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Structural results on circular-arc graphs and circle graphs: A survey and the main open problems

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ABSTRACT

Circular-arc graphs are the intersection graphs of open arcs on a circle. Circle graphs are the intersection graphs of chords on a circle. These graph classes have been the subject of much study for many years and numerous interesting results have been reported. Many subclasses of both circular-arc graphs and circle graphs have been defined and different characterizations formulated. In this survey, we summarize the most important structural results related to circular-arc graphs and circle graphs and present the main open problems.

1. Introduction

The aim of this article is to summarize the most important known structural results on circular-arc graphs and circle graphs. We hope this survey can be helpful to those researchers who work on subjects related to these graph classes. In this introductory section, some remarkable structural results are briefly presented.

Circular-arc graphs are the intersection graphs of a set $\mathcal{S}$ of arcs on a circle; such a set $\mathcal{S}$ is called a circular-arc model. The first works about this class of graphs were published by Hadwiger et al. in 1964 [36] and Klee in 1969 [44]. Nevertheless, the first researcher who dealt with the problem of characterizing by forbidden subgraphs this family of graphs was Tucker in his Ph.D. thesis in 1969 [66]. He introduced and managed to characterize by forbidden induced subgraphs two subclasses of circular-arc graphs, namely *unit circular-arc graphs* and *proper circular-arc graphs*. The first subclass consists of those circular-arc graphs having a circular-arc model with all its arcs having the same length and the second one consists of those circular-arc graphs having a circular-arc model without any arc contained in another. The first polynomial-time recognition algorithm for circular-arc graphs was devised by Tucker in 1980 [70]. In 1995, Hsu presented a $O(mn)$-time recognition algorithm. A linear-time recognition algorithm was proposed by McConnell in 2003 [53].

Characterizing by forbidden induced subgraphs the whole class of circular-arc graphs is a long standing open problem [44,65,69]. Nevertheless, several authors have presented some advances in this direction. Trotter and Moore gave

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a characterization by forbidden induced subgraphs within the class of co-bipartite graphs [65]; i.e., they found the complete list of induced subgraphs that have to be forbidden in a co-bipartite graph in order to ensure that such a graph is circular-arc. Bang-Jensen and Hell presented a structural theorem for proper circular-arc graphs within the class of chordal graphs [3] that implies the characterization by forbidden induced subgraphs for proper circular-arc graphs within the class of chordal graphs. In [4] characterizations by minimal forbidden induced subgraphs of circular-arc graphs were presented, in the case where the graph belongs to any of the following four different classes: P_4-free graphs, paw-free graphs, claw-free chordal graphs and diamond-free graphs.

Circular-arc graphs are a generalization of the family of the intersection graphs of intervals in the real line, called *interval graphs*. Interval graphs were characterized by Lekkerkerker and Boland in 1962 [47]. The whole list of forbidden induced subgraphs that characterizes interval graphs was successfully found via a different characterization by means of asteroidal triples presented by the same authors. Any set of intervals in the real line satisfies the Helly property; i.e., any set of pairwise intersecting intervals in the real line have a common point. Consequently, a subclass of circular-arc graphs that naturally generalizes interval graphs are the *Helly circular-arc graphs*; i.e., those circular-arc graphs having an intersection model of arcs such that any subset of pairwise intersecting arcs has a common point. Lin and Szwarcfiter presented a characterization by forbidden structures for this class within the class of circular-arc graphs [50]. Such a characterization yields a linear-time recognition algorithm for the class of Helly circular-arc graphs. [18] introduced the class of proper Helly circular-arc graphs, those graphs having a circular-arc model which is simultaneously proper and Helly. This class was characterized by forbidden induced subgraphs in [49].

A circular-arc graph having a circular-arc model without two arcs covering the whole circle is called a *normal circular-arc graph*. This terminology was introduced in [51]. Hell and Huang proved that the complements of interval bigraphs are exactly those co-bipartite graphs having a normal circular-arc model [39]. A bipartite graph H, with a fixed partition (X, Y), is an *interval bigraph* if the vertices of H can be represented by a family of intervals I_v, $v \in X \cup Y$, so that, for $x \in X$ and $y \in Y$, x and y are adjacent in H if and only if I_x and I_y intersect. Generalizing circular-arc graphs, Alcón et al. introduced the class of loop graphs [1].

Fulkerson and Gross [25] characterized interval graphs in terms of their clique matrices. They were able to prove that the clique matrix of interval graphs satisfies the consecutive 1s property for rows. Following this line of work, Roberts [60] characterized proper interval graphs as those graphs whose augmented adjacency matrix has the consecutive 1s property for columns; i.e., its rows can be permuted in such a way that in each column the 1s appear consecutively. Results in this direction were obtained by Tucker and Gavril for circular-arc graphs and proper interval graphs in [68,30].

A graph is defined to be *circle* if it is the intersection graph of a set $\mathcal{C}$ of chords on a circle, such a set is called a circle model. Circle graphs were introduced by Even and Itai in [21] to solve an ordering problem with the minimum number of parallel stacks without the restriction of loading before unloading is completed, proving that the problem can be translated into the problem of finding the chromatic number of a circle graph. Unfortunately, this problem turns out to be NP-complete [28].

Naji characterized circle graphs in terms of the solvability of a system of linear equations, yielding an $O(n^7)$-time recognition algorithm for this class [54]. The *local complement* of a graph G with respect to a vertex $u \in V(G)$ is the graph $G*u$ that arises from G by replacing the induced subgraph $G[N_G(u)]$ by its complement. Two graphs G and H are *locally equivalent* if and only if G arises from H by a finite sequence of local complementations. Bouchet proved that circle graphs are closed under local complementation, as well as that a graph is circle if and only if every locally equivalent graph contains none of three prescribed graphs as induced subgraphs [8]. Inspired by this result, Geelen and Oum [31] gave a new characterization of circle graphs in terms of *pivoting* (see Section 4.2).

A circle graph with a circle model having all its chords of the same length is called a *unit circle graph*. It is well known that the class of proper circular-arc graphs is properly contained in the class of circle graphs. Furthermore, the class of unit circular-arc graphs and the class of unit circle graphs are the same [19].

Let G_1 and G_2 be two graphs such that $|V(G_i)| \geq 3$, for each $i = 1, 2$, and assume that $V(G_1) \cap V(G_2) = \emptyset$. Let v_i be a distinguished vertex of G_i, for each $i = 1, 2$. The *split composition* of G_1 and G_2 with respect to v_1 and v_2 is the graph $G_1 \circ G_2$ whose vertex set is $V(G_1 \circ G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$ and whose edge set is $E(G_1 \circ G_2) = E(G_1 - \{v_1\}) \cup E(G_2 - \{v_2\}) \cup \{uv : u \in N_{G_1}(v_1) \text{ and } v \in N_{G_2}(v_2)\}$. The vertices v_1 and v_2 are called the *marker vertices*. We say that G has a *split decomposition* if there exist two graphs G_1 and G_2 with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 \circ G_2$ with respect to some pair of marker vertices. If so, G_1 and G_2 are called the *factors* of the split decomposition. Those graphs that do not have a split decomposition are called *prime graphs*. The concept of split decomposition is due to Cunningham [15]. Circle graphs turned out to be closed under this decomposition [6] and in 1994 Spinrad presented a quadratic-time recognition algorithm for circle graphs that exploits this peculiarity [63]. Also based on split decomposition, Paul [58] presented an $O((n+m)\alpha(n+m))$-time algorithm for recognizing circle graphs, where α is the inverse of the Ackermann function.

Circle graphs are a superclass of *permutation graphs*. Indeed, permutation graphs can be defined as those circle graphs having a circle model such that a chord can be added in such a way that this chord meets all the chords belonging to the circle model. On the other hand, permutation graphs are those comparability graphs whose complement graph is also a comparability graph [22]. Since comparability graphs have been characterized by forbidden induced subgraphs [27], such a characterization implies a forbidden induced subgraphs characterization for the class of permutation graphs.

Helly circle graphs are those graphs having a circle model whose chords satisfy the Helly property; i.e., every set of pairwise adjacent chords has a common point. This family of graphs was introduced in [18,19]. It was also conjectured there that a circle graph is Helly circle if and only if it contains no induced diamond. Recently, this conjecture was positively settled [16]. Nevertheless, Helly circle graphs have not been fully characterized by forbidden induced subgraphs yet.

The characterization of circle graphs by forbidden induced subgraphs is also an open problem. In [5], circle graphs were characterized within linear domino graphs by profiting from the closure of the class of circle graphs under split decomposition. Besides, in the same work, characterizations by forbidden induced subgraphs within two superclasses of cographs (i.e., P_4-free) namely, tree-cographs and P_4-tidy graphs, are presented as an application of Gallai's characterization of comparability graphs [27]. Also in [5], the class of *unit Helly circle graphs*, which consists of those circle graphs having a circle model which is simultaneously Helly and unit, is introduced and characterized.

An interesting earlier survey on circular-arc graphs and their subclasses with more emphasis on recognition algorithms is presented in [52]. Instead, our focus is on the structural characterizations of this class and the class of circle graphs, specially by forbidden induced subgraphs. Given the difficulty of these problems in general, many authors formulated partial characterizations. In this survey, in addition to showing these structural characterizations, we present the main techniques employed in the corresponding proofs.

The paper is organized as follows. In Section 2, we give some preliminary definitions. Sections 3 and 4 are devoted to circular-arc graphs and circle graphs, respectively. Finally, in Section 5, we give a list of the main open problems which could serve as a starting point for researchers interested in working on these topics.

2. Definitions

All graphs in this article are without loops and without multiple edges. Let G be a graph and denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. We denote the complement of G by $\overline{G}$. The set of vertices adjacent to a vertex $v \in V$ is called the *neighborhood* of v and denoted by $N_G(v)$. The *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. The *degree* $d_G(v)$ of v is $|N_G(v)|$. Vertices with degree 0 and $|V| - 1$ are called *isolated vertex* and *universal vertex*, respectively. A *pendant vertex* is a vertex of degree one. A graph $H = (V', E')$ is said to be a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If, in addition, $E' = \{uv \in E : u, v \in V'\}$, H is called an *induced subgraph* of G and we say that *the vertex set* $V(H)$ *induces the graph* H. Given a subset $A \subseteq V(G)$, $G[A]$ stands for the subgraph induced by A. Two vertices $u, v \in V$ are said to be *false twins* if $N(v) = N(w)$ and they are said to be *true twins* if $N[v] = N[w]$. Let $A, B \subseteq V(G)$. We say that $A \subseteq V$ *is complete to* $B \subseteq V$ if every vertex of A is adjacent to every vertex of B.

A *forbidden induced subgraph* for a graph class $\mathcal{G}$ is a graph H such that no graph of $\mathcal{G}$ contains an induced H. A graph class $\mathcal{G}$ is said to be *hereditary* if for each $G \in \mathcal{G}$, each induced subgraph of G belongs to $\mathcal{G}$. A graph H is a *minimal forbidden induced subgraph* of a hereditary graph class $\mathcal{G}$ if and only if H is a forbidden induced subgraph of $\mathcal{G}$ but each induced subgraph of H different from H belongs to $\mathcal{G}$.

A *path* is a linear sequence of different vertices $P = v_1, \ldots, v_k$ such that v_i is adjacent to v_{i+1} for $i = 1, \ldots, k-1$. The *internal vertices* of the path are $v_2, \ldots, v_{n-1}$. Sums in this paragraph should be understood modulo k. If there is no edge v_iv_j such that $|i-j| \geq 2$ (i.e., all its internal vertices have degree two), P is said to be either a *chordless path* or *induced path*. A *cycle* C is a linear sequence of vertices $C = v_1, \ldots, v_k, v_1$ such that v_i is adjacent to v_{i+1} for $i = 1, \ldots, k$. If there is no edge v_iv_j such that $|i-j| \geq 2$, C is said to be either a *chordless cycle* or *induced cycle*. By P_n and C_n we denote a induced path and an induced cycle on n vertices, respectively. A *stable set* is a subset of pairwise non-adjacent vertices. A *complete set* is a set of pairwise adjacent vertices. A *clique* is an inclusion-wise maximal complete set in a graph. A *complete graph* is a graph whose vertex set is a complete set. The complete graph on n vertices is denoted by $K_n (n \geq 1)$. The graph K_3 is also called a *triangle*. A *diamond* is the graph obtained from a complete K_4 by removing exactly one edge. A *paw* is the graph obtained from a triangle T by adding a pendant vertex to T. A graph G is *bipartite* if $V(G)$ can be partitioned into two stable sets V_1, V_2; if, in addition, V_1 is complete to V_2, G is *complete bipartite*. Denote by $K_{r,s}$ the complete bipartite graph with $|V_1| = r$ and $|V_2| = s$. A *claw* is the complete bipartite graph $K_{1,3}$. A graph G is defined to be *chordal* if G does not contain any induced cycle with at least four vertices.

Let G_1 and G_2 be two graphs and assume that $V(G_1) \cap V(G_2) = \emptyset$. The *disjoint union* of G_1 and G_2 is the graph $G_1 \cup G_2$ such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. We denote by $G_1 + G_2$ the *join graph* of G_1 and G_2, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Let G be a graph and let A be a vertex set that induces a P_4 in G. A vertex v of G is said a *partner* of A if $G[A \cup \{v\}]$ contains at least two induced P_4s. If each vertex set A that induces a P_4 in G has at most one partner, G is called P_4*-tidy* [32]. The class of P_4-tidy graphs is an extension of the class of cographs and it contains many other graph classes defined by bounding the number of P_4s according to different criteria; e.g., P_4-sparse graphs [40], P_4-lite graphs [41], and P_4-extendible graphs [42]. A *spider* [40] is a graph whose vertex set can be partitioned into three sets S, C, and R, where $S = \{s_1, \ldots, s_k\}$ $(k \geq 2)$ is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; s_i is adjacent to c_j if and only if $i = j$ (a *thin spider*), or s_i is adjacent to c_j if and only if $i \neq j$ (a *thick spider*); R is allowed to be empty and if it is not, then all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. The triple (S, C, R) is called the *spider partition*. Clearly, the complement of a thin spider is a thick spider, and vice versa. A *fat spider* is obtained from a spider by adding a true or false twin of a vertex $v \in S \cup C$. *Tree-cographs* [64] are another generalization of cographs. They are defined recursively as follows: trees are tree-cographs; the disjoint union of tree-cographs is a tree-cograph; and the complement of a tree-cograph is also a tree-cograph. It is immediate from the definition that, if G is a tree-cograph, then G or $\overline{G}$ is disconnected, or G or $\overline{G}$ is a tree.

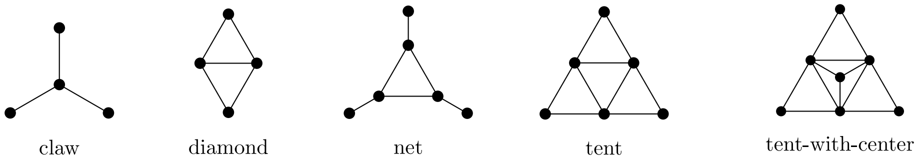

Fig. 1. Some small graphs.

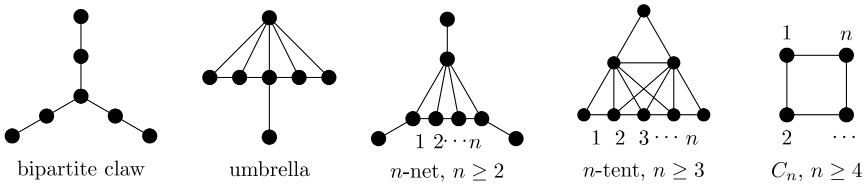

Fig. 2. Minimal forbidden induced subgraphs for the class of interval graphs.

For some small graphs to be referred to in what follows, see Fig. 1.

3. Circular-arc graphs

A graph $G = (V, E)$ is a *circular-arc graph* if there exists a family $\mathcal{M}$ of open arcs on a circle; i.e., there exists a one-to-one correspondence $f : V \to \mathcal{M}$ such that $uv \in E$ if and only if $f(u)$ and $f(v)$ intersect and $u \neq v$. If so, $\mathcal{M}$ is called a *circular-arc model* of G. Clearly, if the arcs of the family $\mathcal{M}$ do not cover the whole circle, then G is an interval graph. Conversely, if G is an interval graph then G is a circular-arc graph having a model whose arcs do not cover the whole circle.

The structure of this section is as follows. In Section 3.1 we present some characterizations of an important subclass of circular-arc graphs: the class of interval graphs. In Section 3.2 we survey characterizations of some subclasses of circular-arc graphs obtained by imposing restrictions on the models. In Section 3.3 we revisit some characterizations of circular-arc graphs within different graph classes.

3.1. Interval graphs

A graph $G = (V, E)$ is an *interval graph* if it is the intersection graph of a set $\mathcal{I}$ of open intervals on the real line; i.e., there exists a one-to-one correspondence $f : V \to \mathcal{I}$ such that $uv \in E$ if and only if $f(u)$ and $f(v)$ intersect and $u \neq v$. Such a family of intervals $\mathcal{I}$ is called an *interval model* of G.

Before stating the well-known forbidden induced subgraph characterization for interval graphs, we will introduce a tool that plays a very important role in this characterization. Three vertices in a graph G form an *asteroidal triple* if every two of them are connected by a path avoiding the third and its neighbors. Lekkerkerker and Boland [47] characterized interval graph by forbidden induced subgraphs. They managed to do so by characterizing interval graphs as those chordal graphs not containing asteroidal triples.

Theorem 1 ([47]). *A graph is an interval graph if and only if it contains no induced bipartite-claw, umbrella, n-net for any $n \geq 2$, n-tent for any $n \geq 3$, or C_n for any $n \geq 4$ (see Fig. 2).*

The proof due to Lekkerkerker and Boland consists in finding the minimally non-asteroidal-triple-free graphs within chordal graphs. The complete list of minimally non-asteroidal-triple-free graphs was found by Köehler [46].

A *proper interval graph* is an interval graph having an interval model such that none of its intervals is properly contained in any other; such an interval model is called a *proper interval model*. Proper interval graphs were introduced by Roberts [60], who also characterized which interval graphs are proper interval.

Theorem 2 ([60]). *Let G be an interval graph. G is proper interval if and only if G does not contain an induced claw.*

A *unit interval graph* is an interval graph having an interval model with all its intervals having the same length; such an interval model is called a *unit interval model*. Wegner [72] and Roberts [60] introduced unit interval graphs. Notice that every unit interval graph is a proper interval graph. Roberts proved that the converse is also true; i.e., the classes of proper interval graphs and unit interval graphs coincide. Consequently, by combining Theorems 1 and 2, the theorem below follows.

Theorem 3 ([60]). *Let G be a graph. The following assertions are equivalent:*

(i) *G is a unit interval graph.*
(ii) *G is a proper interval graph.*
(iii) *G contains no induced claw, net, tent, or C_n for any $n \geq 4$.*

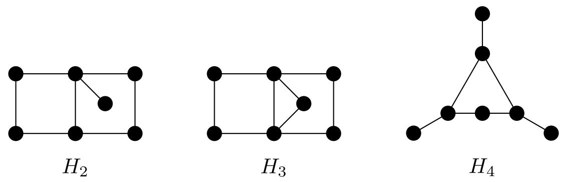

Fig. 3. Some graphs whose complements are not proper circular-arc graphs.

In this direction, Proskurowski and Telle introduced the class of *q-proper interval graphs* where q is a non-negative integer [59], defined as those interval graphs having an interval model such that any interval is contained in at most q intervals. Since 0-proper interval graphs are exactly the proper interval graphs, these graph classes generalize the class of proper interval graphs. For each positive integer k, let T_k be the graph obtained by adding $k-1$ true twins to the vertex of degree 3 of $K_{1,3}$. For instance, the graph T_1 is exactly the claw. It is easy to see that T_{q+1} is not a q-proper interval graph for any non-negative integer k. Moreover, T_{q+1} is the only minimally non-(q-proper interval) graph, as follows from the generalization of Theorem 2.

Theorem 4 (*[59]*). *Let G be an interval graph and q a non-negative integer. G is a q-proper interval graph if and only if G contains no induced* T_{q+1}.

Let G be a graph whose vertex set $V(G)$ is $\{v_1, \dots, v_n\}$ and whose cliques are $\{Q_1, \dots, Q_k\}$. A *clique-matrix* $Q(G)$ of G is the $k \times n$ such that $Q_{ij} = 1$ if and only if $v_j \in Q_i$ and $Q_{ij} = 0$ otherwise. Fulkerson and Gross [25] characterized interval graphs as those graphs whose clique-matrix has the *consecutive 1s property for rows*; i.e., there is a permutation of the columns of $Q(G)$ such that in each row the 1s appear consecutively. An *adjacency matrix* $M(G)$ of G is defined as the $n \times n$ matrix such that $M_{ij} = 1$ if vertices v_i and v_j are adjacent and $M_{ij} = 0$ otherwise. Note that $M(G)$ is symmetric and has only 0s on the main diagonal. The *augmented adjacency matrix* of G is defined as the matrix $M^*(G) := I + M(G)$ where I is the $n \times n$ identity matrix. Proper interval graphs were characterized by Roberts [60] as those graphs whose augmented adjacency matrix has the *consecutive 1s property for columns*; i.e., its rows can be permuted in such a way that in each column the 1s appear consecutively.

3.2. Circular-arc graphs having restricted models

Similarly to the case of interval graphs, a circular-arc graph G is called a *proper circular-arc graph* if and only if G admits a circular-arc model where no arc is properly contained in any other. Tucker [69] characterized proper circular-arc graphs by forbidden induced subgraphs as follows.

Theorem 5. *Let G be a graph. Then, G is a proper circular-arc graph if and only if G contains no induced co-(bipartite-claw), net,* $\overline{H_2}, \overline{H_3}, \overline{H_4}$, *tent* $\cup K_1$, $\overline{C_j \cup K_1}$ *for any* $j \geq 4$, $\overline{C_{2j}}$ *for any* $j \geq 3$, *and* $\overline{C_{2j+1} \cup K_1}$ *for any* $j \geq 1$ *(see Fig. 3).*

A digraph D is an *orientation* of a graph G if for every $u, v \in V(G)$ it holds that $uv \in E(G)$ if and only if either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. A *tournament* is an orientation of a complete graph. A *local tournament* [2] is an oriented graph for which the in-neighbors as well as the out-neighbors of any vertex induce a tournament. Skrien [61] proved that the class of graphs orientable as local tournaments is exactly the class of proper circular-arc graphs.

Theorem 6 (*[61]*). *A graph G is a proper circular-arc graphs if and only if G is orientable as a local tournament.*

A (0, 1)-matrix M is said to have the *circular 1s property for columns* if the rows of M can be permuted so that the 1s in each column are circular, that is, appear in a circularly consecutive fashion. The consecutive and circular 0s properties for columns are similarly defined. Note that a (0, 1)-matrix has the circular 1s property for columns if and only if it has the circular 0s property for columns. Let M be a symmetric (0, 1)-matrix with 1s on the main diagonal. Let U_i be the circular ordered set of 1s in column i starting at the main diagonal and going down until a 0 is encountered. Let V_i be the analogous set of 1-entries in row i starting at the main diagonal and going right until a 0 is encountered. M is said have the *quasi-circular 1s* property if the U_i and F_i contain all the 1s in M.

Tucker [68] characterized circular-arc graphs as those admitting an augmented adjacency matrix having the quasi-circular 1s property.

Theorem 7 (*[68]*). *Let G be a graph. G is a circular-arc graph if and only if the vertices of G can be indexed in such a way that* $M^*(G)$ *has the quasi-circular 1s property.*

This result can be reformulated in terms of a special ordering of the vertices of the graph G as follows.

Theorem 8 (*[67]*). *Let G be a graph. G is a circular-arc graph if and only if the vertices of G can be sorted in a circular ordering* $v_1, \dots, v_n$ *such that for* $i < j$ *if* $v_i v_j \in E(G)$ *then either* $v_{j+1}, \dots, v_j \in N(v_i)$ *or* $v_{j+1}, \dots, v_i \in N(v_j)$.

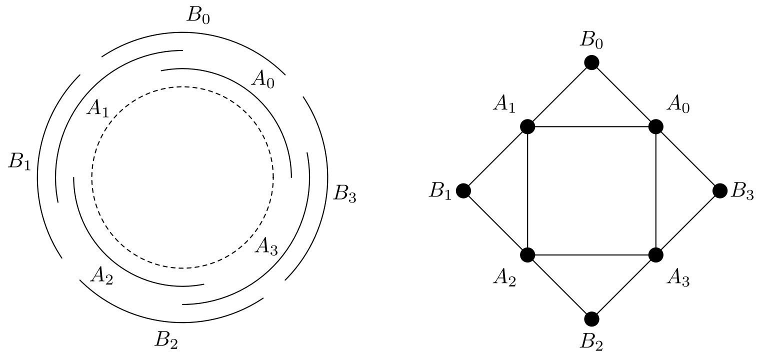

Fig. 4. On the left, a circular-arc model of CI(4, 1). On the right, its intersection graph, the graph CI(4, 1).

Proper circular-arc graphs were also characterized by Tucker in [68] using (0, 1)-matrices. To state his result, we need to define the (0, 1)-matrices having the circularly compatible 1s property. Let M be a symmetric (0, 1)-matrix. M is said to have *circularly compatible 1s* if the 1s in each column are circular and if, after inverting and/or cyclically permuting the order of the rows and (corresponding) columns, the last 1 (in cyclically descending order) of the circular set in the second column is always at least as low as the last 1 of the circular set in the first column unless one of these columns is all 1s or all 0s.

Theorem 9 (*[68]*). *Let G be a graph. G is a proper circular-arc graph if and only if there is an arrangement of $M^*(G)$ having circularly compatible 1s.*

In [69] Tucker also introduced unit circular-arc graphs as those graphs G admitting a circular-arc model whose arcs all have the same length. Clearly, unit circular-arc graphs are proper circular arc graphs. In the same work, he characterized which proper circular-arc graphs are unit circular-arc graphs by forbidden induced subgraphs. In order to state his characterization we need to introduce the following family of graphs. Let $\mathrm{CI}(j, k)$ $(j > k)$ be the circular-arc graph admitting the following circular-arc model on the unit circle: there are j arcs $A_0, A_1, \ldots, A_{j-1}$ each of which is of length $2\pi k/j + \epsilon$ such that A_i starts at $2\pi i/j$ and ends at $2\pi(i + k)/j + \epsilon$ and there are j arcs of length $2\pi k/j - \epsilon$ such that B_i starts at $(2\pi i + \pi)/j$ and ends at $(2\pi(i + k) + \pi)/j - \epsilon$, where $\epsilon > 0$ is a sufficiently small positive quantity. See, for instance, the circular-arc model of CI(4, 1) depicted in Fig. 4.

Theorem 10 (*[69]*). *Let G be a graph. Then, G is a unit circular-arc graph if and only if G is a proper circular-arc graph and G contains no induced* $\mathrm{CI}(j, k)$ *for relatively prime j and k and* $j > 2k$.

The first polynomial-time algorithm for recognizing the class of unit circular-arc graphs was proposed in [20]. The time complexity of that algorithm is $O(n^2)$, where n is the number of vertices of the input graph. The running time was improved to linear-time by an algorithm due to Lin and Szwarcfiter by relating unit circular-arc graphs to a generalization of Eulerian digraphs [51].

Define *semicircular graphs* [4] to be the intersection graphs of open semicircles on a circle. Notice that semicircular graphs are unit circular-arc graphs, but the converse is clearly not true; for example, P_4 is a unit circular-arc graph but not a semicircular graph. The following theorem gives a complete characterization of this class of graphs.

Theorem 11 (*[4]*). *Let G be a graph. Then, G is a semicircular-arc graph if and only if G is* $\{P_4, 3K_1\}$*-free.*

A family of sets has the *Helly property* if every nonempty subfamily of pairwise intersecting members has nonempty intersection. A circular-arc graph G is a *Helly circular-arc graph* if G admits a circular-arc model having the Helly property. Equivalently, G is a Helly circular-arc graph if and only if G admits a circular-arc model where the arcs corresponding to each clique of G have nonempty intersection. These graphs were introduced by Gavril [30] who proved that a graph is Helly circular-arc if and only if its clique-matrix has the circular 1s property for rows. In the same work, an $O(n^3)$-time recognition algorithm is given. In [43], Lin and Szwarcfiter gave the following forbidden induced subgraph characterization of which circular-arc graphs are Helly circular-arc graphs. A graph H is an *obstacle* if H contains a clique $Q = \{v_1, v_2, \ldots, v_t\}$ where $t \geq 3$ and such that for each $i = 1, \ldots, t$, at least one of the following assertions holds (where in both assertions, w_{t+1} means w_1):

(O_1) $N(w_i) \cap Q = Q \setminus \{v_i, v_{i+1}\}$, for some $w_i \in V(H) \setminus Q$.
(O_2) $N(u_i) \cap Q = Q \setminus \{v_i\}$ and $N(z_i) \cap Q = Q \setminus \{v_{i+1}\}$, for some adjacent vertices $u_i, z_i \in V(H) \setminus Q$.

Two examples of obstacles are depicted in Fig. 5. With this definition, the characterization of those circular-arc graphs that are Helly circular-arc graphs runs as follows.

Theorem 12 (*[43]*). *Let G be a circular-arc graph. Then, G is a Helly circular-arc graph if and only if G contains no induced obstacle.*

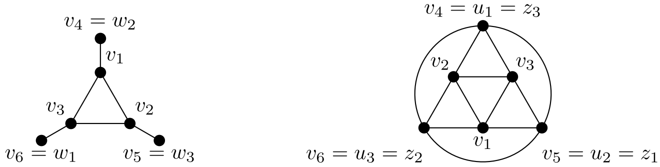

Fig. 5. Two obstacles.

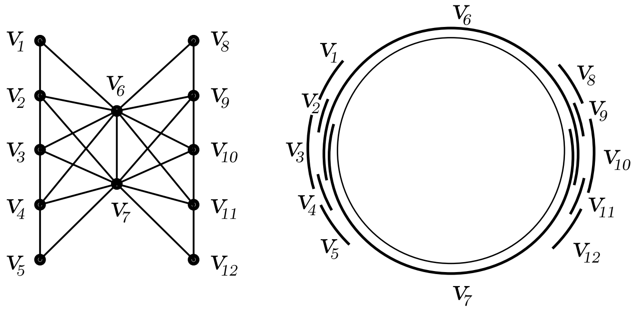

Fig. 6. Minimally non-normal circular-arc graph that is *CA*, and its circular-arc model.

A graph G is a *proper Helly circular-arc graph* if G admits a circular-arc model which is simultaneously proper and Helly. Motivated by an application in a periodic allocation problem [13], in [18] these graphs were first considered and it was shown that they are a proper subclass of the intersection of the classes of proper and Helly circular-arc graphs. Moreover, in [49], proper Helly circular-arc graphs were characterized by forbidden induced subgraphs.

Theorem 13 ([49])**.** *A graph G is a proper Helly circular-arc graph if and only if G is a proper circular-arc graph and G contains no induced* 4*-wheel and no induced tent.*

Similarly, *unit Helly circular-arc graphs* were introduced in [49] as those graphs admitting a circular-arc model which is simultaneously unit and Helly. The corresponding forbidden induced subgraph characterization is as follows.

Theorem 14 ([49])**.** *A graph G is a unit Helly circular-arc graph if and only if G is a unit circular-arc graph and G contains no induced 4-wheel.*

A circular-arc graph is a *normal circular-arc graph* if it admits a circular-arc model such that no two arcs cover the whole circle. For example, interval graphs and semicircular graphs are normal circular-arc graphs. An example of a circular-arc graph which is not normal is given in Fig. 6. This concept was studied in [20,33,39], but the terminology 'normal' was introduced in [51]. The characterization of normal circular-arc graphs by minimal forbidden induced subgraphs is still open.

3.3. Circular-arc graphs restricted to different graph classes

Recall that the class of interval graphs was characterized by forbidden induced subgraphs by Lekkerkerker and Boland (cf. Theorem 1). As mentioned in the Introduction, the corresponding problem of characterizing circular-arc graphs by forbidden induced subgraphs is a long-standing open problem [44,65,69]. Some known minimally non-circular-arc graphs are depicted in Fig. 7. In this subsection, we survey some partial results in this direction obtained by restricting the problem to different graph classes.

A graph is *co-bipartite* if its complement is bipartite or, equivalently, if its vertex set is the set union of two complete sets. Co-bipartite circular-arc graphs arise as an interesting subclass of circular-arc graphs. In [70], Tucker was the first to observe that in any circular-arc model of a co-bipartite circular-arc graph there exist two points p and p' of the circle such that each arc of the model contains at least one of them. This property was generalized by Hell and Huang [38] to the whole class of circular-arc graphs.

Theorem 15 ([38])**.** *Let G be a circular-arc graph whose vertex set can be partitioned into k completes and let $\mathcal{M}$ be a circular-arc model of G. Then, there are k points on the circle such that every arc of $\mathcal{M}$ contains at least one of these k points.*

Based on Tucker's observation, Spinrad [62] proved that for any partition into two cliques C_1 and C_2 of a co-bipartite circular-arc graph, there exists a circular-arc model $\mathcal{M}$ and two points c_1 and c_2 on the circle such that each arc whose corresponding vertex belongs to C_1 meets c_1 and not c_2, and each arc whose corresponding vertex belongs to C_2 meets c_2 and not c_1. Hell and Huang [38] observed that this fact can be used to color the edges of a co-bipartite circular-arc graph red and blue in such a way that no two opposite edges of an induced four-cycle are colored with the same color. Let R and B be

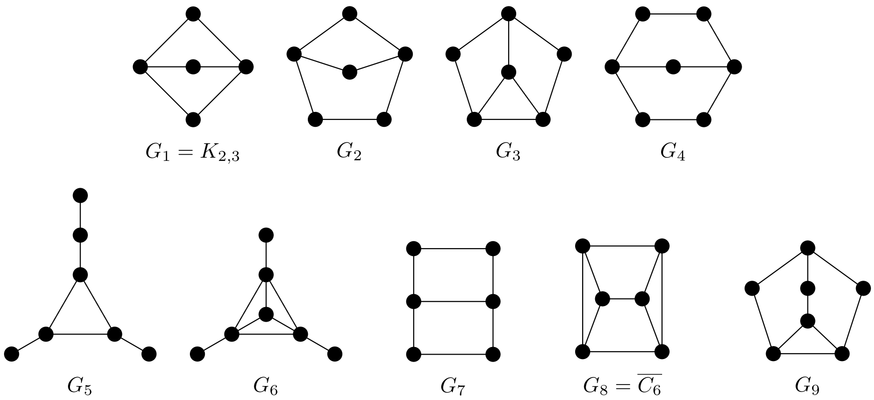

Fig. 7. Some minimally non-circular-arc graphs.

the two parts into which the points c_1 and c_2 divide the circle. In fact, if U and U' are two cliques covering the vertices of a co-bipartite circular-arc graph G, the edges of G can be colored red and blue as follows: let the edges joining two vertices in U be colored red, the edges joining two vertices in U' be colored blue, and each of the remaining edges be colored red or blue depending on whether the arcs in $\mathcal{M}$ corresponding to its endpoints have some common point in R or B, respectively (if the arcs contain common points in both R and B, the edges would be colored by red). Conversely, using a lexicographic order technique developed in [37], Hell and Huang proved that the only co-bipartite graphs admitting such a bicoloring of the edges are precisely those that are circular-arc.

Theorem 16 (*[38]*). *If G is a co-bipartite graph, then G is a circular-arc graph if and only if its edges can be colored red and blue so that no induced four-cycle has two opposite edges of the same color.*

A forbidden induced subgraph characterization of circular-arc graphs within co-bipartite graphs was given by Trotter and Moore [65]. They employed the following notation to describe the complements of the corresponding families of forbidden induced subgraphs. Let $\mathcal{F} = \{\Gamma_j : 1 \leq j \leq k\}$ be a family of subsets of $\{1, 2, \ldots, l\}$. Define $G_{\mathcal{F}}$ to be the bipartite graph whose stable set partition is (X, Y), $X = \{x_1, \ldots, x_l\}$ and $Y = \{y_1, \ldots, y_k\}$ such that $x_i y_j \in E(G_{\mathcal{F}})$ if and only if $i \in \Gamma_j$. For example, if $n \geq 3$ and $\mathcal{F}$ is the family $\mathcal{C}_n$ of Table 1, then $G_{\mathcal{F}} = C_{2n}$. Note that each of the families described in Table 1 is infinite, with the only exception of $\{\mathcal{G}_n\}$. Trotter and Moore used the theory of partially ordered sets to deduce the list of all minimal forbidden induced subgraphs for the class of circular-arc graphs within the complements of bipartite graphs.

Theorem 17 (*[65]*). *If G is a co-bipartite graph, G is a circular-arc graph if and only if G contains no induced complement of the graph $G_{\mathcal{F}}$ for any of the families $\mathcal{F}$ listed in Table 1.*

List homomorphisms generalize list colorings in the following way: Given graphs G, H, and lists $L(v) \subseteq V(H)$, $v \in V(G)$, a *list homomorphism of G to H with respect to the lists L* is a mapping $f : V(G) \to V(H)$, such that $f(u)f(v) \in E(H)$ for all $uv \in E(G)$, and $f(v) \in L(v)$ for all $v \in V(G)$. The list homomorphism problem for a fixed graph H asks whether or not an input graph G together with lists $L(v) \subseteq V(H)$, $v \in V(G)$, admits a list homomorphism with respect to L. In [23], Feder et al. proved that the list homomorphism problem is polynomial-time solvable if the complement of H is a co-bipartite circular-arc graph, and is NP-complete otherwise. For the purpose of the proof they provided a new characterization of co-bipartite circular-arc graphs. To present such a result we need to introduce new concepts. An *edge-asteroid* in a bipartite graph with a stable set partition (X, Y) is a set of $2k + 1$ edges $(k \geq 1) u_0v_0, u_1v_1, \ldots, u_{2k}v_{2k}$ $(u_i \in X, v_i \in Y)$ and $2k + 1$ paths $P_{0,1}, P_{1,2}, \ldots, P_{2k,0}$ where each $P_{i,i+1}$ joins u_i to u_{i+1}, such that there is no edge between $\{u_i, v_i\}$ and $\{v_{i+k}, v_{i+k+1}\} \cup V(P_{i+k,i+k+1})$ (subscripts should be understood modulo $2k + 1$). An edge-asteroid having no edge between $\{u_0, v_0\}$ and $\{v_1, \ldots, v_{2k}\} \cup P_{1,2} \cup P_{2,3} \cup \cdots \cup P_{2k-1,2k}$ is called a *special edge-asteroid*. A bipartite graph is said to be *chordal bipartite* [34] if it does not contain any induced cycle of length at least six. Note that these graphs are not chordal, because induced C_4s are allowed. Feder, Hell, and Huang proved the characterization below by showing that each co-bipartite graph containing any of the forbidden induced subgraphs for the class of circular-arc graphs in the statement of Theorem 17 contains an special edge-asteroid.

Theorem 18 (*[23]*). *Let G be a bipartite graph. $\overline{G}$ is a circular-arc graph if and only if G is chordal bipartite and contains no special edge-asteroids.*

A graph G is a *multiple* of another graph H if G can be obtained from H by replacing each vertex x of H by a complete graph Θ_x and joining all possible vertices of different complete graphs Θ_x, Θ_y if and only if x and y are adjacent in H. Bang-Jensen and Hell [3] proved by induction the following structural result.

Theorem 19 (*[3]*). *Let G be a connected graph containing no induced claw, net, C_4, or C_5. If G contains a tent as induced subgraph, then G is a multiple of a tent.*

Table 1
List of the families that allow us to define the minimal forbidden induced subgraphs of circular-arc graphs within complements of bipartite graphs. Each family $\mathcal{F}$ in this list determines a bipartite graph $G_{\mathcal{F}}$ (as defined just before Theorem 17), whose complement is a minimally non-circular-arc graph.

$\mathcal{C}_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$
$\mathcal{C}_4 = \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$
$\mathcal{C}_5 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}\}$
…
$\mathcal{T}_1 = \{\{1,2\},\{2,3\},\{3,4\},\{2,3,5\},\{5\}\}$
$\mathcal{T}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{2,3,4,6\},\{6\}\}$
$\mathcal{T}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,6\},\{2,3,4,5,7\},\{7\}\}$
…
$\mathcal{W}_1 = \{\{1,2\},\{2,3\},\{1,2,4\},\{2,3,4\},\{4\}\}$
$\mathcal{W}_2 = \{\{1,2\},\{2,3\},\{3,4\},\{1,2,3,5\},\{2,3,4,5\},\{5\}\}$
$\mathcal{W}_3 = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,2,3,4,6\},\{2,3,4,5,6\},\{6\}\}$
…
$\mathcal{D}_1 = \{\{1,2,5\},\{2,3,5\},\{3\},\{4,5\},\{2,3,4,5\}\}$
$\mathcal{D}_2 = \{\{1,2,6\},\{2,3,6\},\{3,4,6\},\{4\},\{5,6\},\{2,3,4,5,6\}\}$
$\mathcal{D}_3 = \{\{1,2,7\},\{2,3,7\},\{3,4,7\}\{4,5,7\},\{5\},\{6,7\},\{2,3,4,5,6,7\}\}$
…
$\mathcal{M}_1 = \{\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,4,6\},\{2,4\},\{2,5\}\}$
$\mathcal{M}_2 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,7\}\}$
$\mathcal{M}_3 = \{\{1,2,3,4,5,6,7,8,9\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,5,6,8,10\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,9\}\}$
…
$\mathcal{N}_1 = \{\{1,2,3\},\{1\},\{1,2,4,6\},\{2,4\},\{2,5\},\{6\}\}$
$\mathcal{N}_2 = \{\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,7\},\{8\}\}$
$\mathcal{N}_3 = \{\{1,2,3,4,5,6,7\},\{1,2,3,4,5\},\{1,2,3\},\{1\},\{1,2,3,4,5,6,8,10\},\{1,2,3,4,6,8\},\{1,2,4,6\},\{2,4\},\{2,9\},\{10\}\}$
…
$\mathcal{G}_1 = \{\{1,3,5\},\{1,2\}\{3,4\},\{5,6\}\}$
$\mathcal{G}_2 = \{\{1\},\{1,2,3,4\},\{2,4,5\},\{2,3,6\}\}$
$\mathcal{G}_3 = \{\{1,2\},\{3,4\},\{5\},\{1,2,3\},\{1,3,5\}\}$

Since each multiple of a tent is a proper circular-arc graph, Theorem 19 allowed them to provide the following description of all the minimal forbidden induced subgraphs for proper circular-arc graphs within the class of connected chordal graphs.

Theorem 20 (*[3]*). *Let G be a connected chordal graph. Then, G is a proper circular-arc graph if and only if it contains no induced claw or net.*

The characterization of Lekkerkerker and Boland of interval graphs by minimal forbidden induced subgraphs (cf. Theorem 1) yields some minimal forbidden induced subgraphs for the class of circular-arc graphs as follows. Given a minimal forbidden induced subgraph H for the class of interval graphs, if H is a non-circular-arc graph, then H is minimally non-circular-arc. Otherwise, if H is a circular-arc graph, then $H \cup K_1$ is a minimally non-circular-arc graph, and furthermore all disconnected minimally non-circular-arc graphs are obtained this way. Since the umbrella, net, n-tent for all $n \geq 3$, and C_n for all $n \geq 4$ are circular-arc graphs, but the bipartite claw and n-net for all $n \geq 3$ are not, this observation and Theorem 1 lead to the following result.

Corollary 21 (*[65]*). *The following graphs are minimally non-circular-arc graphs: bipartite claw, net $\cup K_1$, n-net for all $n \geq 3$, umbrella $\cup K_1$, n-tent $\cup K_1$ for all $n \geq 3$, and $C_n \cup K_1$ for every $n \geq 4$. Any other minimally non-circular-arc graph is connected.*

In [4], the graphs listed in Corollary 21 are called *basic* minimally non-circular-arc graphs. Any other minimally non-circular-arc graph is called *non-basic*. The following result, which gives a structural property for all non-basic minimally non-circular-arc graphs, can be deduced from Theorem 1 and Corollary 21.

Corollary 22 (*[4]*). *If G is a non-basic minimally non-circular-arc graph, then G has an induced subgraph H that is isomorphic to an umbrella, a net, a j-tent for some $j \geq 3$, or C_j for some $j \geq 4$. In addition, each vertex v of $G - H$ is adjacent to at least one vertex of H.*

The theorem below gives necessary conditions for some minimally non-circular-arc graph having an induced cycle H of length at least 4 regarding the neighborhoods of those vertices of the graph not belonging to H.

Theorem 23 (*[4]*). *Let G be a minimally non-circular-arc graph. If G is not isomorphic to $K_{2,3}$, G_2, G_3, G_4, or $C_j \cup K_1$, for $j \geq 4$, then for every induced cycle H of length at least 4 of G and for each vertex v of $G - H$, either v is complete to H or $N_H(v)$ induces a nonempty path in H.*

The above results are used to characterize those cographs that are circular-arc graphs. Notice that the only basic minimally non-circular-arc cograph is $C_4 \cup K_1$. By Corollary 22, it follows that if H is a non-basic minimally non-circular-arc cograph it contains an induced C_4. In [4] it is proved that if H is a cograph whose all proper induced subgraphs are circular-arc and H is different from $K_{2,3}$ and $C_4 \cup K_1$, then H is a $\{3K_1.P_4\}$-free graph. Since $\{3K_1, P_4\}$-free graphs have simple circular-arc models using semicircular arcs, the following characterization is proved.

Theorem 24 ([4]). *Let G be a cograph. Then, G is a circular-arc graph if and only if G contains no induced $K_{2,3}$ or $C_4 \cup K_1$.*

A *paw-free* graph is a graph with no induced paw. Paw-free graphs were studied in [55]. In order to characterize which paw-free graphs are circular-arc graphs notice that the only basic minimally non-circular-arc paw-free graphs are the bipartite claw and $C_j \cup K_1$ for each $j \geq 4$. In addition, by Theorem 23, if H is a non-basic minimally non-circular-arc, then H contains an induced C_j, for some $j \geq 4$. In [4], it was proved that if H is a paw-free graph containing an induced C_j, for some $j \geq 4$, such that all proper induced subgraphs of H are circular-arc graphs and H is different from the bipartite claw, $K_{2,3}$, G_2, G_4, G_7, and $C_j \cup K_1$, for each $j \geq 4$, then either H is $\{3K_1, P_4\}$-free or a multiple of a graph that arises from C_j by adding pendant vertices adjacent to the C_j. Since it is not hard to show that such graphs H have a circular-arc model, the following holds.

Theorem 25 ([4]). *Let G be a paw-free graph. Then, G is a circular-arc graph if and only if G contains no induced bipartite claw, $K_{2,3}$, G_2, G_4, G_7, or $C_j \cup K_1$, for any $j \geq 4$.*

A graph is *claw-free chordal* if it contains no induced claw and it is chordal. Claw-free graphs are widely studied in the literature, see for example [12] or [57]. Besides, as proper circular-arc graphs are claw-free, the study of claw-free chordal graphs arises naturally from the characterization of proper circular-arc graphs within the class of chordal graphs. Notice that the only basic minimally non-circular-arc claw-free chordal graphs are net $\cup K_1$ and tent $\cup K_1$. In addition, by Corollary 22, if H is a non-basic minimally non-circular-arc graph, then H contains an induced net or tent. On the one hand, Bang-Jensen and Hell [3] proved that if a graph is claw-free chordal and contains no induced net but contains an induced tent, then G is the multiple of a tent. On the other hand, it was proved in [4] that if G is a claw-free chordal graph containing an induced net, G is different from net $\cup K_1$, G_5, and G_6, and each proper induced subgraph of G is a circular-arc graph, then G is a multiple of the net or a multiple of the graph that arises from the tent by attaching a pendant vertex adjacent to one of the vertices of degree 2 of the tent. Since the multiple of a circular-arc graph is clearly also a circular-arc graph, the following statement holds.

Theorem 26 ([4]). *Let G be a claw-free chordal graph. Then, G is a circular-arc graph if and only if G contains no induced tent $\cup K_1$, net $\cup K_1$, G_5 or G_6.*

A *diamond-free* graph is a graph with no induced diamond. Diamond-free graphs have been extensively studied. (See, for example, [10,14,71].) By Theorem 1, if G is a diamond-free graph that is not an interval graph, it must contain an induced bipartite claw or an induced cycle of length at least 4. Since the bipartite claw is not a circular-arc graph, this means that each forbidden induced subgraph of the class of circular-arc graphs different from the bipartite claw contains an induced net or a cycle of length at least 4. By analyzing the possible neighborhoods of the vertices not belonging to such a net or cycle using Theorem 23, the following is proved in [4].

Theorem 27 ([4]). *Let G be a diamond-free graph. Then, G is a circular-arc graph if and only if G contains no induced bipartite claw, tent $\cup K_1$, $K_{2,3}$, G_2, G_3, G_4, G_5, G_6, G_7, $\overline{C_6}$, G_9, or $C_j \cup K_1$, for any $j \geq 4$.*

The proofs in [4] show that, for the classes analyzed there (cographs, paw-free graphs, claw-free chordal graph, and diamond-free graphs), all circular-arc graphs involved are also normal. So, the characterizations obtained for circular-arc graphs also hold for normal circular-arc graphs. Moreover, the following result can be deduced.

Corollary 28 ([4]). *If H is a circular-arc graph but minimally non-normal, then H contains an induced diamond, an induced P_4, an induced paw, and either an induced claw or an induced cycle of length at least 4.*

4. Circle graphs

A graph G is a *circle graph* if there exists a one-to-one function $f : V \to L$ $(f(v) = C_v)$, where $L = \{C_v\}_{v \in V(G)}$ is a family of chords on a circle, whose extremes are all different, such that $uv \in E$ if and only if $u \neq v$ and $C_u \cap C_v \neq \emptyset$. L is called a *circle model* of G.

In [21] Even and Itai studied the problem of realizing a given permutation through networks of queues in parallel and through a network of stacks in parallel. In that work, circle graphs are defined for the first time. The problem of determining the number of queues needed to realize the given permutation can be translated into that of determining the chromatic number of a permutation graph (a subclass of circle graphs) and thus this problem is polynomial-time solvable [33]. Determining the minimum number of parallel stacks necessary to realize the given permutation depends on whether or not unloading before a completion of the parallel stacks is accepted. If it is accepted, the problem is also modeled by means of permutation graphs, translating the original problem into that of computing the chromatic number of a permutation graph. If unloading is not accepted, the problem is translated into the problem of determining the chromatic number of a circle graph. It is worth pointing out that determining the chromatic number of a circle graph is an NP-complete problem [28].

Naji presented in [54] an $O(n^7)$-time recognition algorithm for circle graphs based on solving an associated system of equations on GF(2) (the two-element field). An $O(n^5)$-time recognition algorithm strongly based on split decomposition was presented by Bouchet in [6]. Following this line of work Gabor et al. [26] and Spinrad [63] presented quadratic-time

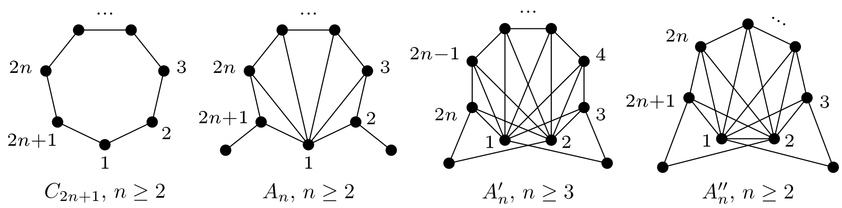

Fig. 8. Some minimal forbidden induced subgraphs for comparability graphs.

recognition algorithms improving Bouchet's result. Recently, Paul [58] presented an $O((n+m)\alpha(n+m))$-time algorithm also based on split decomposition, where α is the inverse Ackermann function.

Circle graphs are equivalent to two classes of intersection graphs, namely alternance graphs and overlap graphs. A *double occurrence word* is a finite string of symbols in which each symbol appears precisely twice. Let $\Gamma = (\pi_1, \pi_2, \ldots, \pi_{2n})$ be a double occurrence word. The *shift* operation on Γ transforms Γ into $(\pi_{2n}, \pi_1, \pi_2, \ldots, \pi_{2n-1})$. The *reverse* operation transforms Γ into $\widetilde{\Gamma} = (\pi_{2n}, \pi_{2n-1}, \ldots, \pi_2, \pi_1)$. With each double occurrence word Γ we associate a graph $G[\Gamma]$ whose vertices are the symbols in Γ and in which two vertices are adjacent if and only if the corresponding symbols appear precisely once between the two occurrences of the other. Clearly, a graph is circle if and only if it is isomorphic to $G[\Gamma]$ for some double occurrence word. Those graphs that are isomorphic to $G[\Gamma]$ for some double occurrence Γ are also called *alternance graphs*. A graph G is *overlap interval* if there exists a bijective function $f : V \to \mathcal{I}$ $(f(v) = I_v)$ where $\mathcal{I} = \{I_v\}_{I \in V(G)}$ is a family of intervals on the real line, such that $uv \in E$ if and only if I_u and I_v overlap; i.e., $I_u \cap I_v \neq \emptyset$, $I_u \not\subseteq I_v$ and $I_v \not\subseteq I_u$. It is well known that circle graphs and overlap interval graphs are the same class (see, for instance, [33, p. 244]).

If G is a circle graph isomorphic to $G[\Gamma]$ for some double occurrence word Γ, then G is said to be *uniquely representable* if for any double occurrence word Δ such that G is isomorphic to $G[\Delta]$, Δ arises from Γ by applying shift and reverse operations. A circle graph G is a prime graph (under the split decomposition) if and only if it is uniquely representable [6]. For instance, all induced cycles are prime circle graphs.

In Section 4.1, we review the forbidden induced subgraph characterization of an important subclass of the class of circle graphs: permutation graphs. In Section 4.2, we present Bouchet's characterization of circle graphs in terms of forbidden induced subgraphs via local complementation, and Geelen and Oum's characterization in terms of pivoting. In Section 4.3, we revisit Fraysseix's characterizations of circle graphs as cocyclic-path intersection graphs and fundamental graphs of planar graphs. In Section 4.4, Naji's characterization of circle graphs by means of a system of equations in GF(2) is presented. In Section 4.5, we revisit different subclasses of circle graphs arising by restricting the circle model in different ways. Finally, in Section 4.6, we present some partial solutions to the problem of characterizing circle graphs via forbidden induced subgraphs by restricting it to different graph classes.

4.1. Permutation graphs

Since comparability graphs are strongly related to permutation graphs we will first introduce them. Recall that a digraph D is an *orientation* of a graph G if for every $u, v \in V(G)$ it holds that $uv \in E(G)$ if and only if either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. An orientation is *transitive* if it is a transitive binary relation on $A(D)$; i.e., if $(u, v) \in A(D)$ and $(v, w) \in A(D)$, then $(u, w) \in A(D)$. A graph is said to be *comparability* if it has a transitive orientation. Comparability graphs were characterized by Gallai by means of a list of forbidden induced subgraphs [27].

Theorem 29 ([27]). *A graph is a comparability graph if and only if it does not contain as an induced subgraph any graph in Fig. 8 and its complement does not contain as an induced subgraph any graph in Fig. 9.*

Let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation of $V_n = \{1, \ldots, n\}$; i.e., π is a one-to-one function. By $G(\pi)$ we denote the graph whose vertex set is V_n and whose edge set is formed by those unordered pairs ij satisfying $i < j$ and $\pi^{-1}(i) > \pi^{-1}(j)$. A graph G is defined to be a *permutation graph* if there exists a permutation π such that the graph $G(\pi)$ is isomorphic to G. Notice that if you place $\{1, \ldots, n\}$ in two parallel vertical copies of the real line and join i of the line on the left with $\pi(i)$ in the line on the right, the intersection graph of these segments is isomorphic to $G(\pi)$. Therefore, permutation graphs are a subclass of circle graphs. Even et al. in [22] presented a characterization of permutation graphs that shows the relationship between this class and comparability graphs.

Theorem 30 ([22]). *A graph G is a permutation graph if and only if G and $\overline{G}$ are comparability graphs.*

Therefore, the characterization of comparability graphs in [27] leads immediately to a forbidden induced subgraph characterization of permutation graphs.

Corollary 31 ([22,27]). *A graph G is a permutation graph if and only if G and $\overline{G}$ do not contain as an induced subgraph any graph in Figs. 8 and 9.*

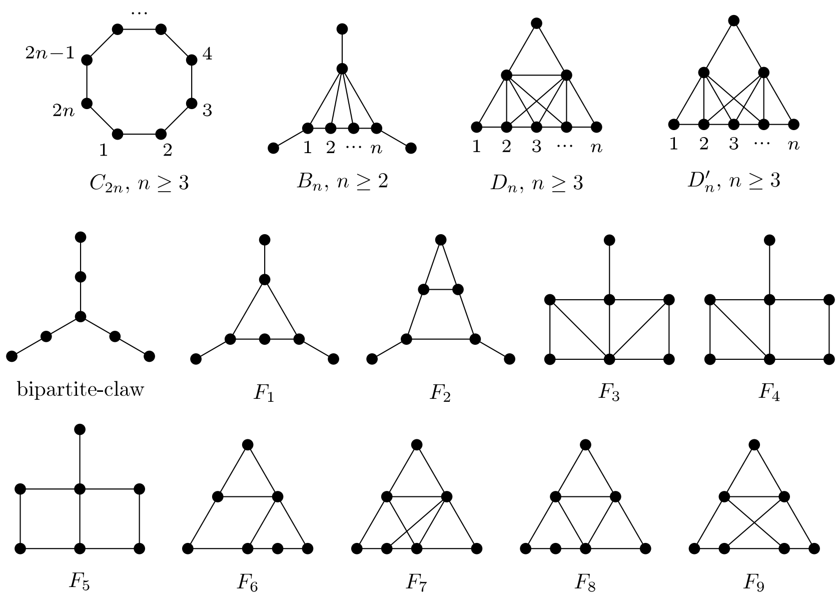

Fig. 9. Some graphs whose complements are minimal forbidden induced subgraphs for comparability graphs.

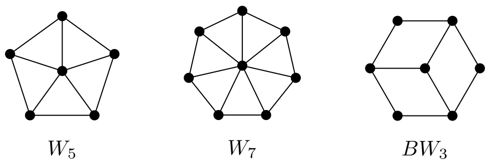

Fig. 10. W_5, W_7, and BW_3.

The following result characterizes permutation graphs as circle graphs having a special circle model.

Theorem 32 (*[33, P. 252]*)**.** *Permutation graphs are exactly those circle graphs that have a circle model admitting an equator, i.e. an additional chord meeting all the chords of the model.*

As an immediate consequence, we obtain the following corollary.

Corollary 33. *$G + K_1$ is a circle graph if and only if G is a permutation graph.*

The following result is a consequence of the corollary above.

Corollary 34. *The join $G = G_1 + G_2$ is a circle graph if and only if both G_1 and G_2 are permutation graphs.*

4.2. Circle graphs characterizations via local complementation

Recall that alternance graphs and circle graphs are equivalent classes. Given a double alternance word Γ, we denote by $\widetilde{\Gamma}$ the word that arises by traversing Γ from right to left. For instance, if $\Gamma = abcadcd$, then $\widetilde{\Gamma} = dcdacba$.

Given a graph G and a vertex v of G. The *local complement of G at v*, denoted by $G * v$, is the graph that arises from G by replacing $N(v)$ by its complementary graph. This operation is strongly linked with circle graphs; namely, if G is a circle graph, then $G * v$ is a circle graph. This is because, if a represents the vertex v in Γ and $\Gamma = AaBaC$ where A, B and C are subwords of Γ, then $G[Aa\widetilde{B}aC]$ is a double alternance model for $G * v$. Bouchet proved the following theorem.

Theorem 35 (*[8]*)**.** *Let G be a graph. G is a circle graph if and only if any graph locally equivalent to G has no induced subgraph isomorphic to W_5, W_7, or BW_3 (see Fig. 10).*

It was also Bouchet who proved the following property of circle graphs. Let $G = (V, E)$ and let $A = \{A_{vw} : v,\ w \in V\}$ be an antisymmetric integral matrix [7]. For $W \subseteq V$, we denote $A[W] = \{A_{vw} : v,\ w \in V\}$. Matrix A satisfies *property α* if the following property (related to unimodularity) holds: $\det(A[W]) \in \{-1, 0, 1\}$ for all $W \subseteq V$. Graph G is *unimodular* if there is an orientation of G such that the resulting digraph satisfies property α. Bouchet proved that every circle graph admits such an orientation [7].

Geelen and Oum [31] gave a new characterization of circle graphs in terms of pivoting. The result of pivoting a graph G with respect to an edge uv is the graph $G \times uv = G * u * v * u$ (where $*$ stands for local complementation). A graph G' is *pivot equivalent* to G if G' arises from G by a sequence of pivoting operations. They proved, with the aid of a computer, that G is a circle graph if and only if each graph that is pivot equivalent to G contains none of 15 prescribed induced subgraphs.

4.3. Circle graphs as cocyclic-path intersection graphs and fundamental graphs of planar graphs

In [17] Fraysseix presented a characterization of circle graphs, which leads to a novel interpretation of circle graphs as the intersection graphs of induced paths of a given graph. A *cocycle* of a graph G with vertex set V is the set of edges joining a vertex of V_1 to a vertex of V_2 for some bipartition (V_1, V_2) of V. A *cocyclic-path* is an induced path whose set of edges constitutes a cocycle. A *cocyclic-path intersection graph* [17] is a simple graph with vertex set being a family of cocyclic-paths of a given graph, two vertices being adjacent if and only if the corresponding cocyclic-paths have an edge in common. Notice that the definition is restricted to those graphs covered by cocyclic-paths any two of which have at most a common edge. Fraysseix proved the following characterization of circle graphs as cocyclic-path intersection graphs.

Theorem 36 (*[17]*). *Let G be a graph. G is a circle graph if and only if G is a cocyclic-path intersection graph.*

The *only if part* of the proof is straightforward and will be sketched next. Let G be a circle graph and let $\mathcal{C}(G)$ be a circle model of it. Assume, without loss of generality, that there are no three chords of $\mathcal{C}(G)$ having a point in common. It is easy to prove by induction that the faces defined by the chords in $\mathcal{C}(G)$ can be bicolored, say black and white, in such a way that faces with a segment in common receive different colors. Let us construct from G the cocyclic-path intersection graph $\widetilde{G}$. Let the vertices of $\widetilde{G}$ be the faces colored black and each intersection between two chords in $\mathcal{C}(G)$ give rise to an edge in $\widetilde{G}$ whose endpoints are the black faces having in their boundaries the intersection point of the two chords. The union of the edges representing the intersections of a given chord with other chords is a cocyclic-path. Consequently, the union of these cocyclic-paths is the graph $\widetilde{G}$ and the intersection graph of these cocyclic-paths of $\widetilde{G}$ is isomorphic to G.

Let E be a set and A a collection of subsets of E. The minimal nonempty subsets of E that can be expressed as symmetric differences of subsets in A define the circuits of a binary matroid. We refer to that matroid as the *matroid generated by A*. For definitions and results on matroids, the reader is referred to [56]. Given a graph G and a spanning tree T of G, the *fundamental graph of G restricted to T* is the bipartite graph whose vertex set is formed by the edges of G and two vertices e and e' are adjacent if and only if $e \in E(T)$, $e \notin E(T)$ and $T - e + e'$ is a spanning tree of G. Fraysseix proved in [17] that the binary matroid generated by the sets $\{N(x) : x \in X\}$ and $\{N(y) : y \in Y\}$ of a bipartite graph G whose classes are X and Y is graphic if and only if G is a circle graph. This result is equivalent to the following theorem.

Theorem 37 (*[17]*). *Let G be a bipartite graph. G is a circle graph if and only if G is a fundamental graph of a planar graph.*

4.4. Circle graphs as solutions of systems of equations in $GF(2)$

We will present a result that follows from an application of Bouchet's result already reviewed. Naji characterized circle graphs by solving a system of linear equations in $GF(2)$, obtaining the first polynomial time recognition algorithm for the class [54]. The proof of this characterization is hard. However, Gasse [29] presented a shorter and elegant proof of this result by using Bouchet's characterization for circle graphs (see Theorem 35). Given a graph G we define a system $\mathcal{S}(G)$ whose variable set is $\{\alpha(x, y) : x, y \in V(G), x \neq y\}$ as follows:

$$\mathcal{S}(G) = \begin{cases} \alpha(x, y) + \alpha(y, x) = 1 & \forall xy \in E(G), \\ \alpha(x, y) + \alpha(x, z) = 0 & \forall xy \notin E(G),\ xz \notin E(G),\ yz \in E(G), \\ \alpha(x, y) + \alpha(x, z) + \alpha(y, z) + \alpha(z, y) = 1 & \forall xy \in E(G),\ xz \in E(G),\ yz \notin E(G). \end{cases}$$

A graph G is said to be consistent if $\mathcal{S}(G)$ admits a solution in GF(2). Let G be a circle graph and $\mathcal{C}$ a circle model for G. We orient each chord in $\mathcal{C}$. The orientation of a chord γ of $\mathcal{C}$ allows us to define an initial point γ_i and an ending point γ_e. Let γ be a chord of $\mathcal{C}$ and a point p of the circle. We say that p is *on the left of* γ if p is met when traversing the circle from γ_i to γ_e in the clockwise direction; otherwise p is said to be *on the right of* γ. Let v be a vertex of G, we denote by γ_v the chord of $\mathcal{C}$ corresponding to v. We can build up a solution of $\mathcal{S}(G)$ as follows:

- Given the edge $xy \in E(G)$. Set $\alpha(x, y) = 1$ if and only if the initial point of γ_y is on the left of γ_x.
- Given the edge $xy \notin E(G)$. Set $\alpha(x, y) = 1$ if and only if both points of γ_y are on the left of γ_x.

Gasse proved the following two lemmas.

Lemma 38 (*[29]*). *W_5, W_7 and BW_3 are not consistent.*

Lemma 39 (*[29]*). *Let G be a consistent graph and v a vertex of G, then $G * v$ is consistent.*

Naji's theorem below follows immediately from Theorem 35 and Lemmas 38 and 39.

Theorem 40 (*[54]*). *Let G be a graph. $\mathcal{S}(G)$ is consistent if and only if G is a circle graph.*

Notice that the above theorem together with Theorem 37 give a test to determine if a bipartite graph G is planar. Construct a fundamental graph of G, then check if the system $\mathcal{S}(G)$ is consistent.

Consider the reduced system $\mathcal{S}_R(G)$ formed only by the first type of equations of $\mathcal{S}(G)$. Note that if $\mathcal{S}(G)$ and $\mathcal{S}_R(G)$ were equivalent systems, then the complement graph of any circle graph would be a circle graph. Nevertheless, this assertion is false, because, for example, C_6 is a circle graph and $\overline{C_6}$ is not a circle graph. In [9], Bouchet was able to prove the following refinement of Naji's theorem for bipartite graphs.

Theorem 41 ([9]). *Let G be a bipartite graph. If $\mathcal{S}_R(G)$ has a solution then $\mathcal{S}(G)$ also has a solution.*

As a corollary of Theorem 41, the result below is deduced.

Corollary 42 ([9]). *Let G be a bipartite graph. If G is a circle graph, then $\overline{G}$ is a circle graph.*

Naji's theorem is hard to prove and Theorem 41 and Corollary 42 were deduced from it. Also in [9], Bouchet posed the following problem: Is there a direct proof of Corollary 42?

4.5. Circle graphs having restricted circle models

A graph G is a *unit circle* graph if it admits a circle model in which all the chords have the same length. This class coincides with the class of unit circular-arc graphs (i.e., the intersection graphs of a family of arcs on a circle, all of the same length) [19]. Tucker [69] gave a characterization by minimal forbidden induced subgraphs for this class (see Theorem 13). Recently, linear and quadratic-time recognition algorithms for the recognition of this class have been proposed [51,20].

The concept of *Helly circle graph* is presented in [19]. A graph belongs to this class if it has a circle model whose chords are pairwise different and satisfy the Helly property (i.e., every subset of pairwise intersecting chords has a common point). In [19], it was conjectured that a circle graph is a Helly circle graph if and only if it is diamond-free. This conjecture was recently settled affirmatively in [16], yielding a polynomial-time recognition algorithm for Helly circle graphs.

Let G be a circle graph isomorphic to $G[\Gamma_G]$ where Γ_G is a double occurrence word. Let H be a subgraph G induced by a subword Γ_H of Γ_G. H is said to be *convex* if, for every subword *abccba* of Γ_G, if a and c are letters of Γ_H then b is a letter of Γ_H. A *clique* is defined to be *non-trivial* if and only if it has at least two vertices. An induced subgraph H of G is *clique maximal* if every non-trivial clique of H is a clique of G. An induced subgraph H of G is *almost component maximal* if at most one connected component of H is not a maximal component of G. An induced subgraph H of G is said to be *convenient* if it is convex, clique maximal, and almost component maximal. A *mixed Helly model* (G, H) is a circle model of G where the induced circle model of H is Helly.

Given a diamond-free circle Helly graph, the proof of the conjecture follows from the fact that (G, G) admits a mixed Helly model. To show that (G, G) admits such a model, the authors of [16] consider a maximum induced subgraph H of G such that H is convenient and (G, H) admits a mixed circle model and assume, by way of contradiction, that H is different from G. Finally, they prove that there exists a convenient induced subgraph H' containing H as a proper induced subgraph. The result is based on the following technical lemma.

Lemma 43 ([16]). *Let G be a graph and H a convenient proper induced subgraph of G. Then, there exists a vertex $u \in V(G) - V(H)$ such that $G[V(H) \cup \{u\}]$ remains convex. Furthermore, if H has a component that is a proper subgraph of a connected component C of G, then there exists such a vertex u in C.*

Finally the authors distinguish two cases: the case where every component of H is a component of G and the case where one connected component of H is a proper subgraph of a component of G. Recall that G is a diamond-free circle graph. In the first case, they consider a vertex $u \in V(G) - V(H)$ such that $H' = G[V(H) \cup \{u\}]$ is convex. It is clear that H' is almost component maximal. Since the non-trivial cliques of H' are exactly the non-trivial cliques of H, H' is clique maximal and thus convenient. Then, since the non-trivial cliques of H' are exactly the non-trivial cliques of H, it is clear that the mixed Helly model of (G, H) gives rise to a Helly model of (G, H').

Thus, it can be assumed that H has a connected component that is a proper subgraph of a connected component of G. Lemma 43 ensures that there exists at least one vertex $u \in V(G) - V(H)$ such that $H' = G[V(H) \cup \{u\}]$ is convex and almost component maximal (i.e., u has neighbors in H). The proof ends when it is proved that (G, H') admits a mixed Helly model. Therefore, the theorem below follows.

Theorem 44 ([16]). *Let G be a circle graph. G is Helly circle if and only if G is diamond-free.*

In connection with the class of Helly circle graphs, the class of *unit Helly circle graphs* was introduced in [5] as the class of those circle graphs G admitting a circle model which are simultaneously unit and Helly. The theorem below characterizes unit Helly circle graphs by minimal forbidden induced subgraphs.

Theorem 45 ([5]). *Let G be a graph. Then, the following assertions are equivalent:*

(i) *G is a unit Helly circle graph.*
(ii) *G contains no induced claw, paw, diamond, or $C_n \cup K_1$ for any $n \geq 3$.*
(iii) *G is a chordless cycle, a complete graph, or a disjoint union of chordless paths.*

The proof is of a geometric nature and relies on properties of tangent lines to a circle.

4.6. Partial characterizations by forbidden induced subgraphs

Recall that the problem of characterizing circle graphs by forbidden induced subgraphs remains open. In this section, we present some partial characterization of this class within different graph classes.

As a consequence of Theorem 35, the following result can be proved.

Theorem 46 (*[5]*). *Let G be a graph. If G is not a circle graph, then any graph H that arises from G by edge subdivisions is not a circle graph.*

A *prism* is a graph that consists of two disjoint triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ linked by three vertex disjoint paths P_1, P_2, P_3 where P_i links a_i and b_i for $i = 1, 2, 3$, and such that all the internal vertices of P_1, P_2, and P_3 have degree 2. The graph $\overline{C_6}$ is a prism where each triangle is linked by induced path P_1, P_2 and P_3 having just one edge each. This graph is locally equivalent to W_5, so by Theorem 35, $\overline{C_6}$ is not a circle graph. Besides, since every prism arises from $\overline{C_6}$ by edge subdivision, Theorem 46 implies that prisms are not circle graphs.

A graph is a *linear domino* [45] if and only if it contains no induced diamond and no induced claw. The following theorem characterizes those linear domino graphs that are circle graphs.

Theorem 47 (*[5]*). *Let G be a linear domino graph. Then, G is a circle graph if and only if G contains no induced prisms.*

The proof given in [5] is based on the fact that circle graphs are closed under split decomposition [6]. As a corollary of the above theorem, the following partial characterization of Helly circle graphs is obtained.

Corollary 48 (*[5]*). *Let G be a claw-free graph. Then, G is a Helly circle graph if and only if G contains no induced prism and no induced diamond.*

A *theta* is a graph arising from $K_{2,3}$ by edge subdivision. Chudnovsky and Kapadia [11] gave a polynomial-time algorithm that decides whether a graph contains a theta or a prism as induced subgraphs. Since linear domino graphs contain no induced theta, the characterization above and the existence of polynomial-time algorithms for recognizing circle graphs imply alternative polynomial-time algorithms to decide the existence of an induced theta or prism restricted to linear domino graphs. Interestingly enough, the problem of deciding whether a graph contains an induced prism is NP-complete in general [48].

The following two results characterize by forbidden induced subgraphs the class of circle graphs within two superclasses of the class of cographs; namely, P_4-tidy graphs and tree-cographs.

Theorem 49 (*[5]*). *Let G be a P_4-tidy graph. Then, G is a circle graph if and only if G contains no W_5, net $+ K_1$, tent $+ K_1$, or tent-with-center as induced subgraph.*

Theorem 50 (*[5]*). *Let G be a tree-cograph. Then, G is a circle graph if and only if G contains no induced (bipartite-claw) $+ K_1$ and no induced co-(bipartite-claw).*

The proofs of the two above results rely on Gallai's forbidden induced subgraph characterization of permutation graphs (see Corollary 31).

5. Some open problems

The main structural open problem regarding the classes of circular-arc graphs and circle graphs is that of characterizing them by forbidden induced subgraphs. We conclude this survey putting forward some other structural open problems related to these classes.

Problem 1. Give a forbidden induced subgraph characterization for circular-arc graphs within the class of chordal graphs. This would extend the characterizations in [4,24] of circular-arc graphs within claw-free chordal graphs and $5K_1$-free chordal graphs, respectively.

Problem 2. Give a forbidden induced subgraph characterization for circular-arc graphs within the class of H-free graphs where H is a four-vertex graph. This problem was solved in [4] when H is P_4, the paw, or the diamond.

Problem 3. A particularly interesting case of the preceding problem is to characterize circular-arc graphs within the class of claw-free graphs by forbidden induced subgraphs. A good starting point could be to characterize circular-arc graphs within the class of graphs with stability number at most two.

Problem 4. Find a characterization by forbidden induced subgraphs for normal circular-arc graphs. A starting point could be the characterization of normal Helly circular-arc graphs in [35].

Problem 5. Find a characterization by forbidden induced subgraphs for Helly circular-arc graphs. There is a characterization of Helly circular-arc graphs by forbidden induced subgraphs in [50] within circular-arc graphs. That characterization is not in terms of minimal forbidden induced subgraphs. Therefore, it remains to characterize Helly circular-arc graphs by minimal forbidden induced subgraphs within the class of circular-arc graphs and also to characterize Helly circular-arc graphs by forbidden induced subgraphs not assuming that the graph is circular-arc.

Problem 6. Find a characterization by forbidden induced subgraphs for q-proper circular-arc graphs (defined analogously to q-proper interval graphs). The corresponding characterization for q-proper interval graphs was given in [59].

Problem 7. Characterize circle graphs by forbidden induced subgraphs within the class of chordal graphs. Block graphs are a subclass of chordal graphs and are circle graphs. However, not every chordal graph is a circle graph.

Problem 8. Find a decomposition such that Helly circle graphs are closed under this decomposition. This would be analogous to the split decomposition for circle graphs [8].

Problem 9. Characterize Helly circle graphs by forbidden induced subgraphs. The class of Helly circle graphs was characterized by forbidden induced subgraphs within circle graphs in [16].

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