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RADIAL SYMMETRY OF GROUND STATES FOR A REGIONAL FRACTIONAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. The aim of this paper is to study radial symmetry properties for ground state solutions of elliptic equations involving a regional fractional Laplacian, namely

$$(-\Delta)^{\alpha}_{\rho}u + u = f(u) \ \text{ in } \ \mathbb{R}^n, \ \text{ for } \ \alpha \in (0,1). \tag{1}$$

In [9], the authors proved that problem (1) has a ground state solution. In this work we prove that the ground state level is achieved by a radially symmetry solution. The proof is carried out by using variational methods jointly with rearrangement arguments.

1. **Introduction.** In this paper we study symmetry properties of ground states of the nonlinear Schrödinger equation with a non-local regional diffusion

$$\begin{aligned} (-\Delta)^{\alpha}_{\rho}u + u = f(u) \quad &\text{in } \mathbb{R}^n, \\ u \in H^{\alpha}(\mathbb{R}^n), & \end{aligned} \tag{2}$$

where $0 < \alpha < 1$, $n \geq 2$ and $f : \mathbb{R} \to \mathbb{R}$ is super-linear and has sub-critical growth. The operator $(-\Delta)^{\alpha}_{\rho}$ is a non-local regional Laplacian, which is implicitly defined as

$$\int_{\mathbb{R}^n} (-\Delta)^{\alpha}_{\rho} u(x) v(x) dx = \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{[u(x+z) - u(x)][v(x+z) - v(x)]}{|z|^{n+2\alpha}} dz dx,$$

for all $u, v \in H^{\alpha}(\mathbb{R}^n)$, where the range of scope of the operator is determined by the function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$. There have been another definition of non-local regional Laplacian in the work by Bogdan, Burdzy and Chen [5] and Guan [10], where the operator there is non-variational and its range of scope is independent of x . Our version is a variational adaptation of the operator defined by Ishii and Nakamura in [11], where an x-dependent range of scope is considered.

In a recent paper [8], the study of positive solutions of the nonlinear fractional Schrödinger equation

$$(-\Delta)^{\alpha}u + u = f(x,u) \ \text{ in } \ \mathbb{R}^n, \tag{3}$$

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was considered, where the fractional Laplacian is defined as

$$(-\Delta)^{\alpha}u(x) = \frac{1}{2}\int_{R^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N-2\alpha}}dy, \quad x \in R^n.$$

The authors obtained existence of ground state solutions using the mountain pass theorem and a comparison argument devised by Rabinowitz in [19], for $\alpha = 1$. They also analyzed regularity, decay and symmetry properties of these solutions. Cheng in [6], considered the problem with a potential V and with pure power nonlinearity $f(t) = |t|^{p-1}t$. Ground states are found by imposing a coercivity assumption on V

$$\lim_{|x|\to\infty} V(x) = +\infty.$$

In [20], Secchi provides a generalization of the main result of [6] to an equation of the form

$$(-\Delta)^{\alpha}u + V(x)u = f(x,u), \quad x \in \mathbb{R}^n, \tag{4}$$

for a more general non-linearity. He obtained the existence of a ground state by the method used in [8].

Motivated by these previous works, the authors considered in [9] the nonlinear Schrödinger equation with nonlocal regional diffusion

$$(-\Delta)^{\alpha}_{\rho}u + u = f(u) \ \text{ in } \mathbb{R}^n. \tag{5}$$

Following some ideas in [8], they obtained the existence of a ground state by the mountain pass theorem and a comparison argument. Moreover they also analyzed the concentration phenomena occurring when the diffusion parameter approaches zero and they analyze the role of the scope function ρ in the concentration.

Dipierro, Palatucci and Valdinoci [7], consider the existence of radially symmetric solutions of (4) when V and f do not depend explicitly on the space variable x. For the first time, using rearrangement tools and following the ideas of Berestycki and Lions [3], the authors prove existence of a nontrivial, radially symmetric, solution to

$$\begin{aligned}(-\Delta)^{\alpha}u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,\\ u \in H^{\alpha}(\mathbb{R}^n),\end{aligned} \tag{6}$$

when $\alpha \in (0,1)$ and p is subcritical.

Solutions of (6) can be obtained by finding critical points of the Euler-Lagrange functional I defined in the fractional Sobolev spaces $H^{\alpha}(\mathbb{R}^n)$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2\alpha}}dzdx + \int_{\mathbb{R}^n}\left(\frac{1}{2}|u(x)|^2 - \frac{1}{p+1}|u(x)|^{p+1}\right)dx.$$

They used a basic rearrangement inequality to conclude that

$$I(u^*) \le I(u), \tag{7}$$

where u^* is the symmetric rearrangement of u. Therefore, if there is a minimizer of I then there is also a radially symmetric minimizer. Their proof is based on a minimization method, working with an appropriate constraint. This constraint is useful in this case because of the autonomous and homogeneous character of (6). In [8], symmetry properties of positive solutions of the nonlinear fractional Schrödinger equation

$$(-\Delta)^{\alpha}u + u = f(u) \ \text{ in } \ \mathbb{R}^n,$$

where studied, using the integral form of the moving planes method. They assumed that the function f has a super-linear and sub-critical growth and

(F) $f \in C^1(\mathbb{R})$, increasing and there exists $\tau > 0$ such that

$$\lim_{v \to 0} \frac{f'(v)}{v^\tau} = 0.$$

Taking advantage of the representation formula for u given by

$$u(x) = (\mathcal{K} * f(u))(x), \quad x \in \mathbb{R}^n, \tag{8}$$

where $\mathcal{K}$ is the kernel associated to the linear part of the equation, they apply the moving planes argument.

In view of the previous works we just described, it is natural to ask for radial symmetry of ground states for the equation (2). Next we present in detail the hypotheses required for our results to be proved. Let us assume that ρ satisfies the following conditions:

(ρ_1) $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$, there are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$\rho(x) \geq \rho_0 \ \ \forall x \in \mathbb{R}^n \ \ \textit{and} \ \ \lim_{|x| \to \infty} \rho(x) = \rho_\infty.$$

(ρ_2) When $\rho_\infty = \infty$, we further assume that

$$\lim_{|x| \to \infty} \frac{\rho(x)}{|x|} \leq \frac{1}{2}.$$

(ρ_3) ρ is radially symmetric.

Regarding f we asume:

(f_1) $f(t) \geq 0$ if $t \geq 0$ and $f(t) = 0$ if $t \leq 0$.

(f_2) The function $t \to \frac{f(t)}{t}$ is increasing for $t > 0$ and $\lim_{t \to 0} \frac{f(t)}{t} = 0$.

(f_3) $\exists \theta > 2$ such that $\forall t > 0$

$$0 < \theta F(t) \leq t f(t), \quad \text{where} \quad F(t) = \int_0^t f(s) ds.$$

(f_4) $\exists C > 0$ such that

$$|f(t)| \leq C(1 + |t|^p), \ \ 1 < p < \frac{n + 2\alpha}{n - 2\alpha}.$$

Under the hypotheses given above, the solutions of (2) are the critical points of the functional functional I_ρ defined on $H^\alpha(\mathbb{R}^n)$ by

$$I_\rho(u) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{|u(x) - u(z)|^2}{|x - z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} \left(\frac{1}{2} |u(x)|^2 - F(u(x)) \right) dx.$$

Now we state the main theorem in our paper.

Theorem 1.1. *Suppose that* $(\rho_{1)} - (\rho_3)$ *and* $(f_1) - (f_4)$ *hold. Then the mountain pass value of* I_ρ *is achieved by a radially symmetric solution of (2).*

To prove this theorem we proceed by using rearrangements and variational methods. The idea is to replace the path γ in the mountain pass setting, by its symmetrization γ^* and then the critical point would be near of the set $\gamma^*([0, 1])$. This idea works since rearrangements are continuous in $H^\alpha(\mathbb{R}^n)$, see [1], however it cannot be used directly when $\alpha = 1$ and $n > 1$ since rearrangements are not continuous in $H^1(\mathbb{R}^n)$. See more details in the work by Van Schaftingen [23].

We observe that the approach used in [8] is not possible to be used here, since a representation formula like (8) is not available in general for $(-\Delta)^\alpha_\rho$. On the other hand we cannot use the approach in [7] since our problem is x-dependent and so rescaling is not available.

Rearrangements have long been a basic tool in the calculus of variations and in the theory of partial differential equations arising as Euler-Lagrange equations of variational problems. The basic Polya-Szegö inequality claims that the symmetric decreasing rearrangement diminishes the L^2-norm of the gradient of a function u:

$$\int_{\mathbb{R}^n} |\nabla u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^2 dx \tag{9}$$

where u^* represent the symmetric decreasing rearrangement of u, see [13].

The inequality (9), together with its several variants [13], is a powerful key to a number of variational problems of geometric and functional nature, concerning extremal properties of domains and functions. Besides optimal Sobolev embeddings, classical isoperimetric inequalities in mathematical physics and sharp eigenvalue inequalities fall within these results; a priori estimates for solutions to elliptic problems in sharp form are also a closely related topic [14].

The fractional version of (9), namely

$$\int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} u^*(x)|^2 dx \leq \int_{\mathbb{R}^n} |(-\Delta)^{\alpha/2} u(x)|^2 dx, \tag{10}$$

was proved by Almgren and Lieb [1] using a rearrangement inequality for convex integrands. Recently Park [18] proved this inequality using Fourier analysis, based on arguments by Beckner [2] and Almgren and Lieb [1].

We notice that inequality (10) is a principal key to get a radially symmetric minimizer. So, an interesting problem is to prove the following inequality

$$\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u^*(x+z)-u^*(x)|^2}{|z|^{n+2\alpha}} dzdx \leq \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx, \tag{11}$$

This is a second main goal in this paper. We want to get a regional version of Riesz and Polya-Szegö inequality with a radial symmetric positive scope function $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$. In section §3, following the ideas of Almgren and Lieb [1] we prove the following preliminary inequality: let u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ vanishing at infinity, then

$$\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} u(x)v(x-z)w(z)dydx \leq \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} u^*(x)v^*(x-z)w^*(z)dzdx, \tag{12}$$

and (11).

This paper is organized as follows. In section §2 we recall some fact of fractional Sobolev spaces and in section §3 we recall the main inequalities from rearrangement theory and we prove inequalities (11) and (12). In section §4 we prove our main symmetry result, Theorem 1.1.

2. **Preliminaries.** Let $0 < \alpha < 1$ and $n \geq 1$. The fractional Sobolev space of order α on $\mathbb{R}^n$ is defined by

$$H^\alpha(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) / \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} dydx < \infty \right\},$$

endowed with the norm

$$\|u\|^2_{H^\alpha} = \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} dydx.$$

For the reader's convenience, we review the main embedding result for fractional Sobolev spaces.

Theorem 2.1 ([17]). *Let $\alpha \in (0,1)$, then there exists a positive constant $C = C(n,\alpha)$ such that*

$$\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2\alpha}} dydx \tag{13}$$

and then we have that $H^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for all $q \in [2, 2^*_\alpha]$.

Moreover, $H^\alpha(\mathbb{R}^n) \hookrightarrow L^q(\Omega)$ is compact for any bounded set $\Omega \subset \mathbb{R}^n$ and for all $q \in [2, 2^*_\alpha)$, where $2^*_\alpha = \frac{2n}{n-2\alpha}$ is the critical exponent.

We introduce a new fractional Sobolev Hilbert space

$$H^\alpha_\rho(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) / \quad \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|}{|z|^{n+2\alpha}} dzdx < +\infty \right\}$$

endowed with the norm

$$\|u\|^2_{H^\alpha_\rho} = \int_{\mathbb{R}^n} |u(x)|^2 dx + \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}} dzdx,$$

which is induced by the following inner product

$$\langle u, v\rangle_{H^\alpha_\rho} = \int_{\mathbb{R}^n} u(x)v(x)dx \; + \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx.$$

The following proposition, which is proved in [9], is crucial in our analysis.

proposition 1. *If ρ satisfies (ρ_1), then there exists a constant $C = C(n,\alpha,\rho)$ such that*

$$\|u\|^2_{H^\alpha} \leq C\|u\|^2_{H^\alpha_\rho}.$$

This proposition implies that $H^\alpha_\rho(\mathbb{R}^n)$ and $H^\alpha(\mathbb{R}^n)$ have equivalent norms.

3. **Symmetry rearrangements.**

In this section first we recall some facts regarding rearrangement of sets and functions. Then we present a new regional Riesz and Polya-Szegö inequality when the range of scope determined is a radially symmetric function.

Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set and denote the measure of A by $|A|$. Define the symmetrization A^* of A to be the closed ball centered at the origin such with the same measure as A. Thus in one dimension

$$A^* = [-\frac{|A|}{2}, \frac{|A|}{2}]$$

and in n dimensions, if we define $\omega(n)$ to be the volume of the unit ball in $\mathbb{R}^n$, then for $A \subset \mathbb{R}^n$

$$A^* = B(0, (|A|/\omega(n))^{1/n}).$$

Let $u : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable function, then u is said to vanish at infinity if

$$|\{x : |u(x)| > t\}| < \infty \text{ for all } t > 0.$$

The symmetric decreasing rearrangement of a characteristic function χ_A is defined as

$$\chi^*_A := \chi_{A^*}.$$

We now recall that any non negative function can be expressed as an integral of the characteristic functions of the sets $\{u \geq t\}$ (which is a standard abbreviation for $\{x : u(x) \geq t\}$) as follows

$$u(x) = \int_0^{u(x)} 1dt = \int_0^\infty \chi_{\{u\geq t\}}(x)dt. \tag{14}$$

Notice that this, along with Fubini's theorem, implies

$$\int_{\mathbb{R}^n} u(x)dx = \int_{\mathbb{R}^n}\int_0^\infty \chi_{\{u\geq t\}}(x)dtdx = \int_0^\infty |\{x : u(x) \geq t\}|dt.$$

Now if $u : \mathbb{R}^n \to \mathbb{R}$ is a Borel measurable function vanishing at infinity we define the rearrangement of u as

$$u^*(x) = \int_0^\infty \chi^*_{\{|u|\geq t\}}(x)dt. \tag{15}$$

The rearrangement u^* has a number of properties, see [15]:

(i) u^* is nonnegative.

(ii) u^* is radially symmetric and non-increasing, i.e.:

$$|x| \leq |y| \text{ implies } u^*(y) \leq u^*(x).$$

(iii) u^* is a lower semicontinuos function.

(iv) The level sets of u^* are simply the rearrangement of the level sets of u, i.e.

$$\{x : u^*(x) > t\} = \{x : |u(x)| > t\}^*.$$

An important consequence of this is the equimeasurability of the function u and u^*, that is

$$|\{u^* > t\}| = |\{|u| > t\}|, \text{ for all } t > 0.$$

(v) For any positive monotone function ϕ, we have

$$\int_{\mathbb{R}^n} \phi(|u(x)|)dx = \int_{\mathbb{R}^n} \phi(u^*(x))dx.$$

In particular, $u^* \in L^p(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and

$$\|u\|_{L^p} = \|u^*\|_{L^p}.$$

(vi) Let $V(|x|) \geq 0$ be a radially symmetric increasing function on $\mathbb{R}^n$. If u is a nonnegative function on $\mathbb{R}^n$, vanishing at infinity then

$$\int_{\mathbb{R}^n} V(|x|)|u^*(x)|^2dx \leq \int_{\mathbb{R}^n} V(|x|)|u(x)|^2dx.$$

(vii) **Riesz' rearrangement inequality.** Let u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ that vanish at infinity. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x)v(x-y)w(y)dydx \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u^*(x)v^*(x-y)w^*(y)dydx.$$

Now we will present and prove a new type of rearrangements inequalities, following the ideas of Almgren and Lieb [1]. First we prove the regional Riesz inequality and for this we need the following lemma

lemmama 3.1. *Let u be nonnegative measurable function on $\mathbb{R}^n$ that vanish at infinity and $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetric function. Given $x \in \mathbb{R}^n$, we let $w(y) = u(y)\chi_{B(0,\rho(|x|))}(y)$. Then, for each $x \in \mathbb{R}^n$*

$$w^*(y) \leq u^*(y)\chi_{B(0,\rho(|x|))}(y). \tag{16}$$

Proof. First we notice that for measurable sets A and B we have

$$(A \cap B)^* \subset A^* \cap B^*. \tag{17}$$

In fact, since $A \cap B \subset A$ and $A \cap B \subset B$, then $|A \cap B| \leq |A|$ and $|A \cap B| \leq |A|$. Therefore

$$\begin{aligned}
&B(0, (|A \cap B|/w(n))^{1/n}) \subset B(0, (|A|/w(n))^{1/n}) \ \text{ and} \\
&B(0, (|A \cap B|/w(n))^{1/n}) \subset B(0, (|B|/w(n))^{1/n}).
\end{aligned}$$

This implies

$$B(0, (|A \cap B|/w(n))^{1/n}) \subset B(0, (|A|/w(n))^{1/n}) \cap B(0, (|B|/w(n))^{1/n}),$$

hence $(A \cap B)^* \subset A^* \cap B^*$. Now, we notice that (16) follow from:

$$\{(u(y)\chi_{B(0,\rho(|x|))}(y))^* > t\} \subseteq \{[u^*(y)\chi_{B(0,\rho(|x|))}(y)] > t\} \ \text{ for all } \ t.$$

In fact, by (iv) and (17) we have

$$\begin{aligned}
\{(u\chi_{B(0,\rho(|x|))})^* > t\} =& \{u\chi_{B(0,\rho(|x|))} > t\}^* \\
=& [\{u > t\} \cap B(0, \rho(|x|))]^* \\
\subseteq& \{u > t\}^* \cap B(0, \rho(|x|)) \\
=& \{u^* > t\} \cap B(0, \rho(|x|)) \\
=& \{u^* \chi_{B(0,\rho(|x|))} > t\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(u\chi_{B(0,\rho(|x|))}\right)^*(y) =& \int_0^\infty \chi_{\{(u\chi_{B(0,\rho(|x|))})^* > t\}}(y)dt \\
\leq& \int_0^\infty \chi_{\{u^*\chi_{B(0,\rho(|x|))}\} > t}(y)dt \\
=& (u^*\chi_{B(0,\rho(|x|))})(y).
\end{aligned}$$

This proves our inequality. $\square$

With this lemma we are ready to prove our regional Riesz inequality

Theorem 3.2 (Regional Riesz Rearrangement Inequality)**.** *Let u, v, w be nonnegative measurable functions on $\mathbb{R}^n$ that vanish at infinity and $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetric function. Then*

$$\int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u(x)v(x-y)w(y)dydx \leq \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u^*(x)v^*(x-y)w^*(y)dydx \tag{18}$$

Proof. By Riesz rearrangement inequality and Lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u(x)v(x-y)w(y)dydx \\
&\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x)v(x-y)w(y)\chi_{B(0,\rho(|x|))}(y)dydx \\
&\quad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x)v^*(x-y)(w\chi_{B(0,\rho(|x|))})^*(y)dydx \\
&\quad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u^*(x)v^*(x-y)w^*(y)\chi_{B(0,\rho(|x|))}(y)dydx \\
&\quad = \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} u^*(x)v^*(x-y)w^*(y)dydx. \qquad \square
\end{aligned}$$

Now we are going to prove the regional version of Polya-Szegö inequality. We consider the functional $E_\rho : H^\alpha_\rho(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$E_\rho[u] = \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} \frac{|u(x+z)-u(x)|^2}{|z|^{n+2\alpha}}dzdx.$$

We will get a representation of E_ρ that will be useful for our purpose. We start with the representation formula

$$\frac{1}{|z|^{n+2\alpha}} = \frac{1}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\int_0^\infty e^{-t|z|^2} t^{\frac{n+2\alpha}{2}-1}dt, \tag{19}$$

whose proof can be found in [1]. Using (19) and Fubini's theorem we have

$$\begin{aligned}
E[u] &= \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 \frac{1}{|z|^{n+2\alpha}}dzdx \\
&= \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 \frac{1}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\int_0^\infty e^{-t|z|^2} t^{\frac{n+2\alpha}{2}-1}dtdzdx \\
&= \frac{1}{\Gamma\left(\frac{n+2\alpha}{2}\right)}\int_0^\infty\int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 e^{-t|z|^2}dzdx\, t^{\frac{n+2\alpha}{2}-1}dt.
\end{aligned}$$

From here we define

$$I_t[u] = \int_{\mathbb{R}^n}\int_{B(0,\rho(|x|))} |u(x+z)-u(x)|^2 e^{-t|z|^2}dzdx, \quad t>0. \tag{20}$$

Theorem 3.3 (Regional Polya-Szegö inequality)**.** *Let $0<\alpha<1$ and $u \in H^\alpha_\rho(\mathbb{R}^n)$ and $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ be a positive radially symmetric function. Then*

$$E_\rho[u^*] \leq E_\rho(u). \tag{21}$$

Proof. Without loss of generality we may assume that u is non-negative since $|u(x+z)-u(x)| \geq ||u(x+z)|-|u(x)||$. Furthermore, in view of (20) we only need to prove that

$$I_t[u^*] \leq I_t[u], \quad \forall t>0. \tag{22}$$

Let $\phi(t) = |t|^2$ and let us write $\phi = \phi_+ + \phi_-$, where

$$\phi_\pm(t) = \begin{cases} \phi(t), & \text{if } \pm t \geq 0, \\ 0, & \text{if } \pm t \leq 0. \end{cases}$$

We decompose $I_t = I_t^+ + I_t^-$ accordingly. Below we prove the assertion of the theorem with I_t replaced by I_t^+. The assertion for I_t^- is similar and hence the result for the original I_t follows. Since $\phi_+(0)=0$, we have that

$$\begin{aligned}
\phi_+(u(x+z)-u(x)) &= \int_0^{u(x+z)-u(x)} \phi_+'(t)dt \\
&= \int_{u(x)}^{u(x+z)} \phi_+'(u(x+z)-t)dt \\
&= \int_0^\infty \phi_+'(u(x+z)-t)\chi_{\{u\leq t\}}(x)dt.
\end{aligned}$$

Then, by Fubini's theorem

$$\begin{aligned} I_t^+[u] &= \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} \phi'_+(u(x+z)-u(x))e^{-t|z|^2}dzdx \\ &= \int_0^\infty \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} \phi'_+(u(x+z)-t)e^{-t|z|^2}\chi_{\{u\leq t\}}(x)dzdxdt. \end{aligned} \tag{23}$$

Now

$$\begin{aligned} &\int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} \phi'_+(u(x+z)-t)e^{-t|z|^2}\chi_{\{u\leq t\}}(x)dzdx \\ =& \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} \phi'_+(u(x+z)-t)e^{-t|z|^2}(1-\chi_{\{u>t\}}(x))dzdx \\ =& \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} \phi'_+(u(x+z)-t)e^{-t|z|^2}dzdx \\ &- \int_{\mathbb{R}^n} \int_{B(0,\rho(|x|))} \phi'_+(u(x+z)-t)e^{-t|z|^2}\chi_{\{u>t\}}(x)dzdx. \end{aligned} \tag{24}$$

We notice that the function $g(s) = \phi'_+(s-t)$ is increasing and $\left(e^{-t|z|^2}\right)^* = e^{-t|z|^2}$, for all $t > 0$, then by property (v) we have

$$\left(\phi'_+(u(x)-t)\right)^* = \phi'_+(u^*-t).$$

From here, (23), (24) and Theorem 3.2 we find that

$$I_t^+[u^*] \leq I_t^+[u], \quad \forall t > 0.$$

Since we also have $I_t^-[u^*] \leq I_t^-[u]$, we conclude. □

Remark 3.1. Theorem 3.3 implies the non-expansivity of symmetric decreasing rearrangement of the regional fractional Sobolev norm in $H^\alpha_\rho(\mathbb{R}^n)$, that is, for $u \in H^\alpha_\rho(\mathbb{R}^n)$

$$\|u^*\|_{H^\alpha_\rho} \leq \|u\|_{H^\alpha_\rho}. \tag{25}$$

Finally we recall a result proved by Almgren and Lieb in [1], which is a crucial ingredient to prove our main theorem in the next section.

Theorem 3.4. *For each $0 < \alpha < 1$ and each $n \geq 1$, the map $\mathfrak{R} : H^\alpha(\mathbb{R}^n) \to H^\alpha(\mathbb{R}^n)$, defined as $\mathfrak{R}u = u^*$, is continuous and, as a consequence, $\mathfrak{R} : H^\alpha_\rho(\mathbb{R}^n) \to H^\alpha_\rho(\mathbb{R}^n)$ is also continuous.*

4. **Symmetry results: proof of Theorem 1.1.** In this section we provide a proof of Theorem 1.1 and we also extent the ideas to prove a symmetry result in the case of the equation

$$\begin{aligned} (-\Delta)^\alpha u + V(|x|)u = f(u) \ \text{ in } \mathbb{R}^n, \\ u \in H^\alpha(\mathbb{R}^n). \end{aligned} \tag{26}$$

In order to prove Theorem 1.1 we consider the functional $I_\rho : H^\alpha_\rho(\mathbb{R}^n) \to \mathbb{R}$ defined by

$$I_\rho(u) = \frac{1}{2}\|u\|^\alpha_\rho - \int_{\mathbb{R}^n} F(u(x))dx, \tag{27}$$

It is a simple exercise to check that I_ρ is well-defined and of class C^1.

Proof of Theorem 1.1. Under (f_1)-(f_4), (ρ_1)-(ρ_2), in [9] we have proved that I_ρ satisfies the mountain pass geometry condition with mountain pass level

$$c_\rho = \inf_{\gamma\in\Gamma_\rho} \sup_{t\in[0,1]} I_\rho(\gamma(t)),$$

where $\Gamma_\rho = \{\gamma \in C([0,1], H_\rho^\alpha(\mathbb{R}^n))/\ \ \gamma(0) = 0,\ I_\rho(\gamma(1)) < 0\}$. By definition of c_ρ, for any $n \in \mathbb{N}$, there is $\gamma_n \in \Gamma_\rho$ such that

$$\sup_{t\in[0,1]} I_\rho(\gamma_n(t)) \le c_\rho + \frac{1}{n^2}. \tag{28}$$

Now, defining $\gamma_n^*(t) = [\gamma_n(t)]^*$, we see that Theorem 3.4 and the fact that $I_\rho(\gamma_n^*(1)) \le I_\rho(\gamma_n(1)) < 0$ imply that $\gamma_n^* \in \Gamma_\rho$. Moreover, by the regional Polya-Zsegö inequality proved in Theorem 3.3, we have

$$I_\rho(\gamma_n^*(t)) \le I_\rho(\gamma_n(t)),\ \ \forall t \in [0,1].$$

So

$$\sup_{t\in[0,1]} I_\rho(\gamma_n^*(t)) \le c_\rho + \frac{1}{n^2}. \tag{29}$$

Then by Theorem 4.3 of [16], there is a sequence $u_n \in H_\rho^\alpha(\mathbb{R}^n)$ and $\xi_n \in [0,1]$ such that

$$\|u_n - \gamma_n^*(\xi_n)\|_{H_\rho^\alpha} \le \frac{1}{n}, \tag{30}$$

$$I_\rho(u_n) \in (c_\rho - \frac{1}{n^2}, c_\rho + \frac{1}{n^2}) \quad \text{and} \tag{31}$$

$$\|I_\rho'(u_n)\|_{(H_\rho^\alpha)'} \le \frac{1}{n}. \tag{32}$$

Following the ideas of the proof of Theorem 1.1 of [9] we can show that: $u_n \to u$ in $H_\rho^\alpha(\mathbb{R}^n)$, $I_\rho(u) = c_\rho$, $I_\rho'(u) = 0$ and

$$\lim_{n\to\infty} \|u - \gamma_n^*(\xi_n)\|_{H_\rho^\alpha} = 0. \tag{33}$$

The last equality shows that $u = u^*$. □

We can use the same ideas with equation (26). In [20], Secchi studied (26) with an $x-$dependent nonlinearity $f(x,t)$. Using the approach of Rabinowitz in [19], namely a comparison argument, Secchi proved the existence of a ground state solution of (26), when $f(x,t)$ is super-linear and has a sub-critical growth. On the other hand, in [21] Secchi considered the existence of radially symmetric solutions of (26) under some weaker conditions on f, using the monotonicity trick of Struwe and Jeanjean [12].

Now, our purpose is to prove the symmetry result for (26) using the approach discussed above. For that purpose we consider that the nonlinearity f satisfies (f_1)-(f_4) and regarding the potential V we assume

(V_1) $V \in C^0(\mathbb{R}^n)$ and $\inf_{\mathbb{R}^n} V(x) = V_0 > 0$
(V_2) $\lim_{|x|\to\infty} V(x) = V_\infty$.
(V_3) V is radially symmetric and increasing.

The solutions of (26) are the critical points of the functional

$$I_V(u) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{R^n} \frac{|u(x+z) - u(x)|^2}{|z|^{n+2\alpha}} dzdx + \int_{\mathbb{R}^n} \frac{1}{2}V(x)|u(x)|^2 - F(u(x))dx$$

defined on the Sobolev space

$$H^\alpha_V(R^n) = \left\{ u \in H^\alpha(R^n) \,/\, \int_{R^n} V(x)|u(x)|^2 dx < \infty \right\},$$

endowed with the inner product

$$\langle u, v\rangle_{H^\alpha_V} = \int_{\mathbb{R}^n} V(x)u(x)v(x)dx \; + \int_{\mathbb{R}^n}\int_{R^n} \frac{[u(x+z)-u(x)][v(x+z)-v(x)]}{|z|^{n+2\alpha}} dzdx.$$

Now we state our result.

Theorem 4.1. *Suppose that $(f_1) - (f_4)$ and $(V_1) - (V_3)$ hold. Then the mountain pass value of I_V is achieved by a radially symmetric solution of (26).*

Proof. Under (f_1)-(f_4), (V_1)-(V_2) we find that I_V satisfies the mountain pass geometry conditions, using the proof of Secchi in [20] with minor modifications. The mountain pass level for I_V is given by

$$c_V = \inf_{\gamma\in\Gamma_V} \sup_{t\in[0,1]} I_V(\gamma(t)),$$

where Γ_V is defined as usual. By definition of c_V, for any $n \in \mathbb{N}$, there is $\gamma_n \in \Gamma_V$ such that

$$\sup_{t\in[0,1]} I_V(\gamma_n(t)) \le c_V + \frac{1}{n^2}. \tag{34}$$

Now, let $\gamma_n^*(t) = [\gamma_n(t)]^*$. By the continuity of rearrangements in $H^\alpha_V(\mathbb{R}^n)$ we have that $\gamma_n^* \in \Gamma_V$. Moreover, by the fractional Polya-Szegö inequality and taking into account that V satisfies (V_3), we have

$$I_V(\gamma_n^*(t)) \le I_V(\gamma_n(t)), \quad \forall t \in [0,1].$$

So

$$\sup_{t\in[0,1]} I_V(\gamma_n^*(t)) \le c_V + \frac{1}{n^2}. \tag{35}$$

By Theorem 4.3 in [16], there is a sequence $u_n \in H^\alpha_\rho(\mathbb{R}^n)$ and $\xi_n \in [0,1]$ such that

$$\|u_n - \gamma_n^*(\xi_n)\|_{H^\alpha_V} \le \frac{1}{n}, \tag{36}$$

$$I_V(u_n) \in (c_V - \frac{1}{n^2}, c_V + \frac{1}{n^2}), \tag{37}$$

$$\|I'_V(u_n)\|_{(H^\alpha_V)'} \le \frac{1}{n}. \tag{38}$$

Following the ideas of the proof of Theorem 5.2 of [20], we can show that $u_n \to u$, $I_V(u) = c_V$, $I'_V(u)u = 0$ and finally that

$$\lim_{n\to\infty} \|u - \gamma_n^*(\xi_n)\|_{H^\alpha_V} = 0, \tag{39}$$

concluding the proof. □

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