{A}\times\mathcal{V}$ of Φ: $(\bar{a},v)\in\Phi(\bar{a},v)$. It readily follows that, for each $i\in I$ and each $s\in S$, $(\bar{a},v)$ satisfies conditions (A1) and (A2). Lemma A.1 implies that $\bar{a}$ is a Markov perfect equilibrium.

Appendix B. Proof of Corollary 2

We will define a new game and apply Theorem 1 to this new game. Denote the set of probability measures on A by $\mathcal{P}\left(A^i\right)$. We endow the set $\mathcal{P}\left(A^i\right)$ with the weak* topology; consult Chapter 15 in Aliprantis and Border (2006) for details. This space is contained

[17] Observe that if S were not countable, the sets $\mathcal{A}^i$ and $\mathcal{V}^i$ need not be metric spaces. The fact that these sets are metric spaces is used in the proof of Lemma A.8.

in a linear space. Given a profile $\mu = (\mu^i)_{i \in I} \in \prod_{i \in I} \mathcal{P}\left(A^i\right)$, we can define

$$\bar{\pi}^i(\mu, s) = \int \pi^i(a, s)\left(\mu^1 \otimes \ldots \otimes \mu^{|I|}\right)(da).$$

Analogously, we can extend the transition function defining

$$\bar{Q}\left(s^{'}; \mu, s\right) = \int Q\left(s^{'}; a, s\right)\left(\mu^1 \otimes \ldots \otimes \mu^{|I|}\right)(da).$$

Under our assumptions, these extensions are well defined. We can therefore consider a new dynamic stochastic game, called the extended game, in which player i's set of actions is $\mathcal{P}\left(A^i\right)$ and payoffs and transitions, $\bar{\pi}^i$ and $\bar{Q}$, are evaluated taking expectations. Observe that a Markov perfect equilibrium of the extended game is a behavior Markov perfect equilibrium of the original game. Because A^i is compact, $\mathcal{P}\left(A^i\right)$ is as well (Aliprantis and Border, 2006, Theorem 15.11). It can be shown that since $Q(s';a,s)$ is continuous in a and $\pi^i(a,s)$ is upper semi-continuous in a and lower semi-continuous in a^{-i}, the extensions inherit these properties (Aliprantis and Border, 2006, Theorem 15.5). Because payoffs $\bar{\pi}^i$ are linear in μ^i, the extended game has convex best replies. From Theorem 1, the extended game possesses a Markov perfect equilibrium which in turn yields the desired result. □

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