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On equilibrium existence with endogenous restricted financial participation☆

Abdelkrim Seghir[a,*], Juan Pablo Torres-Martínez[b]

[a] The American University in Cairo, School of Business, Economics Department, AUC Avenue, P.O. Box 74, New Cairo 11835, Egypt
[b] Department of Economics, University of Chile, Diagonal Paraguay 257 Office 1405-A, Santiago, Chile

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ABSTRACT

Without requiring either financial survival assumptions or linear spanning conditions over financial spaces, we prove equilibrium existence in an abstract incomplete market economy with endogenous restricted financial participation. We apply our results to general financial structures including nominal, real and collateralized asset markets.

1. Introduction

Modern financial markets restrict borrowers' participation in terms of which assets they can trade. For instance, financial restrictions may appear when borrowers are required to constitute collateral guarantees or when credit markets offer segmented products, such as students loans or resources to first-home-buyers. Financial participation may also be restricted due to non-economic motives, as countries with different accesses to credit markets due to political issues. The objective of this paper is to study restricted financial participation from a general equilibrium perspective, allowing for non-perishable commodities and various financial structures, including incomplete markets with nominal, real and collateralized assets.

In the general equilibrium literature, restricted financial participation were modeled in two ways. The first one assumes that financial restrictions are exogenously given. For such a framework, Angeloni and Cornet (2006) prove equilibrium existence in real financial markets assuming that portfolio sets are convex and compact, containing a neighborhood of zero at least for one agent (this last requirement is called *financial survival assumption*). More recently, Aouani and Cornet (2009) show equilibrium existence for the numeraire and the nominal cases assuming financial survival assumptions for all agents and requiring that portfolio sets can be defined by finitely many linear inequalities.[1] Without imposing the latter condition, the same authors prove equilibrium existence for more general financial structures under a *nonredundancy-type hypothesis*.[2] Furthermore, when agents' portfolio sets are closed and convex containing zero, Cornet and Gopalan (2010) show equilibrium existence for nominal financial markets using a *spanning condition* on the set of admissible portfolios, which requires the closed cone generated by the union of portfolio sets to be a linear space. The second way of modeling restricted financial participation is to assume that these constraints emerge endogenously due to regulatory, institutional or budgetary considerations that may depend on market prices and/or commodity purchases. Such a structure was considered by Cass et al. (2001) and more recently by Carosi et al. (2009). Cass et al. (2001) prove equilib-

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* Corresponding author.
E-mail addresses: kseghir@aucegypt.edu (A. Seghir), juan.torres@fen.uchile.cl (J.P. Torres-Martínez).

[1] These sets are called *convex polyhedral*.

[2] See Aouani and Cornet (2009, Assumption F3). In the case of nominal assets and unrestricted participation, the nonredundancy-type assumption is equivalent to the classical hypothesis that the payoff matrix has full rank.

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rium existence for nominal assets, where admissible portfolio sets are described by functions that depend only on asset prices and satisfy some differentiability and regularity assumptions. Carosi et al. (2009) show equilibrium existence for numeraire financial markets, where restricted participation are given by functions that depend on commodity and asset prices and satisfy some homogeneity, differentiability and regularity assumptions.

In our model, restrictions on financial participation are endogenous, in the sense that they may depend on commodity purchases, as in mortgage markets where physical guarantees need to be held to obtain a loan. More precisely, portfolio participation constraints are represented by a general correspondence whose values are not necessarily given by inequalities determined by differentiable or regular functions. With neither survival financial assumptions nor linear spanning conditions over financial spaces, we prove equilibrium existence in an *abstract economy* where preferences satisfy a *property of impatience*: any reduction on future consumption can be compensated by an increment of consumption today. In particular, this property is satisfied by preferences that are representable by utility functions that are unbounded on first period consumption. In addition, and for technical purposes, we assume that admissible debts, in the abstract economy, belong to a compact set. This hypothesis will be endogenously satisfied when we apply our existence result to either nominal or real assets markets. Since we allow portfolio constraints to depend on purchases of commodities, we can also apply our main result to extend the model of collateralized asset markets of Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007) to allow for endogenous restricted participation. As we do not impose any financial survival assumption, the presence of exclusive collateralized loans (i.e., credit opportunities that may be negotiated only by some agents) is compatible with equilibrium existence.

The remaining of the paper is organized as follows. Section 2 presents our abstract economy and states the associated equilibrium existence theorem. In Section 3 we apply this result to extend the classical models of nominal, real and collateralized assets to allow for restricted financial participation. Technical proofs are given in Appendix A.

2. An abstract financial economy

We consider an exchange economy with two periods $t\in\{0,1\}$ and uncertainty about which state of nature of a finite set $\mathcal{S}:=\{1,\dots,S\}$ will prevail at $t=1$. Denote by $s=0$ the state of nature (known with certainty) at period $t=0$ and let $\mathcal{S}^*=\{0\}\cup\mathcal{S}$ be the set of all states of nature in the economy.

There is a set $\mathcal{L}=\{1,\dots,L\}$ of perfectly divisible commodities that can be traded in spot markets at any state of nature $s\in\mathcal{S}^*$. The commodity space is $\mathbb{R}_+^{L(S+1)}$ and $p=(p_s;s\in\mathcal{S}^*)$ denotes the unitary commodity prices. We allow for depreciation, durability and transformation of commodities into other goods between periods. More precisely, we assume that any bundle x consumed at the first period is transformed into a bundle $Y_s x$ at state of nature $s\in\mathcal{S}$, where Y_s is an $(L\times L)$-matrix with non-negative entries.

Financial markets consist of a finite set $\mathcal{J}=\{1,\dots,J\}$ of assets. Each asset $j\in\mathcal{J}$ can be traded at the first period at a unitary price q_j and delivers state-contingent payments, $(V_j^s(p_s);s\in\mathcal{S})\in\mathbb{R}_+^S$, at the second period. Let us denote by $q=(q_j;j\in\mathcal{J})$ the vector of unitary asset prices and by $V:\mathbb{R}_+^{L(S+1)}\to\mathbb{R}_+^{S\times J}$ the map that associates to each p the vector $V(p)=(V_j^s(p_s);(s,j)\in\mathcal{S}\times\mathcal{J})$.

There is a finite number H of agents. Each agent $h\in\mathcal{H}=\{1,\dots,H\}$ is characterized by a consumption space $X^h=\mathbb{R}_+^{L(S+1)}$, a utility function $u^h:X^h\to\mathbb{R}$ and physical endowments $w^h=(w_s^h;s\in\mathcal{S}^*)\in\mathbb{R}_+^{L(S+1)}$. Agent h's vector of accumulated endowments is denoted by $W^h=(W_0^h,(W_s^h;s\in\mathcal{S})):=(w_0^h,(w_s^h+Y_s w_0^h;s\in\mathcal{S}))\in\mathbb{R}_+^{L(S+1)}$.

At the first period, each agent $h\in\mathcal{H}$ chooses a portfolio $\theta^h-\varphi^h$, where $\theta^h=(\theta_j^h;\, j\in\mathcal{J})\in\mathbb{R}_+^J$ (resp. $\varphi^h=(\varphi_j^h;\, j\in\mathcal{J})\in\mathbb{R}_+^J$) are the quantities of assets he purchases (resp. sells). In addition, at each state of nature $s\in\mathcal{S}^*$, agent h chooses a consumption bundle $x_s^h\in\mathbb{R}_+^L$. We denote by $x^h=(x_s^h;s\in\mathcal{S}^*)$ the consumption plan of agent $h\in\mathcal{H}$.

Financial positions may be restricted, in the sense that, each agent h is constrained to choose short-sales $\varphi^h\in\Phi^h(x_0^h)\subset\mathbb{R}_+^J$, where the correspondence $\Phi^h:\mathbb{R}_+^L\twoheadrightarrow\mathbb{R}_+^J$ associates first period commodity purchases with admissible debts. Thus, we allow credit opportunities to depend on commodity purchases. Moreover, since survival assumptions and spanning conditions over admissible portfolio sets are not required, agents may have access only to some credit contracts. That is, there may exist a set of canonical vectors of $\mathbb{R}^J$, $A=\{e(j);j\in\mathcal{J}'\}$, where $\mathcal{J}'\subset\mathcal{J}$, such that $\Phi^h(x_0^h)\cap\langle A\rangle=\varnothing$, for some $x_0^h\in\mathbb{R}_+^L$.[3]

Given prices (p,q), the budget set $B^h(p,q)$ of agent $h\in\mathcal{H}$ is the set of plans $(x^h,\theta^h,\varphi^h)\in\mathbb{E}:=X^h\times\mathbb{R}_+^J\times\mathbb{R}_+^J$ such that $\varphi^h\in\Phi^h(x_0^h)$ and

$$p_0x_0^h+\sum_{j\in\mathcal{J}}q_j(\theta_j^h-\varphi_j^h)\le p_0w_0^h;$$

$$p_sx_s^h\le p_sw_s^h+p_sY_sx_0^h+\sum_{j\in\mathcal{J}}V_j^s(p_s)(\theta_j^h-\varphi_j^h).$$

Definition. An equilibrium of our economy is given by a vector of prices $(\bar p,\bar q)\in\mathbb{R}_+^{L(S+1)}\times\mathbb{R}_+^J$ jointly with allocations $((\bar x^h,\bar\theta^h,\bar\varphi^h);h\in\mathcal{H})\in\mathbb{E}^H$ such that:

(i) For each agent $h\in\mathcal{H}$,
$(\bar x^h,\bar\theta^h,\bar\varphi^h)\in\operatorname{Argmax}\{u^h(x^h);(x^h,\theta^h,\varphi^h)\in B^h(\bar p,\bar q)\}$.
(ii) Physical and asset markets clearing conditions hold,

$$\sum_{h\in\mathcal{H}}(\bar x^h,\bar\varphi^h)=\sum_{h\in\mathcal{H}}(W^h,\bar\theta^h).$$

Our equilibrium existence result is:

Theorem. *Suppose that the following assumptions hold*:

(A1) *For each $h\in\mathcal{H}$, $u^h:X^h\to\mathbb{R}$ is continuous, strongly quasi-concave and strictly increasing.*[4]
(A2) *For each agent $h\in\mathcal{H}$, given a plan of consumption $x=(x_s;s\in\mathcal{S}^*)\gg 0$, for any $\theta\in(0,1)$ there is a bundle $\tau^h(\theta,x)\in\mathbb{R}_+^L$ such that,*

$$u^h(x_0+\tau^h(\theta,x),(\theta x_s;s\in\mathcal{S}))>u^h(x_0,(x_s;s\in\mathcal{S})).$$

(A3) *For each $h\in\mathcal{H}$, accumulated endowments $W^h\in\mathbb{R}_{++}^L$.*
(A4) *The map $p\mapsto V(p)=(V_j^s(p_s);(s,j)\in\mathcal{S}\times\mathcal{J})$ is continuous. In addition, given $j\in\mathcal{J}$, for each $(p_s;s\in\mathcal{S})\gg 0$, the vector $(V_j^s(p_s);s\in\mathcal{S})$ is different from zero.*
(A5) *For each $h\in\mathcal{H}$,*

[3] The set $\langle A\rangle$ denotes the linear space generated by A.

[4] Given a convex set $X\subset\mathbb{R}^k$, a function $f:X\to\mathbb{R}$ is *strongly quasi-convave* if $f(\lambda x+(1-\lambda)y)>\min\{f(x),f(y)\}$, for any $(x,y)\in X\times X$ such that $f(x)\neq f(y)$. This property is weaker than strictly quasi-concavity, which requires $f(\lambda x+(1-\lambda)y)>\min\{f(x),f(y)\}$, for any $(x,y)\in X\times X$ such that $x\neq y$.

(i) *the correspondence* $\Phi^h : \mathbb{R}^L_+ \twoheadrightarrow \mathbb{R}^J_+$ *has a closed and convex graph.*
(ii) *for each* $x_0^h \in \mathbb{R}^L_+$, $0 \in \Phi^h(x_0^h)$ *and* $\Phi^h(x_0^h) \subseteq \Phi^h(x_0^h + y)$, $\forall y \in \mathbb{R}^L_+$.

(A6) *For each* $x_0 \in \mathbb{R}^L_{++}$ *there exists* $\delta_{x_0} > 0$ *such that* $\delta_{x_0}(1, \ldots, 1) \in \sum_{h \in \mathcal{H}} \Phi^h(x_0)$.
(A7) *For each* $h \in \mathcal{H}$, *the correspondence* Φ^h *has compact values.*

Then, our economy has an equilibrium.

Assumption (A1) is classical. Indeed, any agent $h \in \mathcal{H}$ whose preferences $\succcurlyeq_h$ are complete, rational, continuous and strictly increasing, has a continuous and strictly increasing utility function. In addition, if preferences satisfy $((x, y) \in X^h \times X^h : x \succ_h y) \Rightarrow (\forall \lambda \in (0, 1] : \lambda x + (1-\lambda) y \succ_h y)$, then the utility function of agent h is strongly quasi-concave.

Angeloni and Cornet (2006) and Aouani and Cornet (2009) require financial survival assumptions to guarantee that budget sets have nonempty interior when commodity and asset prices, $(p_0, (q_j; j \in J))$, are normalized in the $(L+J-1)$-dimensional simplex.[5] In our model, no financial survival is required and, therefore, budget sets may have empty interior implying that budget set correspondences are not necessarily lower-hemicontinuous. In such a case, the continuity of agents' demand correspondences does not necessarily hold, as Berge's Maximum Theorem cannot be applied. To circumvent this technical problem, we normalize commodity prices p_0 in the $(L-1)$-dimensional simplex and, using Assumption (A2), we derive endogenous upper bounds for asset prices $(q_j; j \in J)$ (see Lemma 2). Assumption (A2) is an impatience condition on agents' preferences, which requires that, for any agent, any reduction in future consumption can be compensated by an increment in the consumption at the first period. This condition does not depend on the representation of individuals' preferences. Assumption (A2) is satisfied, for instance, by preferences which are representable by utility functions that are unbounded on the first-period consumption, such as von-Neumann utility functions with quasi-linear, Cobb–Douglas or Leontief kernels.

Assumption (A3) assumes that the initial accumulated endowment of each agent is positive at each state of nature. For a perishable commodity, it is equivalent to require that initial endowment of that commodity is positive at each state of nature. However, for a durable good, (A3) requires the interiority of individual endowments in that commodity at the first period only. This assumption is used to guarantee the lower hemicontinuity of the budget correspondences (see Lemma 1).

It is well known since Radner (1972) and Hart (1975) pioneering papers that the continuity of the individuals' demand correspondences may fail as the rank of the matrix $(V_j^s(p_s))_{s,j}$ may drop when $(p_s; s \in \mathcal{S})$ changes. Assumption (A4) assures that budget set correspondences have closed graphs. Using Berge's Maximum Theorem, Assumptions (A4), (A5)(i) and (A7) imply that individuals' demand correspondences are continuous, although the rank of the matrix of financial payments may still drop when commodity prices change. This issue was first addressed, in the context of smooth economies, by Duffie and Shafer (1985).

Assumption (A4) also guarantees that, when commodity prices are strictly positive, asset prices are non-trivial. Otherwise, an agent may invest at zero cost in assets with non-trivial payments (see Lemma 3).

Assumption (A5)(i) allows to prove that the budget set correspondences have convex values. Assumption (A5)(ii) assumes that credit opportunities do not decrease as purchases of durable goods increase. The intuition is that the ownership of durable goods may increase credit opportunities as (depreciated) durable commodities may serve as a partial debt recovery. That is, agents with higher accumulated wealth are more likely to be solvent in the second period and, therefore, have larger debt opportunities. Hypothesis (A5)(ii) implies that agents can increase the consumption of any commodity without changing their portfolio of debt, which guarantees that commodity prices are positive in equilibrium (see Lemma 3).

Assumption (A6) requires that each asset can be sold short by at least one agent, independently of the consumption level. Thus, financial trading is not prevented *ex ante*.

Finally, Assumption (A7) is purely technical. It assures that in our *abstract economy* budget set correspondences have compact values, which is required to prove that budget set correspondences are continuous (as explained above). Moreover, hypotheses (A7) and (A5)(ii) assure the existence of endogenous Radner upper bounds on short-sales. Thus, using market feasibility, we will determine natural upper bounds to truncate admissible plans in our generalized game (see Appendix A).

In the applications discussed below, Assumption (A7) will be satisfied endogenously (as in the case of nominal assets) or may be obtained as a consequence of some characteristics of financial markets and trading rules (as in the case of real and collateralized assets). In addition, unrestricted financial participation is a particular case of our framework, at least when assets are nominal. As emphasized by Radner (1972) and Hart (1975), even when financial participation is unrestricted, equilibrium may fail to exist for other financial structures.

3. Applications

3.1. Nominal asset markets

Suppose that assets are nominal. That is, for each $(s, j) \in \mathcal{S} \times \mathcal{J}$ there is a non-negative number $R_{s,j}$ such that $V_j^s(p_s) = R_{s,j}$ for any vector of prices $p_s \in \mathbb{R}^L_+$. Then, Assumption (A4) is satisfied when $(R_{s,j}; s \in \mathcal{S}) \neq 0$, for any $j \in \mathcal{J}$. In addition, assume that Assumptions (A1)–(A3) and (A5)–(A6) hold. In such a case, using monotonicity of preferences and Cramer's rule, we can find endogenous bounds on short-sales.[6] More precisely, there is $\alpha > 0$ such that, any budgetary feasible debt satisfies $\varphi_j^h < \alpha$, for any $(h, j) \in \mathcal{H} \times \mathcal{J}$. Without loss of generality, one can restrict financial participation to $\varphi^h \in \Phi^h(x_0^h) \cap [0, \alpha]^J$. Moreover, by redefining the correspondence of admissible financial positions Φ^h to incorporate the set $[0, \alpha]^J$, we can guarantee that Assumption (A7) also holds. Then, as a consequence of the theorem in the previous section, there is an equilibrium for nominal asset markets, even when commodi-

[5] The $(L+J-1)$-dimensional simplex is given by $\left\{ (p, q) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+ : \sum_{l \in \mathcal{L}} p_{0,l} + \sum_{j \in \mathcal{J}} q_j = 1 \right\}$.

[6] Indeed, using the strict monotonicity of preferences, we can assure that second period budget constraints are satisfied as equalities for any optimal plan. Therefore, using Cramer's rule, we obtain that financial portfolios $\theta^h - \varphi^h$ can be represented as a continuous function of commodity prices and consumption bundles of states of nature $s \in \mathcal{S}$, $((p_s, x_s^h); s \in \mathcal{S})$. Taking commodity prices in the $(L-1)$-dimensional simplex, it follows that financial portfolios are also bounded, as consumption bundles are non-negative and bounded from above by aggregated endowments. Finally, without loss of generality, we conclude that, we can restrict the admissible portfolios to those that satisfy an explicit short-sale constraint, as there is no real effect on selling and buying an asset $j \in \mathcal{J}$ simultaneously.

ties may be durable and financial participation is endogenously restricted.

3.2. Real asset markets with endogenous short-sales constraints

Under Assumptions (A1)–(A3) and (A5), suppose that assets are real. That is, for any $j \in \mathcal{J}$, there are bundles $(A_j^s; s \in \mathcal{S}) \in \mathbb{R}_+^{LS} \setminus \{0\}$ such that, $V_j^s(p_s) = p_s A_j^s$, $\forall s \in \mathcal{S}$. In addition, assume that for any $x_0^h \in \mathbb{R}_+^L$, $\Phi^h(x_0^h) \subseteq \{\varphi \in \mathbb{R}_+^J : \varphi \le m^h(x_0^h)\}$, where $m^h : \mathbb{R}_+^L \to \mathbb{R}_+^J$ is a continuous, non-decreasing and concave function. Finally, for each $j \in \mathcal{J}$, there is some agent $h \in \mathcal{H}$ such that $m_j^h(x_0^h) > 0$ for all $x_0^h \in \mathbb{R}_{++}^L$. Then, all assumptions of the theorem above hold. Therefore, there exists an equilibrium in real financial markets with durable goods, as long as participation constraints assure that short-sales are bounded.

3.3. Collateralized asset markets

Let $(A_j^s; s \in \mathcal{S}) \in \mathbb{R}_+^{LS} \setminus \{0\}$ be the plan of promises of the real asset $j \in \mathcal{J}$. As in Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007), we assume that each asset is subject to default and backed by physical resources. More precisely, let $C_j \in \mathbb{R}_+^L$ be the bundle of commodities that a borrower of one unit of asset j has to constitute at the first period as a collateral guarantee. In the absence of any payment enforcement over collateral repossession, asset payments satisfy $V_j^s(p_s) = \min\{p_s A_j^s, p_s Y_s C_j\}$. Assume that, for any $j \in \mathcal{J}$, there is $s \in \mathcal{S}$ such that $\min\{||A_j^s||_L, ||Y_s C_j||_L\} > 0$.[7] Then, Assumption (A4) holds.

Since borrowers are required to constitute collateral guarantees, for any $(h, x_0^h) \in \mathcal{H} \times \mathbb{R}_+^L$, we assume that $\Phi^h(x_0^h) = \left\{ \varphi^h \in \Omega^h : \sum_{j \in \mathcal{J}} C_j \varphi_j^h \le x_0^h \right\}$, where Ω^h is a closed and convex subset of $\mathbb{R}_+^J$ containing the vector zero. Also, suppose that there is $\delta > 0$ such that $\delta(1, \dots, 1) \in \sum_{h \in \mathcal{H}} \Omega^h$. It then follows that Assumptions (A5)–(A7) hold too.

Therefore, if we suppose that preferences and endowments satisfy Assumptions (A1)–(A3), then an equilibrium exists in Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007) models of collateralized loans, even when agents have restricted access to some loans.

Note that restricted financial participation is determined by the sets $(\Omega^h; h \in \mathcal{H})$. As we said above, we are particularly interested in the case where borrowers may not have access to credit in some assets, i.e.: Ω^h are positive cones generated by some but not all the canonical vectors of $\mathbb{R}^J$. This kind of restricted participation is not allowed in models with survival financial assumptions, as this type of hypotheses requires that agents have access to all credit markets, independently of prices.

4. Concluding remarks

With neither financial survival assumptions nor spanning conditions on financial admissible sets, we prove the existence of equilibrium in a two-period abstract economy with restricted financial participation. Essentially, in order to overcome technical problems that may appear when agents do not have any access to some credit markets, we assume that preferences satisfy some impatience condition. As applications of our result, we prove existence of equilibrium with incomplete markets, durable goods and restricted financial participation in nominal and real asset markets.

Our financial participation restrictions are endogenous, as they depend on individuals consumptions. This allows us to extend the model of Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007) to include exclusive collateralized loans, that is, debt contracts designed ex ante for a subset of potential borrowers. This type of exclusion is not compatible with the traditional financial survival assumptions.

As matter of future research, our result could be extended to multi-period economies where restricted financial participation depend on the history of individual decisions. Moreover, financial participation could depend on prices, as in Cass et al. (2001) and Carosi et al. (2009). Such extensions will allow to analyze financial markets where short-sales depend on the constitution of margin requirements, which in turn depend on past decisions and future prices. These margin requirements may act as financial collateral in case of default and also as a mechanism to endogenously bound short-sales.

Appendix A.

We prove equilibrium existence using a generalized game approach. To this end, we will truncate the set of admissible consumption bundles and financial positions. More precisely, given $n \in \mathbb{N}$, let

$$K(n) = \{\left((\theta_j : j \in \mathcal{J}), (\varphi_j; j \in \mathcal{J})\right) \in \mathbb{R}_+^J \times \mathbb{R}_+^J : \\ \forall j \in \mathcal{J},\ \ \varphi_j \le 2\kappa(n) \ \wedge\ \theta_j \le 2\kappa(n)H\},$$

be a truncated set of financial positions, which depends on the parameter

$$\kappa(n) := \max_{h \in \mathcal{H}} \max_{x_0 \in [0,n]^L} \max_{\varphi \in \Phi^h(x_0)} \sum_{j \in \mathcal{J}} \varphi_j = \max_{h \in \mathcal{H}} \max_{\varphi \in \Phi^h(n,\dots,n)} \sum_{j \in \mathcal{J}} \varphi_j,$$

where the last equality follows from Assumption (A5)(ii). Note that, Assumption (A7) assures that $\kappa(n)$ is well defined and Assumption (A5)(ii) implies that $\kappa(n)$ is non-decreasing in n. It follows from Assumption (A6) that $\kappa(n) > 0$ for any $n > 0$.

In the generalized game below we restrict players to choose plans (x, θ, φ) in the box $\mathbb{Y}(n) := [0, n]^L \times [0, 2W]^{SL} \times K(n)$, where $W = \max_{(s,l) \in \mathcal{S}^* \times \mathcal{L}} \sum_{h \in \mathcal{H}} W_{s,l}^h$ is an upper bound for accumulated physical resources in our economy. Moreover, at any state of nature, we restrict our attention to commodity prices in the $(L-1)$-dimensional simplex $\Delta := \left\{ p \in \mathbb{R}_+^L : \sum_{l \in \mathcal{L}} p_l = 1 \right\}$.

Given a pair $(n, m) \in \mathbb{N} \times \mathbb{N}$, consider a generalized game $\mathcal{G}(n, m)$ with $H+S+1$ players. In this game, for each agent $h \in \mathcal{H}$ there is a player a_h who takes as given prices $(p, q) \in \Delta^{S+1} \times [0, m]^J$ and chooses a plan $(x^h, \theta^h, \varphi^h) \in B_n^h(p, q) := B^h(p, q) \cap \mathbb{Y}(n)$ in order to maximize his objective function $v^h : \mathbb{Y}(n) \to \mathbb{R}$, where $v^h(x^h, \theta^h, \varphi^h) = u^h(x^h)$.

Moreover, there is a player a_0 who takes as given plans $((x^h, \theta^h, \varphi^h); h \in \mathcal{H}) \in \mathbb{Y}(n)^H$ and chooses prices $(p_0, q) \in \Delta \times [0, m]^J$ in order to maximize the function

$$p_0 \sum_{h \in \mathcal{H}} (x_0^h - w_0^h) + \sum_{j \in \mathcal{J}} q_j \sum_{h \in \mathcal{H}} (\theta_j^h - \varphi_j^h).$$

[7] The symbol $||\cdot||_L$ denotes the Euclidean norm of $\mathbb{R}_+^L$.

Finally, for any state of nature $s \in \mathcal{S}$, there is a player a_s who takes as given plans $((x^h,\theta^h,\varphi^h); h \in \mathcal{H}) \in \mathbb{Y}(n)^H$ and chooses prices $p_s \in \Delta$ in order to maximize the function $p_s \sum_{h\in\mathcal{H}} (x_s^h - (w_s^h + Y_s x_0^h))$.

Definition. A Nash equilibrium for the generalized game $\mathcal{G}(n,m)$ is given by a vector of strategies, $((\bar{p},\bar{q}); ((\bar{x}^h,\bar{\theta}^h,\bar{\varphi}^h); h \in \mathcal{H})) \in \Delta^{S+1} \times [0,m]^J \times \mathbb{Y}(n)^H$, which associates to each player an optimal response to the strategies of the other players.

Lemma 1. *Under Assumptions* (A1)–(A7), *for each* $(n,m) \in \mathbb{N}\times\mathbb{N}$, *the game* $\mathcal{G}(n,m)$ *has a Nash equilibrium.*

Proof. For each $s \in \mathcal{S}^*$, the objective function of player a_s is continuous in all variables and quasi-concave in the own strategy. In addition, for these players, the correspondences of admissible strategies are constant with non-empty, convex and compact values.[8] Thus, these correspondences are also continuous.

On the other hand, it follows from Assumption (A1) that the objective function of each player a_h, with $h \in \mathcal{H}$, is continuous and quasi-concave in the own strategy. The correspondence B_n^h of admissible strategies for player a_h is upper hemicontinuous, since it is closed and has non-empty values that are contained in the compact set $\mathbb{Y}(n)$. The lower hemicontinuity of B_n^h follows from Assumptions (A3) and (A5)(ii), since B_n^h is the closure of the interior truncated budget set correspondence, denoted by $\overset{\circ}{B}{}_n^h$, which is lower-hemicontinuous.[9] Also, by (A5)(i), B_n^h has convex values.

The existence of a Nash equilibrium follows from the fact that: (i) players' objective functions are continuous and quasi-concave in their own strategy, and (ii) correspondences of admissible strategies are continuous with compact, convex and non-empty values. More precisely, under (i) and (ii), it follows from Berge's Maximum Theorem that players' best response correspondences are upper-hemocontinuous with non-empty, compact and convex values. Thus, applying Kakutani's fixed point theorem to the product of best response correspondences, we get a Nash equilibrium as a fixed point. □

Lemma 2. *Let* $((\bar{p},\bar{q}); ((\bar{x}^h,\bar{\theta}^h,\bar{\varphi}^h); h \in \mathcal{H}))$ *be a Nash equilibrium of the game* $\mathcal{G}(n,m)$. *Under Assumptions* (A1)–(A7), *if the bundle* $\bar{x}_0^h \le W(1,\ldots,1)$ *for all* $h \in \mathcal{H}$, *then for n large enough there exists* $\bar{m} \in \mathbb{N}$ *such that*, $\max_{j\in\mathcal{J}} \bar{q}_j < \bar{m}$.

Proof. It follows from Assumptions (A1), (A2) and (A3) that there exists $\tau \in \mathbb{R}_+^L$ such that, for each $h \in \mathcal{H}$, $u^h(w_0^h+\tau,(0.7W_s^h; s\in\mathcal{S})) > u^h(W(1,\ldots,1),(2W(1,\ldots,1); s\in\mathcal{S}))$. Indeed, following the notation of Assumption (A2), it is sufficient to take $\tau = W(1,\ldots,1) + \sum_{k\in\mathcal{H}} \tau^k(\theta,z)$, where $z=(W(1,\ldots,1),(2W(1,\ldots,1); s\in\mathcal{S}))$ and $\theta\in(0,1)$ is chosen so that $0.7W_{s,l}^k > 2W\theta$, for any $(k,s,l)\in\mathcal{H}\times\mathcal{S}\times\mathcal{L}$.

Moreover, it follows from the strict monotonicity of preferences, the definition of $\mathbb{Y}(n)$ and the fact that $\bar{x}_0^h \le W(1,\ldots,1)$ for all $h\in\mathcal{H}$, that

$$u^h(w_0^h+\tau,(0.7W_s^h; s\in\mathcal{S})) > u^h(\bar{x}_0^h,(\bar{x}_s^h; s\in\mathcal{S})).$$

Therefore, given $h\in\mathcal{H}$, $(w_0^h+\tau,(0.7W_s^h; s\in\mathcal{S})) \notin B_n^h(\bar{p},\bar{q})$. In particular, player a_h cannot buy the bundle $\tau\in\mathbb{R}_+^L$ with the resources obtained from a financial promise that can be honored, at any state of nature $s\in\mathcal{S}$, by selling the bundle $0.3W_s^h$.

On the other hand, Assumption (A6) assures that, given an asset $j\in\mathcal{J}$, there exists $h(j)\in\mathcal{H}$ such that, for some $\delta_j>0$, we have $\delta_j e(j) \in \Phi^{h(j)}(w_0^h+\tau)$, where $e(j)$ denotes the canonical vector of $\mathbb{R}^J$ on the jth component. Then, suppose that player $a_{h(j)}$ chooses the portfolio $(\hat{\theta}^{h(j)},\hat{\varphi}^{h(j)}) = (0,\eta_j e(j))$, where $\eta_j \in (0, \min\{\kappa(1),\delta_j\})$ satisfies $(\max_{(p,s)\in\Delta\times\mathcal{S}} V_j^s(p))\eta_j < 0.3 \min_{(s,l)\in\mathcal{S}\times\mathcal{L}} W_{s,l}^{h(j)}$.[10] Note that, η_j depends only on primitive parameters of the economy.

If n is large enough, the consumption plan $((w_0^h+\tau; (0.7W_s^h; s\in\mathcal{S}))$, jointly with the financial positions $(\hat{\theta}^{h(j)},\hat{\varphi}^{h(j)})$, belongs to $\mathbb{Y}(n)$. However, as we pointed out above, player $a_{h(j)}$ cannot finance the consumption of the bundle τ with the resources obtained by short-selling η_j units of asset j. Thus, $\bar{q}_j\eta_j < \bar{p}_0\tau \le ||\tau||_\Sigma$, which assures the existence of an upper bound for $\bar{q}_j$ that only depends on primitives of the economy. We conclude the proof by choosing $\bar{m} = ||\tau||_\Sigma \max_{j\in\mathcal{J}}(1/\eta_j)$. □

Since the bundle $\tau\in\mathbb{R}_+^L$ depends only on primitive parameters of the economy, we can define $n^* = W + ||\tau||_\Sigma$.

Lemma 3. *Under Assumptions* (A1)–(A7), *and for* $n > n^*$, *a Nash equilibrium of* $\mathcal{G}(n,\bar{m})$ *is an equilibrium for our economy.*

Proof. Let $((\bar{p},\bar{q}); ((\bar{x}^h,\bar{\theta}^h,\bar{\varphi}^h); h\in\mathcal{H})) \in \Delta^{S+1}\times[0,\bar{m}]^J\times\mathbb{Y}(n)^H$ be a Nash equilibrium of the generalized game $\mathcal{G}(n,\bar{m})$. Adding first period budget constraints of players a_h, with $h\in\mathcal{H}$, we get

$$\bar{p}_0\sum_{h\in\mathcal{H}}(\bar{x}_0^h - w_0^h) + \sum_{j\in\mathcal{J}}\bar{q}_j\sum_{h\in\mathcal{H}}(\bar{\theta}_j^h - \bar{\varphi}_j^h) \le 0.$$

It follows that the optimal value of the objective function of player a_0 is nonpositive and, therefore, for each $l\in\mathcal{L}$, $\sum_{h\in\mathcal{H}}(\bar{x}_{0,l}^h - w_{0,l}^h) \le 0$. Indeed, otherwise, player a_0 would choose a price equal to one for commodity $l\in\mathcal{L}$ and a zero price for the other commodities and assets, obtaining a positive value for his objective function, a contradiction with the definition of Nash equilibrium. Therefore, for each $h\in\mathcal{H}$, $\bar{x}_0^h \le W(1,\ldots,1)$. On the other hand, if for some $j\in\mathcal{J}$, $\sum_{h\in\mathcal{H}}(\bar{\theta}_j^h - \bar{\varphi}_j^h) > 0$, then $\bar{q}_j = \bar{m}$, which contradicts Lemma 2 for $n > n^*$. Thus, $\sum_{h\in\mathcal{H}}(\bar{\theta}^h - \bar{\varphi}^h) \le 0$.

Since $\bar{x}_{0,l}^h < n$ for all $(h,l)\in\mathcal{H}\times\mathcal{L}$, it follows from Assumption (A5)(ii) that first period budget constraints are saturated. Therefore, $\sum_{h\in\mathcal{H}}(\bar{x}_0^h - w_0^h) = 0$. In fact, otherwise, some commodity at $t=0$ has a

[8] That is, the correspondences that associate to each plan in $\mathbb{Y}(n)^H$ the set of admissible prices.

[9] The correspondence $\overset{\circ}{B}{}_n^h : \Delta^{S+1}\times[0,m]^J \twoheadrightarrow \mathbb{Y}(n)$ associates to each (p,q) the allocations in $B_n^h(p,q)$ that satisfy state-contingent budget constraints as strict inequalities. This correspondence has non-empty values, since the consumption bundle $(0.5w_0^h,(0.25W_s^h; s\in\mathcal{S}))$ jointly with the zero financial portfolio always belongs to $\overset{\circ}{B}{}_n^h(p,q)$, independently of the vector of prices $(p,q)\in\Delta^{S+1}\times[0,m]^J$ (a consequence of Assumption (A5)(ii)). Also, given $(p,q)\in\Delta^{S+1}\times[0,m]^J$ and a sequence $((p_k,q_k); k\in\mathbb{N}) \subset \Delta^{S+1}\times[0,m]^J$ that converges to (p,q), for any $(x^h,\theta^h,\varphi^h)\in\overset{\circ}{B}{}_n^h(p,q)$ there exists $N\in\mathbb{N}$ such that $(x^h,\theta^h,\varphi^h)\in\overset{\circ}{B}{}_n^h(p_k,q_k)$ for any $k\ge N$. Then, it follows from the sequential characterization of lower-hemicontinuity that $\overset{\circ}{B}{}_n^h$ is a lower-hemicontinuous correspondence.
Given any $(x^h,\theta^h,\varphi^h)\in B_n^h(p,q)$ and $\lambda\in(0,1)$, one has $((\lambda x_0^h+(1-\lambda)(\omega_0^h/2),(\lambda x_s^h; s\in\mathcal{S})),\lambda\theta^h,\lambda\varphi^h)\in\overset{\circ}{B}{}_n^h(p,q)$ (since Assumption (A5) assures that Φ^h has a convex graph and $0\in\Phi^h(0)$). Thus, taking the limit as λ goes to zero, we show that (x^h,θ^h,φ^h) belongs to the closure of $\overset{\circ}{B}{}_n^h(p,q)$. As $\overset{\circ}{B}{}_n^h(p,q)\subseteq B_n^h(p,q)$, it follows that B^h is equal to the closure of the interior truncated budget set correspondence.

[10] Since $(\kappa(n); n\in\mathbb{N})$ is a non-decreasing and strictly positive sequence of n, to make η_j a feasible debt for player $a_{h(j)}$ in the game $\mathcal{G}(n,m)$, i.e. $\eta_j<\kappa(n)$, it is sufficient to assure that $\eta_j<\kappa(1)$.

zero price, a contradiction with the existence of an interior optimal plan under Assumptions (A1) and (A5)(ii). Analogously, if $\sum_{h\in\mathcal{H}}(\bar{\theta}_j^h - \bar{\varphi}_j^h) < 0$, then $\bar{q}_j = 0$. Then, to guarantee financial market feasibility it is sufficient to prove that asset prices are strictly positive.

Summing up the budget constraints of all players a_h at state of nature $s\in\mathcal{S}$, it follows from $\sum_{h\in\mathcal{H}}(\bar{\theta}^h - \bar{\varphi}^h) \le 0$ that,

$$\bar{p}_s\sum_{h\in\mathcal{H}}(\bar{x}_s^h - (w_s^h + Y_s\bar{x}_0^h)) \le 0.$$

That is, the optimal value of player a_s's objective function is nonpositive. This implies that $\sum_{h\in\mathcal{H}}(\bar{x}_s^h - (w_s^h + Y_s\bar{x}_0^h)) \le 0$ and, therefore, $\bar{x}_s^h < 2W(1,\ldots,1)$. By monotonicity of preferences, it follows that $\bar{p}_s \gg 0$. Since it holds for any state of nature $s\in\mathcal{S}$, Assumption (A4) implies that asset payments are non-trivial. Thus, $\bar{q} \gg 0$. This property assures financial market feasibility, $\sum_{h\in\mathcal{H}}(\bar{\theta}^h - \bar{\varphi}^h) = 0$.

It follows from Assumption (A1) that second period budget constraints are satisfied as equalities. Then, $\bar{p}_s\sum_{h\in\mathcal{H}}(\bar{x}_s^h - (w_s^h + Y_s\bar{x}_0^h)) = 0$. Since $\bar{p}_s \gg 0$, we conclude that $\sum_{h\in\mathcal{H}}(\bar{x}_s^h - (w_s^h + Y_s\bar{x}_0^h)) = 0$.

That is, market clearing conditions are satisfied.

On the other hand, for each agent $h\in\mathcal{H}$, the plan $(\bar{x}^h, \bar{\theta}^h, \bar{\varphi}^h) \in B_n^h(\bar{p}, \bar{q}) \subset B^h(\bar{p}, \bar{q})$ belongs to $\text{int}(\mathbb{Y}(n))$ (relative to $\mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^J \times \mathbb{R}_+^J$). Therefore, the strong quasi-concavity of u^h, jointly with the convexity of budget sets, implies that $(\bar{x}^h, \bar{\theta}^h, \bar{\varphi}^h)$ is also optimal in $B^h(\bar{p}, \bar{q})$. $\square$

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