Constrained Consumptions, Lipschitzian Demands, and Regular Economies[1]

J. M. Bonnisseau[2] and J. Rivera-Cayupi[3]

Abstract. We consider an exchange economy where the consumers face linear inequality constraints on consumption. We parametrize the economy with the initial endowments and constraints. We exhibit sufficient conditions on the constraints implying that the demand is locally Lipschitzian and continuously differentiable on an open dense subset of full Lebesgue measure. Using this property, we show that the equilibrium manifold is lipeomorphic to an open, connected subset of an Euclidean space and that the lipeomorphism is almost everywhere continuously differentiable. We prove that regular economies are generic and that they have a finite odd number of equilibrium prices and local differentiable selections of the equilibrium prices.

Key Words. Demand functions, regular economies, Lipschitz behaviors.

1. Introduction

This paper deals with economic equilibrium sensitivity analysis, which is closely related to the study of the equilibrium manifold, the graph of the equilibrium correspondence. The global analysis of the economic equilibrium rests mainly upon a differentiable approach, which requires the differentiability of the demand functions (see Refs. 1–4 and references therein). This differentiability is derived often from well-known assumptions on the utility functions, particularly on its boundary behavior: the indifference curves are supposed not to cross the boundary of the consumption set. Our main objective is to provide a global analysis in the presence of constraints on consumptions (see Refs. 3, 5, 6). Thus, although the demand may be nondifferentiable, we can overcome this setback of

---

[1] This work was partially supported by CCE, ECOS, and ICM Sistemas Complejos de Ingeniería.

[2] Professor, CERMSEM, UMR CNRS 8095, Department of Applied Mathematics, Université Paris 1, Paris, France.

[3] Assistant Professor, FACEA, Department of Economics, Universidad de Chile, Santiago, Chile.

differentiability to get the usual results, namely, the index formula, the existence of equilibria, the finite number of equilibria and the local selections of the equilibrium prices for regular economies, and the genericity of regular economies.

The existence of exogenous restrictions on consumption has been recognized in economics for a long time as Debreu points out in Ref. 7. We can find a detailed discussion on the relevance of restrictions on the consumption in economy in the book of Deaton and Muellbauer (Ref. 5, Chapter 1).

In our framework, the demand depends on the price and income but also on the level of the constraints. We show that it is not differentiable if the strict complementary slackness condition does not hold at this point. We prove that the demand is locally Lipschitz continuous under reasonable conditions. For example, our hypothesis hold true if, for each consumer, there is no constraint on at least one commodity.

Then, we prove a fundamental property: the demand is continuously differentiable on an open dense subset of full Lebesgue measure. In the second part of the paper, this property is the key fact to study the properties of the equilibrium manifold, which is then a differentiable manifold almost everywhere. Furthermore, it gives us the existence of differentiable selections of the prices around a regular economy.

Following the traditional approach in economy, we study also the properties of the demand when the level of constraints is fixed. We obtain the same result at the cost of a stronger assumption on the constraints. Indeed, this is not a trivial consequence of the first result, since the genericity in the product space of endowments and constraint levels does not necessarily imply the genericity when one parameter is fixed. The proof of the demand properties relies on the result of Cornet and Vial (Ref. 8) on the Lipschitz behavior of the solution of a mathematical programming problem, but we need also to decompose the demand on several auxiliary functions to get the result. We prove that the demand is continuously differentiable around any point where it is differentiable by showing that differentiability is equivalent to the strict complementary slackness conditions on the first-order conditions. Then, we use a result from Fiacco and McCormick (Ref. 9).

There are a lot of contributions to the optimization literature on the continuity properties of solutions of a mathematical programming problem or more generally of variational inequalities. Nevertheless, we point out two things. First, we deal with the Lipschitz behavior of the demand, which is a stronger property than the upper-Lipschitz behavior as Rockafellar and Wets noticed in Ref. 10, page 420. So, we cannot derive our result from recent contributions like the ones of Klatte (Ref. 11), Levy (Ref. 12), or Shapiro (Ref. 12). Second, we prove actually that the demand is continuously differentiable on an open set of full Lebesgue measure. This property is much stronger than almost everywhere differentiability, which is a direct consequence of the Lipschitz behavior. As far as we know, this almost everywhere smooth behavior of the demand is not studied in the optimization literature nor in the economic literature.

At this point, we have to mention several previous contributions to the economic literature. In Rader (Ref. 14) and Shannon (Ref. 15), the authors are mainly concerned with the finiteness of the number of equilibria as in the seminal paper of Debreu (Ref. 2). Then, the almost everywhere differentiability of the demand, together with a property on the image of a null set are enough. In Rader (Ref. 14), sufficient conditions on the preferences are given to ensure this result whereas in Shannon (Ref. 15), this is an assumption. In Villanacci (Ref. 6), only fixed positivity constraints are considered and his analysis requires an additional assumption on the existence of a strictly positive Pareto optimal allocation, which is not stated by using only the basic description of an economy. We do not need this assumption to obtain our result.

In Section 2, we present the model, the assumptions, and we study the demand. Section 3 is devoted to the equilibrium manifold and the regular economies with respect to constraints and initial endowments. In Section 4, the case of fixed constraints is studied. Finally, Section 5 is the Appendix where some technical proofs are given.

2. Properties of the Demand Functions

Let us consider an economy with positive integers ℓ of commodities and m of consumers. Let $L \equiv \{1, \dots, \ell\}$ and $M \equiv \{1, \dots, m\}$[4]. Given an agent $i \in M$, we assume that her/his preferences are represented by a utility function $u_i : R^\ell_{++} \to R$, which satisfies the following assumption.

Assumption A1. For each $i \in M, u_i$ is a C^2 mapping. For each $x \in R^\ell_{++}$, $\nabla u_i(x) \in R^\ell_{++}$, $D^2 u_i(x)$ is negative definite on $\nabla u_i(x)^\perp$, the orthogonal complement of the vector $\nabla u_i(x)$. For all $x \in R^\ell_{++}$, $\{x' \in R^\ell_{++} \mid u_i(x) \le u_i(x')\}$ is a closed set of R^ℓ.

The consumption may be limited also by additional physical constraints. Formally, for each consumer $i \in M$, one has a $k_i \times \ell$ matrix A_i. For a given parameter $b_i \in R^{k_i}$, the possible consumptions are given by

$$X_i(b_i) = \left\{x \in R^\ell_{++} | A_i x \le b_i\right\}.$$

In the following, $a_{ik} \in R^\ell$ denotes the kth row of A_i and, if K is a subset of $\{1, \dots, k_i\}$, A_{iK} is the submatrix of A_i obtained by keeping only the rows

[4] We use the following notation: if $x = (x_i), y = (y_i) \in R^n$, $x \le (\ll)\, y$ means $x_i \le (<)\, y_i$ for each $i = 1, \dots, n$. $x \cdot y = \sum_{i=1}^m x_i y_i$ denotes the inner product of x and y.

$a_{ik}, k \in K$. A^*_{iK} is the transpose of A_{iK}. If $b \in R^{k_i}$, b_K is the restriction of b to the components in K. Let

$$C_i = \{\xi \in R^\ell_+ \backslash \{0\} | A_i \xi \leq 0\},$$

$$C_{iK} = \{\xi \in R^\ell_+ \backslash \{0\} | A_{iK} \xi | \leq 0\}.$$

Assumption A2.

(a) For each $i \in M, C_i = \{\xi \in R^\ell_+ \backslash \{0\} | A_i \xi \leq 0\}$ is nonempty. For all nonempty subset K of $\{1, \ldots, k_i\}$ such that the cardinality of K is less than or equal to $\ell - 1$ and $(a_{ik})_{k \in K}$ are linearly independent,

$$\{\mu \in R^K | \mu \cdot A_{iK} \xi < 0, \forall \xi \in C_{iK}\} \subset R^K_+.$$

(b) $\cup_{i \in M} C_i = R^\ell_+ \backslash \{0\}$.

(c) For each $i \in M$, for all $K \subset \{1, \ldots, k_i\}$ such that $(a_{ik})_{k \in K})$ are positively linearly independent,[5] then they are linearly independent.

Part (a) implies that the consumption set $X_i(b_i)$ is not bounded. Part (b) means that, for each nonnegative direction, at least one consumer can increase her/his consumption along this direction. So, each commodity is always desirable. Part (c) is devoted to the case of fixed constraint levels.

Now, we give some simple cases under which Assumption A2 is satisfied. In the following, $\mathbf{1}^h$ is the hth vector of the canonical basis of R^ℓ. The proof is given in Section 5 (Appendix).

Lemma 2.1.

(i) Assumption A2(a) holds true under any one of the following conditions:
  (a) There exists $\xi \in R^\ell_+ \backslash \{0\}$ such that $A_i \xi = 0$.
  (b) Each row of A_i is either equal to the transpose of $\mathbf{1}^h$ or of $-\mathbf{1}^h$ for some commodity h, all rows are different; for some commodity h_0, the transpose of $\mathbf{1}^{h_0}$ is not a row of A_i.
  (c) The rows of A_i does not belongs to R^ℓ_+ and the sets $L_k = \{h \in L | a_{ikh} \neq 0\}, k = 1, \ldots, k_i$, are pairwise disjoint.

(ii) Assumption A2(c) holds true under condition (b) or (c) or if the matrix A_i has full row rank.

Finally, we assume that each consumer has an initial endowment of commodities denoted $e_i \in R^\ell_{++}$. We define an economy as a point $((e_i), (b_i))_{i \in M}$ in the set of economies E defined by

$$E = \left\{((e_i), (b_i))_{i \in M} \in (R^L_{++})^M \times \prod\nolimits_{i \in M} R^{k_i} | \forall i \in M, A_i e_i \ll b_i \right\}.$$

[5] This means that, for all $(t_k) \in R^K_+$, if $\sum_{k \in K} t_k a_{ik} = 0$, then $t_k = 0$ for all $k \in K$.

From the monotony of the utility functions and Assumption A2, it appears that the equilibrium prices are always positive. Since they are also defined only up to multiplication by a positive real number, we consider only the normalized price in

$$S = \left\{ p \in R^{\ell}_{++} \,\middle|\, \sum_{h \in L} p_h = 1 \right\}.$$

For a given level of constraint $b_i \in R^{k_i}$, a given price $p \in S$ and a wealth $r_i \in R_{++}$, the demand $f_i(p, r_i, b_i)$ of the ith consumer is the solution, if it exists, of the following optimization problem:[6]

$$\max u_i(x), \quad \text{s.t. } p \cdot x \leq r_i, x \in X_i(b_i).$$

Let Ω_i be the open subset of $S \times R \times R^{k_i}$ defined by

$$\Omega_i \equiv \{(p, r, b_i) \in S \times R \times R^{k_i} \mid \exists x \in R^{\ell}_{++}, p \cdot x < r_i, \ A_i x \ll b_i\}.$$

On Ω_i, the set $X_i(b_i)$ has a nonempty interior and the income r_i is not the minimal one with respect to p. The main goal of this section is to prove the following proposition.

Proposition 2.1. Under Assumptions A1 and A2(a), (b), for each $(p, r_i, b_i) \in \Omega_i$, $f_i(p, r_i, b_i)$ is a singleton and the demand function $f_i : \Omega_i \to R^{\ell}_{++}$ is locally Lipschitz continuous. Furthermore, there exists an open subset Ω_i° of Ω_i such that $\Omega_i \backslash \Omega_i^{\circ}$ is a null set with respect to the Lebesgue measure and f_i is continuously differentiable on Ω_i°.

Proof. We omit the subscript i in the proof to simplify the notation. The continuity, the strict quasiconcavity, the local nonsatiation, the boundary behavior of u given by Assumption A1 together with the maximum theorem of Berge imply that $f(p, r, b)$ is a singleton and f is a continuous mapping. Indeed, for all $(p, r, b) \in \Omega$, there exists $x \in R^{\ell}_{++}$ such that $p \cdot x < r$ and $Ax \ll b$. Thus, the interior of the budget set is never empty; hence, the budget set is lower and upper semicontinuous with respect to (p, r, b). Note also that it is also bounded since p is strictly positive. The remainder of the proof is divided in three steps.

Step 1. Let $x = f(p, r, b)$. The first-order necessary conditions of optimality implies that there exist $\lambda \geq 0$ and $\mu \in R^k_+$ such that

$$\nabla u(x) = \lambda p + A^* \mu, \quad \mu \cdot (Ax - b) = 0, \quad \lambda(p \cdot x - r) = 0.$$

[6] If a solution exists, it is unique from the convexity of the admissible set and the strict quasiconcavity of u_i.

Let

$$K(x) = \{\kappa \in \{1, \dots, k\} | A_\kappa \cdot x = b_\kappa\}$$

be the set of binding constraints at x. We recall that the strict complementarity slackness condition holds if the vectors $((a_\kappa)_{\kappa \in K(x)}, p)$ are linearly independent, $\lambda > 0$ and $\mu_{K(x)} \gg 0$.

Assumption A2(a) implies that

$$A^* R^k_+ \cap R^\ell_{++} = \emptyset.$$

Indeed,

$$C = \{\xi \in R^\ell_+ \backslash \{0\} \mid A\xi \leq 0\}$$

is nonempty; hence, there exists $w \in R^\ell_+ \backslash \{0\}$ such that $Aw \leq 0$. Then, if there exists $\beta \in R^k_+$ such that $A^*\beta \in R^\ell_{++}$, one has

$$0 < w \cdot A^*\beta = Aw \cdot \beta \leq 0,$$

which leads to a contradiction.

Since

$$A^* R^k_+ \cap R^\ell_{++} = \emptyset,$$

from the Caratheodory theorem, there exists a (possibly empty) subset K of $K(x)$, $\lambda' > 0$, and $\mu' \in R^K_{++}$ such that the family $((a_\kappa)_{\kappa \in K}, p)$ is linearly independent and[7]

$$\nabla u(x) = \lambda' p + A^*_K \mu', \quad A_K x - b_K = 0, \quad p \cdot x = r.$$

Hence, either $x = f_\emptyset(p, r, b)$, where $f_\emptyset(p, r, b)$ is the solution of the maximization of u on the set $\{\xi \in R^\ell_{++} | p \cdot \xi \leq r\}$ or $x = f_K(p, r, b)$, where $f_K(p, r, b)$ is the solution of the maximization of u on the set $\{\xi \in R^\ell_{++} | p \cdot \xi \leq r, A_K \xi \leq b_K\}$.

Step 2. We already know (see for example Refs. 1, 3) that $f_\emptyset$ is continuously differentiable on Ω. Now, we prove that, for all $K \subset \{1, \dots, k\}$, such that the cardinality of K is less or equal to $\ell - 1$ and the vectors $(a_\kappa)_{\kappa \in K}$ are linearly independent, the mapping f_K is locally Lipschitz continuous on Ω. If $(p, r, b) \in \Omega$, then there exists $\xi \in R^\ell_{++}$ such that $p \cdot \xi < r$ and $A\xi \ll b$, which implies $A_K x \ll b_K$. Consequently, the argument given in the first step applies to show that f_K is a continuous mapping.

Let $x' = f_K(p', r', b')$. Then, the binding constraints at x' are the budget constraint (since $A^* R^k_+ \cap R^\ell_{++} = \emptyset$) and the constraints $A_{K'} x \leq b_{K'}$ for a (possibly empty) subset K' of K. From Cornet and Vial (Ref. 8), f_K

[7] This implies that that the cardinality of K is less or equal to $\ell - 1$. We recall that, if K is empty, one simply has $\nabla u(x) = \lambda' p$ and $p \cdot x = r$.

is locally Lipschitz at (p', r', b') if $((a_\kappa)_{\kappa\in K'}, p')$ is a linearly independent family. If K' is empty, the result is obvious since $p' \neq 0$. If not, since the vectors $(a_\kappa)_{\kappa\in K}$ are linearly independent, there exists $\nu \in R^{K'}\backslash\{0\}$ such that $p' = A^*_{K'}\nu$. If we complete ν by zero coordinates for $\kappa \in K\backslash K'$, one obtains a vector $\tilde{\nu}$ such that $p' = A^*_K\tilde{\nu}$. For all $\xi \in C_K$, $0 < p' \cdot \xi = A^*_K\tilde{\nu} \cdot \xi = \tilde{\nu} \cdot A_K\xi$. From Assumption A2(a), $\tilde{\nu} \in -R^K_+$, and, from the construction of $\tilde{\nu}$, one deduces that $\nu \in -R^{K'}_+\backslash\{0\}$.

Let $\xi \in R^\ell_{++}$ such that $p' \cdot \xi < r'$ and $A_K\xi \ll b'_K$. Due to $A_{K'}\xi \ll b'_{K'}$, one has

$$p' \cdot \xi = A^*_{K'}\nu \cdot \xi = \nu \cdot A_{K'}\xi > \nu \cdot b_{K'}.$$

But since $A_{K'}x' = b'_{K'}$, one has $p' \cdot \xi \geq p' \cdot x' = r'$, which contradicts $p' \cdot \xi < r'$. Hence, the vectors $((a_\kappa)_{\kappa\in K'}, p')$ are linearly independent and f_K is locally Lipschitz continuous.

We now apply the following result (Mas-Colell, Ref. 3, Chapter 1): if $(g_j)_{j\in J}$ is a finite collection of locally Lipschitz continuous mappings from an open subset Ω to R^ℓ and if g is a continuous mapping from Ω to R^ℓ such that, for all $\omega \in \Omega$, $g(\omega) \in \{g_j(\omega) | j \in J\}$, then g is locally Lipschitz continuous. We conclude finally that f is locally Lipschitz continuous on Ω.

Step 3. The Rademacher's theorem implies that f is almost everywhere differentiable. Let Ω° be the subset of Ω on which f is differentiable. If the strict complementarity slackness condition holds true at $f(p, r, b)$, Assumption A1 implies that f is continuously differentiable on a neighborhood of (p, r, b); see Fiacco and McCormick (Ref. 9). Thus, the proposition is a consequence of the following result: if the demand function is differentiable at (p, r, b), then the strict complementarity slackness condition holds true at $\bar{x} = f(p, r, b)$.

Let $(p, r, b) \in \Omega$ such that f is differentiable at this point. First, we prove that the vectors $((a_\kappa)_{\kappa\in K(\bar{x})}, p)$ are linearly independent. If it is false, there exist $K \subset K(\bar{x})$, $K \neq K(\bar{x})$, $\lambda' > 0$, and $\mu' \in R^K_{++}$ such that the vectors $((a_\kappa)_{\kappa\in K}, p)$ are linearly independent and

$$\nabla u(x) = \lambda' p + A^*_K\mu', \quad A_Kx - b_K = 0, \quad p \cdot x = r.$$

Let $\bar{\kappa} \in K(\bar{x})\backslash K$. Let b' be such that $b_\kappa = b'_\kappa$ if $\kappa \neq \bar{\kappa}$. From the above first-order necessary and sufficient condition, we have that $f(p, r, b') = f(p, r, b) = \bar{x}$ if $b'_{\bar{\kappa}} > b_{\bar{\kappa}}$. Hence, $a_{\bar{\kappa}} \cdot f(p, r, b') = a_{\bar{\kappa}} \cdot \bar{x} = b_{\bar{\kappa}}$. For b' such that $b'_{\bar{\kappa}} < b_{\bar{\kappa}}$, one has $a_{\bar{\kappa}} \cdot f(p, r, b') \leq b'_{\bar{\kappa}}$. Then the mapping $a_{\bar{\kappa}} \cdot f(p, r, b')$ is not differentiable with respect to $b'_{\bar{\kappa}}$ and thus $f(p, r, b')$ is not differentiable at b.

To end the proof, we show that the multipliers are positive if f is differentiable. Since $A_K^* R_+^K \cap R_{++}^\ell = \emptyset$, λ is positive. Let us assume that $\mu_{\bar{\kappa}} = 0$ for some $\bar{\kappa} \in K(\bar{x})$. Since the vectors $(a_\kappa)_{\kappa \in K(\bar{x})}$ are linearly independent, there exists a vector $\xi \in R^\ell$ such that $a_\kappa \cdot \xi = 0$ for $\kappa \neq \bar{\kappa}$ and $a_{\bar{\kappa}} \cdot f = 1$.

For $t \in R$, close to 0, let

$$\begin{aligned} x(t) &= \bar{x} + t\xi, \\ p(t) &= (1/\lambda)\big(\nabla u(x(t)) - A^*_{K(\bar{x})}\mu_{K(\bar{x})}\big), \\ r(t) &= p(t) \cdot x(t). \end{aligned}$$

For $t < 0$, $f(p(t), r(t), b) = x(t)$, since $a_{\bar{\kappa}} \cdot x(t) = b_{\bar{\kappa}} + t < b_{\bar{\kappa}}$ and the first order necessary conditions are satisfied with the multipliers λ and $\mu_{K(\bar{x})}$. For $t > 0$, one has $a_\kappa \cdot f(p(t), r(t), b) \leq b_{\bar{\kappa}}$. Consequently, the function $a_{\bar{\kappa}} \cdot f(p(t), r(t), b)$ is not differentiable at $t = 0$, which implies that $f(p(t), r(t), b)$ is not differentiable at 0, hence f is not differentiable with respect to (p, r), thus with respect to (p, r, b). □

We now come to a more precise result under Assumption A2(c) when the constraint levels b_i are fixed. Let

$$\Omega_i^{b_i} \equiv \{(p, r_i) \in S \times R \,|\, (p, r_i, b_i) \in \Omega_i\}.$$

In the following statement, Ω_i° is the subset of Ω_i given by Proposition 2.1.

Proposition 2.2. Under Assumptions A1 and A2, for each $b_i \in R^{k_i}$, $\Omega_i^{\circ b_i} = \{(p, r_i) \in \Omega_i^{b_i} \,|\, (p, r_i, b_i) \in \Omega_i^\circ\}$ is open and $\Omega_i^{b_i} \backslash \Omega_i^{\circ b_i}$ is a null set.

Proof. Again, we omit in the proof the subscript i. From the proof of the previous proposition, $f(.,.,b)$ is not differentiable if the vectors $(a_\kappa)_{\kappa \in K(\bar{x})}$ are linearly independent and the multipliers $\mu_{K(\bar{x})}$ are not all positive for the binding constraints. Now, since Ω° is an open subset, its trace $\Omega^{\circ b}$ is open. Furthermore, for each fixed $b \in R^k$, $f(.,.,b)$ is locally Lipschitz continuous on Ω^b, when it is nonempty. The proof is then a consequence of Rademacher's theorem if one shows that $f(.,.,b)$ differentiable at (p, r) implies that $(p, r) \in \Omega^{\circ b}$.

Let $(p, r) \in \Omega^b$ such that $f(.,.,b)$ is differentiable at (p, r). Let $x = f(p, r, b)$ and let $K(x)$ be the set of binding constraints at x. We start by showing that $(a_\kappa)_{\kappa \in K(x)}$ are positively linearly independent. Indeed, since $(p, r, b) \in \Omega$, there exists $\xi \in R^\ell_{++}$ such that $A\xi \ll b$ and $p \cdot \xi < r$. If there exists $\mu \in R_+^{K(x)} \backslash \{0\}$ such that $A^*_{K(x)}\mu = 0$, one has

$$0 = A^*_{K(x)}\mu \cdot \xi = \mu \cdot A_{K(x)}\xi < \mu \cdot b_{K(x)} = \mu \cdot A_{K(x)}x = A^*_{K(x)}\mu \cdot x = 0.$$

Thus, we get a contradiction. From Assumption A2(c), the vectors $(a_\kappa)_{\kappa \in K(x)}$ are linearly independent. Using the same argument as at the end of the proof of the last proposition, one shows that $f(.,.,b)$ is not differentiable at (p,r) if there exist nonpositive multipliers. In particular, this is the case if the vectors $((a_\kappa)_{\kappa\in K(x)}, p)$ are linearly dependent. Consequently, one can conclude that $f(.,.,b)$ is differentiable at (p,r) implies that the strict complementarity slackness condition holds true and thus, f is differentiable at (p,r,b), which means that $(p,r,b)\in\Omega^\circ$ or equivalently $(p,r)\in\Omega^{\circ b}$. $\square$

3. Equilibrium Manifold

In this section, we study the equilibrium price vectors associated with an economy $((e_i),(b_i))$ from a global point of view as in Balasko (Ref. 1) or Mas-Colell (Ref. 3). We assume that Assumptions A1 and A2 (a) and (b) hold true. A price vector $p\in S$ is an equilibrium price for the economy $((e_i),(b_i))\in E$ if the total demand at $p\in S$ is equal to the supply, that is,

$$\sum_{i\in M} f_i(p, p\cdot e_i, b_i) = \sum_{i\in M} e_i.$$

In that case, we shall say that $(p,(e_i),(b_i))\in S\times E$ is an equilibrium point and the equilibrium manifold $E_{eq}\subseteq S\times E$ is defined as the set of equilibrium points in $S\times E$.

In the following, $\mathbf{1}$ is the vector of R^ℓ with every coordinate equal to 1, $\mathbf{1}^\perp$ is the orthogonal space to $\mathbf{1}$, and proj is the orthogonal projection on it.

Let $\mathcal{U}$ be the open subset of $S\times R^m\times(\mathbf{1}^\perp)^{m-1}\times\prod_{i\in M}R^{k_i}$ defined as follows: $(p,(r_i),(\eta_i)_{i=1}^{m-1},(b_i))\in\mathcal{U}$ if, for each $i\in M$, $A_ie_i\ll b_i$ and $0\ll e_i$, with $e_i=\eta_i+(r_i-p\cdot\eta_i)\mathbf{1}$ for $i\le m-1$ and $e_m=f_m(p,r_m,b_m)+\sum_{i=1}^{m-1}(f_i(p,r_i,b_i)-e_i)$. In other words, the initial endowments e_i defined by the above formula are in the interior of the consumption sets for the parameters b_i and the global demand $\sum_{i=1}^m f_i(p,r_i,b_i)$ is equal to the total initial endowment $\sum_{i=1}^m e_i$. Note that $(p,(r_i),(\eta_i)_{i=1}^{m-1},(b_i))\in\mathcal{U}$ implies $(p,r_i,b_i)\in\Omega_i$ since $p\cdot e_i=r_i$ for all $i\in M$. Thus, $f_i(p,r_i,b_i)$ is well defined.

The sketch of the proof of Proposition 3.1 is given in the Appendix.[8]

Proposition 3.1. $\mathcal{U}$ is an open connected subset of $S\times R^m\times(\mathbf{1}^\perp)^{m-1}\times\Pi_{i\in M}R^{k_i}$.

Let us now define the mappings $\theta:\mathcal{U}\to E_{eq}$ and $\phi:E_{eq}\to\mathcal{U}$ as

$$\theta(p,(r_i),(\eta_i)_{i=1}^{m-1},(b_i))\equiv(p,(e_i),(b_i)),$$

[8] For the complete proof, the reader is referred to the working paper (Ref. 16).

with

$$e_i = \eta_i + (r_i - p \cdot \eta_i)\mathbf{1}, \quad \text{for } i \le m-1,$$

$$e_m = f_m(p, r_m, b_m) + \sum_{i=1}^{m-1} (f_i(p, r_i, b_i) - e_i),$$

$$\phi(p, (e_i), (b_i)) \equiv \big(p, (p \cdot e_i)_{i=1}^m, (\text{proj } e_i)_{i=1}^{m-1}, (b_i)\big).$$

The definition of $\theta : \mathcal{U} \to E_{eq}$ and $\phi : E_{eq} \to \mathcal{U}$ are borrowed from Balasko (Ref. 1) and extended to take into account the parameters (b_i).

Let $\mathcal{V}$ the subset of $\mathcal{U}$ defined by: $(p, (r_i), (\eta_i)_{i=1}^{m-1}, (b_i))$ belongs to $\mathcal{V}$ if, for each $i \in M$, (p, r_i, b_i) belongs to Ω_i°. The next proposition is a direct consequence of Proposition 2.1 and previous definitions.

Proposition 3.2.

(i) θ and ϕ are one-to-one and onto; moreover, $\theta^{-1} = \phi$.
(ii) θ and ϕ are locally Lipschitz continuous mappings.
(iii) $\mathcal{U} \backslash \mathcal{V}$ is a closed null set.
(iv) θ is continuously differentiable on $\mathcal{V}$.
(v) E_{eq} is lipeomorphic to $\mathcal{U}$.

The extended projection Π from $\mathcal{U}$ to E is defined as $\Pi \equiv \pi \circ \theta$, where π is the ordinary projection from $E_{eq} \subset S \times E$ to E; that is,

$$\begin{aligned} &\Pi\big(p, (r_i), (\eta_i)_{i=1}^{m-1} (b_i)\big) \\ &= \big((\eta_i + (r_i - p \cdot \eta_i))\mathbf{1}\big)_{i=1}^{m-1}, f_m(p, r_m, b_m) \\ &\quad + \sum_{i=1}^{m-1} (f_i(p, r_i, b_i) - \eta_i - (r_i - p \cdot \eta_i)\mathbf{1}), (b_i)). \end{aligned}$$

The proof of the following proposition is also given in the Appendix (Section 5).

Proposition 3.3. The extended projection $\Pi : \mathcal{U} \to E$ is a proper, locally Lipschitz continuous mapping and is continuously differentiable on $\mathcal{V}$. $\Pi(\mathcal{U} \backslash \mathcal{V})$ is a closed null set.

Definition 3.1. An economy $((e_i), (b_i)) \in E$ will be called regular if it does not belong to $\Pi(\mathcal{U} \backslash \mathcal{V})$ and if it is not the image of a critical point of $\Pi | \mathcal{V}$. An economy is singular if it is not regular. We denote the set of singular [resp. regular] economies by E^s [resp. E^r].[9]

[9] Similar notions of regular [resp. singular] values are used in the literature dealing with nonsmooth mappings (see Refs. 14–15).

Since $\mathcal{V}$ and E have the same dimension, the Sard theorem and the properness of Π imply the following result.

Proposition 3.4. E^s is a closed null subset of E.

From standard results of differential topology, we have the following result.

Theorem 3.1.

(i) For all $((e_i), (b_i)) \in E^r$, there exists a finite number of equilibrium prices.
(ii) Let $((e_i), (b_i)) \in E^r$ and let $p \in S$ be an equilibrium price for this economy. Then, there exists a neighborhood $\mathcal{N}$ of $((e_i), (b_i))$, a neighborhood $\mathcal{N}'$ of $p \in S$, and a differentiable mapping $q : \mathcal{N} \to \mathcal{N}'$ such that
  (a) $q((e_i), (b_i)) = p$,
  (b) for all $((e'_i), (b'_i)) \in \mathcal{N}$, $q((e'_i), (b'_i))$ is the unique equilibrium price of $((e'_i), (b'_i))$ in $\mathcal{N}'$.

We end this section by computing the degree[10] of Π.

Theorem 3.2.

(i) Π is of degree 1 and then onto.
(ii) For all $((e_i), (b_i)) \in E$, there exists an equilibrium.
(iii) For all $((e_i), (b_i)) \in E^r$, there exists a finite odd number of equilibrium prices.

Proof. Due to the connectedness of $\mathcal{U}$ and the properness of Π, it is sufficient to compute the degree for one value, that is, for one economy $((e_i), (b_i))$. We define the reference economy as follows: let $f_{i\emptyset}$ be the unconstrained demand associated to the utility function u_i, let $(r_i) \in R^M_{++}$, and let $p \in S$. For each consumer, let $e_i = f_{i\emptyset}(p, r_i)$ and let b_i large enough such that $A_i e_i \ll b_i$. This economy, with or without constraint, has a unique equilibrium $(p, (e_i))$; see Balasko (Ref. 1).

For the prices in a neighborhood of p, no constraints are binding at the demand thanks to the continuity of the demand functions. So, the demand f_i coincides locally with $f_{i\emptyset}$. Consequently, the demand does not depend on the constraint levels. Thus, the determinant of the Jacobian matrix of Π at a point $(p, (r_i), (\eta_i)_{i=1}^{m-1}, (b_i))$, with $r_i = p \cdot e_i$ and $\eta_i = \text{proj } e_i$, is the same as the determinant of the mapping $\widetilde{\Pi}$ defined by

$$\begin{aligned}
&\widetilde{\Pi}\big(p, (r_i), (\eta_i)_{i=1}^{m-1}\big) \\
&= ((\eta_i + (r_i - p \cdot \eta_i)\mathbf{1})_{i=1}^{m-1}, f_{m\emptyset}(p, r_m) \\
&\quad + \sum_{i=1}^{m-1}(f_{i\emptyset}(f, r_i) - \eta_i - (r_i - p \cdot \eta_i)\mathbf{1})).
\end{aligned}$$

[10] We consider the degree for a continuous mapping as defined in Deimling (Ref. 17).

Note that $\widetilde{\Pi}$ is exactly the natural projection studied in Balasko (Ref. 1) composed by the local diffeomorphism θ. Thus, this mapping is a local diffeomorphism; thanks to the uniqueness of the equilibrium price, one gets that the degree of Π is equal to 1. $\square$

4. Fixed Constraints

In this section, we use the previous analysis to study the case where the consumption constraints are fixed, that is, when the parameters (b_i) are fixed. Let $\widetilde{b} = (\widetilde{b}_i)$ and let Assumptions A1 and A2 hold true. Then, the economy depends on only the initial endowments which lie on the set

$$E^{\widetilde{b}} = \{(e_i) \in (R^{\ell}_{++})^m \mid \forall i \in M, A_i e_i \ll \widetilde{b}_i)\}.$$

The set $E^{\widetilde{b}}$ may be empty for some value of $\widetilde{b}$. That is why we define

$$\mathcal{B} = \left\{ \widetilde{b} \in \prod_{i \in M} R^{k_i} | E^{\widetilde{b}} \neq \emptyset \right\}.$$

We can then define the sets $E^{\widetilde{b}}_{eq}$, $E^{\widetilde{b}r}$, $\mathcal{U}^{\widetilde{b}}$, $\mathcal{V}^{\widetilde{b}}$ and the mappings $\theta^{\widetilde{b}}$, $\phi^{\widetilde{b}}$, and $\Pi^{\widetilde{b}}$ by merely considering the parameter $\widetilde{b}$ as fixed. Note that Proposition 2.2 implies that $(p, (r_i), (\eta_i)_{i=1}^{m-1}, (\widetilde{b}_i))$ belongs to $\mathcal{V}$ if $(p, (r_i), (\eta_i)_{i=1}^{m-1})$ belongs to $\mathcal{V}^{\widetilde{b}}$. All the results given in Proposition 3.1 to Theorem 3.1 still hold except the connectedness of $\mathcal{U}^{\widetilde{b}}$.

The argument used to compute the degree of Π does not work, since we consider an initial endowment that is Pareto optimal without constraint. For this, we choose the constraint levels large enough. Since they are now fixed, we cannot use the same method.

Proposition 4.1. For each $\widetilde{b} = (\widetilde{b}_i) \in \mathcal{B}$, $\Pi^{\widetilde{b}}$ is of degree 1 (and then onto); for all $(e_i) \in E^{\widetilde{b}r}$, there exists a finite odd number of equilibrium prices.

Proof. Given $(\widetilde{b}_i) \in \mathcal{B}$ and $(e_i) \in E^{\widetilde{b}r}$, let $P(e_i)$ be the finite set of equilibrium price vectors in S associated to (e_i), which is also the finite set of equilibrium price vectors in S associated to $((e_i), (\widetilde{b}_i))$. For all $p \in P(e_i)$, let

$$\mu = \left(p, (r_i), (\eta_i)_{i=1}^{m-1}\right) = \phi^{\widetilde{b}}(p, (e_i)) \in \mathcal{U}^{\widetilde{b}}.$$

Note that

$$(\mu, (\widetilde{b}_i)) \in \phi(p, (e_i), (\widetilde{b}_i)).$$

From the definition of a regular economy, $\mu \in \mathcal{V}^{\widetilde{b}}$ for each $p \in P(e_i)$; hence, from Proposition 2.2, $(\mu, (\widetilde{b}_i))$ belongs to $\mathcal{V}$. This implies that Π is differentiable in a neighborhood of $(\mu, (\widetilde{b}_i))$.

From the definition of Π and $\Pi^{\widetilde{b}}$, the Jacobian matrix of Π at $(\mu, \widetilde{b}_i)$ is a $(m\ell + \sum_{i\in I} k_i)$ square matrix, which has the following structure:

$$D\Pi(\mu) = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix},$$

where A is the Jacobian matrix of $\Pi^{\widetilde{b}}$ at μ and I is the $\sum_{i\in I} k_i$ identity matrix. Now, since (e_i) is a regular economy in $E^{\widetilde{b}}$ and since

$$\det[A] = \det[D\Pi(\mu)],$$

we can deduce that $((e_i), (\widetilde{b}_i))$ is a regular economy in E and

$$\sum_{\mu\in(\Pi^{\widetilde{b}})^{-1}(e_i)} \mathrm{sign}(\det[D\Pi^{\widetilde{b}}(\mu)]) = \sum_{\mu\in\Pi^{-1}((e_i),(\widetilde{b}_i))} \mathrm{sign}(\det[D\Pi(\mu)]),$$

which implies that

$$\deg(\Pi^{\widetilde{b}}) = \deg(\Pi) = 1. \qquad \square$$

5. Appendix: Proofs

Proof of Lemma 2.1.

(a) One easily checks that $\xi \in C_i$, $\xi \in C_{iK}$ for all K and $A_{iK}\xi = 0$. Thus, $\{\mu \in R^K | \mu \cdot A_{iK}\xi < 0, \forall \xi \in C_{iK}\} = \emptyset$, hence the conclusion holds.

(b) Note first that $\mathbf{1}^{h_0} \in C_i$. Let K be a subset of $\{1, \ldots, k_i\}$ such that the cardinality of K is less than or equal to $\ell - 1$. Then, there exists $h \in L$ such that $a_{ik} \neq \mathbf{1}^h$ and $a_{ik} \neq -\mathbf{1}^h$ for all $k \in K$. Consequently, $A_{iK}\mathbf{1}^h = 0$, hence $\mathbf{1}^{h_0} \in C_{iK}$. Thus,

$$\{\mu \in R^K | \mu \cdot A_{iK}\xi < 0, \forall \xi \in C_{iK}\} = \emptyset,$$

which leads to the result.

(c) For each $k \in \{1, \ldots, k_i\}$, there exists $h(k) \in L_k$ such that the $h(k)$ component of the row a_{ik}, denoted $a_{ikh(k)}$, is negative. Then, since the subsets $(L_k)_{k=1}^{k_i}$ are disjoint, for all k, $\mathbf{1}^{h(k)} \in C_i$. For all $k \in K \subset \{1, \ldots, k_i\}$, $\mathbf{1}^{h(k)} \in C_{iK}$ and $\mu \cdot A_{iK}\mathbf{1}^{h(k)} = \mu_k a_{ikh(k)}$. Since $a_{ikh(k)} < 0$, if μ satisfies $\mu \cdot A_{iK}\mathbf{1}^{h(k)} < 0$, then $\mu_k > 0$, hence $\mu \in R_+^K$. $\square$

Proof of Proposition 3.1. The openness is a direct consequence of the definition and the continuity of the demand mappings f_i. Now, we give a sketch of the proof of the connectedness of $\mathcal{U}$. Let $\mu^* = (p^*, (r_i^*), (\eta_i^*)_{i=1}^{m-1}, (b_i^*))$ be an element of $\mathcal{U}$. Let (e_i^*) be such that $\theta(\mu^*) = (p^*, (e_i^*), (b_i^*))$. Since $\mu^* \in \mathcal{U}$, for all $i \in M$, one has $0 \ll e_i^*$ and $A_i e_i^* \ll b_i^*$. Since, for all $i \in M$, the set $\{x \in R_+^\ell | p^* \cdot x \leq r_i^*\}$ is compact, there exists $\bar{b}_i$ such that $b_i^* \ll \bar{b}_i$ and $A_i \xi \ll \bar{b}_i$ for all $\xi \geq 0$ satisfying $p^* \cdot \xi \leq r_i^*$.

The first step shows that μ^* is connected to $\bar{\mu} = (p^*, (r_i^*), (\bar{\eta}_i)_{i=1}^{m-1}, (\bar{b}_i))$, with $\bar{\eta}_i = \text{proj } f_i(p^*, r_i^*, \bar{b}_i)$ for all $i \in M$. Let $(\bar{e}_i)$ such that $\theta(\bar{\mu}) = (p^*, (\bar{e}_i), (\bar{b}_i))$. One checks easily that, for all i, $\bar{e}_i = f_i(p^*, r_i^*, \bar{b}_i)$ and $A_i \bar{e}_i \ll \bar{b}_i$. This implies that $f_i(p^*, r_i^*, \bar{b}_i)$ is the standard (without constraint) demand at (p^*, r_i^*) for the utility function u_i. The second step shows that there exists a continuous path in $\mathcal{U}$ between $\bar{\mu}$ and μ' for all $\mu' = (p', r', (\eta_i')_{i=1}^{m-1}, b') \in \mathcal{U}$ such that $\eta_i' = \text{proj} f_i(p', r_i', b_i')$ and $A_i f_i(p', r_i', b_i') \ll b_i'$ for all $i \in M$. □

Proof of Proposition 3.3. Except for the properness, the properties of Π are direct consequences of the properties of θ and ϕ. From the definition of Π, it suffices to show that π is proper, since θ is an homeomorphism. Let $\mathcal{E}$ be a compact subset of E and let $(p^\nu, (e_i^\nu), (b_i^\nu))_{\nu \leq 1}$ be a sequence of $\pi^{-1}(\mathcal{E}) \subset E_{eq}$. Let (x_i^ν) be the sequence defined by $x_i^\nu = f_i(p^\nu, p^\nu \cdot e_i^\nu, b_i^\nu)$. Since (x_i^ν) is an attainable allocation, it follows that the sequence $(p^\nu, (e_i^\nu), (b_i^\nu), (x_i^\nu))$ remains in a compact set. Thus, it has a converging subsequence and we denote its limit by $(p, (e_i), (b_i), (x_i))$. Note that the closedness of $\mathcal{E}$ implies that $((e_i), (b_i))$ belongs to $\mathcal{E}$.

We show that $(p, (x_i))$ is an equilibrium of the economy $((e_i), (b_i))$, which implies that $(p, (e_i), (b_i)) \in \pi^{-1}(\mathcal{E})$. Since $((e_i), (b_i)) \in \mathcal{E} \subset E$, for all $i \in M$, $e_i \in R_{++}^\ell$ and $A_i e_i \ll b_i$. Consequently, $p \cdot e_i > \inf p \cdot X_i(b_i)$. Now, we prove that, for all $i \in M$, x_i is a solution of

$$\max\ u_i(x), \quad \text{s.t. } p \cdot x \leq p \cdot e_i, x \in X_i(b_i).$$

If this is not true, then there exists $x_i' \in X_i(b_i)$ such that $p \cdot x_i' \leq p \cdot e_i$ and $u_i(x_i') > u_i(x_i)$. Since $((e_i), (b_i)) \in E$, there exists $\xi_i \in R_{++}^\ell$ such that $A_i \xi_i \ll b_i$ and $p \cdot \xi_i, < p \cdot e_i$. Thus, from the continuity of u_i, moving slightly x_i' toward ξ_i, there exists x_i'' such that $x_i'' \in R_{++}^\ell$, $A_i x_i'' \ll b_i$, $p \cdot x_i'' < p.e_i$, and $u_i(x_i'') > u_i(x_i)$. Then, for ν large enough, $x_i'' \in X_i(b_i^\nu)$, $p^\nu \cdot x_i'' \leq p^\nu \cdot e_i^\nu$, and $u_i(x_i'') > u_i(x_i^\nu)$, which contradicts $x_i^\nu = f_i(p^\nu, p^\nu \cdot e_i^\nu, b_i^\nu)$.

We end the proof by showing that $p \in S$. From Assumption A2(b), for all $\xi \in R_+^\ell \backslash \{0\}$, there exists $i \in M$ such that $A_i \xi \leq 0$. Consequently, for $t > 0$ small enough, $x_i + t\xi \in X_i(b_i)$. From the strict monotonicity of u_i, $u_i(x_i + t\xi) > u_i(x_i)$. Hence, since x_i is a solution of the above problem, one deduces that $x_i + t\xi$ does not satisfy the budget constraint, which implies $p \cdot \xi > 0$. Since this inequality holds true for all $\xi \in R_+^l \backslash \{0\}$, one gets $p \in S$. Since Π is locally Lipschitz continuous, $\Pi(\mathcal{U} \backslash \mathcal{V})$ is a null set (see Ref. 18). Since Π is proper, $\Pi(\mathcal{U} \backslash \mathcal{V})$ is closed. □

References

1. BALASKO, Y., *Foundations of the General Equilibrium Theory*, Academic Press, New York, NY, 1988.
2. DEBREU, G., *Economies with a Finite Set of Equilibria*, Econometrica, Vol. 38, pp. 387–392, 1970.
3. MAS-COLELL, A., *The Theory of General Economic Equilibrium: A Differentiable Approach*, Cambridge University Press, Cambridge, UK, 1985.
4. SMALE, S., *Global Analysis and Economics*, Handbook of Mathematical Economics, Edited by K. Arrow and M. Intriligator, North-Holland, Amsterdam, Netherlands, Vol. 1, Chapter 8, pp. 331–370, 1981.
5. DEATON, A., and MUELLBAUER, J., *Economics and Consumer Behavior*, Cambridge University Press, Cambridge, UK, 1992.
6. VILLANACCI, A., *A Proof of Existence and Regularity for an Exchange Economy with Nonnegativity Constraints*, DIMEFAS Discussion Paper, University of Florence, Florence, Italy, 1993.
7. DEBREU, G., *Theory of Value*, John Wiley and Sons, New York, NY, 1959.
8. CORNET, B., and VIAL, J. P., *Lipschitzian Solutions of Perturbed Nonlinear Programming Problems*, SIAM Journal on Control and Optimization, Vol. 24, pp. 1123–1137, 1986.
9. FIACCO, A., and MCCORMICK, G., *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, Classics in Applied Mathematics, SIAM, Philadelphia, Pennsylvania, Vol. 4, 1990.
10. ROCKAFELLAR, R. T., and WETS, R. J. B., *Variational Analysis*, Springer Verlag, Berlin, Germany, 1998.
11. KLATTE, D., *Upper Lipschitz Behavior of Solutions to Perturbed $C^{1,1}$ Programs*, Mathematical Programming, Vol. 88B, pp. 285–311, 2000.
12. LEVY, A. B., *Lipschitzian Multifunctions and a Lischitzian Inverse Mapping Theorem*, Mathematics of Operations Research, Vol. 26, pp. 105–118, 2001.
13. SHAPIRO, A., *Sensitivity Analysis of Parametrized Variational Inequalities*, Mathematics of Operations Research, Vol. 30, pp. 109–126, 2005.
14. RADER, T., *Nice Demand Functions*, Econometrica, Vol. 41, pp. 913–935, 1973.
15. SHANNON, C., *Regular Nonsmooth Equations*, Journal of Mathematical Economics, Vol. 23, pp. 147–166, 1994.
16. BONNISSEAU, J. M., and RIVERA-CAYUPI, J., *Constrained Consumptions and Regular Economies*, Cahier de la Maison des Sciences Economiques 2003-51, Université Paris 1 Panthéon-Sorbonne, 2003; see also ftp://mse.univ-paris1.fr/pub/mse/cahiers2003/B03051.pdf.
17. DEIMLING, K., *Nonlinear Functional Analysis*, Springer Verlag, Berlin, Germany, 1985.
18. FEDERER, H., *Geometric Measure Theory*, Springer Verlag, Berlin, Germany, 1969.