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RESEARCH ARTICLE

Fiat money and the value of binding portfolio constraints

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Abstract We establish necessary and sufficient conditions for the individual optimality of a consumption-portfolio plan in an infinite horizon economy where agents are uniformly impatient and fiat money is the only asset available for intertemporal transfers of wealth. Next, we show that fiat money has a positive equilibrium price if and only if for some agent the zero short sale constraint is binding and has a positive shadow price (now or in the future). As there is always an agent that is long, it follows

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Previous working paper versions of this article appeared under the titles: "On the role of debt constraints in monetary equilibrium" and "Welfare improving debt constraints". Although in this article we avoid short-sales of money, more general debt constraints were studied in the previous working paper versions (see, for instance, Páscoa, Petrassi and Torres-Martínez (2008)). M. Petrassi wants to disclaim that the views expressed in this work do not necessarily reflect those of the Banco Central do Brasil or its members.

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that marginal rates of intertemporal substitution never coincide across agents. That is, monetary equilibria are never full Pareto efficient. We also give a counter-example illustrating the occurrence of monetary bubbles under incomplete markets in the absence of uniform impatience.

Keywords Binding portfolio constraints · Fundamental value of money · Asset pricing bubbles

JEL Classification D50 · D53 · E41 · E44

1 Introduction

The uniform impatience assumption (see Hernández and Santos (1996, Assumption C.3), or Magill and Quinzii (1996, Assumptions B2 and B4)), together with borrowing constraints, is a usual requirement for existence of equilibrium in economies with infinite lived agents. This condition is satisfied whenever utilities are separable over time and across states of nature so long as (a) the intertemporal discounted factor is constant, (b) individual endowments are uniformly bounded away from zero, and (c) the aggregate endowment is uniformly bounded from above.

As Santos and Woodford (1997, Theorem 3.3) showed, the assumption of uniform impatience has important implications for asset pricing: it rules out speculation in assets in positive net supply for deflator processes in the non-arbitrage pricing kernel,[1] which yield finite present values of aggregate wealth. The well-known example of a positive price of fiat money by Bewley (1980) highlighted the importance of the finiteness of the present value of aggregate wealth.

What happens if we use as deflators the agents' inter-nodes marginal rates of substitution? These deflators may fail to be in the asset pricing kernel when some portfolio constraints are binding. For these Kuhn-Tucker deflators, assets in positive net supply may be priced above the series of deflated dividends and the difference may be due to the presence of shadow prices rather than to a bubble. Giménez (2007) already made this comment and Araujo et al. (2009) worked along these lines when addressing the pricing of long-lived collateralized assets.

To consider a simple and provocative case, we look, as in Bewley (1980), at economies with a single asset, paying no dividends and in positive net supply. As usual, we call this asset fiat money (or simply money), although we are quite aware that we are just looking at its role as a store of value, i.e., as an instrument to transfer wealth across time and states of nature. In this context and under uniform impatience, we show that money can and will only be positively valued as a result of agents' desire to

[1] Notice that under inequality constraints on portfolios, non-arbitrage (from one node to its immediate successors) is equivalent to the existence of a positive vector of state prices solving a linear system of inequalities relating asset prices and returns (see Jouini and Kallal (1995) or Araujo et al. (2005)). The state prices that make all inequalities hold as equalities constitute the asset pricing kernel of the linear operator that defines the system, but there may be other solutions (for example, those given by the Kuhn-Tucker multipliers).

take short positions that they cannot. That is, under uniform impatience, the positive price of money must be due to the presence of shadow prices of binding constraints.

This result does not collide with the example by Bewley (1980) or the results by Santos and Woodford (1997). It complements these results. Under uniform impatience, a positive price of money implies that the present value of the aggregate wealth must be infinite for any deflator in the asset pricing kernel (Theorem 3.3 of Santos and Woodford (1997)). However, when the price of fiat money is positive, for any Kuhn-Tucker deflator process of a certain agent, the present value of the aggregate wealth may be finite, but this deflator fails to be in the asset pricing kernel (because this agent has binding portfolio constraints).

In Bewley (1980) example, the two uniformly impatient agents were not allowed to take short positions and the economy had no uncertainty. The positive price of money was a bubble for the unique deflator process in the asset pricing kernel and for this deflator the present value of aggregate wealth was infinite. However, the zero short-sales constraint was binding infinitely often. Thus, for the Kuhn-Tucker deflator process of each agent, the fundamental value of money was positive, consisting of the shadow prices of debt constraints.

Hence, we obtain a result that may seem surprising: credit frictions create room for welfare improvements through transfers of wealth that become possible only when money has a positive price. However, monetary equilibria are always Pareto inefficient. Otherwise, by definition, agents' rates of intertemporal substitution would coincide. However, as money is in positive net supply, at least one agent must go long, having a zero shadow price. Thus, the shadow prices of all agents should be zero and, therefore, the price of money could not be positive.

To clarify our results, we prove that when money has a positive value, there exists a deflator, but not one of the Kuhn-Tucker deflators, under which the discounted value of aggregated wealth is *infinite* and a bubble appears. That is, in our framework (that includes Bewley (1980) model) it is always possible to interpret monetary equilibrium as a bubble. However, when we focus on Kuhn-Tucker multipliers—deflators that make financial Euler conditions compatible with physical Euler conditions—the positive price of money is always a consequence of a positive fundamental value.

We close the paper with an example of a stochastic economy that does not satisfy the uniform impatience assumption. Money is positive valued in equilibrium, although shadow prices of zero short-sale constraints are equal to zero. For the Kuhn-Tucker deflator processes of both agents, aggregate wealth has a finite present value.

Our main mathematical tool is a duality approach to dynamic programming problems that was already used in the context of long-lived collateralized assets by Araujo et al. (2009). This approach allows us to characterize non-interior solutions and the respective Kuhn-Tucker multiplier processes. A recent related paper by Rincón-Zapatero and Santos (2009) addresses the uniqueness of this multiplier process and the differentiability of the value function, without imposing the usual interiority assumptions.

The paper is organized as follows. Section 2 characterizes uniform impatience. Section 3 presents the basic model. In Sect. 4, we develop the necessary mathematical tools: a duality theory of individual optimization. In Sect. 5, we define the concepts of fundamental value of money and asset pricing bubbles. Finally, Sect. 6 presents the results on monetary equilibria and Sect. 7 an example of monetary equilibrium in an economy without uniform impatience. Some proofs are left to the Appendix.

2 Characterizing uniform impatience when utilities are separable

In this section, we recall the assumption of uniform impatience and characterize it for separable utilities in terms of intertemporal discount factors. As a consequence, we show that the uniform impatience assumption does not hold for agents with hyperbolic intertemporal discounting (see Laibson 1998).

Consider an infinite horizon discrete time economy where the set of dates is $\{0, 1, \ldots\}$ and there is no uncertainty at $t = 0$. Given a history of realizations of the states of nature for the first $t-1$ dates, with $t \geq 1$, $\bar{s}_t = (s_0, \ldots, s_{t-1})$, there is a finite set $S(\bar{s}_t)$ of states that may occur at date t. A vector $\xi = (t, \bar{s}_t, s)$, where $t \geq 1$ and $s \in S(\bar{s}_t)$, is called a *node*. The only node at $t = 0$ is denoted by ξ_0. Let D be the (countable) *event-tree*, i.e., the set of all nodes.

Given $\xi = (t, \bar{s}_t, s)$ and $\mu = (t', \bar{s}_{t'}, s')$, we say that μ is a *successor* of ξ, and we write $\mu > \xi$, if $t' > t$ and the first $t+1$ coordinates of $\bar{s}_{t'}$ are $(\bar{s}_t, s)$. We write $\mu \geq \xi$ to say that either $\mu > \xi$ or $\mu = \xi$ and we denote by $t(\xi)$ the date associated with a node ξ. Let $\xi^+ = \{\mu \in D : (\mu \geq \xi) \wedge (t(\mu) = t(\xi) + 1)\}$ be the set of immediate successors of ξ. The (unique) predecessor of $\xi > \xi_0$ is denoted by ξ^- and $D(\xi) := \{\mu \in D : \mu \geq \xi\}$ is the sub-tree with root ξ.[2] The set of nodes with date T in $D(\xi)$ is denoted by $D_T(\xi)$, and $D^T(\xi) = \bigcup_{k=t(\xi)}^{T} D_k(\xi)$ denotes the set of successors of ξ with date less than or equal to T. When $\xi = \xi_0$ notations above will be shorten to D_T and D^T.

At any node $\xi \in D$, a finite set of perishable commodities, denoted by L, is available for trade. There is a finite set of infinite-lived agents, H. Each agent $h \in H$ has at any $\xi \in D$ a physical endowment $w^h(\xi) \in \mathbb{R}_+^L$ and has preferences over consumption plans, $(x(\xi); \xi \in D) \in \mathbb{R}_+^{L \times D}$, which are represented by a utility function $U^h : \mathbb{R}_+^{L \times D} \to \mathbb{R}_+ \cup \{+\infty\}$.[3] Aggregated physical endowments at a node ξ are given by $W(\xi) \in \mathbb{R}_{++}^L$.

Assumption 1 (*Separability of preferences*) For any agent $h \in H$, her utility function is separable over time and across states of nature. That is, $U^h((x(\xi); \xi \in D)) = \sum_{\xi \in D} u^h(\xi, x(\xi))$, where for any $\xi \in D$, $u^h(\xi, \cdot) : \mathbb{R}_+^L \to \mathbb{R}_+$ is a continuous, concave and strictly increasing function. Also, $\sum_{\xi \in D} u^h(\xi, W(\xi))$ is finite.

Given a consumption plan $x = (x(\mu); \mu \in D)$, a node $\xi \in D$ and a vector $(\pi, \Delta) \in (0, 1) \times \mathbb{R}_{++}^L$, let $z(x, \xi, \pi, \Delta)$ be the consumption plan that coincides with x at any node in $D \setminus D(\xi)$, is equal to $x(\xi) + \Delta$ at ξ, and is equal to $\pi x(\mu)$ at any successor node μ of ξ.

Assumption 2 (*Uniform impatience*) There are $\pi \in (0, 1)$ and $(\Delta(\mu); \mu \in D) \in \mathbb{R}_+^{L \times D}$ such that, given a consumption plan $x = (x(\mu); \mu \in D)$, with $0 \leq x(\mu) \leq$

[2] The symbol ":=" means "defined by".

[3] Since the event-tree D is countable and there is a finite number of commodities, the set $\mathbb{R}_+^{L \times D}$ of sequences of consumption bundles $(x(\xi); \xi \in D)$, with $x(\xi) \in \mathbb{R}_+^L$ for any $\xi \in D$, is well defined.

$W(\mu)$ for any $\mu \in D$, we have,

$$U^h\left(z(x,\xi,\pi',\Delta(\xi))\right) > U^h(x), \qquad \forall h \in H, \quad \forall(\xi,\pi') \in D \times [\pi, 1).$$

Moreover, there is $\kappa > 0$ such that, $w^h(\xi) \geq \kappa\Delta(\xi) > 0, \quad \forall \xi \in D$.

The requirements of impatience above depend on both preferences and physical endowments. As particular cases, we obtain the assumptions imposed by Hernández and Santos (1996), where for any $\mu \in D$, $\Delta(\mu) = W(\mu)$, and Magill and Quinzii (1996), where initial endowments are uniformly bounded away from zero by a bundle $\underline{w} \in \mathbb{R}^L_{++}$, and $\Delta(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Our characterization of uniform impatience is as follows.

Proposition 1 *Suppose that aggregate endowments $(W(\xi); \xi \in D)$ are bounded and that there is $\underline{w} \in \mathbb{R}^L_+ \setminus \{0\}$ such that, individual initial endowments satisfy $w^h(\xi) \geq \underline{w}$, $\forall \xi \in D$. Moreover, assume that, for some agent $h \in H$, there exists a continuous and strictly increasing function $f^h : \mathbb{R}^L_+ \to \mathbb{R}_+$ such that,*

$$U^h((x(\xi); \xi \in D)) = \sum_{\xi \in D} \beta^h_{t(\xi)}\rho^h(\xi) f^h(x(\xi)),$$

where $\beta^h_{t(\xi)} \in (0,1)$ is a discount factor and $\rho^h(\xi)$ denotes the probability to reach node ξ, which satisfies $\rho^h(\xi) = \sum_{\mu \in \xi^+} \rho^h(\mu)$, with $\rho^h(\xi_0) = 1$. Finally, for each $t \geq 0$, let $a^h_t = \frac{1}{\beta^h_t}\sum_{r=t+1}^{+\infty} \beta^h_r$.

Then, the function U^h satisfies uniform impatience (Assumption 2) if and only if the sequence $(a^h_t)_{t\geq 0}$ is bounded.

Proof Assume that $(W(\xi); \xi \in D)$ is a bounded consumption plan. That is, there is $\overline{W} \in \mathbb{R}^L_+$ such that, $W(\xi) \leq \overline{W}$, $\forall \xi \in D$. If $(a^h_t)_{t\geq 0}$ is bounded, then there exists $\overline{a}^h > 0$ such that, $a^h_t \leq \overline{a}^h$, for each $t \geq 0$. Since $\mathbb{F} := \{x \in \mathbb{R}^L_+ : x \leq \overline{W}\}$ is compact, the continuity of f^h assures that there is $\pi \in [0,1)$ such that $f^h(x) - f^h(\pi' x) \leq \min_{z\in\mathbb{F}}(\frac{f^h(z+\underline{w})-f^h(z)}{2\overline{a}^h})$, $\forall x \in \mathbb{F}$, $\forall \pi' \in [\pi, 1)$.

Thus, uniform impatience follows by choosing $\kappa = 1$ and $\Delta(\xi) = \underline{w}$, $\forall \xi \in D$. Indeed, given a plan $(x(\mu); \mu \in D) \in \mathbb{R}^{L\times D}_+$ such that $x(\mu) \leq W(\mu)$ $\forall \mu \in D$, for any $(\xi, \pi') \in D \times [\pi, 1)$ we have that,

$$\begin{aligned}
&\sum_{\mu>\xi} \beta^h_{t(\mu)}\rho^h(\mu) f^h(x(\mu)) - \sum_{\mu>\xi} \beta^h_{t(\mu)}\rho^h(\mu) f^h(\pi' x(\mu)) \\
&\quad \leq \frac{\beta^h_{t(\xi)} a^h_t}{2\overline{a}^h}\rho^h(\xi) \min_{z\in\mathbb{F}}\left(f^h(z+\underline{w}) - f^h(z)\right) \\
&\quad < \beta^h_{t(\xi)}\rho^h(\xi) f^h(x(\xi)+\Delta(\xi)) - \beta^h_{t(\xi)}\rho^h(\xi) f^h(x(\xi)).
\end{aligned}$$

Reciprocally, suppose that uniform impatience property holds. Then, there are $(\pi, \kappa) \in [0,1) \times \mathbb{R}_{++}$ and $(\Delta(\mu); \mu \in D) \in \mathbb{R}^{L\times D}_+$ satisfying, for any $\xi \in$

$D, w^h(\xi) \geq \kappa\Delta(\xi)$, such that: given $(x(\mu); \mu \in D) \in \mathbb{R}_+^{L\times D}$ with $x(\mu) \leq W(\mu)$, $\forall \mu \in D$, we have that, for any node $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}\rho^h(\xi)} \sum_{\mu>\xi} \beta^h_{t(\mu)}\rho^h(\mu)\left[f^h(x(\mu)) - f^h(\pi x(\mu))\right] < f^h(x(\xi)+\Delta(\xi)) - f^h(x(\xi)).$$

It follows that, for any node ξ,

$$\frac{1}{\beta^h_{t(\xi)}\rho^h(\xi)}\left[\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)f^h(\underline{w}) - \sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)f^h(\pi\underline{w})\right] < f^h\left(\left(1+\frac{1}{\kappa}\right)\overline{W}\right).$$

Therefore, we conclude that, for any $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}}\left(f^h(\underline{w}) - f^h(\pi\underline{w})\right)\sum_{t=t(\xi)+1}^{+\infty}\beta^h_t < f^h\left(\left(1+\frac{1}{\kappa}\right)\overline{W}\right),$$

which implies that the sequence $(a^h_t)_{t\geq 0}$ is bounded. □

Under the conditions of Proposition 1, if intertemporal discount factors are constant, i.e. $\exists\lambda^h \in \mathbb{R}_{++} : \frac{\beta^h_{t(\xi)+1}}{\beta^h_{t(\xi)}} = \lambda^h$, $\forall\xi \in D$, then both $\lambda^h < 1$ and $a^h_t = \frac{\lambda^h}{1-\lambda^h}$, for each $t \geq 0$. In this case, the utility function U^h satisfies the uniform impatience condition.

However, even with bounded plans of endowments, uniform impatience is a restrictive condition when intertemporal discount factors are time varying. For instance, if we consider *hyperbolic intertemporal discount factors*, i.e $\beta^h_t = (1+\theta t)^{-\frac{\tau}{\theta}}$, where $(\tau, \theta) \gg 0$, then the function U^h, as defined in the statement of Proposition 1, satisfies Assumption 1 and the sequence $(a^h_t)_{t\geq 0}$ goes to infinity as t increases. Therefore, in this case, uniform impatience does not hold.

3 A monetary model with uniform impatience agents

In the context of the economy defined in the previous section, let $p(\xi) := (p(\xi, l); l \in L)$ be the commodity price at $\xi \in D$ and $p := (p(\xi); \xi \in D)$.

Assume that there is only one asset, *money*, that can be traded along the event-tree. Although this security does not deliver any physical payment, it can be used to make intertemporal transfers. Let $q = (q(\xi); \xi \in D)$ be the plan of monetary prices. We assume that money is in positive net supply that neither disappears from the economy nor deteriorates. Denote money endowments at a node $\xi \in D$ by $e^h(\xi) \in \mathbb{R}_+$ and let $z^h(\xi) \in \mathbb{R}_+$ be the quantity of money that h negotiates at ξ.

Given prices (p, q), let $B^h(p, q)$ be the choice set of agent $h \in H$, i.e. the set of plans $(x, z) := ((x(\xi); \xi \in D), (z(\xi); \xi \in D)) \in \mathbb{R}_+^{L \times D} \times \mathbb{R}_+^D$, such that, at any $\xi \in D$, the following budget constraint holds,

$$g_\xi^h(y(\xi), y(\xi^-); p, q) := p(\xi)\left(x(\xi) - w^h(\xi)\right) + q(\xi)\left(z(\xi) - e^h(\xi) - z(\xi^-)\right) \leq 0,$$

where $y(\xi) = (x(\xi), z(\xi))$, $y(\xi_0^-) := (x(\xi_0^-), z(\xi_0^-)) = 0$.

Agent's h individual problem is to choose a plan $y^h = (x^h, z^h)$ in $B^h(p, q)$ in order to maximize her utility function $U^h : \mathbb{R}_+^{L \times D} \to \mathbb{R}_+ \cup \{+\infty\}$.

Definition 1 An equilibrium for our economy is given by a vector of prices (p, q) jointly with individual plans $\left((x^h, z^h); h \in H\right)$, such that,

(a) For each $h \in H$, the plan $(x^h, z^h) \in B^h(p, q)$ is optimal at prices (p, q).
(b) At any $\xi \in D$, physical and asset markets clear

$$\sum_{h \in H} x^h(\xi) = W(\xi), \qquad \sum_{h \in H} z^h(\xi) = \sum_{\xi_0 \leq \mu \leq \xi} \sum_{h \in H} e^h(\mu).$$

Note that a pure spot market equilibrium, i.e. an equilibrium with zero monetary price, always exists provided that preferences satisfy Assumption 1 above.

4 Duality theory for individual optimization

In this section, we determine necessary and sufficient conditions for individual optimality.

Some previous definitions and notations are necessary. By normalization, we assume that prices (p, q) belong to $\mathbb{P} := \{(p, q) \in \mathbb{R}_+^{L \times D} \times \mathbb{R}_+^D : \sum_{l \in L} p(\xi, l) + q(\xi) = 1, \ \forall \xi \in D\}$. Given a concave function $f : X \subset \mathbb{R}^L \to \mathbb{R} \cup \{-\infty\}$ the super-differential at $x \in X$, denoted by $\partial f(x)$, is defined as the set of vectors $p \in \mathbb{R}^L$ such that, for all $x' \in X$, $f(\xi, x') - f(\xi, x) \leq p(x' - x)$.

Definition 2 Under Assumption 1, given $(p, q) \in \mathbb{P}$ and $y^h = (x^h, z^h) \in B^h(p, q)$, we say that $(\gamma^h(\xi); \xi \in D) \in \mathbb{R}_+^D$ constitutes a family of Kuhn-Tucker multipliers (associated to y^h) if there exist, for each $\xi \in D$, super-gradients $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ such that,

(a) For every $\xi \in D$, $\gamma^h(\xi)\, g_\xi^h(y^h(\xi), y^h(\xi^-); p, q) = 0$.
(b) The following Euler conditions hold:

$$\gamma^h(\xi)p(\xi) - u'(\xi) \geq 0,$$
$$\left(\gamma^h(\xi)p(\xi) - u'(\xi)\right) x^h(\xi) = 0,$$
$$\gamma^h(\xi)q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu) \geq 0,$$
$$\left(\gamma^h(\xi)q(\xi) - \sum_{\mu \in \xi^+} \gamma^h(\mu)q(\mu)\right) z^h(\xi) = 0.$$

(c) The following transversality condition holds:

$$\lim_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi)q(\xi)z^h(\xi) = 0.$$

First of all, we want to note that, when Kuhn-Tucker multipliers exist and are used as intertemporal deflators, the discounted value of individual endowments is finite.

Proposition 2 (Finite discounted value of individual endowments) *Let there be a plan* $(p, q) \in \mathbb{P}$ *and* $y^h = (x^h, z^h) \in B^h(p, q)$ *such that* $U^h(x^h) < +\infty$. *If Assumption* 1 *holds then for any family of Kuhn-Tucker multipliers associated to* y^h, $(\gamma^h(\xi);\ \xi \in D)$, *we have*

$$\sum_{\xi \in D} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi)e^h(\xi)\right) < +\infty.$$

Since we only know that, for any plan $(p, q) \in \mathbb{P}$, the choice set $B^h(p, q)$ belongs to $\mathbb{R}_+^{D \times L} \times \mathbb{R}_+^D$, it is not obvious that a plan of Kuhn-Tucker multipliers will exist. Thus, as individual admissible plans are determined by countably many inequalities, we will prove the existence of these multipliers by direct construction, using the Kuhn-Tucker Theorem for Euclidean spaces.

Proposition 3 (Characterization of individual optimality) *Under Assumption* 1, *let there be a plan* $(p, q) \in \mathbb{P}$ *and* $y^h = (x^h, z^h) \in B^h(p, q)$. *If* $U^h(x^h) < +\infty$ *and* y^h *is an optimal plan for agent* $h \in H$ *at prices* (p, q), *then there exists a family of Kuhn-Tucker multipliers associated to* y^h. *Reciprocally, if there exists a family of Kuhn-Tucker multipliers associated to* y^h, *then* y^h *is an optimal plan for agent* h *at prices* (p, q).

We say that *debt constraints induce frictions over agent* h *in* $\tilde{D} \subset D$ if the plan of shadow prices $(\eta^h(\mu);\ \mu \in \tilde{D})$ defined, at each $\mu \in \tilde{D}$, by

$$\eta^h(\mu) = \gamma^h(\mu)q(\mu) - \sum_{\nu \in \mu^+} \gamma^h(\nu)q(\nu)$$

is different from zero, where $\mu^{+} = \{\nu \in D : (\nu \geq \mu) \wedge (t(\nu) = t(\mu) + 1)\}$ is the set of immediate successors of μ and $(\gamma^h(\mu); \mu \in D)$ is the plan of Kuhn-Tucker multipliers of agents h. Note that, as a consequence of financial Euler conditions, at any node $\mu \in \tilde{D}$, $\eta^h(\mu) z^h(\mu) = 0$.

The shadow price $\eta^h(\xi)$ measures, for an agent h not purchasing money, the desire to violate the debt constraint and take an arbitrarily small short position and, therefore, marginally increase her utility today at a rate $\gamma^h(\xi)q(\xi)$ while loosing tomorrow a marginal amount of utility at a rate $\sum_{\mu \in \xi^{+}} \gamma^h(\mu)q(\mu)$. Thus, for someone not purchasing money at node ξ, the price of money might exceed the sum of deflated prices at the immediately following nodes if there is a positive incremental desire to advance wealth from the next nodes to node ξ. The incremental desire to transfer wealth to the node ξ is, in some sense, the price the agent would be willing to pay to get credit with fiat money at this node (i.e. by short selling money) and, for this reason, we can refer to the shadow price as the value that the agent places on the credit services of money (which are being ruled out here and might be allowed but bounded under more general short sales constraints—see, for instance, Páscoa et al. (2008)).

Intuitively and focussing on the simplest case where each node $\mu \in D$ has a unique successor, denoted by $\mu + 1$, if an agent h wishes to compensate a relative lack of resources at $\mu + 1$ by purchasing money at μ, she should do it until (the marginal benefit) $\gamma^h(\mu + 1)q(\mu + 1)$ equals (the marginal cost) $\gamma^h(\mu)q(\mu)$, or equivalently, until the personal marginal rate of substitution $\gamma^h(\mu)/\gamma^h(\mu + 1)$ matches the market rate $q(\mu + 1))/q(\mu)$ (which is the one plus the interest rate).

However, if agent h wished to compensate a relative lack of resources at μ by short selling money the analogous equalities could not be attained as short sales are ruled out. That is, the equalities would be attained at the desired but unfeasible consumption plan (with a lower $\gamma^h(\mu)$ and a higher $\gamma^h(\mu + 1)$) but at the constrained optimal plan we have (the marginal benefit) $\gamma^h(\mu)q(\mu)$ higher than (the marginal cost) $\gamma^h(\mu + 1)q(\mu + 1)$ with the equality holding once the shadow price of money credit services is included on marginal costs. Equivalently, the personal marginal rate of substitution $\gamma^h(\mu)/\gamma^h(\mu + 1)$ exceeds the market rate $q(\mu + 1)/q(\mu)$ but the equality holds once the shadow price is taken into account.

5 Fundamental values and bubbles

The recursive use of the financial Euler conditions leads us to define the fundamental value of an asset at node ξ as the deflated sum of the value of the asset's deliveries and of the incremental desires to take positions prevented by portfolio constraints (i.e. the shadow prices of these constraints). That is, the fundamental value of an asset at a node ξ is the deflated sum of the net value added at each future node μ, which is given by $q(\mu) - \sum_{\eta \in \mu^{+}} \frac{\gamma^h(\eta)}{\gamma^h(\mu)} q(\eta)$.

Hence, using agent h Kuhn Tucker multipliers as deflators, $\gamma^h := (\gamma^h(\xi); \xi \in D)$, the fundamental value of an asset at a node ξ, denoted by $F(\xi, q, \gamma^h)$, is

$$F(\xi, q, \gamma^h) := \sum_{\mu \geq \xi} \frac{\gamma^h(\mu)}{\gamma^h(\xi)} \left(q(\mu) - \sum_{\eta \in \mu^+} \frac{\gamma^h(\eta)}{\gamma^h(\mu)} q(\eta) \right).$$

In a frictionless world, i.e. where the financial debt constraints are non-saturated, the fundamental value of an asset coincides with the discounted value of future deliveries that an agent will receive for one unit of the asset that she buys and keeps forever (a consequence of financial Euler conditions). However, when frictions occur, the fundamental value may include the shadow prices of binding portfolio constraints.

For instance, in our context and as a consequence of the financial Euler conditions, the fundamental value of fiat money is the discounted sum of the shadow prices associated with the zero short-sales constraints,

$$F(\xi, q, \gamma^h) = \frac{1}{\gamma^h(\xi)} \sum_{\mu \in D(\xi)} \eta^h(\mu).$$

That is, the fundamental value is the discounted sum of *incremental desires of reallocating consumption to the present at the expense of tomorrow*, a measure of the effects that the absence of monetary loans has on agent h.

Proposition 4 *Under Assumption 1, given an equilibrium* $[(p, q); ((x^h, z^h); h \in H)]$, *at each node* $\xi \in D$ *and for any* $h \in H$, $q(\xi) \geq F(\xi, q, \gamma^h)$.

Therefore, the fundamental value of fiat money is always well defined and it is immediate to see that the price $q(\xi)$ is equal to $F(\xi, q, \gamma^h)$ plus a residual value, $\lim_{T \to +\infty} \sum_{\mu \in D_T(\xi)} \frac{\gamma^h(\mu)}{\gamma^h(\xi)} q(\mu)$, called the asset price bubble at node ξ. When $q(\xi) > F(\xi, q, \gamma^h)$ we say that fiat money has a bubble at ξ under γ^h.

6 Characterizing monetary equilibria

Let us see under what conditions can we have equilibria with positive price of money, also called *monetary equilibria*.

Theorem 1 *Suppose that Assumption 1 holds and let* $[(p, q); ((x^h, z^h); h \in H)]$ *be an equilibrium. If at some node* $\xi \in D$ *the monetary price* $q(\xi)$ *is strictly positive and agents are uniform impatient, then debt constraints induce frictions over each agent in* $D(\xi)$. *Reciprocally, if debt constraints induce frictions over some agent in the sub-tree* $D(\xi)$, *then* $q(\xi) > 0$.

Some observations

(i) Intuitively, our Theorem asserts that a positive price of fiat money may only appear as a consequence of a positive incremental desire of uniformly impatient agents to use future wealth to consume more today. Moreover, even without uniform impatience, a positive desire for monetary loans assures a positive price of fiat money.

(ii) This theorem is related to the result in Santos and Woodford (1997, Theorem 3.3), that asserted that, under uniform impatience, assets in positive net supply are free of price bubbles for deflators, in the asset pricing kernel, that yield finite present values of aggregate wealth. However, our Theorem asserts that we may have a positive price of money due to the presence of shadow prices in the Kuhn-Tucker deflator process. Of course, in this case, for any kernel deflator, the present value of aggregate wealth will be infinite.

(iii) A direct consequence of our Theorem is that, if fiat money has a positive value, then all agents use it to transfer resources from the future to the present. Thus, under uniform impatience (which, as we prove below, avoids monetary bubbles), monetary equilibria are incompatible with agents consuming, in equilibrium, their endowments throughout their life.

Proof of the Theorem 1 By definition, if for some uniformly impatient agent $h \in H$, $(\eta^h(\mu); \mu \geq \xi) = 0$ then $F(\xi, q, \gamma^h) = 0$, where $\gamma^h := (\gamma^h(\xi); \xi \in D)$ is a family of Kuhn-Tucker multipliers associated to (x^h, z^h) (Proposition 3 assures its existence). Thus, a monetary equilibrium is a *bubble* for this agent. However, under uniform impatience (Assumption 2), bubbles are ruled out in equilibrium for the deflators given by her Kuhn-Tucker multipliers. Indeed, at each $\xi \in D$, $q(\xi)z^h(\xi) \geq 0$. Therefore, by the impatience property, $0 \leq (1-\pi)q(\xi)z^h(\xi) \leq p(\xi)\Delta(\xi)$. Moreover, as money is in positive net supply and all agents are uniform impatient, it follows that $(\frac{q(\xi)}{p(\xi)\Delta(\xi)}; \xi \in D)$ is uniformly bounded.

Since by Proposition 2, $\sum_{\xi \in D} \gamma^h(\xi)p(\xi)w^h(\xi) < +\infty$, it follows from Assumption 2 that $\sum_{\xi \in D} \gamma^h(\xi)q(\xi) < +\infty$, and, therefore, agent h does not observe bubbles in equilibrium. Therefore, we conclude that, if $q(\xi) > 0$ then $(\eta^h(\mu); \mu \geq \xi) \neq 0$, for each agent $h \in H$.

Reciprocally, by definition, if debt constraints induce frictions over some agent in the sub-tree $D(\xi)$, then the fundamental value of money at ξ is strictly positive, which implies that $q(\xi) > 0$. □

If we discount the future using other non-arbitrage deflators, instead of the Kuhn-Tucker multipliers, then fiat money may have a bubble. Remember that non-arbitrage deflators are plans of state prices $(\nu(\xi); \xi \in D)$ that satisfy the Euler financial conditions. When the Euler financial conditions are satisfied with equality at any node then we say that the deflator is in the (non-arbitrage) asset-pricing kernel.

Corollary 1 *Under Assumption 1, given a monetary equilibrium* $[(p, q); ((x^h, z^h); h \in H)]$, *there always exists a plan of non-arbitrage deflators in the asset-pricing kernel,* $(\nu(\xi); \xi \in D)$, *for which the fundamental value of money is zero and, therefore, the price of money is a bubble.*

Proof Fix an agent $h \in H$. It follows from Proposition 3 that there is a family of Kuhn-Tucker multipliers associated with the plan (x^h, z^h), namely $(\gamma^h(\xi); \xi \in D)$. Also, by Euler conditions, if $q(\xi) = 0$ at some node $\xi \in D$, then $q(\mu) = 0$, $\forall \mu > \xi$.

Define $\nu := (\nu(\xi) : \xi \in D)$ by $\nu(\xi_0) = 1$, and

$$\nu(\mu) = 1, \qquad \forall \mu \in D \text{ such that } \exists \xi \in D : (\mu > \xi) \wedge (q(\xi) = 0),$$
$$\frac{\nu(\mu)}{\nu(\mu^-)} = \frac{\gamma^h(\mu)}{\gamma^h(\mu^-) - \frac{\eta^h(\mu^-)}{q(\mu^-)}}, \quad \forall \mu \in D\setminus\{\xi_0\} \text{ such that } q(\mu^-) > 0.$$

Financial Euler conditions on $(\gamma^h(\xi);\ \xi \in D)$ imply that, for each $\xi \in D$, $\nu(\xi)q(\xi) = \sum_{\mu \in \xi^+} \nu(\mu)q(\mu)$. Therefore, using the plan of deflators ν, *financial Euler conditions* hold and the positive price of money is a bubble. □

Under Assumption 2, the plan $(\frac{q(\xi)}{p(\xi)\Delta(\xi)}; \xi \in D)$ is uniformly bounded along the event-tree and, therefore, the existence of a bubble for a plan of deflators $(\nu(\xi); \xi \in D)$ implies that the deflated value of future individual endowments, $\sum_{\xi \in D} \nu(\xi)p(\xi)w^h(\xi)$, has to be infinite for any agent $h \in H$.[4] This plan of deflators, which is incompatible with physical Euler conditions, is compatible with zero shadow prices and our observation conforms to the results by Santos and Woodford (1997, Theorem 3.3): a monetary bubble may only occur, for a plan of deflators in the asset pricing kernel, if the associated present value of aggregate wealth is infinite.[5]

Some remarks:

- Adding an (unrestricted) asset that pays returns would not necessarily make fiat money useless, since the degree of market incompleteness may still leave room for a spanning role of money. On the other hand, if we allow for an increasing number of non-redundant (unrestricted) securities in order to assure that aggregated wealth can be replicated by the deliveries of a portfolio trading plan, money will have zero price. Indeed, in this context, independently of the non-arbitrage kernel deflator, the discounted value of future wealth must be finite (see Santos and Woodford 1997, Lemma 2.4). Therefore, if money has a positive value, we obtain a contradiction, since as we say above, associated to any monetary equilibrium we may construct a deflator in the asset pricing kernel under which the discounted value of aggregated wealth is *infinite*. However, the issue of new assets in order to achieve that property of the financial markets can be too costly.
- In models addressing the role of money as a medium of exchange, starting with Clower (1967), it is instead liquidity frictions that become crucial to have a positive price of money. Grandmont and Younés (1972) consider a temporary equilibrium model where fiat money is the only store of value and prove that equilibrium exists as a consequence of some "viscosity" in the exchange process. In a recent work along those lines, but in the general equilibrium context, Santos (2006) showed

[4] In other case, Assumption 2 assure that $\sum_{\xi \in D} \nu(\xi)q(\xi) < +\infty$, which is incompatible with bubbles.

[5] Actually, under any non-arbitrage deflator obtained as a strict convex combination of a Kuhn-Tucker deflator and the deflator in Corollary 1, fiat money has both a positive fundamental value and a bubble, but the present value of wealth is infinite. Such indeterminacy is absent when a Kuhn-Tucker deflator is used, as the positive price of money is just a consequence of a positive fundamental value, and, moreover, deflated wealth is finite.

that monetary equilibrium only arises when cash-in-advance constraints are binding infinitely often for all agents. Also, in a cashless economy with zero short-sales restrictions, Giménez (2007) provided examples of monetary bubbles that can be reinterpreted as positive fundamental values.

Next, we show that given a monetary equilibrium allocation, there is always another feasible allocation that makes some agent better off without hurting others, i.e. monetary equilibria are inefficient in the Pareto sense. This claim is shown by noticing that, in our context, Pareto efficiency requires marginal rates of intertemporal substitution to be equal across agents.[6]

Proposition 5 *Under Assumption 1, if for each $\xi \in D$, $u^h(\xi, \cdot)$ is differentiable in $\mathbb{R}^L_{++}$, and for any $l \in L$, $\lim_{x_l \to 0^+} \nabla u^h(\xi, x) = +\infty$, then any monetary equilibrium is Pareto inefficient.*

Proof Suppose that there exists an efficient monetary equilibrium. Since $\lim_{x_l \to 0^+} \nabla u^h(\xi, x) = +\infty$, $\forall (h, \xi, l) \in H \times D \times L$, all agents have interior consumption along the event-tree. Positive net supply of money implies that there exists, at each $\xi \in D$, at least one lender. Therefore, by the efficiency property, it follows that *all* individuals have zero shadow prices. Thus, it follows from the transversality condition of Definition 2, jointly with Proposition 4, that $q(\xi) = 0$ for any node $\xi \in D$. A contradiction. □

The inefficiency of monetary equilibrium was previously addressed in the context of temporary equilibrium models with cash-in-advance constraints by Grandmont and Younés (1973). Also, as was analyzed by Hahn (1973) (see also Starret 1973), the existence of transactions costs may lead to inefficient allocations.

7 Monetary equilibrium in the absence of uniform impatience

To highlight the role of uniform impatience, we adapt Example 1 in Araujo et al. (2009) in order to show that without this assumption money may have a bubble for deflators that give a finite present value of aggregate wealth, even when the deflator process is given by Kuhn-Tucker multipliers.

Essentially this happens because individuals will believe that, as time goes on, the probability that the economy may fall in a path in which endowments increase without an upper bound converges to zero fast enough. In our example, the Kuhn-Tucker deflator process is a non-arbitrage kernel deflator yielding a finite present value of aggregate wealth, but we know that the supremum over all asset pricing kernel deflators of the present value of aggregate wealth is infinite (see Santos and Woodford 1997, Theorem 3.1 and Corollary 3.2).

[6] More formally, an allocation $((\overline{x}^i(\xi))_{\xi \in D},\ i \in H)$ is Pareto efficient if for each agent $i \in H$ it maximizes the utility of the agent, $U^i((x^i(\xi))_{\xi \in D})$, among the allocations $((x^j(\xi))_{\xi \in D},\ j \in H)$ that satisfy both $U^j((x^j(\xi))_{\xi \in D}) \geq U^j((\overline{x}^j(\xi))_{\xi \in D}),\ \forall j \neq i$; and $\sum_{j \in H} x^j(\xi) \leq W(\xi),\ \forall \xi \in D$. Under the conditions of Proposition 5, the necessary Kuhn-Tucker conditions for these problems imply that the marginal rates of intertemporal substitution must be equal across agents.

Example Assume that each $\xi \in D$ has two successors: $\xi^+ = \{\xi^u, \xi^d\}$. There are two agents $H = \{1, 2\}$ and only one commodity. Each $h \in H$ has physical endowments $(w^h(\xi))_{\xi \in D}$, receives financial endowments $e^h \geq 0$ only at the first node, and has preferences represented by the utility function $U^h(x) = \sum_{\xi \in D} \beta^{t(\xi)} \rho^h(\xi)\, x(\xi)$, where $\beta \in (0, 1)$ and the plan $(\rho^h(\xi))_{\xi \in D} \in (0, 1)^D$ satisfies $\rho(\xi_0) = 1$, $\rho^h(\xi) = \rho^h(\xi^d) + \rho^h(\xi^u)$ and

$$\rho^1(\xi^u) = \frac{1}{2^{t(\xi)+1}} \rho^1(\xi), \qquad \rho^2(\xi^u) = \left(1 - \frac{1}{2^{t(\xi)+1}}\right) \rho^2(\xi).$$

Suppose that agent $h = 1$ is the only one endowed with the asset, i.e. $(e^1, e^2) = (1, 0)$ and that, for each $\xi \in D$,

$$w^1(\xi) = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in D^{du}, \\ 1 & \text{otherwise}; \end{cases}$$
$$w^2(\xi) = \begin{cases} 1 + \beta^{-t(\xi)} & \text{if } \xi \in \{\xi_0^d\} \cup D^{ud}, \\ 1 & \text{otherwise}; \end{cases}$$

where D^{du} is the set of nodes attained after going down followed by up, i.e. $D^{du} = \{\phi \in D : \exists \xi,\ \phi = (\xi^d)^u\}$ and D^{ud} denotes the set of nodes reached by going up and then down, i.e. $D^{ud} = \{\phi \in D : \exists \xi,\ \phi = (\xi^u)^d\}$.

Agents will use positive endowment shocks in low probability states to buy money and sell it later in states with higher probabilities. Let prices be $(p(\xi), q(\xi))_{\xi \in D} = (\beta^{t(\xi)}, 1)_{\xi \in D}$ and suppose that consumption of agent h is given by $x^h(\xi) = w^{h'}(\xi)$, where $h \neq h'$. It follows from budget constraints that, at each ξ, the portfolio of agent h must satisfy $z^h(\xi) = \beta^{t(\xi)}(w^h(\xi) - w^{h'}(\xi)) + z^h(\xi^-)$, where $z^h(\xi_0^-) := e^h$ and $h \neq h'$.

Thus, consumption allocations above jointly with the financial positions given by $(z^1(\xi_0), z^1(\xi^u), z^1(\xi^d)) = (1, 1, 0)$ and $(z^2(\xi))_{\xi \in D} = (1 - z^1(\xi))_{\xi \in D}$ are budget and market feasible. Finally, given $(h, \xi) \in H \times D$, let $\gamma^h(\xi) = \rho^h(\xi)$ be the candidate for Kuhn-Tucker multiplier of agent h at node ξ. It follows that conditions below hold and they assure individual optimality (see Proposition 3),

$$(\gamma^h(\xi) p(\xi), \gamma^h(\xi) q(\xi)) = (\beta^{t(\xi)} \rho^h(\xi), \gamma^h(\xi^u) q(\xi^u) + \gamma^h(\xi^d) q(\xi^d)),$$
$$\sum_{\{\phi \in D:\, t(\phi) = T\}} \gamma^h(\phi) q(\phi) z^h(\phi) \longrightarrow 0, \quad \text{as } T \to +\infty.$$

By construction, the plan of shadow prices associated to zero short-sales constraints is zero. Therefore, money has a zero fundamental value and a bubble under Kuhn-Tucker multipliers.

Intuitively, the price of the consumption good is falling at a rate that compensates the intertemporal impatience of the agents. Thus, since along the event-tree the purchase power of fiat money is rising, it is not worthy for agents to have short positions on money and, therefore, binding zero short-sale constraints do not induce frictions.

Also, the diversity of individuals' beliefs about the uncertainty (probabilities $\rho^h(\xi)$) implies that both agents perceive a finite present value of aggregate wealth.[7] Finally, Assumption 2 is not satisfied, because aggregated physical endowments were unbounded along the event-tree.[8] □

8 Conclusion

It is well known that, under uniform impatience, positive net supply assets are free of bubbles for non-arbitrage kernel deflators which always yield finite present values of wealth. However, this does not mean that prices cannot be above the series of deflated dividends for the deflators given by the agents' marginal rates of substitution, which also yield finite present values of individual wealth.

In this paper, we showed that binding zero short-sales constraints lead to positive prices of fiat money, due to positive fundamental values that consist of the deflated shadow values of the credit services that money might but cannot provide. These

[7] Using agent' h Kuhn-Tucker multipliers as deflators, the present value of aggregated wealth at $\xi \in D$, denoted by $PV^h(\xi)$, satisfies,

$$\begin{aligned}
PV^h(\xi) &= \sum_{\mu \geq \xi} \frac{\gamma^h(\mu)}{\gamma^h(\xi)}\, p(\mu)\, W(\mu) \\
&= \frac{2}{\rho^h(\xi)} \sum_{\mu \geq \xi} \rho^h(\mu)\beta^{t(\mu)} + \frac{1}{\rho^h(\xi)} \sum_{\{\mu \geq \xi : \mu \in D^{ud} \cup D^{du} \cup \{\xi_0^d\}\}} \rho^h(\mu) \\
&= 2\frac{\beta^{t(\xi)}}{1-\beta} + \sum_{\{\mu \geq \xi : \mu \in D^{ud} \cup D^{du} \cup \{\xi_0^d\},\, t(\mu) \leq t(\xi)+1\}} \frac{\rho^h(\mu)}{\rho^h(\xi)} \\
&\quad + \sum_{s=t(\xi)+1}^{+\infty} \left[\frac{1}{2^{s+1}} \left(1 - \frac{1}{2^s}\right) + \left(1 - \frac{1}{2^{s+1}}\right) \frac{1}{2^s} \right] \\
&= 2\frac{\beta^{t(\xi)}}{1-\beta} + \frac{3}{2}\frac{1}{2^{t(\xi)}} - \frac{1}{3}\frac{1}{4^{t(\xi)}} + \frac{1}{\rho^h(\xi)} \sum_{\{\mu \geq \xi : \mu \in D^{ud} \cup D^{du},\, t(\mu) \leq t(\xi)+1\}} \rho^h(\mu) < +\infty.
\end{aligned}$$

[8] If Assumption B holds, there are $(\kappa, \pi) \in \mathbb{R}_{++} \times (0, 1)$ such that, for any $\xi \in D^{uu} := \{\mu \in D : \exists \phi \in D; \mu = (\phi^u)^u\}$,

$$\frac{1}{\kappa} = \frac{w^h(\xi)}{\kappa} > \frac{1-\pi}{\beta^{t(\xi)}\rho^h(\xi)} \sum_{\mu > \xi} \rho^h(\mu)\beta^{t(\mu)} W(\mu), \qquad \forall h \in H.$$

Thus, for all $(\xi, h) \in D^{uu} \times H$, $\beta^{t(\xi)} \left(\frac{1}{\kappa(1-\pi)} + W(\xi)\right) > PV^h(\xi)$. On the other hand, given $\xi \in D^{uu}$,

$$PV^1(\xi) \geq \frac{1}{\rho^1(\xi)} \sum_{\{\mu \geq \xi : \mu \in D^{ud} \cup D^{du},\, t(\mu) \leq t(\xi)+1\}} \rho^1(\mu) = 1 - \frac{1}{2^{t(\xi)+1}}.$$

Therefore, as for any $T \in \mathbb{N}$ there exists $\xi \in D^{uu}$ with $t(\xi) = T$, we conclude that, $\beta^T \left(\frac{1}{\kappa(1-\pi)} + 2\right) > 0.5$, for all $T > 0$. A contradiction.

monetary equilibria improve upon the allocations that would prevail when money was not available, but are still Pareto inefficient, as the marginal rates of substitution will not be the same for an agent purchasing money and for an agent who is being prevented from doing the desired shorting of money.

Appendix

Proof of Proposition 2 Let $\mathscr{L}^h_\xi : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the function defined by $\mathscr{L}^h_\xi(y(\xi), y(\xi^-)) = v^h(\xi, y(\xi)) - \gamma^h(\xi)\, g^h_\xi(y(\xi), y(\xi^-); p, q)$, where $y(\xi) = (x(\xi), z(\xi))$ and $v^h(\xi, \cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is given by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{otherwise.} \end{cases}$$

It follows from Assumption 1 and Euler conditions that, for each $T \geq 0$,

$$\sum_{\xi \in D^T} \mathscr{L}^h_\xi(0, 0) - \sum_{\xi \in D^T} \mathscr{L}^h_\xi(y^h(\xi), y^h(\xi^-)) \leq -\sum_{\xi \in D_T} \gamma^h(\xi) q(\xi)(0 - z^h(\xi)).$$

Therefore, as for each $\xi \in D$, $\gamma^h(\xi)\, g^h_\xi(y^h(\xi), y^h(\xi^-); p, q) = 0$, we have that, for any $S \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \sum_{\xi \in D^S} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi) e^h(\xi)\right) \\ &\leq \limsup_{T \to +\infty} \sum_{\xi \in D^T} \gamma^h(\xi) \left(p(\xi)\, w^h(\xi) + q(\xi) e^h(\xi)\right) \\ &\leq U^h(x^h) + \limsup_{T \to +\infty} \sum_{\xi \in D_T} \gamma^h(\xi) q(\xi) z^h(\xi). \end{aligned}$$

It follows from the transversality condition in the definition of Kuhn-Tucker multipliers that the last term in the right-hand side of the inequality above is equal to zero. Therefore, as $U^h(x^h)$ if finite, we conclude the proof. □

Proof of Proposition 3 Suppose that $(y^h(\xi))_{\xi \in D}$ is optimal for agent $h \in H$ at prices (p, q). For each $T \in \mathbb{N}$, consider the truncated optimization problem,

$$(P^{h,T}) \quad \begin{aligned} &\max \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ &\text{s.t.} \begin{cases} g^h_\xi\left(y(\xi), y(\xi^-);\ p, q\right) \leq 0, \quad \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ (x(\xi), z(\xi)) \geq 0, \quad \forall \xi \in D^T. \end{cases} \end{aligned}$$

It follows that, under Assumption 1, each truncated problem $P^{h,T}$ has a solution $(y^{h,T}(\xi))_{\xi \in D^T}$.[9]

Define $v^h(\xi, \cdot) : \mathbb{R}^L \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$v^h(\xi, y(\xi)) = \begin{cases} u^h(\xi, x(\xi)) & \text{if } x(\xi) \geq 0; \\ -\infty & \text{in other case.} \end{cases}$$

where $y(\xi) = (x(\xi), z(\xi))$. Given a multiplier $\gamma \in \mathbb{R}$, let $\mathscr{L}^h_\xi(\cdot, \gamma; p, q) : \mathbb{R}^{L+1} \times \mathbb{R}^{L+1} \to \mathbb{R} \cup \{-\infty\}$ be the Lagrangian at node ξ, i.e.

$$\mathscr{L}^h_\xi(y(\xi), y(\xi^-), \gamma;\ p, q) = v^h(\xi, y(\xi)) - \gamma\, g^h_\xi(y(\xi), y(\xi^-); p, q).$$

It follows from Rockafellar (1997, Theorem 28.3, p. 281) that there exist non-negative multipliers $(\gamma^{h,T}(\xi))_{\xi \in D^T}$ such that the following saddle point property

$$\begin{aligned} &\sum_{\xi \in D^T} \mathscr{L}^h_\xi(y(\xi), y(\xi^-), \gamma^{h,T}(\xi);\ p, q) \\ &\quad \leq \sum_{\xi \in D^T} \mathscr{L}^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-), \gamma^{h,T}(\xi);\ p, q) \end{aligned} \tag{1}$$

is satisfied, for each plan $(y(\xi))_{\xi \in D^T} = (x(\xi), z(\xi))_{\xi \in D^T} \in \mathbb{R}^{L \times D^T}_+ \times \mathbb{R}^{D^T}_+$. Moreover, at each node $\xi \in D^T$, multipliers satisfy

$$\gamma^{h,T}(\xi)\, g^h_\xi(y^{h,T}(\xi), y^{h,T}(\xi^-); p, q) = 0.$$

[9] In fact, as $(y^h(\xi))_{\xi \in D}$ is optimal and $U^h(x^h) < +\infty$, it follows that there exists a solution for $P^{h,T}$ if and only if there exists a solution for the problem,

$$(\tilde{P}^{h,T}) \qquad \begin{aligned} &\max \sum_{\xi \in D^T} u^h(\xi, x(\xi)) \\ &\text{s.t.} \begin{cases} g^h_\xi\left(y(\xi), y(\xi^-);\ p, q\right) \leq 0, & \forall \xi \in D^T, \text{ where } y(\xi) = (x(\xi), z(\xi)), \\ z(\xi) \geq 0, & \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0 \\ z(\xi) = 0, & \text{if } \left[\xi \in D^{T-1} \text{ and } q(\xi) = 0\right] \text{ or } \xi \in D_T, \\ x(\xi) \geq 0, & \forall \xi \in D^T. \end{cases} \end{aligned}$$

Indeed, it follows from the existence of an optimal plan which gives finite utility that if $q(\xi) = 0$ for some $\xi \in D$, then $q(\mu) = 0$ for each successor $\mu > \xi$. Also, budget feasibility assures that,

$$z(\xi) \leq \frac{p(\xi) w^h(\xi)}{q(\xi)} + z(\xi^-), \quad \forall \xi \in D^{T-1} \text{ such that } q(\xi) > 0.$$

As $z(\xi_0^-) = 0$, the set of feasible financial positions is bounded in the problem $(\tilde{P}^{h,T})$. Thus, budget feasible consumption allocations are also bounded and, therefore, the set of admissible strategies is compact. As the objective function is continuous, there is a solution for $(\tilde{P}^{h,T})$.

Since the optimality of $(y^h(\xi))_{\xi\in D}$ in the original problem implies that $U^h(x^h)$ is greater than or equal to $\sum_{\xi\in D^T} u^h(\xi, x^{h,T}(\xi))$,[10] it follows that, for any plan $(y(\xi))_{\xi\in D^T} = (x(\xi), z(\xi))_{\xi\in D^T} \in \mathbb{R}_+^{L\times D^T} \times \mathbb{R}_+^{D^T}$

$$\sum_{\xi\in D^T} \mathscr{L}_\xi^h(y(\xi), y(\xi^-), \gamma^{h,T}(\xi);\ p, q) \le U^h(x^h).$$

Analogous arguments to those made in Claims A1–A3 in Araujo et al. (2009) implies that,

Claim *Under Assumption 1, the following conditions hold:*

(i) *For each* $t < T$,

$$0 \le \sum_{\xi\in D^t} \gamma^{h,T}(\xi)\left(p(\xi)w^h(\xi) + q(\xi)e^h(\xi)\right) \le U^h(x^h).$$

(ii) *For each* $0 < t < T$,

$$\sum_{\xi\in D_t} \gamma^{h,T}(\xi)q(\xi)z^h(\xi^-) \le \sum_{\xi\in D\setminus D^{t-1}} u^h(\xi, x^h(\xi)).$$

(iii) *For each* $\xi \in D^{T-1}$ *and for any* $y(\xi) = (x(\xi), z(\xi)) \ge 0$,

$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \le \left(\gamma^{h,T}(\xi)p(\xi);\ \gamma^{h,T}(\xi)q(\xi) - \sum_{\mu\in\xi^+}\gamma^{h,T}(\mu)q(\mu)\right) \times (y(\xi) - y^h(\xi)) + \sum_{\eta\in D\setminus D^T} u^h(\eta, x^h(\eta)).$$

We know that, at each $\xi \in D$, $\underline{w}^h(\xi) := \min_{l\in L} w^h(\xi, l) > 0$. Also, as a consequence of monotonicity of $u^h(\xi)$, $||p(\xi)||_\Sigma > 0$. Thus, item (i) above guarantees that, for each $\xi \in D$,

$$0 \le \gamma^{h,T}(\xi) \le \frac{U^h(x^h)}{\underline{w}^h(\xi)\,||p(\xi)||_\Sigma}, \quad \forall T > t(\xi).$$

Therefore, the sequence $(\gamma^{h,T}(\xi))_{T\ge t(\xi)}$ is bounded, node by node. As the event-tree is countable, there is a common subsequence $(T_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and non-negative

[10] In fact, the plan $(\tilde{y}_\xi)_{\xi\in D}$ that equals to $\tilde{y}_\xi = y_\xi^{h,T}$, if $\xi \in D^T$, and equals to $\tilde{y}_\xi = 0$, if $\xi \in D \setminus D_T$, is budget feasible in the original economy and, therefore, the allocation $(y^{h,T}(\xi))_{\xi\in D^T}$ cannot improve the utility level of agent h.

multipliers $(\gamma^h(\xi))_{\xi\in D}$ such that, for each $\xi \in D$, $\gamma^{h,T_k}(\xi) \to_{k\to+\infty} \gamma^h(\xi)$, and

$$\gamma^h(\xi) g_\xi^h(y^h(\xi), y^h(\xi^-); p, q) = 0\,; \tag{2}$$

$$\lim_{t\to+\infty} \sum_{\xi\in D_t} \gamma^h(\xi) q(\xi) z^h(\xi^-) = 0\,, \tag{3}$$

where Eq. (2) follows from the strictly monotonicity of $u^h(\xi)$, and Eq. (3) is a consequence of item (ii) (taking the limit as T goes to infinity and, afterwards, the limit in t).

Moreover, using item (iii), and taking the limit as T goes to infinity, we obtain that, for each $y(\xi) = (x(\xi), z(\xi)) \geq 0$,

$$u^h(\xi, x(\xi)) - u^h(\xi, x^h(\xi)) \leq \left(\gamma^h(\xi) p(\xi)\,;\ \gamma^h(\xi) q(\xi) - \sum_{\mu\in\xi^+} \gamma^h(\mu) q(\mu)\right) \times (y(\xi) - y^h(\xi)).$$

It follows that $(\gamma^h(\xi) p(\xi)\,;\ \gamma^h(\xi) q(\xi) - \sum_{\mu\in\xi^+} \gamma^h(\mu) q(\mu))$ belongs to the super-differential set of the function $v^h(\xi,\cdot) + \delta(\cdot, \mathbb{R}_+^L \times \mathbb{R}_+)$ at point $y^h(\xi)$, where $\delta(y, \mathbb{R}_+^L \times \mathbb{R}_+) = 0$, when $y \in \mathbb{R}_+^L \times \mathbb{R}_+$ and $\delta(y, \mathbb{R}_+^L \times \mathbb{R}_+) = -\infty$, otherwise. Notice that, for each $y \in \mathbb{R}_+^L \times \mathbb{R}_+$, $\varsigma \in \partial\delta(y, \mathbb{R}_+^L \times \mathbb{R}_+) \Leftrightarrow 0 \leq \varsigma(y' - y),\ \ \forall y' \in \mathbb{R}_+^L \times \mathbb{R}_+$. Last condition holds if and only if both $\varsigma \geq 0$ and $\varsigma \cdot y = 0$.

Thus, by Theorem 23.8 in Rockafellar (1997, p. 223),[11] for any vector $y \in \mathbb{R}_+^L \times \mathbb{R}_+$, if $v'(\xi)$ belongs to $\partial\left[v^h(\xi, y) + \delta(y, \mathbb{R}_+^L \times \mathbb{R}_+)\right]$ then there exists $\tilde{v}'(\xi) \in \partial v^h(\xi, y)$ such that $v'(\xi) - \tilde{v}'(\xi) \in \partial\delta(y, \mathbb{R}_+^L \times \mathbb{R}_+)$, that is, $v'(\xi) - \tilde{v}'(\xi) \geq 0$ and $(v'(\xi) - \tilde{v}'(\xi)) \cdot (x, z) = 0$, where $y = (x, z)$. Therefore, it follows that there exists, for each $\xi \in D$, a super-gradient $\tilde{v}'(\xi) \in \partial v^h(\xi, y^h(\xi))$ such that,

$$\left(\gamma^h(\xi) p(\xi)\,;\ \gamma^h(\xi) q(\xi) - \sum_{\mu\in\xi^+} \gamma^h(\mu) q(\mu)\right) - \tilde{v}'(\xi) \geq 0,$$

$$\left[\left(\gamma^h(\xi) p(\xi)\,;\ \gamma^h(\xi) q(\xi) - \sum_{\mu\in\xi^+} \gamma^h(\mu) q(\mu)\right) - \tilde{v}'(\xi)\right] \cdot (x^h(\xi), z^h(\xi)) = 0.$$

As $\tilde{v}'(\xi) \in \partial v^h(\xi, y^h(\xi))$ if and only if there is $u'(\xi) \in \partial u^h(\xi, x^h(\xi))$ such that $\tilde{v}'(\xi) = (u'(\xi), 0)$, it follows from last inequalities that Euler conditions hold.

On the other side, item (i) in claim above guarantees that,

$$\sum_{\xi\in D} \gamma^h(\xi)(p(\xi) w^h(\xi) + q(\xi) e^h(\xi)) < +\infty$$

[11] This result assures that the super-gradient set of the sum of a finite number of functions is equal to the sum of the super-gradient sets of each function.

and, therefore, Eqs. (2) and (3) assure that,

$$\begin{aligned}&\lim_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi)\\&\quad\leq\lim_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)\left(p(\xi)w^h(\xi)+q(\xi)e^h(\xi)+q(\xi)z^h(\xi^-)\right)\\&\quad\leq\lim_{t\to+\infty}\sum_{\xi\in D_t}\gamma^h(\xi)q(\xi)z^h(\xi^-)=0,\end{aligned}$$

which implies that transversality condition holds.

Reciprocally, it follows from Euler conditions that, for each $T\geq 0$,

$$\begin{aligned}\sum_{\xi\in D^T}\mathscr{L}^h_\xi(y(\xi),y(\xi^-),\gamma^h(\xi);\ p,q)-\sum_{\xi\in D^T}\mathscr{L}^h_\xi(y^h(\xi),y^h(\xi^-),\gamma^h(\xi);\ p,q)\\\leq-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi)).\end{aligned}$$

Moreover, as at each node $\xi\in D$ we have that $\gamma^h(\xi)g^h_\xi(y^h(\xi),y^h(\xi^-);\,p,q)=0$, each budget feasible plan $y=((x(\xi),z(\xi));\,\xi\in D)$ must satisfy

$$\sum_{\xi\in D^T}u^h(\xi,x(\xi))-\sum_{\xi\in D^T}u^h(\xi,x^h(\xi))\leq-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi)).$$

Since the sequence $(\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z^h(\xi))_{T\in\mathbb{N}}$ converges, it is bounded. Thus,

$$\begin{aligned}&\limsup_{T\to+\infty}\left(-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)(z(\xi)-z^h(\xi))\right)\\&\quad\leq\limsup_{T\to+\infty}\left(-\sum_{\xi\in D_T}\gamma^h(\xi)q(\xi)z(\xi)\right)\leq 0\end{aligned}$$

Therefore,

$$U^h(x)=\limsup_{T\to+\infty}\sum_{\xi\in D^T}u^h(\xi,x(\xi))\leq U^h(x^h),$$

which guarantees that the plan $(x^h(\xi),z^h(\xi))_{\xi\in D}$ is optimal. □

Proof of Proposition 4. By Proposition 3, there are, for each agent $h \in H$, non-negative shadow prices $(\eta^h(\xi); \xi \in D)$, satisfying for each $\xi \in D$,

$$0 = \eta^h(\xi) z^h(\xi);$$
$$\gamma^h(\xi) q(\xi) = \sum_{\mu \in \xi^+} \gamma^h(\mu) q(\mu) + \eta^h(\xi).$$

Therefore,

$$\gamma^h(\xi) q(\xi) = \sum_{\mu \geq \xi} \eta^h(\mu) + \lim_{T \to +\infty} \sum_{\mu \in D_T(\xi)} \gamma^h(\mu) q(\mu).$$

As multipliers and monetary prices are non-negative, the infinite sum in the right hand side of equation above is well defined because its partial sums are increasing and bounded by $\gamma^h(\xi) q(\xi)$. This also implies that the limit of the (discounted) asset price exists. □

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