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The impossibility of effective enforcement mechanisms in collateralized credit markets[☆]

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ABSTRACT

We analyze the possibility of the simultaneous presence of two key features in price-taking sequential economies: collateralized credit operations and effective additional enforcement mechanisms, i.e. those implying payments besides the value of collateral guarantees.

We show that these additional mechanisms, instead of strengthening, actually weaken the restrictions that collateral places on borrowing. In fact, when collateral requirements are not large enough in relation to the effectiveness of the additional mechanisms, lenders anticipate payments exceeding the value of the collateral requirements. Thus, by non-arbitrage, they lend more than the value of these guarantees. In turn, in the absence of other market frictions such as borrowing constraints, agents may indefinitely postpone the payment of their debts, implying the collapse of the agent's maximization problem and of such credit markets.

1. Introduction

In modern financial markets, collateral guarantees play an important role in enforcing borrowers not to entirely default on their financial promises. These guarantees are used in several credit operations, from corporate bonds to Collateralized Mortgages Obligations,[1] allowing markets to reduce credit risk and increase portfolio diversification. However, to protect investors from the excess of losses induced by large negative shocks in the value of collateral guarantees, financial markets may create and implement additional enforcement mechanisms against default. In this paper, we focus on the theoretical effects of such a policy on the agent's maximization problem and, consequently, on the price-taking credit market.

In the infinite horizon context, the incentives provided by the collateralization of financial contracts are mostly addressed when the only enforcement mechanism against default is the seizure of the associated guarantees.[2] Particularly, for incomplete market economies, Araujo et al. (2002) prove the existence of equilibrium independent of the choice of collateral

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[1] That is, derivative assets secured by pools of individual mortgages, each of which is backed mostly by real estates.

[2] *Related literature.* The inclusion of collateralized debt in a general equilibrium framework was originally studied by Geanakoplos (1997) and Geanakoplos and Zame (2002). These seminal two-period models were extended to more general securitization structures by Steinert and Torres-Martínez (2007), and to sequential economies with infinite lived agents by Araujo et al. (2002). Kubler and Schmedders (2003) analyze Markovian economies, proving the existence of stationary equilibria and providing an algorithm to approximate such equilibria numerically. Recently, the role of collateralized assets in sequential economies where agents have uncertain lifetimes was addressed by Seghir and Torres-Martínez (2008).

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guarantees. One important consequence is that such simple financial structure keeps the credit market from collapsing. Essentially, since agents default only when the value of the collateral requirements is smaller than that of the respective financial promise, the net payoff of lending is always less than or equal to the one associated to holding the amount of the required collateral. Therefore, by the absence of arbitrage, the value of any loan has to be less than the value of the respective collateral, precluding agents to become leveraged and eliminating Ponzi schemes.

However, Páscoa and Seghir (2009) have shown that the results above may not hold when linear utility penalties for default act as an additional enforcement mechanism besides the seizure of collateral guarantees.[3] They provide examples of deterministic economies where sufficiently harsh penalties induce agents to fully honor their debts, paying more than the depreciated collateral, which, by non-arbitrage, may lead the value of collateral requirements to be persistently lower than that of the loan. In such a context, given any budget feasible plan, agents may improve their utilities by taking new loans, constituting the associated collateral guarantees and increasing their consumption. Therefore, there is no individual's optimal plan and we cannot define a credit market.

In this paper, we analyze how the interaction between collateral guarantees and generic additional enforcement mechanisms can extend the result from Páscoa and Seghir's particular examples to more general economies. Doing so, we identify the effectiveness of these additional mechanisms as an important economic concept responsible for the result. Additionally, we argue that two new features of our approach enable us to reach further conclusions than the ones already presented in the literature. First, instead of using a general equilibrium framework, we only analyze the decision problem of one agent. Second, we work with a reduced form approach to model the inclusion of additional enforcement mechanisms against default.

Focusing on the maximization problem of a price taker agent in an economy analogous to that studied by Araujo et al. (2002), we introduce *effective* additional enforcement mechanisms, i.e. mechanisms enforcing payments besides the value of the collateral guarantees. We represent these additional mechanisms by their effectiveness on enforcing payments besides the value of the collateral requirements. Thus, we do not intend to explicitly model how the market imposes additional payments on borrowers besides the value of collateral guarantees. However, with this reduced form approach, we can concentrate on the pricing and market effects of these additional mechanisms. In fact, we derive an explicit relationship between the primitives of the economy, such as the effectiveness and collateral requirements, implying the collapse of the agent's maximization problem.

Essentially, we only need one agent to reach conclusions about asset pricing in competitive credit markets. Additionally, with the reduced form approach for the additional mechanisms against default such pricing becomes well tractable. Then, as we include enforcement mechanisms in addition to the seizure of collateral requirements, lenders may expect sufficiently large payments for their loans besides the value of the these requirements. In such situation, these additional mechanisms, instead of strengthening, actually weaken the restrictions that collateral places on borrowing. In fact, lenders anticipate that, even in case of default, they still receive more than just the value of the collateral guarantees. Thus, by non-arbitrage, they lend more than the value of these guarantees. On the other hand, borrowers have the incentive and the possibility to take new credits in order to pay their older ones, since there is no debt constraints or monitoring precluding agents to incur in a Ponzi scheme. Their behavior, then, leads to the non-existence of a physical feasible solution for the agent's problem.

Regarding the relationship found between the primitives of the economy, we may view it from two different perspectives. From the first one, given a level of effectiveness of the additional mechanisms, we show that there are strictly positive upper bounds for collateral requirements under which agents have incentives to indefinitely postpone their debts through new credits, leading to the non-existence of an optimal utility maximizing plan. Therefore, the market choice of collateral guarantees becomes relevant. From the second one, we provide theoretical foundations to the examples given by Páscoa and Seghir (2009). That is, given collateral requirements, we show that any sufficiently effective additional enforcement mechanism implies the non-existence of physical feasible individuals' optimal plans. Hence, it is the effectiveness of these mechanisms that brings the main result, not any mechanism *per se*.

The remainder of the paper is organized as follows: Section 2 presents an infinite-horizon economy with assets subject to default and with effective enforcement mechanisms in addition to collateral repossession. In Section 3 we show our main result. Some extensions are discussed in Sections 4 and 5.

2. Model

Consider a discrete-time infinite-horizon economy with uncertainty and symmetric information. Let S be the set of states of nature and $\mathbb{F}_t$ the information available at period $t \in T := \mathbb{N} \cup \{0\}$. $\mathbb{F}_t$ is a partition of S, and if $t' > t$, make $\mathbb{F}_{t'}$ finer than $\mathbb{F}_t$. Summarizing the uncertainty structure, define an event-tree as $D = \{(t, \sigma) \in T \times 2^S : t \in T, \sigma \in \mathbb{F}_t\}$, where a pair $\xi := (t, \sigma) \in D$ is called a node and $t(\xi) := t$ is the associated period of time. For simplicity, at $t = 0$ there is no information, that is $\mathbb{F}_0 := \{S\}$ and, therefore, there is only one node, which is denoted by ξ_0.

A node $\xi' = (t', \psi')$ is a successor of $\xi = (t, \psi)$, denoted by $\xi' \geq \xi$, if $t' \geq t$ and $\psi' \subseteq \psi$. Given $\xi \in D$, the set of its successors is given by the subtree $D(\xi) := \{\mu \in D : \mu \geq \xi\}$. Also, for each $\xi \neq \xi_0$, since $\mathbb{F}_{t(\xi)}$ is finer than $\mathbb{F}_{t(\xi)-1}$, there is only one predecessor, $\xi^- \in D$. We say that $\mu \in D$ is an immediate successor of ξ when it is in the set $\xi^+ := \{\xi' \in D : \xi' \geq \xi, t(\xi') = t(\xi) + 1\}$.

[3] Páscoa and Seghir's model is an extension of Zame (1993) and Dubey et al. (2005) two-period equilibrium models with default and utility penalties, to allow for infinite horizon and collateralized debt.

At each node $\xi \in D$ there is a non-empty and finite set of commodities, L. These commodities may be traded in a competitive market at unitary prices $p_\xi = (p_{(\xi,l)})_{l \in L} \in \mathbb{R}^L_+$ by a non-empty set of consumers. Also, at any node $\xi > \xi_0$, there is a technology represented by a matrix with non-negative entries, $Y_\xi := ((Y_\xi)_{l,l'}; (l, l') \in L \times L)$, which transform commodity bundles consumed at ξ^-, allowing for durable commodities. Thus, for each $(l, l') \in L \times L$, $(Y_\xi)_{l,l'}$ is the amount of commodity l obtained at ξ if one unit of commodity l' is consumed at ξ^-. Also, let $W_\xi \in \mathbb{R}^L_+$ be the aggregate physical resources up to node ξ, while $W = (W_\xi)_{\xi \in D}$ is the plan of such resources.

There is a finite set of real assets $J(\xi)$ at each node $\xi \in D$. Each j in $J(\xi)$ is short-lived, has promises $A_{(\mu,j)} \in \mathbb{R}^L_+$ at each node $\mu \in \xi^+$, and is traded in competitive markets by a unitary price $q_{(\xi,j)} \in \mathbb{R}_+$. Since assets are subject to credit risk, borrowers are burdened to constitute physical collateral guarantees in order to limit lenders' losses. Particularly, for every unit of an asset $j \in J(\xi)$ sold, borrowers must establish—and may consume—a bundle $C_{(\xi,j)} \in \mathbb{R}^L_+ \setminus \{0\}$ that is seized by the market in case of default. For the sake of notation, let $J(D) := \{(\xi, j) \in D \times \cup_{\mu \in D} J(\mu) : j \in J(\xi)\}$ and $J^+(D) := \{(\mu, j) \in D \times \cup_{\eta \in D} J(\eta) : (\mu^-, j) \in J(D)\}$.

Furthermore, additional default enforcement mechanisms may exist. We allow generality in the type of additional enforcement mechanisms assuming that, for each unit of asset $j \in J(\xi)$, borrowers pay, and lenders expect to receive, a fixed percentage of the remaining debt, $\lambda_{(\mu,j)} \in [0, 1]$. More formally, for every unit of asset $j \in J(\xi)$, each borrower pays at each node $\mu \in \xi^+$ an amount

$$F_{(\mu,j)}(p_\mu) := \min\{p_\mu A_{(\mu,j)}, p_\mu Y_\mu C_{(\xi,j)}\} + \lambda_{(\mu,j)} [p_\mu A_{(\mu,j)} - p_\mu Y_\mu C_{(\xi,j)}]^+,$$

where $\lambda_{(\mu,j)} \in [0, 1]$ is the *effectiveness* of additional enforcement mechanisms on asset j at node μ, and, for any $z \in \mathbb{R}$, $[z]^+ := \max\{z, 0\}$.

Our approach allows us to include in our analysis economic (i.e. those induced by legal contracts) and non-economic (e.g. moral sanctions, loss of reputations) default enforcement mechanisms, provided that these mechanisms may be summarized by a family of parameters of effectiveness, $(\lambda_{(\mu,j)})_{(\mu,j) \in J^+(D)}$. However, this last requirement do not induce loss of generality, since traders perfect foresee asset payments. In fact, we can always normalize financial payments as done above.

Definition. Given $(\mu, j) \in J^+(D)$, additional enforcement mechanisms are *effective* on asset j at node μ when $\lambda_{(\mu,j)} A_{(\mu,j)}$ is a non-zero vector. Enforcement mechanisms are *persistently effective* in a subtree $D(\xi)$, if for any $\mu > \xi$, there is $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ.

Definitions above not only depend on parameters $(\lambda_{(\mu,j)})_{(\mu,j) \in J^+(D)}$, but also on the non-triviality of the original promises. Thus, *effective additional enforcement mechanisms* means that, in the case of default, a strictly positive amount of resources is seized besides the depreciated collateral value.

In contrast to any equilibrium model, we focus in the non-existence of a physically feasible solution for the individual's problem. For this reason, it is sufficient to study a decision model where there is an infinitely lived agent, namely i, who perfectly foresees both market prices and the effectiveness of additional enforcement mechanisms.

Agent i has physical endowments $(w^i_\xi)_{\xi \in D} \in \mathbb{R}^{D \times L}_+$ and preferences represented by a utility function $U^i : \mathbb{R}^{D \times L}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. As commodities may be durable, we denote by W^i_ξ the cumulated endowments of agent i up to node ξ, which are recursively defined by: $W^i_\xi = w^i_\xi + Y_\xi W^i_{\xi^-}$, when $\xi > \xi_0$, and $W^i_{\xi_0} = w^i_{\xi_0}$, otherwise. Also, we assume that, for any $\xi \in D$, $W^i_\xi \leq W_\xi$.

Let $x_\xi \in \mathbb{R}^L_+$ be a bundle of autonomous consumption at node ξ (i.e. non-collateralized commodities). Also, define $\theta_{(\xi,j)}$ and $\varphi_{(\xi,j)}$ as quantities of asset $j \in J(\xi)$ purchased and sold at the same node. Given $(p, q) \in \Pi := \mathbb{R}^{D \times L}_+ \times \mathbb{R}^{J(D)}_+$, a plan

$$(x, \theta, \varphi) := ((x_\xi, \theta_{(\xi,j)}, \varphi_{(\xi,j)}); (\xi, j) \in J(D)) \in \mathbb{E} := \mathbb{R}^{D \times L}_+ \times \mathbb{R}^{J(D)}_+ \times \mathbb{R}^{J(D)}_+$$

is *budget feasible* for agent i at prices (p, q) when

$$p_{\xi_0}(x_{\xi_0} - w^i_{\xi_0}) + p_{\xi_0} \sum_{j \in J(\xi_0)} C_{(\xi_0,j)} \varphi_{(\xi_0,j)} + \sum_{j \in J(\xi_0)} q_{(\xi_0,j)} (\theta_{(\xi_0,j)} - \varphi_{(\xi_0,j)}) \leq 0, \tag{1}$$

$$\begin{aligned} &p_\xi(x_\xi - w^i_\xi) + p_\xi \sum_{j \in J(\xi)} C_{(\xi,j)} \varphi_{(\xi,j)} + \sum_{j \in J(\xi)} q_{(\xi,j)} (\theta_{(\xi,j)} - \varphi_{(\xi,j)}) \\ &\quad \leq p_\xi Y_\xi x_{\xi^-} + \sum_{j \in J(\xi^-)} (p_\xi Y_\xi C_{(\xi^-,j)} \varphi_{(\xi^-,j)} + F_{(\xi,j)}(p_\xi)(\theta_{(\xi^-,j)} - \varphi_{(\xi^-,j)})), \quad \forall \xi > \xi_0. \end{aligned} \tag{2}$$

Also, $(x, \theta, \varphi) \in \mathbb{E}$ is *physically feasible* if $x_\xi + \sum_{j \in J(\xi)} C_{(\xi,j)} \varphi_{(\xi,j)} \leq W_\xi$, for any $\xi \in D$. Finally, given $(p, q) \in \Pi$, the objective of agent i is to maximize the utility of his consumption, $U^i \left((x^i_\xi + \sum_{j \in J(\xi)} C_{(\xi,j)} \varphi^i_{(\xi,j)})_{\xi \in D} \right)$, choosing a budget feasible plan $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$.

3. Enforcement mechanisms and the size of collateral bundles

In this section, we prove our main result: in contrast to the polar case studied by Araujo et al. (2002), the market choice of collateral bundles becomes relevant when there are *persistently effective* additional enforcement mechanisms besides collateral repossession. To achieve our objective, we impose the following hypotheses.

Assumption A1. For any $\xi \in D$, $W^i_\xi \gg 0$.

Assumption A2. Given $z = (z_\xi) \in \mathbb{R}^{L\times D}_+$, $U^i(z) = \sum_{\xi \in D} u^i_\xi(z_\xi)$, where for any $\xi \in D$, the function $u^i_\xi : \mathbb{R}^L_+ \to \mathbb{R}_+$ is concave, continuous and strictly increasing. Also, $U^i(W)$ is finite.

Given $\eta \in D$, let $\Omega(\eta)$ be the set of assets $j \in J(\eta)$ on which additional enforcement mechanisms are effective at some node $\mu \in \eta^+$. Note that, given a subtree $D(\xi)$ in which additional enforcement mechanisms are persistently effective, $\Omega(\eta) \neq \varnothing$, $\forall\, \eta \in D(\xi)$.

Theorem. *Under* Assumptions A1 and A2, *suppose that additional enforcement mechanisms are persistently effective in a subtree $D(\xi)$. Independently of the prices $(p, q) \in \Pi$, there are strictly positive upper bounds $(\Psi_\eta)_{\eta \in D(\xi)}$ such that, if collateral bundles satisfy*

$$\min_{j \in \Omega(\eta)} \sum_{l \in L} C_{(\eta,j,l)} < \Psi_\eta, \quad \forall \eta \in D(\xi),$$

then agent i's problem does not have a physically feasible solution. □

Proof. Assume that, for some $(p, q) \in \Pi$, there is a physically feasible solution for agent i's problem, denoted by $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$. It follows from Lemma 1 (see Appendix A) that there are, for every $\eta \in D$, multipliers $\gamma^i_\eta \in \mathbb{R}_{++}$ and super-gradients $v^i_\eta \in \partial u^i_\eta \left(x^i_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi^i_{(\eta,j)} \right)$ such that,[4] for each asset $j \in J(\eta)$, the following asset pricing conditions hold,

$$\gamma^i_\eta p_\eta \geq v^i_\eta + \sum_{\mu \in \eta^+} \gamma^i_\mu p_\mu Y_\mu, \tag{3}$$

$$\gamma^i_\eta q_{(\eta,j)} \geq \sum_{\mu \in \eta^+} \gamma^i_\mu F_{(\mu,j)}(p_\mu). \tag{4}$$

Also, the family of multipliers $(\gamma^i_\eta)_{\eta \in D}$ can always be constructed to satisfy (see Lemma 1)

$$\gamma^i_\eta p_\eta W^i_\eta \leq \sum_{\eta \in D} u^i_\eta(c^i_\eta) \leq \sum_{\eta \in D} u^i_\eta(W_\eta), \tag{5}$$

where $c^i_\eta := x^i_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi^i_\eta$ is the consumption bundle chosen by agent i at node η (the last inequality above follows from Assumption A2 jointly with the physical feasibility of agent i's consumption).

Using these properties, it is possible to find lower and upper bounds for deflated commodity prices. In fact, Assumption A1 and inequalities (5) ensure that, given $(\eta, l) \in D \times L$,

$$\gamma^i_\eta p_{(\eta,l)} \leq \gamma^i_\eta \sum_{l' \in L} p_{(\eta,l')} \leq \bar{\pi}_\eta := \frac{U^i(W)}{\min_{l' \in L} W^i_{(\eta,l')}}.$$

On the other hand, given $(\eta, l) \in D \times L$, it follows from inequality (3) that

$$\begin{aligned} 2\gamma^i_\eta p_{(\eta,l)} W_{(\eta,l)} &= \gamma^i_\eta p_\eta ((c^i_\eta + 2W_{(\eta,l)}\vec{e}_l) - c^i_\eta) \geq v^i_\eta ((c^i_\eta + 2W_{(\eta,l)}\vec{e}_l) - c^i_\eta) \\ &\geq u^i_\eta(c^i_\eta + 2W_{(\eta,l)}\vec{e}_l) - u^i_\eta(c^i_\eta) \geq \min_{0 \leq c \leq W_\eta} (u^i_\eta(c + 2W_{(\eta,l)}\vec{e}_l) - u^i_\eta(c)) > 0, \end{aligned}$$

where the last inequality follows from the strictly monotonicity of function u^i_η and, given $l \in L$, the vector $\vec{e}_l \in \mathbb{R}^L$ is defined by: $\vec{e}_{(l,l')} = 1$ if $l' = l$, and equal to zero in other case.

Thus, for any $(\eta, l) \in D \times L$,

$$\gamma^i_\eta p_{(\eta,l)} \geq \underline{\pi}_{(\eta,l)} := \frac{1}{2W_{(\eta,l)}} \min_{0 \leq c \leq W_\eta} (u^i_\eta(c + 2W_{(\eta,l)}\vec{e}_l) - u^i_\eta(c)) > 0.$$

[4] Given a concave function $f : X \subset \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ and $x \in X$, the super-differential of f at x, $\partial f(x)$, is defined as the set of points $p \in X$, called super-gradients, such that $f(y) - f(x) \leq p(y - x)$, $\forall y \in X$.

For any $\eta \in D$, define $\underline{\pi}_\eta = (\underline{\pi}_{(\eta,l)}; l \in L)$. Under the assumptions on the statement of the Theorem, it follows that, for each $\eta \in D(\xi)$, the number

$$\Upsilon_\eta := \min_{j \in \Omega(\eta)} \sum_{\mu \in \eta^+} \lambda_{(\mu,j)} \underline{\pi}_\mu A_{(\mu,j)}$$

is strictly positive. Therefore, suppose that, at each $\eta \in D(\xi)$,

$$\min_{j \in \Omega(\eta)} \sum_{l \in L} C_{(\eta,j,l)} < \Psi_\eta := \frac{\Upsilon_\eta}{\bar{\pi}_\eta}.$$

With this upper bound over unitary collateral bundles we conclude that, at every node $\eta \in D(\xi)$ there exists $j \in \Omega(\eta)$ such that

$$\begin{aligned}
\gamma_\eta^i (p_\eta C_{(\eta,j)} - q_{(\eta,j)}) &\le \gamma_\eta^i p_\eta C_{(\eta,j)} - \sum_{\mu \in \eta^+} \gamma_\mu^i F_{(\mu,j)}(p_\mu) \\
&\le \underline{\pi}_\eta \sum_{l \in L} C_{(\eta,j,l)} - \sum_{\mu \in \eta^+} \lambda_{(\mu,j)} \gamma_\mu^i p_\mu A_{(\mu,j)} \\
&< \Upsilon_\eta - \sum_{\mu \in \eta^+} \lambda_{(\mu,j)} \underline{\pi}_\mu A_{(\mu,j)} \le 0.
\end{aligned}$$

Finally, using the Lemma 2 in Appendix A, we conclude that agent i's problem does not have a solution, contradicting the optimality of $(x^i, \theta^i, \varphi^i) \in \mathbb{E}$ under prices $(p, q) \in \Pi$.

The result above has two properties that are worth being highlighted.

About upper bounds $(\Psi_\eta)_{\eta \in D(\xi)}$. Regarding the upper bounds on collateral, we can see that, by construction, they depend only on the primitives of the economy and, for computational objectives, can be easily found. Additionally, these upper bounds may converge, when the time period goes to infinity, to any non-negative real number. We illustrate this property with the following example.

Consider an economy satisfying Assumptions A1 and A2, where there are only two commodities and there exists a plan $(W_t)_{t \ge 0} \in \mathbb{R}_{++}^{\mathbb{N}}$ such that, for any $\xi \in D$, $W_\xi = W_{t(\xi)}(1, 1)$ and $W_\xi^i = \sigma W_\xi$, with $\sigma \in (0, 1)$. In other words, aggregate endowments are not affected by the uncertainty and agent i maintain along the event-tree a fixed portion of aggregate physical wealth. Also, there is only one asset at each node $\xi \in D$, and its promises at nodes $\mu \in \xi^+$ are given by a constant bundle $A = (a_1, a_2) \in \mathbb{R}_{++}^2$, i.e. it is a real promise that keep the consumption purchase power along the event-tree. Finally, agent i has additively separable utility with preferences at each node $\xi \in D$ characterized by $u_\xi^i(x_1, x_2) = \beta^{t(\xi)} \rho(\xi)(x_1 + \sqrt{x_2})$, where the intertemporal discount factor β belongs to $(0, 1)$, the probability to reach a node ξ, $\rho(\xi)$, satisfies $\rho(\xi) = \sum_{\mu \in \xi^+} \rho(\mu)$, with $\rho(\xi_0) = 1$.

In this context, if additional enforcement mechanisms have a constant effectiveness of $\lambda \in (0, 1)$ in a subtree $D(\bar{\xi})$, then

$$\Psi_\eta = \lambda \frac{\sigma}{2U^i(W)} \frac{W_{t(\eta)}}{W_{t(\eta)+1}} \beta^{t(\eta)+1} \left(2a_1 W_{t(\eta)+1} + a_2 \left(\sqrt{3W_{t(\eta)+1}} - \sqrt{W_{t(\eta)+1}}\right)\right), \quad \forall \eta \in D(\bar{\xi}).$$

Since $\beta \in (0, 1)$, if the sequence $(W_t)_{t \ge 0}$ is bounded from above and away from zero, then Ψ_η goes to zero as $t(\eta)$ goes to infinity. On the other hand, assume that $(W_t)_{t \ge 0}$ decreases along the time following the recursive rule

$$W_t = \frac{W_0}{\beta^2} \beta^{\vartheta(t)}, \quad \forall t \ge 0,$$

where $\vartheta(0) = \vartheta(1) = 2$ and, for any $t \ge 2$, $\vartheta(t) = \sum_{k=1}^{t-1} 2^k (t-k) + 2^t$. Then, for any $t \ge 1$, $\vartheta(t+1) = 2(\vartheta(t) + t)$, which implies that $W_{t+1} = \frac{1}{W_0} \beta^{2t+2} W_t^2$, $\forall t \ge 1$. Therefore, Ψ_η converges, when $t(\eta)$ goes to infinity, to

$$\zeta(W_0, a_2) := \lambda \frac{\sigma a_2 \sqrt{W_0}}{2U^i(W)} (\sqrt{3} - 1).$$

Since

$$U^i(W) = \sum_{t=0}^{\infty} \beta^t (W_t + \sqrt{W_t}) = \frac{W_0}{\beta^2} \sum_{t=0}^{\infty} \beta^{t+\vartheta(t)} + \frac{\sqrt{W_0}}{\beta} \sum_{t=0}^{\infty} \beta^{t+(\vartheta(t)/2)}.$$

it follows that,

$$\zeta(W_0,a_2):=a_2\left(\lambda\sigma(\sqrt{3}-1)\beta^2\right)\left(2\left(\sqrt{W_0}\sum_{t=0}^{\infty}\beta^{t+\vartheta(t)}+\sum_{t=0}^{\infty}\beta^{t+(\vartheta(t)/2)+1}\right)\right)^{-1}.$$

Thus, the equation $z=\zeta(W_0,a_2)$ has a solution for any $z\in\mathbb{R}_{++}$ and, therefore, Ψ_η may converge, as the period of time goes to infinite, to any positive real number.

Bounds on effectiveness. For convenience of notations, given $\mu\in D$, define $\Omega^+(\mu)=\{j\in J(\mu^-):\lambda_{(\mu,j)}>0\}$. Thus, for any $\xi\in D$: $(\exists\mu\in\xi^+$ such that $j\in\Omega^+(\mu))\Leftrightarrow j\in\Omega(\xi)$.

Given collateral requirements, we can find lower bounds for the effectiveness on a subtree $D(\xi)$, $(\underline{\lambda}_\mu)_{\mu>\xi}$, such that, if $(\lambda_{(\mu,j)})_{\mu>\xi,j\in\Omega^+(\mu)}$ satisfy

$$\min_{j\in\Omega^+(\mu)}\lambda_{(\mu,j)}>\underline{\lambda}_\mu,\quad\forall\mu>\xi,$$

then, independently of prices, and for *any enforcement mechanism* inducing such effectiveness, there is no physical feasible solution for the agents' problem.[5] These lower bounds are informative, i.e. $\underline{\lambda}_\mu\in(0,1)$, only for collateral requirements that are not high enough. In fact, for larger collateral requirements there is no default and, therefore, the market price of collateral requirements is always greater that the loan value.

4. On endogenous effectiveness

A key feature of our model is that we assume that the amount of payments besides the collateral guarantees is independent of the borrowers and does not depend on the history of default. This assumption allowed us to identify sold with purchased assets. Implicitly, we pool the debt contracts into derivatives following a trivial securitization, that is, by identifying prices and payments of debt markets with those of investment markets. However, our analysis may be extended for equilibrium models in which the effectiveness of payment enforcement mechanisms is an endogenous and personalized variable.

For instance, we may suppose that the access to credit markets depend on previous payments. That is, consider a dynamic infinite horizon general equilibrium model in which, at any state of nature, and for every agent, the access to credit securities depend on the history of default. Thus, in this new framework, for default penalties sufficiently restrictive on the access of credit markets, there may be endogenous incentives inducing borrowers to deliver payments larger than the depreciated value of collateral requirements. Also, suppose that financial markets still preserve some features from our original model. That is, each type of credit contract is securitized into only one derivative, primitive and derivative prices are identified, and lenders perfectly foresee the payments of derivatives. Specifically for this last feature, suppose that, in case of default, agents advance any payment in addition to the depreciated collateral as a *percentage* of the remaining debt, facing payment functions with an analogous specification of our $(F_{(\mu,j)})_{(\mu,j)\in J^+(D)}$.

In this new context, under hypotheses on individual' characteristics analogous to Assumptions A1 and A2, there are two conditions under which our Theorem still holds:

(i) For any plan of prices, individual optimal allocations satisfy inequalities analogous to (3)–(5);
(ii) At the moment of the credit operation, borrowers are only required to constitute the associated collateral requirements.

In fact, assume that additional enforcement mechanisms are persistently effective in a sub-event-tree. If there is a physical feasible optimal allocation, using the same arguments of the proof of our theorem, condition (i) implies that if collateral requirements are not high enough, then unitary loan prices persistently exceed the associated collateral value. Thus, by condition (ii), the agent may improve his respective utility increasing borrowing along the event-tree. A contradiction.

Therefore, a natural question arises. When does an economy satisfy conditions (i) and (ii)? Regarding condition (i), it follows from Lemma 2 that any convex model satisfies it.[6] However, some enforcement mechanisms may induce non-convex budget sets. Even in these cases, condition (i) still holds if these non-convexities involve only the borrowers'

[5] Using the same arguments in the proof of the Theorem, it is sufficient to take, for each $\eta\in\mu^+$, with $\mu\geq\xi$,

$$\underline{\lambda}_\eta:=\frac{\bar{\pi}_\mu\min_{j\in\Omega(\mu)}\sum_{l\in L}C_{(\mu,j,l)}}{\min_{j\in\Omega(\mu)}\sum_{\eta'\in\mu^+}\bar{\pi}_{\eta'}A(\eta',j)}.$$

[6] We mean that a model is *convex* when agents' objective functions are concave and, for each vector of prices, budget sets are convex.

deliveries.[7] On the other hand, condition (ii) holds unless there is some restriction on the short sales in addition to collateral requirements.

5. Other extensions

5.1. Long-lived and infinite-lived real assets

Our analysis also holds when long-lived real assets are available for trading. Essentially, non-arbitrage conditions associated to individual's problem (Lemma 1 in Appendix A) are still valid (see Araujo et al., 2008 for detailed arguments).

5.2. About persistent effectiveness

In our main result, we assume that additional enforcement mechanisms are effective in a subtree $D(\xi)$. However, it is possible to weaken this assumption, requiring only effectiveness in a infinite path along the event-tree. For this, we need some definitions.

Given $k \in \mathbb{N} \cup \{+\infty\}$, a *path of uncertainty* is a set $(\mu_n; n \in \mathbb{N},\ n \le k) \subset D$ in which every μ_{n+1} is an immediate successor of μ_n, for each $n < k$. A set $B \subset D$ *does not have finite paths* when for any $\mu \in B$ there exists $\eta \in B$ such that $\eta \in \mu^+$. Additional enforcement mechanisms are *persistently effective in a path of uncertainty* Θ, if for any $\mu \in \Theta$, there is $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ. For any path of uncertainty $\Theta := (\mu_n; n \in \mathbb{N})$ in which additional enforcement mechanisms are persistently effective, define $\mathrm{Eff}(\Theta) \subset D(\mu_1)$ as the maximal connected set containing Θ and having, at each $\mu \in \mathrm{Eff}(\Theta)$, at least one $j \in J(\mu^-)$ on which additional enforcement mechanisms are effective at μ.[8] Note that, given a path of uncertainty Θ, in which additional enforcement mechanisms are persistently effective, if $\mathrm{Eff}(\Theta)$ does not have finite paths, then $\Omega(\eta) \neq \varnothing,\ \forall \eta \in \mathrm{Eff}(\Theta)$. Thus, Lemma 2 holds when additional enforcement mechanisms are persistently effective *only* in a path of uncertainty Θ, provided that $\mathrm{Eff}(\Theta)$ does not have finite paths.[9]

Therefore, our main theorem holds when additional enforcement mechanisms are persistently effective in a path of uncertainty Θ for which $\mathrm{Eff}(\Theta)$ does not have finite paths.

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Appendix A.

Lemma 1. *Let $(p, q) \in \Pi$ and fix a budget and physically feasible plan $z^i := (x^i, \theta^i, \varphi^i) \in \mathbb{E}$. Under* Assumptions A1 and A2, *if z^i is an optimal allocation for agent i's problem at prices (p, q), then for every $\eta \in D$, the function u^i_η is super-differentiable at the point $c^i_\eta := x^i_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi^i_{(\eta,j)}$, there are multipliers $\gamma^i_\eta \in \mathbb{R}_{++}$ and super-gradients $v^i_\eta \in \partial u^i_\eta(c^i_\eta)$ such that, for each $j \in J(\eta)$,*

$$\gamma^i_\eta p_\eta \ge v^i_\eta + \sum_{\mu \in \eta^+} \gamma^i_\mu p_\mu Y_\mu, \tag{6}$$

$$\gamma^i_\eta q_{(\eta,j)} \ge \sum_{\mu \in \eta^+} \gamma^i_\mu F_{(\mu,j)}(p_\mu). \tag{7}$$

Also, the plan of multipliers $(\gamma^i_\eta)_{\eta \in D}$ satisfy

$$\gamma^i_\eta p_\eta W^i_\eta \le \sum_{\eta \in D} u^i_\eta(c^i_\eta). \tag{8}$$

[7] Technically, in this case, the arguments in the proof of Lemma 1 can be remade by redefining the truncated problem $(P^{i,T})$ in such form that, for any $\eta \in D^T$, variables φ_η are fixed and equal to the optimal choices φ^i_η.

[8] A set $B \subset D$ is connected when, for each pair $(\xi, \mu) \in B \times B$, such that $\mu \ge \xi$, the only path of uncertainty connecting ξ to μ is contained in B. Given $\xi \in D$, a set $B \subset D(\xi)$ is maximal relative to a property A when there is no other subset of $D(\xi)$ containing itself and satisfying A.

[9] In fact, if for any $\eta \in \mathrm{Eff}(\Theta)$, there exists $j \in J(\eta)$ for which $p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0$, then agent i may increases his borrowing at the first node of Θ paying his future commitments *either* using new credit at the nodes in which there is effectiveness or delivering depreciated collateral guarantees for the nodes $\mu \notin \mathrm{Eff}(\Theta)$ such that $\mu^- \in \mathrm{Eff}(\Theta)$.

Proof. Given $T \in \mathbb{N}$, define $D_T = \{\eta \in D : t(\eta) = T\}$ and $D^T = \left\{\eta \in D : \eta \in \bigcup_{k=0}^{T} D_k\right\}$. For any $\eta \in D$, let $Z(\eta) = \mathbb{R}_+^L \times \mathbb{R}_+^{J(\eta)} \times \mathbb{R}_+^{J(\eta)}$. For convenience of notations, let $z_{\xi_0^-} := 0 \in Z(\xi_0^-)$, where $Z(\xi_0^-) := \mathbb{R}_+^L$. Consider the optimization problem:

$$
(P^{i,T}) \quad
\begin{array}{l}
\max \sum_{\eta \in D^T} u_\eta^i \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \right) \\
\text{s.t.} \begin{cases}
z_\eta := (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta) & \forall \eta \in D^T, \\
g_\eta^i(z_\eta, z_\eta^-; p, q) \quad \leq 0, & \forall \eta \in D^T, \\
x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \quad \leq 2W_\eta, & \forall \eta \in D^T, \\
z_\eta \quad \in [0, z_\eta^i], & \forall \eta \in D_T.
\end{cases}
\end{array}
$$

where the inequality $g_\eta^i(z_\eta, z_\eta^-; p, q) \leq 0$ represents the budget constraint at node η, that is, inequality (1) or (2), and given $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$, the interval $[x, y] := \{z \in \mathbb{R}^m : \exists a \in [0, 1], z = ax + (1 - a)y\}$. It follows from the existence of an optimal individual plan at prices (p, q) that there exists a solution for $(P^{i,T})$, namely $(z_\eta^{i,T})_{\eta \in D^T}$.[10]

Given $\eta \in D$, define the concave function $v_\eta^i : \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)} \to \mathbb{R} \cup \{-\infty\}$ as

$$
v_\eta^i(z_\eta) = \begin{cases}
u_\eta^i \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \right) & \text{if } x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \geq 0; \\
-\infty & \text{otherwise.}
\end{cases},
$$

where $z_\eta = (x_\eta, \theta_\eta, \varphi_\eta)$. It follows that, for any $T \geq 1$, $\sum_{\eta \in D^T} v_\eta^i(z_\eta^{i,T}) \leq \sum_{\eta \in D} v_\eta^i(z_\eta^i)$.[11]

For each $\eta \in D$ and $\gamma_\eta \in \mathbb{R}_+$, define $\mathcal{L}_\eta^i(\cdot, \gamma; p, q) : \mathbb{Z}(\eta) \times \mathbb{Z}(\eta^-) \to \mathbb{R}$ as

$$
\mathcal{L}_\eta^i(z_\eta, z_{\eta^-}, \gamma_\eta; p, q) = v_\eta^i(z_\eta) - \gamma_\eta g_\eta^i(z_\eta, z_{\eta^-}; p, q).
$$

Given $T \in \mathbb{N}$, for each $\eta \in D^{T-1}$, define the set $\Xi^T(\eta)$ as the family of allocations $(x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta)$ that satisfies $x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta$. Also, for any $\eta \in D_T$, let $\Xi^T(\eta)$ be the set of allocations $(x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta)$ that satisfies both $x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta$ and $(x_\eta, \theta_\eta, \varphi_\eta) \in [0, z_\eta^i]$. Let $\Xi^T := \prod_{\eta \in D^T} \Xi^T(\eta)$.

It follows from Rockafellar (1997, Theorem 28.3), that there exist non-negative multipliers $(\gamma_\eta^{i,T})_{\eta \in D^T}$ such that the following saddle point property holds,

$$
\sum_{\eta \in D^T} \mathcal{L}_\eta^i(z_\eta, z_{\eta_-}, \gamma_\eta^{i,T}; p, q) \leq \sum_{\eta \in D^T} \mathcal{L}_\eta^i(z_\eta^{i,T}, z_{\eta_-}^{i,T}, \gamma_\eta^{i,T}; p, q), \quad \forall (z_\eta)_{\eta \in D^T} \in \Xi^T, \tag{9}
$$

and $\gamma_\eta^{i,T} g_\eta^i(z_\eta^{i,T}, z_{\eta_-}^{i,T}; p, q) = 0$.

---

[10] In fact, define a new problem $(\tilde{P}^{i,T})$,

$$
(\tilde{P}^{i,T}) \quad
\begin{array}{l}
\max \sum_{\eta \in D^T} u_\eta^i \left(x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \right) \\
\text{s.t.} \begin{cases}
z_\eta := (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta) & \forall \eta \in D^T, \\
g_\eta^i(z_\eta, z_{\eta^-}; p, q) \quad \leq 0, & \forall \eta \in D^T, \\
x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \quad \leq 2W_\eta, & \forall \eta \in D^T, \\
z_\eta \in [0, z_\eta^i], & \forall \eta \in D_T, \\
\text{If } q_{(\eta,j)} = 0 \quad \text{then} & \theta_{(\eta,j)} = 0.
\end{cases}
\end{array}
$$

Under Assumption A2 the objective function on $(\tilde{P}^{i,T})$ is continuous, and the set of admissible allocations is compact in $\prod_{\eta \in D^T} Z(\eta)$. Note that, to ensure this it is necessary to have non-zero collateral requirements, otherwise, long and short positions are unbounded.
Thus, there is a solution $(z_\eta^{i,T})_{\eta \in D^T}$. Moreover, this solution for $(\tilde{P}^{i,T})$ is also an optimal choice for $(P^{i,T})$. Essentially, the existence of a *finite* optimum at prices (p, q) for the agent i's problem ensure that, when $q_{(\eta,j)} = 0$, the payments $F_{(\mu,j)}(p_\mu)$ must be equal zero, for each $\mu \in \eta^+$. Thus, when $q_{(\eta,j)} = 0$, choosing positives amounts of $\theta_{(\eta,j)}$ does not induce any gains.

[11] Note that, otherwise, agent i improve his utility in D choosing the allocation $(z_\eta^{i,T})_{\eta \in D^T}$ in the sub-tree D^T, without making any (physical or financial) trade after the nodes with date T.

Claim. *For each $\mu \in D$, the sequence $(\gamma_\mu^{i,T})_{T \geq t(\mu)}$ is bounded. Moreover, given $T > t(\mu)$, for any plan $a_\mu \in \Xi^T(\mu)$ we have that*

$$v_\mu^i(a_\mu) - v_\mu^i(z_\mu^i) \leq \left(\gamma_\mu^{i,T} \nabla_1 g_\mu^i(p,q) + \sum_{\eta \in \mu^+} \gamma_\eta^{i,T} \nabla_2 g_\eta^i(p,q) \right) \cdot (a_\eta - z_\eta^i) + \sum_{\xi \in D \setminus D^{T-1}} v_\xi^i(z_\xi^i),$$

where, for any $\eta \in D$, the vector $(\nabla_1 g_\eta^i(p,q), \nabla_2 g_\eta^i(p,q))$ is defined by

$$\nabla_1 g_\eta^i(p,q) = (p_\eta, q_\eta, (p_\eta C_{(\eta,j)} - q_{(\eta,j)})_{j \in J(\eta)}),$$
$$\nabla_2 g_\eta^i(p,q) = (-p_\eta Y_\eta, (F_{(\eta,j)})_{j \in J(\eta)}, (p_\eta Y_\eta C_{(\eta,j)} - F_{(\eta,j)})_{j \in J(\eta)}).$$

Proof. Given $t \leq T$, substitute the following allocation in inequality (9)

$$z_\eta = \begin{cases} (W_\eta^i, 0, 0), & \forall \eta \in D^{t-1}, \\ (0, 0, 0), & \forall \eta \in D^T \setminus D^{t-1}. \end{cases}$$

We have

$$\sum_{\eta \in D_t} \gamma_\eta^{i,T} p_\eta W_\eta^i \leq \sum_{\eta \in D^T} v_\eta^i(z_\eta^{i,T}) \leq \sum_{\eta \in D} v_\eta^i(z_\eta^i). \tag{10}$$

Assumption A1 ensure that, for each $\eta \in D$, $\min_{l \in L} W_{(\eta,l)}^i > 0$. Also, Assumption A2 implies that $\sum_{l \in L} p_{(\eta,l)} > 0$, guaranteeing that, for each $\mu \in D$, the sequence $(\gamma_\mu^{i,T})_{T > t(\mu)}$ is bounded.

On the other hand, given $(z_\eta)_{\eta \in D^T} \in \Xi^T$, using (9), we have that

$$\sum_{\eta \in D^T} \mathcal{L}_\eta^i(z_\eta, z_{\eta^-}, \gamma_\eta^{i,T}; p, q) \leq \sum_{\eta \in D} v_\eta^i(z_\eta^i).$$

Thus, fix $\mu \in D^{T-1}$ and $a_\mu \in \Xi^T(\mu)$. If we evaluate inequality above in

$$z_\eta = \begin{cases} z_\eta^i, & \forall \eta \neq \mu, \\ a_\mu, & \text{for } \eta = \mu, \end{cases}$$

we obtain

$$v_\mu^i(a_\mu) - \gamma_\mu^{i,T} g_\mu^i(a_\mu, z_{\mu^-}^i; p, q) - \sum_{\eta \in \mu^+} \gamma_\eta^{i,T} g_\eta^i(z_\eta^i, a_\mu; p, q) \leq v_\mu^i(z_\mu^i) + \sum_{\eta \in D \setminus D^T} v_\eta^i(z_\eta^i). \tag{11}$$

Since functions $(g_\xi^i(\cdot\,; p, q); \xi \in D)$ are affine, we have

$$g_\mu^i(a_\mu, z_{\mu^-}^i; p, q) = \nabla_1 g_\mu^i(p,q) \cdot a_\mu - p_\mu \omega_\mu^i + \nabla_2 g_\mu^i(p,q) \cdot z_{\mu^-}^i$$
$$g_\eta^i(z_\eta^i, a_\mu; p, q) = \nabla_1 g_\eta^i(p,q) \cdot z_\eta^i - p_\eta \omega_\eta^i + \nabla_2 g_\eta^i(p,q) \cdot a_\mu, \quad \forall \eta \in \mu^+.$$

Also, budget feasibility of $(z_\eta^i)_{\eta \in D}$ at prices (p,q), jointly with monotonicity of preferences, ensure that,

$$-p_\mu \omega_\mu^i + \nabla_2 g_\mu^i(p,q) \cdot z_{\mu^-}^i = -\nabla_1 g_\mu^i(p,q) \cdot z_\mu^i,$$
$$\nabla_1 g_\eta^i(p,q) \cdot z_\eta^i - p_\eta \omega_\eta^i = -\nabla_2 g_\eta^i(p,q) \cdot z_\mu^i, \quad \forall \eta \in \mu^+.$$

Therefore,

$$\gamma_\mu^{i,T} g_\mu^i(a_\mu, z_{\mu^-}^i; p, q) + \sum_{\eta \in \mu^+} \gamma_\eta^{i,T} g_\eta^i(z_\eta^i, a_\mu; p, q) = \left(\gamma_\mu^{i,T} \nabla_1 g_\mu^i(p,q) + \sum_{\eta \in \mu^+} \gamma_\eta^{i,T} \nabla_2 g_\eta^i(p,q) \right) \cdot (a_\mu - z_\mu^i).$$

Using (11), we conclude the proof.

Since D is countable and, for any node η, the sequence $(\gamma_\eta^{i,T})_{T \geq t(\eta)}$ is bounded, using Tychonoff Theorem (see Aliprantis and Border, 1999, Theorem 2.57), there is a common subsequence $(T_k)_{k \in \mathbb{N}} \subset \mathbb{N}$ and non-negative multipliers, $(\gamma_\eta^i)_{\eta \in D}$, such that, for each $\eta \in D$, $\lim_{k \to \infty} \gamma_\eta^{i,T_k} = \gamma_\eta^i$ and

$$\gamma_\eta^i g_\eta^i(z_\eta^i, z_{\eta_-}^i; p, q) = 0,$$

where, as we said above, the last equation follows from the strictly monotonicity of u_η^i. Moreover, taking the limit as T goes to infinity in inequality (10) we obtain that

$$\sum_{\eta \in D_t} \gamma_\eta^i p_\eta W_\eta^i \leq \sum_{\eta \in D} v_\eta^i(z_\eta^i), \quad \forall t \geq 0. \tag{12}$$

Therefore, Eq. (8) follows.

Since for any $\eta \in D$, $\Xi^{s_1}(\eta) = \Xi^{s_2}(\eta)$ when $\min\{s_1, s_2\} > t(\eta)$, it follows from the inequality in the statement of Claim above, taking the limit as T goes to infinity, that

$$v_\eta^i(a_\eta) - v_\eta^i(z_\eta^i) \leq \left(\gamma_\eta^i \nabla_1 g_\eta^i(p,q) + \sum_{\mu \in \eta^+} \gamma_\mu^i \nabla_2 g_\mu^i(p,q) \right) \cdot (a_\eta - z_\eta^i), \quad \forall a_\eta \in \Xi^{t(\eta)+1}(\eta). \tag{13}$$

Thus,

$$\left(\gamma_\eta^i \nabla_1 g_\eta^i(p,q) + \sum_{\mu \in \eta^+} \gamma_\mu^i \nabla_2 g_\mu^i(p,q) \right) \in \partial \left(v_\eta^i + \delta_{Z_1(\eta)} + \delta_{Z_2(\eta)} \right)(z_\eta^i),$$

where the functions $\delta_{Z_h(\eta)} : \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)} \to \mathbb{R} \cup \{-\infty\}$, $h \in \{1,2\}$, satisfy

$$\delta_{Z_1(\eta)}(x_\eta, \theta_\eta, \varphi_\eta) = \begin{cases} 0, & \text{if } (x_\eta, \theta_\eta, \varphi_\eta) \in Z(\eta), \\ -\infty, & \text{otherwise.} \end{cases}$$

$$\delta_{Z_2(\eta)}(x_\eta, \theta_\eta, \varphi_\eta) = \begin{cases} 0, & \text{if } x_\eta + \sum_{j \in J(\eta)} C_{(\eta,j)} \varphi_{(\eta,j)} \leq 2W_\eta, \\ -\infty, & \text{otherwise.} \end{cases}$$

where $z_\eta = (x_\eta, \theta_\eta, \varphi_\eta) \in \mathbb{R}^L \times \mathbb{R}^{J(\eta)} \times \mathbb{R}^{J(\eta)}$. Since the plan $(z_\eta^i)_{\eta \in D}$ is physically feasible, there exists a neighborhood V of z_η^i such that $\delta_{Z_2(\eta)}(b) = 0$ for every $b \in V$. Then, we have that $\partial \delta_{Z_2(\eta)}(z_\eta^i) = \{0\}$. Also, it follows by Theorem 23.8 and 23.9 in Rockafellar (1997), that there exists $v_\eta^i \in \partial u_\eta^i(c_\eta^i)$ and $\kappa_\eta^i \in \partial \delta_{Z(\eta)}(x_\eta^i, \theta_\eta^i, \varphi_\eta^i)$ such that

$$\gamma_\eta^i \nabla_1 g_\eta^i(p,q) + \sum_{\mu \in \eta^+} \gamma_\mu^i \nabla_2 g_\mu^i(p,q) = \left(v_\eta^i, 0, (C_{(\eta,j)} v_\eta^i)_{j \in J(\eta)} \right) + \kappa_\eta^i. \tag{14}$$

Notice that, by definition, for each $z_\eta \geq 0$, $\kappa \in \partial \delta_{Z(\eta)}(z_\eta) \Leftrightarrow 0 \leq \kappa(y - z_\eta)$, $\forall y \geq 0$, therefore, $\kappa_\eta^i \geq 0$. Thus, the inequalities stated in the lemma hold from Eq. (14). On the other hand, strictly monotonicity of function u_η^i, ensure that $v_\eta^i \gg 0$ and, therefore, it follows from (6), that γ_η^i is *strictly positive*. □

In a context of collateralized assets and *linear utility penalties for default*, (see Remark 3.1 in Páscoa and Seghir (2009)) show that Ponzi schemes could be implemented if there exists a subtree $D(\xi)$ such that, for every node $\mu \geq \xi$, there is always some asset $j \in J(\mu)$ whose price exceeds the respective collateral value, $p_\mu C_{(\mu,j)} - q_{(\mu,j)} < 0$ (see Remark 3.1 in Páscoa and Seghir, 2009). In such event, the individual's problem does not have a finite solution. In our context, the same result follows by analogous arguments.

Lemma 2. *Assume that, given $x \in \mathbb{R}_+^{L \times D}$, if $U^i(x)$ is finite, then $U^i(y) > U^i(x)$ for any $y > x$. Also, suppose that additional enforcement mechanisms are persistently effective in a subtree $D(\xi)$ such that, for any $\eta \in D(\xi)$, there exists $j \in J(\eta)$ for which $p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0$. Then, agent i's individual problem does not have a finite solution, otherwise, Ponzi schemes could be implemented.* □

Proof. Assume there is a budget feasible plan for agent i, $(x^i, \theta^i, \varphi^i)$, that gives a finite optimum. Under the monotonicity condition stated in the Lemma, $p_\eta \gg 0$ for every node $\eta \in D(\xi)$. For each $\eta \in D(\xi)$, let $J^1(\eta) = \{j \in J(\eta) : p_\eta C_{(\eta,j)} - q_{(\eta,j)} < 0\}$. Now, consider the allocation $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$, with

$$\left((x_\mu, \theta_\mu, \varphi_\mu) ; (\theta_\eta, \varphi_{(\eta,j)}) \right)_{\mu \notin D(\xi),\, \eta \in D(\xi)} = \left((x_\mu^i, \theta_\mu^i, \varphi_\mu^i) ; (\theta_\eta^i, \varphi_{(\eta,j)}^i) \right)_{\mu \notin D(\xi),\, \eta \in D(\xi)}, \quad \forall j \in J(\eta) \setminus J^1(\eta)$$

and

$$\begin{aligned}
\varphi_{(\eta,j)} &= \varphi^i_{(\eta,j)} + \delta_\eta, \qquad \forall \eta \in D(\xi), \qquad \forall j \in J^1(\eta),\\
x_{(\eta,l)} &= x^i_{(\eta,l)} + \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j \in J^1(\eta)} (q_{(\eta,j)} - p_\eta C_{(\eta,j)})\delta_\eta, \quad \forall l \in L, \text{ if the node } \eta = \xi,\\
x_{(\eta,l)} &= x^i_{(\eta,l)} + \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j \in J^1(\eta)} (q_{(\eta,j)} - p_\eta C_{(\eta,j)})\delta_\eta - \frac{1}{(\#L)\,p_{(\eta,l)}} \sum_{j \in J^1(\eta^-)} p_\eta A_{(\eta,j)}\,\delta_{\eta^-}, \qquad \forall\, \eta > \xi, \quad \forall l \in L,
\end{aligned}$$

where the plan $(\delta_\eta)_{\eta \in D(\xi)}$ is chosen in such form that the following conditions hold,

$$\sum_{j \in J^1(\xi)} (q_{(\xi,j)} - p_\xi C_{(\xi,j)})\delta_\xi > 0, \tag{15}$$

$$\sum_{j \in J^1(\eta)} (q_{(\eta,j)} - p_\eta C_{(\eta,j)})\delta_\eta > \sum_{j \in J^1(\eta^-)} p_\eta A_{(\eta,j)}\,\delta_{\eta^-}, \quad \forall \eta > \xi. \tag{16}$$

It follows that $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$ is budget feasible at prices (p, q). Moreover, equations above show that Ponzi schemes are possible at prices (p, q). In fact, agent i increases his borrowing at ξ and pays his future commitments by using new credit. It follows that $(x_\xi, \theta_\xi, \varphi_\xi)_{\xi \in D}$ improves the utility level of agent i, contradicting the optimality of $(x^i, \theta^i, \varphi^i)$. □

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