How to Sell to Buyers with Crossholdings*

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October 30, 2008

Abstract. This paper characterizes the optimal selling mechanism in the presence of horizontal crossholdings. We find that the optimal mechanism imposes a discrimination policy against the stronger bidders so that the seller's expected revenue is increasing in both the common crossholding and the degree of asymmetry in crossholdings. Furthermore, it can be implemented by a sequential procedure that includes a price-preferences scheme and the possibility of an exclusive deal with the weakest bidder. We also show that a simple sequential negotiation mechanism, although suboptimal, yields a larger seller's expected revenue than both the first-price and the second-price auctions.

Keywords: optimal auctions, crossholdings, asymmetric auctions, private values

JEL Classification: C72, D44, D82, G32, G34

*I am deeply grateful to M. Angeles de Frutos for her comments and guidance. This work has also benefited from comments by Antonio Cabrales, Luciano de Castro, Juan José Dolado, Juan-José Ganuza, Ángel Hernando, Vijay Krishna, Diego Moreno, Georges Siotis, and the participants in the XXII Jornadas de Economía Industrial (Barcelona, 2006), the 2007 ENTER Jamboree (Mannheim, 2007), and the 22th European Economic Association (EEA) Meeting (Budapest, August 2007). The usual disclaimers apply.

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1 Introduction

Auctions in which bidders have crossholdings in other bidders' surplus are very frequent in practice, as there are many cases that resemble a contest with *horizontal* crossholdings. For instance, it is usual in some markets for competing firms to hold shares in one other, or for an important proportion of a company's ownership to belong to non-controller block shareholders, which in turn also hold a controlling stake in a rival company.[1]

Unlike the standard auctions, the presence of horizontal crossholdings introduces counter-value incentives on bidders because they get a payoff not only when they win, but also when they lose the auction. Since the loser bidder appropriates a proportion of the winning surplus, he cares about the valuation and the price paid by the winner bidder. Thus, losing transforms the bidder into a *minority buyer*, which induces a *less* aggressive bidding behavior from him. That is, the incentive to lose counteracts the natural bidder's incentives to raise his bid in order to obtain the object.

The previous literature has studied this kind of auction in a framework where signals are independently distributed and values may be interdependent. This literature has shown that the less aggressive bidding behavior induced by horizontal toeholds produces the classical result of revenue equivalence between standard auctions (Myerson (1981), Riley and Samuelson (1981)) no longer holding, even when bidders have symmetric crossholdings. A seller interested in maximizing her expected revenue should therefore not be indifferent with respect to the mechanism used to assign the object. Consequently, design of an optimal selling mechanism should be a very relevant question for her.

In this paper, we address this question and characterize the optimal selling mechanism in the presence of horizontal crossholdings. To this end, we follow the mechanism design methodology introduced by Myerson (1981) in a setup with independent private values and independently distributed signals. In addition, our modelling strategy allows us to study issues which have not been considered so far.

[1] For the case of direct cross-ownership, Claessens et al. (1998) document the fact that other companies (non-affiliated) constitute one of the most important blockholders in the corporate ownership in various Asian countries. For the case of indirect cross-ownership, Hansen and Lott (1996) report that the portfolios held by institutional investors in the U.S. include shares in competing firms in some markets like the computer industry and the automobile industry. Similarly, Brunello et al. (2001) and Becht and Roell (1999) describe how the pyramidal groups are a very frequent structure for corporate ownership in Italy, France and Belgium.

Our approach is a *normative* one, instead of a *positive* one, which has been the focus of most of the previous literature. In general, this literature compares some standard auctions in terms of the expected revenue that they yield. As mentioned before, the main conclusion is that the revenue equivalence breaks down in the presence of horizontal crossholdings.[2] In contrast, we do not assume the existence of a particular auction format for exogenous reasons, but characterize how the maximizing expected revenue mechanism should be and how this mechanism could be implemented.

One of the few papers that is normative as ours is that of Chillemi (2005). He characterizes the optimal selling mechanism in the presence of horizontal crossholdings, when bidders have positive and symmetric toeholds. His results show that the optimal mechanism is such that the seller's expected revenue is increasing in the common degree of crossholdings since she can extract a higher surplus from the loser bidder. Our work generalizes these results, as we allow for two types of agents: bidders with asymmetric toeholds and bidders without toeholds. The presence of these bidders results in an optimal allocation rule with a double bias. Firstly, among bidders with positive crossholdings, the optimal mechanism discriminates against the bidder with the highest crossholding; and secondly, this mechanism discriminates in a larger degree against the bidder without crossholding. We conclude that this procedure is such that the seller's expected revenue is increasing not only in the size of a common crossholding, but also in the degree of asymmetry of these toeholds.

Consequently, when we make endogenous the bidders' decision about buying/selling crossholdings, we find that their best decision is to transfer no ownership at all. That is, ex ante identical bidders will prefer to keep this symmetry in order to avoid the discrimination policy imposed by the optimal allocation rule. A similar conclusion emerges when we compare this optimal non-transference of crossholdings with two joint bidding strategies: an illegal bid rigging and a legal consortium. In that case, we show that when the seller can design an optimal mechanism as a reaction to these agreements between bidders, the latter will also prefer to remain as symmetric players whenever the informational advantage of collusion generated by its opacity disappears.

Our results concerning the bias against the stronger bidders are analogous to those of the literature about optimal auctions with bidders asymmetrically informed (Povel and Singh (2004) and Povel and Singh (2005)). For instance, Povel and Singh (2005) analyze the case of takeover contests with a general value

[2] See Chillemi (2005) and Ettinger (2002) for private values; and Dasgupta and Tsui (2004) for private and interdependent values.

model that allows a private and a common value environment. They characterize the optimal selling procedure that a target company should design when it faces outside bidders (without vertical toeholds) who are asymmetrically informed, and also conclude as to the optimality of discriminating against the strongest bidder. Similarly, in this paper we find that in the presence of horizontal crossholdings, the optimal mechanism also imposes a heavier discrimination policy on the stronger players of the game. In our model, the strength of each bidder is given by a stochastic comparative advantage resultant from the degree in which each bidder appropriates of his own surplus. The asymmetric cross-ownership structure here assumed is therefore, a central element in explaining the properties of the optimal discrimination allocation rule, and in particular, the monotonicity of its biases with respect to the ranking of advantaged bidders. As did Povel and Singh (2005), we also prove that the optimal mechanism may also be implemented by a two-stage procedure. In the first stage, the seller invites the stronger bidders to participate in a second stage, in a modified first price auction with personalized reserve prices. If both of them reject participation, the object is awarded to the weakest bidder via an exclusive deal for a price which he will always accept. Otherwise, a modified first price auction takes place with the accepting bidders (which will always include the weakest bidder), where the discrimination policy is implemented through a price-preferences scheme.

A central property of the optimal mechanism is that it has to be able to balance out two opposite effects on seller's revenues properly. Since the discrimination policy induces the stronger bidders with high signals to reveal the truth, this enables the seller to extract more value from these bidders and thus, increase her expected revenue. However, this incentive devise is based on a threat with potential costs in terms of efficiency (and thus in terms of creation of value) if it had to be materialized. If the signals of the stronger bidder(s) are not sufficiently high so as to meet the more demanding requirements of the discrimination policy, the seller will have to carry out this threat and assign the object to a weaker bidder, with the risk that his value be smaller than those of the excluded bidder(s). In consequence, the seller's revenue may decrease due to a less ex post creation of value. Notice that it is analogous to the reserve price practice, although here the negative effect on decreasing the creation of value is less severe. This is because the eventual cost of the threat is only to sell the object to a bidder with a smaller value than the excluded bidder, but with a value larger than the seller's one. In contrast, with a reserve price, the object is withdrawn from the auction and is kept in the seller's hands, which in our model always will be worse in terms of created value.

Finally, it is shown that a more simple sequential negotiation mechanism, although

suboptimal, yields a larger seller's expected revenue than both the first-price and the second-price auctions. This finding is explained by the fact that this procedure considers exclusive deals with a timing that gives priority to the stronger bidders, as an attempt to extract surplus selectively, and thus, to replicate the main property of the optimal mechanism.

The remainder of this paper proceeds as follows. Section 2 constructs a model of auctions with horizontal crossholdings. Section 3 characterizes and discusses the properties of the optimal selling mechanism from the seller's viewpoint. The effects of this procedure on the bidders' participation strategies are analyzed in the next section. The implementation of the optimal mechanism via auctions and negotiations is examined in Section 5. Conclusions and extensions are discussed in Section 6. All the proof are gathered in the Appendix.

2 The Model

We have a seller who wants to sell a single object to one of three risk-neutral bidders. The value of the object to bidder i is t_i, which is private information, but the seller and the other bidders know that it is independently and identically distributed according to the c.d.f. F with support $\left[\underline{t}, \overline{t}\right]$, density f and hazard rate $H(t_i) = f(t_i)/(1 - F(t_i))$.[3] Denote by t_0 the seller's value, which is assumed common knowledge and normalized to zero.

A horizontal crossholding of bidder i is defined as a partial participation of this bidder in another bidder's surplus, and we suppose the following ownership link structure. Bidders 1 and 2 have crossholdings in each other, and bidder 3 has no crossholdings in the other bidders' surplus. The parameter θ_i represents the share of bidder i in bidder j's surplus, for all $i, j = 1, 2$ and $i \neq j$. Thus, $(1 - \theta_j)$ represents the participation of bidder i in his own surplus. Crossholdings are assumed common knowledge, with $1/2 > \theta_1 \geq \theta_2 \geq 0$. Finally, no ownership links between bidders and the seller are considered.

It is worthy to make some remarks about the main assumptions of the model. First, the adoption of the simplest valuation and information environment, i.e. the independent private value framework, has the following justification. Since we want to focus on the effects generated by the asymmetry stemming *only* from the different initial stakes held by each bidder, we abstract away from any other sources of asymmetry such as those caused by the valuation and information

[3] We focus on the regular case, i.e., increasing hazard rates, as it is standard in auction theory.

environment. Consequently, we assume identically distributed signals. For a similar reason, we also work with private valuations instead of interdependent ones. Since the presence of common values introduces an extra source of less aggressive bidding behavior -a different one from that induced by crossholdings-, we prefer to examine a simpler valuation setting in order to establish more clearly the effects of crossholdings on the optimal mechanism.[4]

Second, although at first glance, our modelling strategy regarding the number of bidders and the ownership structure seems to be very *ad hoc*, it indeed allows us to analyze, in a very simple way, matters which have not been considered so far by the received literature. In fact, the scarce literature with a normative approach as our work (e.g. Chillemi (2005)) characterizes the optimal selling procedure when bidders possess *positive* and *symmetric* stakes in their rivals. In contrast, our model generalizes this analysis, as it considers two types of agents: bidders with *asymmetric* crossholdings and bidders *without* crossholdings. We shall see that these novelties concerning the ownership structure are crucial to attaining two remarkable results: to obtain an optimal discriminatory allocation rule and to identify properly the source and nature of the biases imposed by such a policy.

3 The Optimal Selling Mechanism

We restrict our attention to a special class of mechanisms: the *direct revelation mechanisms*. Denote by t the vector of signals realizations, i.e., $t = (t_1, t_2, t_3)$, with support T. Similarly, denote by t_{-i} the vector of signal realizations of all bidders except bidder i and T_{-i} its corresponding support. Let $p_i(t)$ be the probability with which the optimal mechanism allocates the object to bidder i, given the vector of reported signal realizations t, and let $x_i(t)$ be the payment from bidder i to the seller. Let $Q_i(t_i)$ be bidder i's conditional probability of winning given that he observes t_i, i.e., $Q_i(t_i) \equiv \int_{T_{-i}} p_i(t_i, t_{-i}) f(t_{-i}) dt_{-i}$. Bidder i's expected payoff, conditional on signal t_i and announcement $\widehat{t_i}$, is then given by[5]

$$U_i(\widehat{t_i}/t_i) \equiv \int_{T_{-i}} \left[(1-\theta_j)(t_i p_i - x_i) + \theta_i(t_j p_j - x_j)\right] f(t_{-i}) dt_{-i}$$

[4] Although we recognize, of course, the importance of characterizing this mechanism under a richer environment, this constitutes an extension of our basic model that should be the aim of future works.

[5] For simplicity, we have omitted the arguments of p_i and x_i, such that $p_i = p_i(\widehat{t_i}, t_{-i})$ and $x_i = x_i(\widehat{t_i}, t_{-i})$, for all i.

for $i, j = 1, 2,\ i \neq j$, and

$$U_3(\widehat{t_3}/t_3) \equiv \int_{T_{-3}} [t_3 p_3 - x_3]\, f(t_{-3}) dt_{-3}$$

for all $t_i, \widehat{t_i} \in \left[\underline{t}, \overline{t}\right],\ i = 1, 2, 3$. We define the truthtelling payoff as $V_i(t_i) \equiv U_i(t_i/t_i)$ and the seller's expected revenue when all bidders tell the truth as

$$U_0 \equiv \sum_{i=1}^{3} \int_T x_i(t) f(t) dt$$

Following Myerson (1981) (see details in the Appendix), we can rewrite the seller's expected payoff as

$$U_0 \ = \sum_{i=1}^{3} \left[-V_i(\underline{t}) + \int_T c_i(t_i) p_i(t) f(t) dt \right] \tag{1}$$

where $c_i(t_i)$, bidder i's marginal revenue, is defined as[6]

$$c_i(t_i) \equiv \begin{cases} t_i - (1 - \theta_j) \frac{1}{H(t_i)} & \text{for } i, j = 1, 2,\ i \neq j \\ t_3 - \frac{1}{H(t_3)} & \text{otherwise} \end{cases}$$

Hence, the optimal mechanism solves the following problem:

$$\max_{p_i, V_i(\underline{t})} \quad U_0$$

$$s.t.$$

$$V_i(\underline{t}) \geq 0, \text{ for all } i \tag{2}$$

$$Q_i'(t_i) \geq 0 \ \text{ for all } t_i \in \left[\underline{t}, \overline{t}\right] \text{ and for all } i. \tag{3}$$

$$\sum_{i=1}^{3} p_i(t) \leq 1 \text{ and } p_i(t) \geq 0, \text{ for all } i \text{ and for all } t \in T \tag{4}$$

where (2) is a sufficient condition for bidder i's participation constraint, (3) is one of the two sufficient conditions for the incentive compatibility constraints of the bidders and (4) corresponds to the feasibility constraints. Notice that when there exist crossholdings, the bidders' reservation utilities are no longer exogenous. The reason for this is the fact that now what a bidder with positive crossholdings can

[6] See Bulow and Roberts (1989) for an interpretation of the bidder i's marginal revenue concept.

get when refusing to participate in the auction depends on the rule used to assign the object among the active bidders. The seller will then take advantage of this phenomenon by designing an alternative mechanism that induces the participation constraint that maximizes her expected revenues. This can be attained by means of an *optimal threat* that allows us to find the minimum reservation utility of a bidder with crossholdings such that he prefers to participate in the auction. Given our ownership structure, this optimal threat consists of selling for sure the object to the bidder without crossholdings (bidder 3) whenever a bidder with crossholdings (either bidders 1 or 2) decides not to participate in the auction.[7] Notice that such a threat constitutes the maximum punishment against the nonparticipating bidder. In fact, the execution of the threat implies that the seller fully appropriates the nonparticipating surplus stemming from the crossholdings and thus, all bidders exhibit the same *zero* reservation utility. Notice finally that the commitment capacity of the seller is critical to the successful of the procedure, especially because of the materialization of the threat may not be ex post optimal.[8]

3.1 Optimal allocation rule

Lemma 3.1 *The optimal mechanism sets* $V_i(\underline{t}) = 0$ *and*

$$p_i(t) = \begin{cases} 1 & \textit{if } c_i(t_i) > \max\{0, \max_{j \neq i} c_j(t_j)\} \\ 0 & \textit{otherwise} \end{cases}$$

for all i*, and for all* $t \in T$*.*

Notice that bidder i's marginal revenue is higher than bidder j's if and only if $t_i > z_{ij}(t_j) \equiv c_i^{-1}(c_j(t_j))$ for all $i \neq j$. Likewise, we define $t_i^* \equiv c_i^{-1}(0)$ as the threshold signal for which bidder i's marginal revenue is higher than the seller's. Then, since c_i and its inverse function are well-behaved, it is equivalent to say

[7] This result is formally derived in the Appendix A.

[8] The endogenous nature of the reservation utilities and its consequences for the participation constraints can also have other sources. For instance, Jehiel, Moldavanu and Stacchetti (1996, 1999) indentify a similar phenomenon when there are auctions with *externalities* between bidders. They show that a revenue maximazing procedure in this context has to include an optimal threat that induces bidders to participate in the auction by guaranting to the *critical type* (the lowest type in our case) the lowest possible reservation utility. Consequently, if the externalities are *negative*, the seller will threat with selling for sure to the bidder who imposes the worst damage to the nonparticipating bidder. In contrast, if the externalities are *positive*, the optimal threat implies that the seller keeps the object.

that the optimal mechanism sets $V_i(\underline{t}) = 0$ and

$$p_i(t) = \begin{cases} 1 & \text{if } t_i > \max\{t_i^*, \max_{j \neq i} z_{ij}(t_j)\} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

for all i, and for all $t \in T$.

3.2 Properties of the optimal mechanism

With horizontal crossholdings, the optimal rule implies a *discriminatory policy* as $z_{ij}(t_j) \neq t_j$.[9] By analyzing the properties of the functions z_{ij}, one can characterize the nature of the biases involved in the optimal mechanism and find out under which circumstances it is revenue maximizing to sell the object to each bidder. This is the content of the next lemma.

Lemma 3.2 *The discriminatory policy functions z_{ij} have the following properties:*
(i) The functions $z_{3j}(t_j)$ and $z_{12}(t_2)$ are strictly increasing in t_j and t_2, respectively.
(ii) The functions $z_{13}(t_3)$, $z_{23}(t_3)$ and $z_{21}(t_1)$ are non-decreasing in t_3 and t_1, respectively.
(iii) At $t_1 = t_2 = t_3 = \underline{t}$, $z_{32}(\underline{t}) > z_{31}(\underline{t}) > \underline{t}$, $z_{12}(\underline{t}) > z_{13}(\underline{t}) = \underline{t}$, and $z_{21}(\underline{t}) = z_{23}(\underline{t}) = \underline{t}$.
(iv) For all $z_{ij}(t_j)$, there exists a unique signal $t_j = \sigma > \underline{t}$ such that $z_{ij}(\sigma) = \sigma$, which is $\sigma = \bar{t}$.
(v) $z_{32}(t) > z_{31}(t) > t$, $z_{12}(t) > t \geq z_{13}(t)$, and $z_{23}(t) \leq z_{21}(t) \leq t$, for all $t < \bar{t}$.

Lemma 3.2 describes two properties of the optimal mechanism. First, it points out that at the optimal mechanism all bidders must experience some degree of discrimination when facing another rival, either a positive or a negative one. Second, for bidders 3 and 1 there exist a non-zero probability interval of signals with which these bidders lose no matter the signal of their opponents. We discuss now the intuition and implications of these properties.

Bias against the bidder with the highest crossholding. Among the bidders with ownership links, the optimal mechanism is biased against the bidder with the

[9] The analysis here is analogous to McAfee and McMillan (1989). They introduce the concept of an optimal discriminatory function in the context of an asymmetric procurement when a government faces domestic and foreign firms with different comparative advantages in costs.

highest crossholding, because he wins the object only if his signal is sufficiently higher than the signal of the other bidder with crossholding. For instance, if t_1 and t_2 are uniformly distributed in the interval $\left[\underline{t}, \bar{t}\right]$, for bidder 1 to win it is needed that $t_1 > z_{12}(t_2) = (1-\alpha_2)t_2 + \alpha_2\bar{t} > t_2$, where $\alpha_2 \equiv (\theta_1 - \theta_2)/(2-\theta_2)$, $0 < \alpha_2 < 1/2$. The intuition of this bias is that the bidder with the higher crossholding exhibits a larger appropriability of his own surplus, which gives him an informational advantage over his rival with the smaller crossholding.[10] Thus, bidder 1 is the strong player that has more incentives to under-report signals. The optimal mechanism then encourages this bidder to reveal high signals by imposing a discriminatory policy against him.[11]

Bias against the bidder without crossholding. In order to compare the treatment given by the optimal selling mechanism to bidders with and without ownership links, assume that the three bidders receive the same signal, i.e., $t_1 = t_2 = t_3 = t$. It is easy to check that $z_{32}(t) > z_{31}(t) > t$ for all t,[12] which implies that the optimal mechanism imposes a bias against the bidder *without* crossholding, because his probability of winning against the bidders with crossholdings is zero when all of them receive the same signal. The intuition behind this bias is that this bidder exhibits a complete appropriability of his own surplus, and thus, he is the most advantaged player of this game in terms of some informational measure. Since bidder 3 has the largest incentives to under-report high signals, the seller has to force him to tell the truth reducing his winning probability to the largest extent if he reports low signals. Hence, the seller will be more demanding with bidder 3 than any of his rivals when awarding the object.[13]

Exclusion of sufficiently low signals. Notice that for sufficiently low signal reports, the probability of winning for bidder 1 and bidder 3 is null: since z_{12} and z_{3j} are strictly increasing functions, if $t_1 < z_{12}(\underline{t})$ then $p_1(t) = 0$ and if $t_3 < z_{3j}(\underline{t})$ then $p_3(t) = 0$. For instance, when the signals are uniformly distributed in the interval $\left[\underline{t}, \bar{t}\right]$, we have that $z_{12}(\underline{t}) = (1-\alpha_2)\underline{t} + \alpha_2\bar{t}$ and $z_{3j}(\underline{t}) = (1-\beta_j)\underline{t} + \beta_j\bar{t}$, where $\beta_j \equiv \theta_i/2$, $0 < \beta_j < 1/4$, and $i, j = 1, 2$, $i \neq j$. It is clear that these upper bounds are higher than $\underline{t}$, which means that the probability that some types of

[10] Formalization of this intuition in terms of stochastic dominance is provided later on.

[11] An alternative explanation is that the bidder with the higher crossholding enjoys the higher losing surplus, and thus, has more incentives to under-report signals. In line with this interpretation, the seller could extract more losing bidder's surplus from him. Nevertheless, this interpretation no longer holds when we consider the bias imposed against the bidder without crossholding.

[12] Or equivalently, $c_2(t) > c_1(t) > c_3(t)$.

[13] An alternative interpretation is that bidder 3 does not face counter-value incentives because he does not have crossholdings at all. In consequence, he can adopt a less aggressive bidding behavior and still defeat his rivals.

bidders 1 and 3 lose for sure is positive. However, notice that these upper bounds are *not* larger than $\bar{t}$, which implies that the optimal mechanism does not exclude completely any of these two bidders; it only ignores reported signals that are sufficiently low, and encourages them to reveal high ones (and thus to pay high transfers).

Monotonicity of the bias. The optimal rule sets the following ranking of *favored* bidders (in descending order): (1) the bidder with the smallest (positive) crossholding, i.e., bidder 2, (2) the bidder with the highest crossholding, i.e., bidder 1, and (3) the bidder without crossholdings, i.e., bidder 3. Notice that there is an apparent *non-monotonicity* in the discrimination introduced by the optimal rule, as this ranking is not monotonic with the ranking of bidders' crossholdings. The next proposition shows however that indeed one can identify a monotonic relationship between the degree of bias against each bidder and their level of some stochastic advantage in the game.

Proposition 3.1 *At the optimal mechanism, it is verified that:*
(i) The larger the proportion of own surplus appropriated by bidder i, the higher the stochastic advantage of this bidder in terms of hazard rate dominance.
(ii) As a consequence of (i), the larger the proportion of own surplus appropriated by bidder i, the heavier the discriminatory policy imposed against him.

The proof of Proposition 3.1 points out that bidder i will be favored against bidder j if and only if the modified distribution function of his valuations is *hazard rate dominated* by the modified distribution function of his rival's valuations. This means that the higher the stochastic advantage of a bidder, the higher the degree of negative discrimination that the optimal mechanism imposes on this bidder. This interpretation of the problem allow us to restate the standard result that in an asymmetric auction the optimal rule is such that the stronger bidders are more discriminated against. Since in our model, the source of this stochastic asymmetry between bidders is the proportion of their own surplus that they retain, this implies that the optimal mechanism establishes a scheme of biases that is indeed increasing with that proportion.

Extraction versus creation of value. A central property of the optimal mechanism is that it induces a trade-off for the seller between extraction and creation of value. On the one hand, the discrimination policy encourages the stronger bidders with high signals to reveal the truth. This enables the seller to extract more value from these bidders and so, increase her expected revenue. On the other hand, this incentive devise is based on a threat with potential costs

in terms of efficiency (and thus in terms of creation of value) whenever it has to be materialized. If the signals of the less favored bidder(s) are not sufficiently high to meet the more demanding requirements of the discrimination policy, the seller will have to execute this threat and to award the object to another bidder, with the risk that his value be smaller than that of the excluded bidder(s). In consequence, the seller revenues may decrease due to a less ex post creation of value. The optimal mechanism must therefore balance out these two opposite effects properly in order to maximize the seller's revenues. Notice that it has a similar effect to the reserve price practice, although here the negative effect on decreasing the creation of value is less severe. This is because the cost of the threat is only to sell the object to a bidder with a smaller value than the excluded bidder, but with a larger value than the seller's one. In contrast, with a reserve price, the object is withdrawn from the auction and kept in the seller's hands, which in our model is always a loss.

Effects on the seller's expected revenue. The optimal mechanism internalizes the fact that bidders with crossholdings want the object to be sold as they also get a share of the winning surplus whenever they lose the auction and the winner is different from the bidder without crossholdings, i.e., bidder 3. This allows the seller to extract some of the surplus from losing bidders. Furthermore, this mechanism is also sensitive to opportunities for strengthening the optimal discrimination policy given by changes in the ownership structure. Two results follow from these two phenomena. First, the seller increases her expected revenue when the intensity of a common crossholding increases because both the losing bidder's surplus is higher and a more severe bias can be imposed against bidder 3 as the comparative stochastic advantage of this bidder increases. The next proposition formalizes this result.

Proposition 3.2 *If the degree of crossholding is symmetric* $(\theta_1 = \theta_2 = \theta > 0)$*, then the seller's expected revenue is increasing in the common degree of ownership links.*

Moreover, the seller also increases her expected revenue when the degree of asymmetry in the crossholdings is higher because she can strengthen the discrimination policy against the bidder who appropriates his own surplus more, improving her ability to extract surplus *selectively* from each bidder. This is the content of the following proposition.

Proposition 3.3 *Suppose that* $(\theta_1 + \theta_2)$ *is constant. Let us define the degree of asymmetry in crossholdings as* $\Delta \equiv (\theta_1 - \theta_2)$*. Then the seller's expected revenue is strictly increasing in this degree of asymmetry.*

4 Bidders' participation strategies

Whereas the last two propositions stress the positive effect that crossholdings have on the seller's expected revenues, they imply however opposite consequences from the bidders' point of view. In fact, as the game played by the seller and the bidders constitutes a zero-sum game in expected terms, these two properties induce indeed an extreme aversion toward crossholdings in bidders. Thus, if bidders without crossholdings had the alternative to transfer minority stakes of ownership between them in a previous stage to that in which the optimal procedure is implemented, they would prefer to remain with the original ownership structure. This corner solution for the case in which values are uniformly distributed in the unitary interval is established in the next proposition.

Proposition 4.1 *Suppose that t_i is uniformly distributed in the interval $[0,1]$ for all i and consider a game with the following timing:*
Stage 1. *Two of the bidders (say bidders 1 and 2) can unilaterally choose a couple (θ_i, θ_j) for $i, j = 1, 2$, $i \neq j$ with $\theta_i, \theta_j \in [0, 1/2)$.*
Stage 2. *Each bidder observes a realization of his value t_i and participates in an auction which corresponds to the optimal mechanism described by Lemma 3.1.*
Then the Subgame Perfect Nash equilibrium of this game is such that it is optimal for these two bidders to choose $(\theta_i^*, \theta_j^*) = (0, 0)$.

The result of Proposition 4.1 means that when bidders know in advance that the seller will design an optimal mechanism as a response to their ownership structure, they will anticipate this behavior and will prefer to face a mechanism that provides them with a symmetric treatment. That is, in order to avoid the biases imposed by the optimal mechanism when two of the three bidders have crossholdings, they will prefer to continue being symmetric players and thus, it will be optimal to transfer no minority ownership between them.

The next proposition compares this optimal non-transference of crossholdings strategy with two joint bidding strategies: an illegal bidding ring and a legal bidding consortium.

Proposition 4.2 *Suppose that t_i is uniformly distributed in the interval $[0,1]$ for all $i = 1, 2, 3$ and consider a game with the following timing:*
Stage 1. *The seller calls for bidders to participate in an auction mechanism whose rules will be optimally designed in Stage 3.*

Stage 2. *Two of the bidders (say bidders 1 and 2) decide about three possible participation strategies: (i) Forming an illegal, efficient and equal profit-sharing bidding ring (S_1^j), (ii) Forming a legal, efficient and equal profit-sharing bidding consortium (S_2^j), or (iii) Unilaterally choosing a couple (θ_j, θ_k) with $\theta_j, \theta_k \in [0, 1/2)$ (S_3^j); for $j, k = 1, 2$, $j \neq k$.*
Stage 3. *The seller designs and implements the optimal selling mechanism according to the observed bidders' participation strategies.*
Stage 4. *Each bidder observes a realization of his value t_i and participates in the auction designed in the previous stage.*
Then for bidder $j = 1, 2$, it is verified that
1. The bidder j's ex-ante truthtelling payoffs yield from each participation strategy are ranked as follows: $V_j(t_j, S_1^j) > V_j(t_j, S_3^{j*}) > V_j(t_j, S_2^j)$ with $S_3^{j*} = (\theta_j^*, \theta_k^*) = (0, 0)$.
2. The Subgame Perfect Nash equilibrium of this game is such that it is optimal for these two bidders to choose S_1^j if the illegal collusion can not be detected. Otherwise, the optimal decision is S_3^{j*}.

The illegal collusive practice dominates therefore the other strategies so long as the ring is not discovered by the seller, as it allows its members to benefit from an informational advantage.[14] Nevertheless, if this practice can be detected for sure by the seller, she will internalize this asymmetry in the optimal mechanism design stage. Notice that in that case the bidding ring becomes strategically equivalent to the consortium as both of them generate the same informational asymmetry, but the extra advantage given by the opacity of the first collusive arrangement vanishes. In consequence, bidders prefer to remain being symmetric players in order to avoid a discriminatory policy against the stronger one (either the ring or the consortium). As shown in Proposition 4.1, the optimal strategy for bidders in that case is the absolute non-transference of crossholdings.[15]

5 How to sell? Auctions vs. Negotiations

In this section we state two results regarding the implementation of the optimal selling mechanism. First, we show that the optimal allocation rule can be implemented using a sequential procedure based mainly on non-standard *auctions*.[16]

[14] Specifically, since the relevant valuation for the efficient ring is the maximum between t_1 and t_2, the ring's valuation distribution function hazard-rate dominates the one of bidder 3.

[15] This can also happen if the illegal nature of the ring deters the bidders' participation.

[16] For the sake of simplicity, to find such an optimal mechanism we assume that $\underline{t}$ is sufficiently high such that $\underline{t}H(\underline{t}) \geq 1$. This implies that it will never be revenue maximinzing for the seller

Second, as to put this auction-based mechanism into practice may be too much complicated, we propose a simpler procedure based on sequential *negotiations*, which, although suboptimal, replicates the main properties of the optimal one.

5.1 The auction-based selling procedure

Our claim is that the properties of the optimal mechanism can be replicated by the following sequential procedure:

Stage I: Call for strong bidders and a (possible) exclusive deal.

Seller invites the strong bidders (3 and 1) to participate in a first-price auction (FPA) with *personalized* reserve prices $\underline{b_3}$ and $\underline{b_1}$, respectively. If both bidders reject participating, the object is offered exclusively to bidder 2 at a price $\underline{b_2}$ such that he will never reject the deal.

Stage II: Competitive bidding process with the accepting bidders.

In this stage, we may have three cases:

II.1. If in Stage I both bidder 1 and bidder 3 are willing to participate, there is a modified FPA between all bidders such that bidder i wins if and only if $b_i > \max_{j \neq i} \widetilde{z_{ij}}(b_j)$ and loses otherwise. The functions $\widetilde{z_{ij}}$ correspond to *price-preferences* that this modified auction introduces in order to replicate the optimal discrimination policy represented by the functions z_{ij} described in the previous section. Notice that thanks to the revelation principle, the optimal allocation rule is expressed in terms of signals which in practice are not observed by the seller. Thus, the price-preferences play the role of translating the optimal discrimination policy to a procedure based on bidders' information actually observed by the seller, which are the bids.[17]

II.2. If in Stage I only bidder 3 accepts participation, there is a modified FPA between bidder 3 and bidder 2 such that bidder 3 wins if and only if $b_3 > \widetilde{z_{32}}(b_2)$ and bidder 2 wins otherwise.

II.3. If in Stage I only bidder 1 accepts participation, there is a modified FPA between bidder 1 and bidder 2 such that bidder 1 wins if and only if $b_1 > \widetilde{z_{12}}(b_2)$ and bidder 2 wins otherwise.

We call this process a modified FPA not only because of the presence of person-

to set a reserve price, and therefore, she always will assign the object to some bidder.

[17] McAfee and McMillan (1989) also analyze the implementation of the optimal discrimination policy through price-preferences in a model of asymmetric government procurements.

alized reserve prices, but also because the price-preferences $\widetilde{z_{ij}}$ imply that finally the winner may not be the bidder who submits the highest bid.

The optimal participation and bidding strategies of each bidder and for each stage are stated in the Appendix (see Lemma 8.1), where a Bayesian Nash equilibrium of this sequential mechanism is fully characterized. The following proposition shows that the mechanism proposed in fact implements the optimal one as it satisfies two conditions: (i) the lowest type bidder gets his reservation payoff, and (ii) the implicit allocation rule coincides with the optimal one.

Proposition 5.1 *The sequential selling procedure is optimal.*

5.2 The negotiation-based selling procedure

Given the potential complications of putting the auction mechanism suggested into practice, it would be interesting to analyze whether another more simple procedure, although suboptimal, may replicate some of the properties of the optimal one. Furthermore, it would be useful to compare this alternative mechanism with some of the auction formats most used in the real world. In line with that analysis, we show that indeed a more simple sequential *negotiation* procedure generates higher expected revenue for the seller than both the FPA and the SPA. The following proposition illustrates this result with two bidders and uniformly distributed valuations.

Proposition 5.2 *Suppose that t_i is uniformly distributed in the interval $[0,1]$ for all $i = 1, 2$, and $\theta_1 > \theta_2 > 0$. Then, consider the following sequential procedure:*
***Stage I**: Negotiation with bidder 1.*
***I.1.** The seller makes a take it-or-leave it offer ρ_1 to bidder 1.*
***I.2.** Bidder 1 observes a realization of his signal t_1 and accepts or rejects this offer. If he accepts, the object is sold to him and the game ends.*
***Stage II**: Negotiation with bidder 2.*
***II.1.** If bidder 1 rejects the deal, the seller makes a new take it-or-leave it offer ρ_2 to bidder 2.*
***II.2** Bidder 2 observes a realization of his signal t_2 and accepts or rejects this offer. If he accepts, the object is sold to him. Otherwise, the object is kept by the seller.*
Then,
1. The Subgame Perfect Nash equilibrium of this game is such that it is optimal for the seller to set $\rho_1^* > \rho_2^*$.

2. At the equilibrium, this mechanism yields a larger seller's expected revenue than both the FPA and the SPA.

The intuition behind this last finding is straightforward. Since the procedure proposed has a negotiation timing that gives priority to bidders according to their own surplus appropriated, it replicates the main property of the optimal mechanism: to impose a discriminatory policy against the stronger bidders.

From the practical point of view, the sequential procedure exhibits realistic properties, as it is frequent the use of rounds of exclusive and preferential negotiations to sell some items. This situation is especially present in the takeover contests, in which the target firm (the board of directors or a special committee) negotiates sequentially and exclusively with the possible raiders. In general, the timetable of these negotiations favors the buyer who is considered the strongest one because of some advantage like a better knowledge of the firm (for instance, a management buy-out), a participation in the target's ownership (a toehold), or something else. This implies that in the real world the seller is indeed able to commit to the rules of the mechanism, even though this may be inefficient ex post. As Povel and Singh (2006) document for the takeover battles, there exists plenty of protection devices aimed to mitigate the opportunistic behavior from the seller and thus, to sustain the deal that had been done previously.[18]

6 Concluding Remarks

We characterize the optimal selling mechanism in the presence of horizontal crossholdings, in a setting with independent private values and independently distributed signals. In this environment, the strength of each bidder is given by a stochastic comparative advantage resultant from the degree in which each bidder appropriates from his own surplus. The asymmetric cross-ownership structure here assumed is therefore, a central element to explain the properties of the optimal allocation rule. In particular, this asymmetry is crucial to the fact that this

[18] Some of these deal protection devices are termination fees, lock-up clauses and poison pills. Recent cases include the sale of the Norwegian Tandberg Television, and the takeover battle for the Spanish tollway operator Europistas. In both cases, the target paid a compensation for revoking a previous exclusive deal in favor of a subsequent buyer. The termination fees were USD 18 million and € 131 million, respectively (see El País, Negocios, November 19, 2006, p. 3; El Economista, August 9, 2006; Tanderberg Television Recommends Ericsson's Offer, http://www.tanderbergtv.com/newsview.ink?newsid=398; Atlanta Business Chronicle, February 26, 2007, http://atlanta.bizjournals.com/atlanta/stories/2007/daily1.html).

procedure discriminates against both the bidder with the highest crossholding and the bidder without crossholding, with the last bias being the most severe.

Furthermore, at the optimal mechanism the seller's expected revenue is increasing not only in the size of a common crossholding, but also in the degree of asymmetry of these crossholdings. These results have two different consequences for the participants in the auction. For the seller, this implies that she will benefit from larger cross-ownership links as it is possible to extract more surplus from the losing bidders whenever he is a bidder with crossholdings, and improve the selectivity of the discriminatory policy as crossholdings become more asymmetric. From the bidders' point of view, the main implication is that when we make endogenous their decision about buying/selling crossholdings, we find that their best decision is to transfer no ownership between them. One of the possible interpretations of this result is that the crossholdings observed in practical auctions are consequence of the fact that the mechanism used by the seller is different from the optimal one. It is likely that for simplicity, regulation issues or from ignorance, the seller decides to apply a standard auction, which in contrast to the optimal mechanism can benefit (hurt) the bidders (seller) as the cross-ownership links are higher.

We show that the optimal allocation rule may be implemented by a sequential procedure that includes a price-preferences scheme and the possibility of an exclusive deal with the weakest bidder. This selling procedure counterbalances properly two opposite effects on the seller's revenues, which arise from the trade-off between extraction and creation of value induced by the optimal mechanism. Interestingly, it is also found that another more simple sequential negotiation procedure, although suboptimal, replicates the main property of the optimal mechanism and dominates the first-price and the second-price auctions in terms of seller's revenues.

The analysis performed in this paper can be extended, at least, in two directions. First, a natural issue is how the properties of the optimal mechanism could change when more complex valuation and information environments are considered, especially due to the extra source of less aggressive bidding behavior introduced by the winner's curse phenomenon. Finally, since the effects induced by vertical crossholdings on the aggressiveness of bidders are opposite those provoked by horizontal crossholdings, finding out what is the optimal selling mechanism in that case also seems to be a relevant extension.

7 Appendix

Appendix A: The optimal mechanism problem.

The optimal mechanism solves the following problem:

$$\max_{x_i \in \mathbb{R},\ p_i \in [0,1],\ \varphi^i \in [0,1]^2} U_0 \tag{6}$$

$$s.t.$$

$$V_i(t_i) \ \geq \varphi^i \overline{u}^i \quad \forall t_i \in \left[\underline{t}, \overline{t}\right],\ i = 1,2,3 \tag{7}$$

$$V_i(t_i) \ \geq U_i(\widehat{t_i}/t_i) \quad \forall t_i, \widehat{t_i} \in \left[\underline{t}, \overline{t}\right],\ i = 1,2,3 \tag{8}$$

$$\sum_{i=1}^{3} p_i(t) \leq 1 \text{ and } p_i(t) \geq 0,\ i = 1,2,3, \forall t \in T \tag{9}$$

where (6) is the seller's expected revenue, (7) and (8) represent bidders' participation constraints and incentive compatibility constraints, respectively, and (9) corresponds to the feasibility constraints. First, notice that since there exist crossholdings, the original participation constraints consider endogenous reservation utilities that depends on the allocation rule adopted by the seller in case of non-participation of one bidder. This rule is represented by $\varphi^i = (\varphi^i_j, \varphi^i_k)$, the vector of probabilities with which the seller assigns the object to bidder j or bidder k if bidder i does not participate in the auction. Similarly, $\overline{u}^i = (\overline{u_j}^i, \overline{u_k}^i)$ represents the vector of outside opportunity utilities of bidder i when bidder j or bidder k gets the object. Given the cross-ownership structure assumed, it is clear that $\overline{u_2}^1 > \overline{u_3}^1$, $\overline{u_1}^2 > \overline{u_3}^2$ and $\overline{u_1}^3 = \overline{u_2}^3$. Hence, it must be optimal that $\varphi^1 = (\varphi^1_2, \varphi^1_3) = (0,1)$ and $\varphi^2 = (\varphi^2_1, \varphi^2_3) = (0,1)$. As we normalize $\overline{u_3}^1 = \overline{u_3}^2 = \overline{u_1}^3 = \overline{u_2}^3 = 0$, all of this implies that the *zero* reservation utility for all bidders is optimal as well.

Second, following Myerson (1981), it is possible to show that the incentive compatibility constraints are satisfied if and only if

(i) $\frac{\partial V_i(t_i)}{\partial t_i} = \begin{cases} (1-\theta_j)Q_i(t_i) & \text{if } \theta_j, \theta_i > 0 \\ Q_i(t_i) & \text{if } \theta_j, \theta_i = 0 \end{cases}$

for $i, j = 1,2,\ i \neq j$

(ii) $\frac{\partial V_3(t_3)}{\partial t_3} = Q_3(t_3)$

(iii) $\frac{\partial Q_i(t_i)}{\partial t_i} \geq 0$ for all i.

Using these conditions, straightforward computations allow us to rewrite the

seller's expected payoff and to simplify the maximization problem as presented in Section 3.

Appendix B: Proofs.

Proof of Lemma 3.1. Clearly from (1), it is in the seller's interest to make $V_i(\underline{t}) = 0$ for all i because $V_i(\underline{t}) > 0$ is suboptimal and setting $V_i(\underline{t}) < 0$ violates the Participation Constraint. On the other hand, $H'(t_i) > 0$ implies $c_i'(t_i) > 0$ and thus $\frac{\partial p_i(t)}{\partial t_i} \geq 0$, so that $Q_i'(t_i) \geq 0$ is satisfied for all i. Finally, since $t_0 = 0$, the optimal allocation rule is found by comparing for a given $t = (t_1, t_2, t_3)$ the terms $c_1(t_1)$, $c_2(t_2)$ and $c_3(t_3)$ whenever they are positive. The solution sets $p_i(t) = 1$ iff $c_i(t_i) > \max\{0, \max_{j \neq i} c_j(t_j)\}$. $\square$

Proof of Lemma 3.2. (i) We only show the claim for z_{31}; the remaining cases are similar and hence omitted. Notice that by definition, $z_{31}(t_1) \equiv c_3^{-1}(c_1(t_1))$. Then, $z_{31}'(t_1) = c_3^{-1\prime}(c_1(t_1))c_1'(t_1) > 0$ follows from the fact that both c_i and its inverse are strictly increasing functions for all i.

(iii) By definition, $z_{12}(\underline{t}) \equiv c_1^{-1}(c_2(\underline{t})) > c_1^{-1}(c_3(\underline{t})) \equiv z_{13}(\underline{t})$, where the inequality follows from the fact that $c_2(\underline{t}) > c_3(\underline{t})$ and the inverse of c_1 is a strictly increasing function. Notice, however, that $z_{13}(\underline{t}) \equiv c_1^{-1}(c_3(\underline{t})) < c_1^{-1}(c_1(\underline{t})) = \underline{t}$, which is not possible and so, we must impose a truncation such that we define

$$z_{13}(t_3) = \begin{cases} \underline{t} & \text{if } \underline{t} \leq t_3 < z_{31}(\underline{t}) \\ c_1^{-1}(c_3(t_3)) & \text{otherwise} \end{cases}$$

Using the same arguments, we can verify that the other cases also hold, which includes the following definition for bidder 2

$$z_{2j}(t_j) = \begin{cases} \underline{t} & \text{if } \underline{t} \leq t_j < z_{j2}(\underline{t}) \\ c_2^{-1}(c_j(t_j)) & \text{otherwise} \end{cases} \text{, for all } j \neq 2$$

(ii) According to the definitions of the discrimination policy functions provided in the proof of (iii), and using the same arguments applied in (i), the desired result follows directly.

(iv) The claim is only proved for z_{31}. First, from (iii) we know that $z_{31}(\underline{t}) > \underline{t}$. Second, notice that $z_{31}(\bar{t}) \equiv c_3^{-1}(c_1(\bar{t})) = \bar{t}$, where the last equality follows from the fact that $c_1(\bar{t}) = c_3(\bar{t})$. Since z_{31} is a strictly increasing function, all of that implies that z_{31} has a unique fixed point $\sigma = \bar{t}$.

(v) This is a direct consequence of results (i)-(iv). $\square$

Proof of Proposition 3.1. Let us define $J_i(t_i) \equiv \frac{H(t_i)}{s_i}$, the *modified* hazard rate of bidder i's value distribution function, where s_i is the proportion in which bidder

i appropriates his own surplus, i.e., $s_1 = 1-\theta_2$, $s_2 = 1-\theta_1$ and $s_3 = 1$. Denote by G_i its corresponding c.d.f.. Since $c_i(t)$ is increasing with t then $z_{ij}(t) \gtreqless t$ iff $c_i(t) \lesseqgtr c_i(z_{ij}(t)) = c_j(t)$, where the last equality follows from the implicit definition of z_{ij}. It is easy to check that this inequality is equivalent to $J_j(t) \gtreqless J_i(t)$ for all t, and for all $i \neq j$, which means that $z_{ij}(t) < t$ iff $G_j \succ_{HRD} G_i$ (i.e., G_j hazard rate dominates G_i). Since $s_3 > s_1 > s_2$ implies that $G_3 \succ_{HRD} G_1 \succ_{HRD} G_2$, the desired result follows. $\square$

Proof of Proposition 3.2. From (1), when $\theta_1 = \theta_2 = \theta > 0$, we obtain that $\frac{\partial U_0}{\partial \theta} = \sum_{i=1}^{2} \int_T \left[\frac{1}{H(t_i)}\right] p_i(t) f(t) dt \geq 0$ because $H(t_i) > 0$ and $p_i(t) \geq 0$ for all t and i. $\square$

Proof of Proposition 3.3. Given some θ_1 and θ_2, from conditions (i)-(iii) of Appendix A and Lemma 3.1, the seller's expected revenue evaluated according to the optimal mechanism is given by

$$\begin{aligned} V_0^{\theta} &= \sum_{\substack{i=1\\ i\neq j}}^{2} \int_T \left[(1-\theta_j) t_i p_i(t) + \theta_i t_j p_j(t)\right] f(t) dt + \int_T t_3 p_3(t) f(t) dt \\ &\quad - \sum_{\substack{i=1\\ i\neq j}}^{2} (1-\theta_j) \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{t_i} Q_i(s_i) ds_i f(t_i) dt_i - \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{t_3} Q_3(s_3) ds_3 f(t_3) dt_3 \end{aligned}$$

Consider an increase and a decrease of ε in θ_1 and θ_2 respectively, with $0 < \varepsilon < \theta_2$. Then, the seller's expected revenue if the ownership link parameters are $\widetilde{\theta}_1 = \theta_1 + \varepsilon$ and $\widetilde{\theta}_2 = \theta_2 - \varepsilon$, but she follows the optimal allocation rule for θ_1 and θ_2, can be reduced to

$$V_0^{\widetilde{\theta}} = V_0^{\theta} + \varepsilon \int_{\underline{t}}^{\bar{t}} \int_{\underline{t}}^{u} \left[Q_2(s) - Q_1(s)\right] ds f(u) du \tag{10}$$

where $Q_i(s) = \Pr(t_j < z_{ji}(s)) \Pr(t_i^* < s)$ for $i, j = 1, 2 \quad i \neq j$. Note that $t_1^* > t_2^*$ implies that $\Pr(t_2^* < s) \geq \Pr(t_1^* < s)$, and from Lemma 3.2 it follows that $F(z_{21}(s)) < F(z_{12}(s)))$ for all $s \in \left[\underline{t}, \bar{t}\right)$. All of that implies that $Q_1(s) \leq Q_2(s)$ for a given signal s. As long as the exclusion of both bidders is not possible for all s, the last result implies from (10) that $V_0^{\widetilde{\theta}} > V_0^{\theta}$. That is, the expected revenue can become larger as asymmetry increases, without changing the allocation rule. Therefore, the seller may additionally increase her expected revenue by switching to the optimal allocation rule. $\square$

Proof of Proposition 4.1. Applying backward induction, firstly we have to find the Nash equilibrium of Stage 2. Since we know this equilibrium from Lemma 3.1

and the fact that the optimal mechanism induces a truthtelling bidders' strategy via the incentive compatibility constraint, we only concentrate on the Nash equilibrium of the complete game. To this end, we previously need to characterize the objective function for bidder $i = 1{,}2$ in Stage 1, using the equilibrium of Stage 2. From the definition of $Q_i(s_i)$, conditions (i)-(iii) of Appendix A and Lemma 3.1, we obtain that when t_i is uniformly distributed in the unitary interval, the truthtelling payoff is given by $V_i(t_i) = (1-\theta_j)\int_{T_{-i}}[t_i - z_i(t_{-i})]\,1_{\{t_i \geq z_i(t_{-i})\}}dt_{-i}$ where

$$\begin{aligned} z_i(t_{-i}) &= \inf\{s_i : c_i(s_i) \geq 0 \text{ and } c_i(s_i) \geq c_j(t_j) \text{ for all } j \neq i\} \\ &= \max\left\{t_i^*, (1-\alpha_j)t_j + \alpha_j, \frac{t_3}{1-\beta_i} - \frac{\beta_i}{1-\beta_i}\right\} \text{ for } i \neq j \end{aligned}$$

is the infimum of all winning values for i against t_{-i} and $t_i^* = \frac{1-\theta_j}{2-\theta_j}$, with α_j and β_i defined as in Section 3. After integrating, we obtain the truthtelling payoff of bidder i at the interim state. For the sake of presentation, we omit this expression here, but we represent it using the generic function $v_i(t_i, \theta_i, \theta_j)$ for the term $\int_{T_{-i}}[t_i - z_i(t_{-i})]\,dt_{-i}$ as follows:

$$\begin{aligned} V_i(t_i, \theta_i, \theta_j) &= (1-\theta_j)[v_i(t_i, \theta_i, \theta_j)1_{\{t_i > z_i(t_{-i}) = t_i^*\}} \\ &\quad + v_i(t_i, \theta_i, \theta_j)1_{\{t_i > z_i(t_{-i}) = (1-\alpha_j)t_j + \alpha_j\}} \\ &\quad + v_i(t_i, \theta_i, \theta_j)1_{\{t_i > z_i(t_{-i}) = t_3/(1-\beta_i) - \beta_i/(1-\beta_i)\}}] \end{aligned}$$

Taking expectation with respect to t_i, we get the ex-ante truthtelling payoff for bidder i, which we summarize as:[19]

$$V_i(\theta_i, \theta_j) \equiv E_{t_i}V_i(t_i, \theta_i, \theta_j) = \int_{t_i = t_i^*}^{1} V_i(t_i, \theta_i, \theta_j)$$

Hence, at Stage 1 bidder i has to solve the following program:

$$\begin{gathered} \max_{(\theta_i, \theta_j)} \; V_i(\theta_i, \theta_j) \\ s.t. \\ 0 \leq \theta_i < 1/2 \\ 0 \leq \theta_j < 1/2 \end{gathered}$$

[19] The explicit expressions of $V_i(t_i, \theta_i, \theta_j)$ and $V_i(\theta_i, \theta_j)$ are available on request.

for $i, j = 1, 2, i \neq j$. Finally, we can check that $\frac{\partial V_i(\theta_i,\theta_j)}{\partial \theta_i} < 0$ and $\frac{\partial V_i(\theta_i,\theta_j)}{\partial \theta_j} < 0$ for all $\theta_i, \theta_j \in [0, 1/2)$. This implies that this program has only a corner solution such that $\theta_i^* = \theta_j^* = 0$, which completes the proof. $\square$

Proof of Proposition 4.2. Applying backward induction, firstly we need to characterize the Bayesian Nash equilibrium resulting from Stage 4 for the two possible optimal selling mechanisms that can be implemented in Stage 3. Since these mechanisms satisfy the incentive compatibility constraint it will be in the best interest of the bidders to follow a truthtelling strategy. Then, we characterize the possible optimal mechanisms in Stage 3.

First, if bidders 1 and 2 decide to form an efficient consortium, the seller will design a mechanism taking into account that the relevant valuation for the coalition is $t_C = \max\{t_1, t_2\}$. Since the consortium's value distribution hazard-rate dominates the bidder 3's, the seller will design an optimal auction with asymmetric bidders so that it will impose a bias against the strongest player of the game, i.e., the consortium. Following the same methodology in Proof of Proposition 3.4, we obtain that the consortium's truthtelling payoff is given by $V_c(t_c) = \int_{t_3} [t_c - z_c(t_3)]\, 1_{\{t_c \geq z_c(t_3)\}} dt_3$ where[20]

$$z_c(t_3) = \max\left\{\sqrt{\frac{1}{3}}, \frac{2t_3 - 1}{2} + \frac{2\sqrt{t_3(t_3-1)+1}}{3}\right\}$$

After integrating, the consortium's truthtelling payoff at the interim state is given by

$$V_c(t_c) = \begin{cases} t_c - 0.67601 & \text{if } t_c \geq z_c(t_3) \\ 0 & \text{otherwise} \end{cases}$$

and the ex-ante consortium's truthtelling payoff is $E_{t_c} V_c(t_c) = 0.08769$. Under the equal profit-sharing rule, each partner of the consortium gets in expected terms $V_j(t_j, S_j^2) \equiv \frac{1}{2} E_{t_c} V_c(t_c) = 0.043845$.

Second, if bidders 1 and 2 decide to form an efficient (but illegal) bidding ring, the seller (and also bidder 3) is not aware of the existence of this coalition when designing the optimal mechanism. In particular, whereas the seller believes that she faces three symmetric bidders, the ring has an informational advantage similar to the consortium case because its relevant valuation is $t_R = \max\{t_1, t_2\}$. Thus, the seller incorrectly designs a standard optimal mechanism with symmetric bidders (Myerson (1981)). Assuming that this procedure is implemented by

[20] Notice that $c_c^{-1}(c_3(t_3)) = \frac{2t_3-1}{2} \pm \frac{2\sqrt{t_3(t_3-1)+1}}{3}$. For the computations, we only consider the positive root.

a second-price auction with a reserve price, the ring's truthtelling payoff is given by

$$V_R(t_R) = \int_{t_3} \left[t_R p_R(t_R, t_3) - x_R(t_R, t_3)\right] 1_{\{t_R \geq z_R(t_3)\}} dt_3$$

where $z_R(t_3) = \max\left\{\frac{1}{2}, t_3\right\}$,

$$p_R(t_R, t_3) = \begin{cases} 1 & \text{if } t_R \geq z_R(t_3) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x_R(t_R, t_3) = \begin{cases} z_R(t_3) & \text{if } p_R(t_R, t_3) = 1 \\ 0 & \text{otherwise} \end{cases}$$

The ring's truthtelling payoff at the interim state is then given by

$$V_R(t_R) = \begin{cases} \frac{t_R^2}{2} - \frac{1}{8} & \text{if } t_R \geq z_R(t_3) \\ 0 & \text{otherwise} \end{cases}$$

and its corresponding ex-ante truthtelling payoff is $E_{t_R} V_R(t_R) = 0.140625$. Each member of the ring obtains then, in expected terms, $V_j(t_j, S_j^1) \equiv \frac{1}{2} E_{t_R} V_R(t_R) = 0.0703125$.

Third, from Proposition 3.4 we know that both bidders optimally choose zero crossholdings when deciding about the transfer of crossholdings between them, such that $S_3^{j*} = (\theta_j^*, \theta_k^*) = (0, 0)$ for $j, k = 1, 2,\ j \neq k$. Thus, the seller now correctly designs a standard optimal mechanism with symmetric bidders. In that case, each bidder gets $V_i(t_i) = \int_{T_{-i}} \left[t_i - z_i(t_{-i})\right] 1_{\{t_i \geq z_i(t_{-i})\}} dt_{-i}$ where $z_i(t_{-i}) = \max\{1/2, \max_{j \neq i} t_j\}$. The bidder i's truthtelling payoff at the interim state becomes

$$V_i(t_i) = \begin{cases} \frac{t_i^3}{3} - \frac{1}{24} & \text{if } t_i \geq z_i(t_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

and its corresponding ex-ante truthtelling payoff is given by $V_i(t_i, S_3^{i*}) \equiv E_{t_i} V_i(t_i) = 0.05729166$. We can therefore establish the following ranking for bidders 1 and 2: $V_j(t_j, S_1^j) > V_j(t_j, S_3^{j*}) > V_j(t_j, S_2^j)$.

Hence, it follows directly that at the participation decision stage (Stage 2) these two bidders prefer the strategy S_1^j whether the existence of the bidding ring is unknown by the seller, as the coalition can take advantage of the informational asymmetry. However, if the ring can be discovered with certainty, the bidding ring's strategy becomes, from a bidder's viewpoint, similar to the consortium strategy. In that case, it is clear from the ex-ante truthelling payoffs that the zero crossholding strategy dominates both joint bidding practices. □

Proof of Proposition 5.1. We need previously to state the next auxiliary result.

Lemma 7.1 *A Bayesian Nash equilibrium of the sequential procedure is the following one:*
Bidder 3' strategy. *Accept participation in Stage II if and only if* $t_3 \geq z_{32}(\underline{t})$*; and in Stage II bid:*

$$b_3^{II.1}(t_3) = E\left[\max\left\{z_{32}(t_2), z_{31}(t_1)\right\} \mid \max\left\{z_{12}(t_2), t_1\right\} < z_{13}(t_3)\right]$$

and

$$b_3^{II.2}(t_3) = E\left[z_{32}(t_2) | t_2 < z_{23}(t_3)\right]$$

Bidder 1' strategy. *Accept participation in Stage II if and only if* $t_1 \geq z_{12}(\underline{t})$*; and in Stage II bid:*

$$b_1^{II.1}(t_1) = E\left[\max\left\{\frac{z_{12}(t_2) - \Theta_1 t_2}{1-\Theta_1}, z_{13}(t_3)\right\} \mid \max\left\{t_2, z_{23}(t_3)\right\} < z_{21}(t_1)\right]$$

and

$$b_1^{II.3}(t_1) = E\left[\frac{z_{12}(t_2) - \Theta_1 t_2}{1-\Theta_1} | t_2 < z_{21}(t_1)\right]$$

Bidder 2' strategy. *Accept the offer to pay* $\underline{b_2}$ *in Stage I; and in Stage II bid:*[21]

$$b_2^{II.1}(t_2) = \frac{\underline{t}L(\underline{t})}{L(t_2)} + E\left[\max\left\{\frac{z_{21}(t_1) - \Theta_2 t_1}{1-\Theta_2}, z_{23}(t_3)\right\} \mid \max\left\{t_1, z_{13}(t_3)\right\} < z_{12}(t_2)\right]$$

$$b_2^{II.2}(t_2) = \frac{\underline{t}F(z_{32}(\underline{t}))}{F(z_{32}(t_2))} + E\left[z_{23}(t_3) | t_3 < z_{32}(t_2)\right]$$

$$b_2^{II.3}(t_2) = \frac{\underline{t}L_1(\underline{t})}{L_1(t_2)} + E\left[\frac{z_{21}(t_1) - \Theta_2 t_1}{1-\Theta_2} | t_1 \in \left[z_{12}(\underline{t}), z_{12}(t_2)\right]\right]$$

where $\Theta_i \equiv \frac{\theta_i}{1-\theta_j}$ *for* $i, j = 1, 2,\ i \neq j$*. These are equilibrium strategies if the seller designs a modified FPA with the following characteristics:*
$\underline{b_3} = z_{32}(\underline{t})\underline{b_1} = z_{12}(\underline{t})$
$\underline{b_2}$ *such that* $\Gamma_2(\underline{t}) = 0$
$\widetilde{z_{ij}}(b) = b_i(z_{ij}(b_j^{-1}(b)))$
where $\Gamma_i(.)$ *represents the bidder i's expected truthtelling payoff (i.e. the average across all stages) in this sequential mechanism.*

Proof of Lemma 8.1. We only demonstrate that these candidate bidding functions constitute an equilibrium for the most general case: bidder 1. Define:

[21] Notice that $L(t) = F^{1-\Theta_2}(z_{12}(t))F(z_{32}(t))$ and $L_1(t) = F^{1-\Theta_2}(z_{12}(t))$.

$b_1^k(t_1)$, the candidate bidding function for bidder 1 in Stage k, as follows

$$b_1^k(t_1) = \begin{cases} b_1^{II.1}(t_1) & \text{if } t_1 > z_{12}(\underline{t}) \text{ and } t_3 > z_{32}(\underline{t}) \text{ (Stage II.1)} \\ b_1^{II.3}(t_1) & \text{if } t_1 > z_{12}(\underline{t}) \text{ and } t_3 < z_{32}(\underline{t}) \text{ (Stage II.3)} \\ 0 & \text{otherwise (Stage II.2 or Stage I)} \end{cases}$$

$q_i^k(t_i, t_{-i})$ as the probability that bidder i gets the object in Stage k,

$\Lambda_1^k(\widehat{t_1}/t_1)$, bidder 1's expected payoff in Stage k when he observes t_1, but plays the strategy as if his signal were $\widehat{t_1}$, as follows

$$\int_{T_{-1}} \left\{(1-\theta_2)\left[t_1 - b_1^k(\widehat{t_1})\right] q_1^k(\widehat{t_1}, t_{-1}) + \theta_1 \left[t_2 - b_2^k(t_2)\right] q_2^k(\widehat{t_1}, t_{-1})\right\} f(t_{-1}) dt_{-1}$$

$\Gamma_1^k(t_1) \equiv \Lambda_1^k(t_1/t_1)$, bidder 1's truthtelling payoff in Stage k, and

$P_1^k(t_1) \equiv \int_{T_{-1}} q_1^k(t_1, t_{-1}) f(t_{-1}) dt_{-1}$, bidder 1's probability of winning in Stage k, conditional on the realization t_1.

Let us organize this proof in two steps.

Step 1. Notice that the payoff function Γ_1^k corresponds to the particular case of the truthtelling payoff function V_1 defined in Section 3 when the optimal payment is $x_1(t_1, t_{-1}) = b_1^k(t_1)$ and the optimal allocation rule is $p_1(t_1, t_{-1}) = q_1^k(t_1, t_{-1})$.[22] It follows then directly from conditions (i) and (iii) of Appendix A that the incentive compatibility constraint $\Gamma_1^k(t_1) \geq \Lambda_1^k(\widehat{t_1}/t_1)$ for all $t_1, \widehat{t_1} \in \left[\underline{t}, \overline{t}\right]$ and k, is satisfied if $\frac{\partial \Gamma_1^k(t_1)}{\partial t_1} = (1-\theta_2) P_1^k(t_1)$ and $\frac{\partial P_1^k(t_1)}{\partial t_1} \geq 0$ for all k.

Step 2. We show now that the strategy $b_1^k(t_1)$ satisfies these two sufficient conditions and therefore is an equilibrium bidding strategy of this game. First, notice that $b_1^k(t_1)$ is increasing for stages II.1 and II.3 and, since by construction $\widetilde{z_{ji}}$ implements the optimal allocation rule, we have that

$$P_1^k(t_1) = \begin{cases} F(z_{21}(t_1))F(z_{31}(t_1)) & \text{if } t_1 > z_{12}(\underline{t}) \text{ and } t_3 > z_{32}(\underline{t}) \text{ (Stage II.1)} \\ F(z_{21}(t_1)) & \text{if } t_1 > z_{12}(\underline{t}) \text{ and } t_3 < z_{32}(\underline{t}) \text{ (Stage II.3)} \\ 0 & \text{otherwise (Stage II.2 or Stage I)} \end{cases} \tag{11}$$

Notice that $\frac{\partial P_1^k(t_1)}{\partial t_1} \geq 0$ is satisfied both for each stage and across stages, as by assumption $f(z_{i1}(t_1)) > 0$, $F(z_{i1}(t_1)) > 0$ and by Lemma 3.2 $z_{i1}'(t_1) > 0$, for all $t_1 > z_{12}(\underline{t})$. We prove now that the second sufficient condition also holds. If $t_1 >$

[22] In particular, notice that since $b_i^k(t_1) = 0$ when $q_i^k(t_i, t_{-i}) = 0$, we can factorize the surplus of bidder 1 and 2 in terms of $q_i^k(t_i, t_{-i})$.

$z_{12}(\underline{t})$ and $t_3 > z_{32}(\underline{t})$ (Stage II.1), it can be checked after some computations that $\frac{\partial \Gamma_1^{II.1}(t_1)}{\partial t_1} = (1-\theta_2)F(z_{21}(t_1))F(z_{31}(t_1)) = (1-\theta_2)P_1^k(t_1)$, where the second equality follows from (11). Using the same argument, a similar result holds for Stage II.3. Finally, when $t_1 < z_{12}(\underline{t})$, bidder 1 does not participate in the auction. Noting that $z_{21}(t_1) = \underline{t}$ for all $t_1 < z_{12}(\underline{t})$, we can verify that $\frac{\partial \Gamma_1^k(t_1)}{\partial t_1} = 0 = (1-\theta_2)P_1^k(t_1)$, where the second equality follows from (11), which completes the proof. □

We are now prepared to demonstrate Proposition 5.1.

Proof of Proposition 5.1. From Lemma 3.1, we know that a selling procedure is optimal if it satisfies two conditions: (1) the bidder with the lowest possible signal realization gets his reservation payoff (which in Appendix A we have showed to be optimally the same for all bidders and normalized to zero), and (2) it uses the optimal allocation rule. Notice that by construction, the sequential selling procedure satisfies both conditions. First, the payoff for either bidder with signal $\underline{t}$ is zero: (i) bidder 3 does not participate in Stage II (because $z_{32}(\underline{t}) > \underline{t}$) and thus, he gets $\Gamma_3^k(\underline{t}) = 0$ for all stage k; (ii) bidder 2 loses the auction for sure if some other bidder agrees to participate in Stage II and thus, he has a positive expected payoff. Otherwise, he pays $\underline{b_2}$ in the exclusive deal, which by construction, guarantees that the average payoff across all stages in the sequential mechanism for the lowest-type is $\Gamma_2(\underline{t}) = 0$; and (iii) bidder 1 neither participates in Stage II (because $z_{12}(\underline{t}) > \underline{t}$), and result (ii) also ensures that he gets in expected terms (as average across all stages) $\Gamma_1(\underline{t}) = 0$. Second, the allocation rule is the optimal one as we can check that $b_i > \widetilde{z_{ij}}(b_j)$ iff $t_i > z_{ij}(t_j)$ using the definition of $\widetilde{z_{ij}}(.)$. □

Proof of Proposition 5.2. Applying backward induction, firstly we need to characterize the Nash equilibrium resulting from Stage II. In this stage, bidder 2 accepts the offer if $(1-\theta_1)(t_2-\rho_2) > 0$, i.e., if $t_2 > \rho_2$, and rejects otherwise. The seller therefore has to optimally choose the offer given by $\rho_2^* = \arg\max_{\rho_2}(1-\rho_2)\rho_2$. After solving, we get that $\rho_2^* = 1/2$ and the optimal seller's expected revenue from this stage is equal to 1/4.

In Stage I, bidder 1 accepts any seller's offer if his payoff is larger than the expected payoff at the equilibrium of stage II. That is, if

$$\begin{aligned}(1-\theta_2)(t_1-\rho_1) &> E_{t_2}\left[\theta_1(t_2-\rho_2^*)|t_2 > \rho_2^*\right]\Pr(t_2 > \rho_2^*) \\ &= \frac{\theta_1}{8},\end{aligned}$$

which is equivalent to the condition $t_1 > \rho_1 + \theta_1/8(1-\theta_2)$. Thus, the seller's

optimal offer is characterized by

$$\rho_1^* = \arg\max_{\rho_1} \left[(1 - (\rho_1 + \frac{\theta_1}{8(1-\theta_2)}))\rho_1 + (\rho_1 + \frac{\theta_1}{8(1-\theta_2)})\frac{1}{4} \right]$$

The solution is given by $\rho_1^* = 5/8 - \theta_1/16(1-\theta_2)$, which yields an optimal seller's expected revenue equal to $(100 - \lambda(12-\lambda))/256$, where $\lambda \equiv \theta_1/(1-\theta_2)$. Hence, and since $\lambda < 1$, it is easy to verify that $\rho_1^* > \rho_2^*$, which proves the first statement of the proposition.

In order to show the second result, notice that in the presence of asymmetric crossholdings it is not possible to find an analytical expression for the equilibrium bidding strategy in both FPA and SPA, and thereby, it is neither possible to obtain a closed expression for the seller's revenue (see Section 5, Dasgupta and Tsui (2004)). Notice however that we can perform a comparison with both FPA and SPA without crossholdings, which yield a larger expected revenue than their versions with crossholdings due to the fact that these ownership links hurt the seller (see Proposition 1 and Section 4, Chillemi (2005)). So, it is enough to show that the expected revenue of the sequential mechanism proposed exceeds the expected revenue for both FPA and SPA without crossholdings, which thanks to the Revenue Equivalence Theorem is the same for both auction formats and equal to 1/3. Since $\lambda < 1$, the worst case for our sequential negotiation mechanism is when $\lambda \to 1$, in which case the expected revenue for the seller converges to $89/256 > 1/3$, implying that the second part of the proposition holds. □

8 References

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