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RESEARCH ARTICLE

Equilibrium with limited-recourse collateralized loans

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Abstract We address a general equilibrium model with limited-recourse collateralized loans and securitization of debts. Each borrower is required to pledge physical collateral, and bankruptcy is filed against him if claims are not fully honored. Moreover, agents have a positive amount of wealth exempt from garnishment and, for at least a fraction of them, commodities used as collateral are desirable. In this context, equilibrium exists for any continuous garnishment rule and multiple types of reimbursement mechanisms.

Keywords Collateralized assets · Bankruptcy · Limited-recourse loans

JEL Classification D52 · D54

1 Introduction

Financial default on secured debts was introduced into the general equilibrium model with incomplete markets by Dubey et al. (1995) and Geanakoplos and Zame (1997,

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2002, 2007). In that model, the financial sector is linked to physical markets through collateral constraints and, therefore, the scarcity of commodities induce endogenous bounds on short-sales. This avoids discontinuities that may appear on individuals' demands, when the dimension of the space of transfers becomes dependent on prices. Thus, an equilibrium always exists.

On the other hand, with the aim of addressing bankruptcy in incomplete markets with unsecured claims, Araujo and Páscoa (2002) propose two models with nominal assets. In the first model, borrowers are burdened by exogenous short-sales constraints, exemptions are proportional to the amount of wealth, and garnished resources are distributed in proportion to the size of claims. In the second model, short-sales constraints are avoided, garnished resources are distributed giving priority to smaller claims, and exemptions asymptotically vanish as debt increases. In a related result, Sabarwal (2003) addresses a finite horizon model with numeraire assets and default dependent credit constraints. In case of bankruptcy, borrowers may have non-proportional exemptions and garnished resources are distributed in proportion to the size of claims. In these models, the garnishment of wealth induce non-convexities on choice sets. Thus, equilibrium existence was proved in economies with a continuum of agents.

In this article, we include bankruptcy and the garnishment of wealth in a general equilibrium framework with collateralized credit contracts and securitization of debts. We replace credit limits of models with unsecured claims by collateral constraints. Since resources obtained by the seizure of collateral guarantees are delivered to investors, there is no indetermination about the right over physical guarantees, avoiding any risk regarding the repossession of collateral. The garnishment of resources in case of bankruptcy follows exogenous rules that are only required to be continuous. Thus, we allow for exemptions that are proportional to the amount of wealth, that decrease as the amount of debt increases, or that protect poor defaulters (reducing the garnishment to a lower percentage of their wealth). Reimbursement mechanisms are very general, allowing the distribution of garnished resources to be proportional to the size of claims or following a seniority criteria among securities. In this context, we prove equilibrium existence under two key assumptions: the existence of positive exemptions in case of bankruptcy and the desirability of collateral, which ensures that any utility level can be attained through an increment in the consumption of commodities used as collateral.

Our economy is stochastic and has two-time periods. Commodities may be durable, perishable, or may transform into other goods through time. There is a continuum of agents which demand commodities, trade debt contracts, and invest in securities.

Debt contracts are limited-recourse loans backed by physical collateral guarantees. Different to Geanakoplos and Zame (1997, 2002, 2007), we allow for the garnishment of the individual's wealth when promises are not fully paid. Each debtor knows that a financial regulator—whose only objective is to ensure the well operation of the bankruptcy law and the securitization processes—will file bankruptcy against him when promises are not fully honored. Therefore, even when the value of collateral guarantees is lower than the original claim, whole debts can be paid. For instance, an agent pays his debts if the garnishable wealth is enough to cover his claims.

Loans associated to each type of debt contract are pooled and securitized into only one asset. When promises associated to a debt contract are honored, debtors' payments are distributed to holders of the associated security. In case of default on a debt contract, foreclosure occurs and resources obtained by the seizure of collateral guarantees are distributed to investors. Also, bankruptcy is filed and investors are reimbursed with the wealth obtained by garnishment. However, these resources may be insufficient to cover unpaid debts and, therefore, we assume that they are delivered following a pre-fixed mechanism. For instance, we can allow agents to be reimbursed proportionally to the size of claims. Alternatively, we can give priority to some securities to receive garnished resources over others. Thus, investors on a senior security have priority to be reimbursed, independent of the size of their claims.

Since security payments are endogenous, we concentrate our attention to non-trivial equilibria, that is, equilibria where security payments are positive in at least one state of nature. Indeed, as in Steinert and Torres-Martínez (2007), we can trivially prove the existence of equilibrium when security payments are zero, as the economy can be reduced to a pure spot market economy (assuming that debt contracts also have zero prices).

As is usual in the literature regarding large economies, the existence of equilibrium is carried out using the existence of pure strategy Cournot-Nash equilibria in non-convex generalized games. We construct games where each agent maximizes his objective function by choosing bounded allocations, and abstract players choose prices and security payments in a form such that market feasibility conditions hold. By increasing the upper bounds on individuals' allocations, we prove that any sequence of Cournot-Nash equilibria converges to a non-trivial equilibrium of our economy. However, to guarantee this last property, it is necessary to bound the individual's allocations associated with cluster points of the sequence of Cournot-Nash equilibria. In the economy proposed by Sabarwal (2003), and in the first model in Araujo and Páscoa (2002), this was done using exogenous short-sales constraints and the fact that, in any cluster point, commodity and asset prices are strictly positive.[1]

In our context, we can prove that commodity and asset prices are strictly positive in any cluster point of a sequence of Cournot-Nash equilibria. However, we do not have short-sales constraints either exogenously imposed or endogenously induced by market feasibility conditions. Therefore, we need to obtain upper bounds on debt positions from budget sets constraints. For this reason, the most important step of our proof of equilibrium existence is to ensure that, for any type of credit contract, the price of collateral guarantees is greater than the amount of the loans. This is a consequence of two assumptions: the desirability of commodities used as collateral and the existence of minimal protection from excessive losses of wealth by confiscation.

As a byproduct of our analysis, we extend the result of equilibrium existence of Geanakoplos and Zame (1997, 2002, 2007) to non-convex economies. Moreover, we obtain a result of equilibrium existence in economies with unsecured debts and perishable commodities, where short-sales are linked to the amount of consumption to induce endogenous debt constraints as a consequence of the scarcity of physical resources.

The remaining sections of the paper are organized as follows: in Sect. 2 we describe some previous results related with our framework. In Sect. 3, we introduce our model. Our results about equilibrium existence are stated in Sect. 4. In Sect. 5 we give exam-

[1] With these properties, budget constraints induce natural upper bounds on consumption and investment positions.

ples of garnishment rules compatible with the framework. Extensions of our results are discussed in Sect. 6. Finally, the proofs of our results are given in the appendixes.

2 Related literature

Our work constitutes a blend of two different frameworks previously addressed in the literature of general equilibrium: economies with collateralized asset markets, as in Dubey et al. (1995) or Geanakoplos and Zame (1997, 2002, 2007), and models that include bankruptcy in markets with unsecured claims, as in Araujo and Páscoa (2002) or Sabarwal (2003).

The model of mortgage loans of Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007) was the first to address collateralized debts into a general equilibrium framework, allowing for heterogenous agents, aggregated uncertainty, and default. This two-period seminal model gives rise to a growing theoretical literature.

In finite horizon models, Araujo et al. (2000) and Araujo et al. (2005) make extensions to allow for endogenous collateral. Steinert and Torres-Martínez (2007) include CLO markets, where some claims have priority over others to receive resources obtained by the repossession of collateral guarantees. Allowing for asymmetric information, Petrassi and Torres-Martínez (2008) analyze the role of collateral to reduce arbitrage opportunities. In a recent paper, Kilenthong (2011) studies the effectiveness of collateral as a risk sharing mechanism.

In the infinite horizon context, Araujo et al. (2002, 2011) prove equilibrium existence in collateralized asset markets without the need to impose transversality conditions, debt constraints, or uniform impatient assumptions.[2] In the context of Markovian economies, the existence of stationary equilibrium in markets with secured debts was proved by Kubler and Schmedders (2003). Also, Seghir and Torres-Martínez (2008) prove that collateral allows the increase of credit opportunities in economies with incomplete demographic participation.

In those models, the only payment enforcement mechanism is the seizure of collateral guarantees. Therefore, each borrower makes strategic default and, hence, delivers the minimum between the original promise and the associated collateral's value. However, additional payment enforcement mechanisms may appear, for instance, in the form of institutional reactions to a strong fall in the value of collateral guarantees. In this context, Páscoa and Seghir (2009) prove that, when defaulters are punished by harsh linear utility penalties, Ponzi schemes opportunities may appear, and equilibrium with trade may cease to exist in infinite horizon economies. Even more, Ferreira and Torres-Martínez (2010) show that if the value of the collateral suffers negative shocks, then the percentage of unpaid resources recovered by additional payment enforcement mechanisms decreases as well. There is also a positive theory of equilibrium existence in infinite horizon collateralized asset markets when utility penalties for default are

[2] In infinite horizon incomplete markets models without credit risk, transversality conditions (or portfolio constraints) jointly with uniform impatient requirements are imposed to avoid Ponzi schemes. See for instance, Kehoe and Levine (1993), Magill and Quinzii (1994, 1996), Hernandez and Santos (1996), Levine and Zame (1996), Araujo et al. (1996), and Florenzano and Gourdel (1996).

allowed, as the results of Páscoa and Seghir (2009) and Martins-da-Rocha and Vailakis (2011a,b).[3]

On the other hand, Araujo and Páscoa (2002) and Sabarwal (2003) address equilibrium existence in two-period incomplete financial markets with perishable commodities. They assume that garnished resources are distributed in proportion to the size of claims. Also, exogenous short-sales constraints or default dependent credit constraints are imposed. In Araujo and Páscoa (2002), a proportion of agents' wealth is protected from expropriation in case of bankruptcy. Therefore, rich agents have exemptions substantially larger than poor consumers. Alternatively, Sabarwal (2003) allows poor agents to have a greater proportion of their wealth protected from garnishment. In both models, individual endowments are uniformly bounded away from zero and, therefore, exemptions are bounded away from zero too.

To avoid short-sales constraints, Araujo and Páscoa (2002) propose an alternative model where garnishable resources increase as unpaid debt grows. Thus, individuals' exemptions are asymptotically zero as debts increase. Also, they assume that claims are reimbursed through a specific mechanism which gives priority to smaller claims to receive the whole payment (independently of the asset). However, strong assumptions are imposed over endowments and preferences: the utility function of each agent is separable, continuously differentiable, and satisfies the Inada conditions; the family of utility functions in the economy is equicontinuous; the family of partial derivatives of utility functions is equicontinuous; and initial endowments are uniformly bounded away from zero.

The existence of collateral guarantees allow us to overcome exogenous short-sales constraints without the need to impose these strong assumptions on agents' characteristics. Also, although they are required to be continuous, our garnishment rules are quite general. Finally, we allow reimbursement mechanisms that not only include proportional distribution, but also seniority structures among securities.

3 The model

We consider an economy with two periods $t \in \{0, 1\}$. There is no uncertainty at $t = 0$ and one state of nature in a finite set S is reached at $t = 1$. Let $S^* = \{0\} \cup S$ be the set of states of nature in the economy, where $s = 0$ denotes the only state of nature at $t = 0$.

At each state $s \in S^*$, there is a finite set L of perfect divisible commodities, which may be durable between periods. That is, for any $s \in S$, there is a linear function $Y_s : \mathbb{R}^L_+ \to \mathbb{R}^L_+$ that represents an exogenous technology transforming bundles that are consumed at the first period into quantities of contingent commodities at state of nature s. Let $p = (p_s; s \in S^*)$ be the vector of commodity prices in the economy, where $p_s = (p_{s,\ell}; \ell \in L) \in \mathbb{R}^L_+$ are the commodity prices at $s \in S^*$.

There is a measure space of consumers, $\mathcal{H} = ([0, 1], \mathbb{B}, \mu)$, where $\mathbb{B}$ is the Borel σ-algebra of $[0, 1]$ and μ the Lebesgue measure. Each $h \in [0, 1]$ maximizes

[3] These results are also extensions of works on default and punishment with unsecured debts (see Dubey et al. 1989, 2005; Zame 1993; Araujo et al. 1998).

his utility function $u^h : \mathbb{R}_+^{L\times S^*} \to \mathbb{R}$ using physical and financial markets to smooth consumption. Let $w^h := (w_s^h; s \in S^*)$ be the physical endowment of agent h, where $w_s^h = (w_{s,\ell}^h; \ell \in L) \in \mathbb{R}_{++}^L$ is the bundle that he receives at $s \in S^*$.

There is a finite set J of collateralized debt contracts which are available for trade at the first period. When a borrower issues one unit of $j \in J$, he receives a quantity of resources π_j and pledges a physical collateral $C_j \in \mathbb{R}_+^L \setminus \{0\}$. The real promises associated to one unit of debt contract $j \in J$ are given by $(A_{s,j}; s \in S) \in \mathbb{R}_+^{L\times S}$. If an issuer of a debt contract does not honor his promises at some state of nature $s \in S$, the market will seize the associated collateral guarantee and may also implement additional payment enforcement mechanisms. We denote by $\pi = (\pi_j; j \in J)$ the unitary prices of debt contracts.

Each debt contract $j \in J$ is securitized into only one asset. That is, payments made by issuers of a contract $j \in J$ are pooled and delivered to holders of an associated security, which is also denoted by j and has unitary price π_j.[4] Let $(\theta^h, \varphi^h) = ((\theta_j^h, \varphi_j^h); j \in J) \in \mathbb{R}_+^J \times \mathbb{R}_+^J$ be agent h's financial positions in securities and debt contracts at the first period. We denote by $x^h = (x_s^h; s \in S^*) \in \mathbb{R}_+^{L\times S^*}$ the non-collateralized consumption plan of an agent h. Thus, at the first period, the total consumption of agent h is equal to $x_0^h + \sum_{j\in J} C_j\varphi_j^h \in \mathbb{R}_+^L$.

Since collateral guarantees are seized in case of default, an agent h that borrows φ_j^h units of debt contract $j \in J$ delivers, at any state $s \in S$, at least an amount $D_{s,j}(p_s)\varphi_j^h$ of resources, where $D_{s,j}(p_s) = \min\{p_s A_{s,j}, p_s Y_s(C_j)\}$. In addition, if agent h debts are not fully paid, the financial regulator files bankruptcy against him and, therefore, his wealth can be garnished.

However, we assume that the law protect agents from excessive losses by wealth confiscation. More precisely, let $z_0^h = (x_0^h, \theta^h, \varphi^h)$ be the consumption and financial decisions of an agent h at the first period. Then, given $p_s \neq 0$, the amount of resources that h has exempt from garnishment at $s \in S$ is given by $\Lambda_s\left(p_s, w_s^h, \mathcal{W}_s(p_s, R_s, w_s^h, z_0^h)\right)$, where $\Lambda_s : \left(\mathbb{R}_+^L \setminus \{0\}\right) \times \mathbb{R}_{++}^L \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ is a continuous function, $R_s = (R_{s,j}; j \in J)$ are the unitary security payments at the state of nature s, and

$$\mathcal{W}_s(p_s, R_s, w_s^h, z_0^h) = p_s w_s^h + p_s Y_s(x_0^h) + \sum_{j\in J}[p_s Y_s(C_j) - p_s A_{s,j}]^+\varphi_j^h + \sum_{j\in J} R_{s,j}\theta_j^h,$$

is the wealth of agent h after the payment or the foreclosure of his debts, where $[y]^+ := \max\{y, 0\}$. Therefore, the amount of wealth that agent $h \in [0, 1]$ loses when bankruptcy is filed at $s \in S$ is given by the garnishment rule $\Phi_s(p_s, R_s, w_s^h, z_0^h) = \left[\mathcal{W}_s(p_s, R_s, w_s^h, z_0^h) - \Lambda_s\left(p_s, w_s^h, \mathcal{W}_s(p_s, R_s, w_s^h, z_0^h)\right)\right]^+$.

[4] Making a normalization of portfolios and security payments, it is always possible to identify those prices.

As a consequence of monotonicity of preferences, at any $s \in S$ associated with original promises $\sum_{j\in J} p_s A_{s,j}\varphi_j^h$, agent h pays the following amount of resources,

$$M_s(p_s, R_s, w_s^h, z_0^h) = \min\left\{\sum_{j\in J} p_s A_{s,j}\varphi_j^h, \sum_{j\in J} D_{s,j}(p_s)\varphi_j^h + \Phi_s(p_s, R_s, w_s^h, z_0^h)\right\}.$$

It follows that bankruptcy is filed against agent $h \in [0, 1]$ at state of nature $s \in S$ if and only if his remaining debt after the payment or foreclosure of debts,

$$\Psi_s(p_s, \varphi^h) := \sum_{j\in J}[p_s A_{s,j} - p_s Y_s(C_j)]^+\varphi_j^h,$$

is greater than the amount of garnishable resources $\Phi_s(p_s, R_s, w_s^h, z_0^h)$.[5]

In our model, the amount of resources exempt from garnishment is always positive. That is, $\Lambda_s(p_s, w_s^h, \mathcal{W}_s(p_s, R_s, w_s^h, z_0^h)) > 0$, for any $s \in S$ and $p_s \neq 0$. This requirement is compatible with a variety of specifications for $(\Lambda_s; s \in S)$. For instance, we can allow for structures where exempt resources are described as a non-linear function of individuals' wealth (see Sect. 5). Alternatively, we can make the assumption that exemptions are constant, or dependent on the amount of endowments, or are proportional to the amount of wealth. Actually, given $p_s \neq 0$, if $\Lambda_s(p_s, w_s^h, \mathcal{W}_s) = p_s\eta_s$, the bundle $\eta_s \gg 0$ determines a threshold under which it is not allowed to garnish resources from any agent. When $\Lambda_s(p_s, w_s^h, \mathcal{W}_s) = \lambda_s\, p_s w_s^h$, the exemption from garnishment is equal to the market value of a percentage $\lambda_s \in (0, 1]$ of individual endowments $w_s^h \gg 0$. Finally, exemptions are proportional to the amount of wealth when $\Lambda_s(p_s, w_s^h, \mathcal{W}_s) = \lambda_s\mathcal{W}_s$, with $\lambda_s \in (0, 1]$ and $\mathcal{W}_s > 0$.

Since in our two-period model the only enforcement in case of bankruptcy is the garnished of the non-exempt wealth, agents do not care about the distribution of rates of default among different promises. Indeed, each borrower only decides between honor his whole debts or file for bankruptcy. For this reason, we assume that in the case of bankruptcy of an agent h at state of nature s, his garnished resources are distributed to the investors of the associated security through delivery rates $(\beta_{s,j}^h; j \in J) \in [0, 1]^J$ that satisfy,

$$\sum_{j\in J}\beta_{s,j}^h[p_s A_{s,j} - p_s Y_s(C_j)]^+\varphi_j^h = \Phi_s(p_s, R_s, w_s^h, z_0^h),$$

where $\beta_{s,j}^h$ is the proportion of the unpaid claim on debt contract j that is honored by the distribution of agent' h non-exempt wealth.[6]

[5] This comes from the fact that $M_s(p_s, R_s, w_s^h, z_0^h) = \sum_{j\in J} D_{s,j}(p_s)\varphi_j^h + \min\{\Psi_s(p_s, \varphi^h), \Phi_s(p_s, R_s, w_s^h, z_0^h)\}$.

[6] Note that, if we allow for financial participation constraints (in a model with more than two periods) or we impose any kind of non-economic utility penalty for default, then agents may have incentives to decide to honor some promises instead of others. As we highlight in our concluding remarks, this extensions can be matter of future research.

Delivery rates are restricted by the reimbursement mechanism of the economy that determines the rules in which confiscated resources are distributed. More precisely, we assume that, for any $(h, s) \in [0, 1] \times S$, delivery rates $(\beta^h_{s,j}; j \in J)$ belong to a connected set $\mathcal{R}_s \subset [0, 1]^J$, which contains vectors $\{0, e\}$, where $e = (1, \ldots, 1) \in \mathbb{R}^J$.[7] The collection of sets $(\mathcal{R}_s; s \in S)$, which determine restrictions over delivery rates, constitute the reimbursement mechanism of the economy. For instance, if

$$\mathcal{R}_s = \{(\beta_j, j \in J) \in [0, 1]^J : \beta_j = \beta_{j'}, \quad \forall (j, j') \in J \times J\},$$

then garnished resources at the state of nature s are distributed to investors proportional to the size of their claims. Alternatively, given an order on the set of securities, $\{j(1), \ldots, j(\#J)\}$, if

$$\begin{aligned}\mathcal{R}_s = \{&(\beta_j, j \in J) \in [0, 1]^J : \\ &\exists m, \ (\beta_{j(m')} = 1, \ \forall m' < m) \ \wedge \ (\beta_{j(m')} = 0, \ \forall m' > m)\},\end{aligned}$$

then claims of a security $j(r)$ are fully honored before any portion of garnished resources is delivered to securities $(j(r'); r' > r)$. Thus, we have a *seniority structure among securities*. Note that, since markets are anonymous and debts are pooled, we cannot determine a seniority structure among investors.

In equilibrium, the quantity of resources that are invested in a security will match the quantity of resources borrowed. In addition, the distribution of debtors' payments and garnished resources will determine unitary security payments $R_s = (R_{s,j}; j \in J)$ at any $s \in S$. Thus, an agent $h \in [0, 1]$ that buys θ^h_j units of security $j \in J$ will receive, at each $s \in S$, an amount of resources $R_{s,j}\theta^h_j$.[8]

Given $(p, \pi, R) \in \mathbb{V} := \mathbb{R}^{L \times S^*}_+ \times \mathbb{R}^J_+ \times \mathbb{R}^{S \times J}_+$, each $h \in [0, 1]$ maximizes his utility function by choosing consumption and financial positions within his budget set $B^h(p, \pi, R)$, defined as the collection of plans $(x^h, \theta^h, \varphi^h) \in \mathbb{E} := \mathbb{R}^{L \times S^*}_+ \times \mathbb{R}^J_+ \times \mathbb{R}^J_+$ such that,

$$p_0(x^h_0 - w^h_0) + \sum_{j \in J} \pi_j(\theta^h_j - \varphi^h_j) + p_0 \sum_{j \in J} C_j \varphi^h_j \leq 0;$$

[7] These properties ensure that, when an agent h files for bankruptcy at s, there are always delivery rates that allow the distribution of the total amount of garnished resources. Actually, as $f((\beta_j; j \in J)) = \sum_{j \in J} \beta_j [p_s A_{s,j} - p_s Y_s(C_j)]^+ \varphi^h_j$ is continuous, $f(\mathcal{R}_s) = [0, \Psi_s(p_s, \varphi^h)]$. Therefore, as in case of bankruptcy, $\Phi_s(p_s, R_s, w^h_0, z^h_0) \in [0, \Psi_s(p_s, \varphi^h)]$, there is always a vector $(\beta^h_{s,j}; j \in J)$ such that $f((\beta^h_{s,j}; j \in J)) = \Phi_s(p_s, R_s, w^h_0, z^h_0)$.

[8] As we remark above, markets are anonymous and, therefore, investors do not give resources directly to other agents. For this reason, an agent that invests in a security j will receive a maximal unitary payment, i.e., $R_{s,j} = p_s A_{s,j}$, if and only if the reimbursement mechanism of the economy assures that the non-exempt wealth of any issuer of debt j covers his commitments on j, independently of the amount of other financial commitments.

$$p_s(x_s^h - w_s^h) \leq p_s Y_s \left(x_0^h + \sum_{j \in J} C_j \varphi_j^h \right) + \sum_{j \in J} R_{s,j} \theta_j^h - M_s(p_s, R_s, w_s^h, z_0^h).$$

We denote our economy with limited-recourse collateralized loans by

$$\mathcal{E} = \mathcal{E}\left(S^*, L, (Y_s)_{s\in S}, J, (A_{s,j}, C_j)_{(s,j)\in S\times J}, (\Lambda_s, \mathcal{R}_s)_{s\in S}, \mathcal{H}, (u^h, w^h)_{h\in[0,1]}\right).$$

Definition 1 A vector $\left[(\overline{p}, \overline{\pi}, \overline{R}); \left((\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h); h \in [0,1]\right)\right] \in \mathbb{V}\times\mathbb{E}^{[0,1]}$ is an equilibrium of $\mathcal{E}$ if the following conditions hold,

1. For each $h \in [0,1]$,

$$\begin{aligned} &u^h\left(\overline{x}_0^h + \sum_{j\in J} C_j \overline{\varphi}_j^h, \left(\overline{x}_s^h; s \in S\right)\right) \\ &\quad = \max_{(x^h,\theta^h,\varphi^h)\in B^h(\overline{p},\overline{\pi},\overline{R})} u^h\left(x_0^h + \sum_{j\in J} C_j \varphi_j^h, \left(x_s^h; s \in S\right)\right). \end{aligned}$$

2. Physical and financial markets clear. That is,

$$\int_{[0,1]} \overline{x}_0^h \,\mathrm{d}h + \sum_{j\in J} C_j \int_{[0,1]} \overline{\varphi}_j^h \,\mathrm{d}h = \int_{[0,1]} w_0^h \,\mathrm{d}h;$$

$$\int_{[0,1]} \overline{x}_s^h \,\mathrm{d}h = \int_{[0,1]} w_s^h \,\mathrm{d}h + Y_s \left(\int_{[0,1]} \overline{x}_0^h \,\mathrm{d}h + \sum_{j\in J} C_j \int_{[0,1]} \overline{\varphi}_j^h \,\mathrm{d}h \right), \quad \forall s \in S;$$

$$\int_{[0,1]} \overline{\theta}_j^h \,\mathrm{d}h = \int_{[0,1]} \overline{\varphi}_j^h \,\mathrm{d}h, \quad \forall j \in J.$$

3. For any pair $(s, j) \times S \times J$, security payments satisfy $\overline{R}_{s,j} \geq D_{s,j}(\overline{p}_s)$ and

$$\overline{R}_{s,j} \int_{[0,1]} \overline{\theta}_j^h \,\mathrm{d}h = D_{s,j}(\overline{p}_s) \int_{[0,1]} \overline{\varphi}_j^h \,\mathrm{d}h + [\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)]^+ \int_{[0,1]} \overline{\beta}_{s,j}^h \overline{\varphi}_j^h \,\mathrm{d}h,$$

where, for any $(h, s) \in [0,1] \times S$, $(\overline{\beta}_{s,j}^h; j \in J) \in \mathcal{R}_s$ and

$$\sum_{j\in J} \overline{\beta}_{s,j}^h \left[\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)\right]^+ \overline{\varphi}_j^h = \min\left\{\Psi_s(\overline{p}_s, \overline{\varphi}^h), \Phi_s(\overline{p}_s, \overline{R}_s, w_s^h, \overline{z}_0^h)\right\}.$$

It follows that, for any security $j \in J$, equilibrium unitary payments $(\overline{R}_{s,j}; s \in S)$ are non-trivial provided that $D_{s',j}(\overline{p}_s) > 0$, for some $s' \in S$. When commodity prices are strictly positive, the latter condition trivially holds if there is a state of nature at which both $A_{s,j} \neq 0$ and $Y_s(C_j) \neq 0$. Therefore, we can argue that *non-trivial* collateral guarantees avoid over-pessimistic expectations about financial returns, a result previously highlighted by Steinert and Torres-Martínez (2007, Section 3).

4 Equilibrium existence

The following result ensures that a non-trivial equilibrium exists, provided that collateral guarantees do not fully depreciate at all states of nature.

Theorem 1 *Suppose that the following assumptions hold,*

(A1) For each $h \in [0, 1]$, $u^h : \mathbb{R}_+^{L\times S^*} \to \mathbb{R}$ *is continuous and strictly increasing.*
(A2) The function $\phi : [0, 1] \to \mathcal{U}(\mathbb{R}_+^{L\times S^*}) \times \mathbb{R}_+^{L\times S^*}$ *defined by* $\phi(h) = (u^h, w^h)$ *is measurable.*[9]
(A3) There is $\overline{w} \in \mathbb{R}_{++}^{L\times S^*}$ *such that, for each* $h \in [0, 1]$, $0 \ll w^h \leq \overline{w}$.
(A4) For each $s \in S$, $\Lambda_s : \left(\mathbb{R}_+^L \setminus \{0\}\right) \times \mathbb{R}_{++}^L \times \mathbb{R}_{++} \to \mathbb{R}_{++}$ *is continuous.*
(A5) For each $s \in S$, $\mathcal{R}_s \subset [0, 1]^J$ *is compact and connected, with* $\{0, e\} \subset \mathcal{R}_s$.
(A6) Given $j \in J$, *there is* $s \in S$ *such that* $\min\{\|A_{s,j}\|_\Sigma, \|Y_s(C_j)\|_\Sigma\} > 0$.[10]
(A7) For any $j \in J$, *there is a set* $H_j \subseteq [0, 1]$ *with positive measure, such that,*

$$\lim_{\sigma\to+\infty} u^h(y_0 + \sigma C_j, (y_s; s \in S)) > u^h(z),$$

for any $h \in H_j$, *and for each* $((y_s; s \in S^*), z) \in \mathbb{R}_{++}^{L\times S^*} \times \mathbb{R}_+^{L\times S^*}$.

Then, there exists an equilibrium for our economy. Also, we can ensure that prices are strictly positive and unitary security payments are non-trivial.

We impose Assumptions (A1)–(A5) to prove equilibrium existence using large non-convex generalized games. Particularly, (A3)–(A5) help us to prove the lower-hemicontinuity of budget set correspondences, a necessary requirement to ensure the existence of Cournot–Nash equilibrium in our games. However, these Cournot–Nash equilibria are not necessarily equilibria of our economy, because individual allocations are exogenously bounded in the games (a requirement that our theorem does not impose). Thus, as is usual in equilibrium theory for large economies, we increase these upper bounds on allocations in order to obtain an equilibrium for our economy as a cluster point of a sequence of Cournot–Nash equilibria.

To do this asymptotic argument, we prove that equilibrium allocations of generalized games are uniformly bounded.[11] Since individual endowments have an uniform upper bound [Assumption (A3)], we obtain the former property from budget feasibility, because in any cluster point of commodity prices, security prices and the value

9 The set $\mathcal{U}(\mathbb{R}_+^{L\times S^*})$ denotes the collection of functions $u : \mathbb{R}_+^{L\times S^*} \to \mathbb{R}$ endowed with the sup norm topology.

10 The symbol $\|\cdot\|_\Sigma$ denotes the norm of the sum.

11 With this property we can apply the multidimensional Fatou's lemma (see Hildenbrand 1974, p. 69) in order to obtain a cluster point of the sequence of Cournot–Nash equilibria.

of the joint operation of taking a loan and pledging the collateral bundle are strictly positive. In fact, commodity prices are positive by the strict monotonicity of utility functions [Assumption (A1)]. This implies that security payments are non-trivial, as a consequence of Assumption (A6). Consequently, security prices are strictly positive. To prove that the value of collateral is greater than the amount of resources borrowed, we assume that (i) for at least a fraction of agents departing from an interior plan of consumption, any utility level can be attained, provided that the consumption of commodities used as collateral increases [Assumption (A7)]; and (ii) agents have an exemption in case of bankruptcy [Assumption (A4)]. Indeed, if there is an optimal plan for an agent $h \in H_j$ at prices $(\overline{p}, \overline{\pi}, \overline{R})$, then $\overline{p}_0 C_j - \overline{\pi}_j > 0$. In another case, agent h may use credit on asset j to consume at the first period the bundle $w_0^h + \sigma C_j \gg 0$. Since exemptions are positive [Assumption (A4)], independently of the size of σ, there are interior bundles $(\gamma_s; s \in S) \gg 0$ which can be consumed tomorrow.[12] Thus, it follows from Assumption (A7) that there is no optimal solution for agent's h problem, a contradiction (see Lemma 6 in Appendix A).

In Geanakoplos and Zame (2002), where there is a finite number of agents, bounds on short-sales can be obtained from the markets' feasibility conditions. However, in our framework there is a continuum of agents and, therefore, to prove equilibrium, we obtain those bounds from budget constraints. For this reason, it is essential to ensure that borrowers receive less resources than those necessary to buy collateral guarantees, i.e., $(\overline{p}_0 C_j - \overline{\pi}_j; j \in J) \gg 0$.

Since functions $(\Lambda_s; s \in S)$ are only required to be continuous and strictly positive, as a particular case of Theorem 1, we have an extension of Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007) to allow for non-convex preferences. Actually, for any $s \in S$, assume that $\Lambda_s(p_s, w_s^h, \mathcal{W}_s) = \mathcal{W}_s$, for each $(p_s, \mathcal{W}_s) \in \left(\mathbb{R}_+^L \setminus \{0\}\right) \times \mathbb{R}_{++}$. Then, the total amount on individuals' wealth is exempt from garnishment and, hence, the only payment enforcement mechanism is the seizure of collateral guarantees. In this particular case, equilibrium exists even without Assumption (A7). Indeed, in the absence of garnishment of wealth, the strictly monotonicity of preferences it is sufficient to ensure that the collateral's cost is always greater than the amount of borrowing (otherwise, any agent can improve his utility level by increasing the amount of borrowed resources).

On the other hand, Assumption (A6) is essential to prove the non-triviality of security payments. Thus, we cannot have, as a particular case of Theorem 1, a result of equilibrium existence for a model with unsecured debts and perishable commodities, i.e., we cannot assume that $(Y_s; s \in S) \equiv 0$. However, if the reimbursement of garnished resources is proportional to the size of claims and exemptions are bounded from above by a proportion of individuals' wealth, we can ensure that a non-trivial equilibrium exists, even when commodities are perishable.

[12] In this context, agent h only takes financial positions on a debt contract j. Also, when the value of depreciated collateral requirements is greater than or equal to the amount of promises, he pays his debt and consumes his state-contingent endowment. Alternatively, he gives default and bankruptcy is filed against him. However, in this case, exemptions are strictly positive and only depend on prices, state-contingent endowments, and the amount of wealth, which is independent of σ. Thus, it is sufficient to choose, at any $s \in S$, a bundle $\gamma_s \gg 0$ cheaper than both the initial endowment and the referred exemption.

Theorem 2 *Under Assumptions (A1)–(A4) and (A7), suppose that,*

(B1) There is $\underline{w} \in \mathbb{R}^{L\times S^*}_{++}$ *such that, for each* $h \in [0, 1]$, $w^h \geq \underline{w}$.
(B2) For each $s \in S$, $\mathcal{R}_s = \{(\beta_j, j \in J) \in [0, 1]^J : \beta_j = \beta_{j'},\ \forall (j, j') \in J \times J\}$.
(B3) Debt contracts are non-trivial, i.e., for any $j \in J$, $(A_{s,j}; s \in S) \neq 0$.
(B4) For any $s \in S$, *there exists* $\kappa_s \in (0, 1)$ *such that,* $\Lambda_s(p_s, w^h_s, \mathcal{W}_s) \leq \kappa_s \mathcal{W}_s$.

Then, there is an equilibrium for our economy, with positive prices and non-trivial security payments.

This result allows us to contribute to the literature of bankruptcy in financial markets with unsecured debts. Indeed, if commodities are perfectly perishable between periods, then Theorem 2 guarantees that equilibrium exists in a two-period economy with unsecured debts, where garnished resources are distributed proportional to the size of claims. In case of bankruptcy, garnished rules are only required to be continuous and bounded from below by a positive proportion of individual's wealth. However, this result depends on a requirement that links the amount of debt with the amount of consumption (since, in this context, collateral constraints do not determine guarantees to investors, but still associate debts with consumption).[13] Although this link between consumption and debt appears as artificial, it can be viewed as a reduced form of a regulatory mechanism that controls the amount of speculative debt, as it associates borrowed resources to the real sector.

In a recent result, Ferreira and Torres-Martínez (2010) shows that, in infinite horizon convex economies, payment enforcement mechanisms may have low effectiveness in capturing resources over collateral values. In our two-period economy, which is non-convex, a similar situation may happen. That is, the capacity of the garnishment of wealth to obtain resources over collateral values may be compromised, particularly when collateral guarantees are low. We illustrate this possibility through the following example.

Example Suppose that Assumptions (A1)–(A7) hold and that there are two commodities in the economy. One is perishable ($\ell = 1$), while the other is durable ($\ell = 2$). Debt contracts have promises in units of the perishable commodity and have collateral requirements in units of the durable commodity: $((A_{s,j}; s \in S); C_j) = (((d_{s,j}, 0); s \in S); (0, \alpha_j))$, for any $j \in J$. Also, for any $h \in [0, 1]$,

$$u^h\left(x^h_0 + \sum_{j\in J} C_j \varphi^h_j, \left(x^h_s; s \in S\right)\right) = a^h_0 \cdot \left(x^h_0 + \sum_{j\in J} C_j \varphi^h_j\right) + \sum_{s\in S} a^h_s \cdot x^h_s,$$

where vectors $(a^h_s; s \in S^*) \gg 0$.

Assume that there is an equilibrium $\left[(\overline{p}, \overline{\pi}, \overline{R}); \left((\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h); h \in [0, 1]\right)\right]$ in which at least one agent $h_0 \in [0, 1]$ has $\overline{x}^{h_0} \gg 0$ and $\overline{\varphi}^{h_0} = 0$.

Since the set $\{(x^{h_0}, \theta^{h_0}) \in \mathbb{R}^{L\times S^*}_+ \times \mathbb{R}^J_+ : (x^{h_0}, \theta^{h_0}, 0) \in B^h(\overline{p}, \overline{\pi}, \overline{R})\}$ is convex and satisfies the Slater condition, it follows from the Kuhn–Tucker Theorem that there

[13] As in Theorem 1, Assumption (A7) allows us to determine endogenous bounds on short-sales.

are multipliers $(\gamma_s; s \in S^*) \gg 0$ such that, for any $(s,\ell) \in S \times L$ we have

$$\left(\gamma_0 \overline{p}_{0,1}; \gamma_0\, \overline{p}_{0,2}; \gamma_s\, \overline{p}_{s,\ell}\right) = \left(a^{h_0}_{0,1}; a^{h_0}_{0,2} + \sum_{s\in S} \gamma_s \overline{p}_{s,2}; a^{h_0}_{s,\ell}\right);$$
$$\gamma_0 \overline{\pi}_j \geq \sum_{s\in S} \gamma_s \overline{R}_{s,j}, \quad \forall j \in J.$$

Then, as in equilibrium commodity prices are strictly positive,

$$0 < \frac{\overline{p}_0 C_j - \overline{\pi}_j}{\overline{p}_{0,1}} = \frac{\overline{p}_{0,2}\alpha_j - \overline{\pi}_j}{\overline{p}_{0,1}} \leq \frac{a^{h_0}_{0,2}}{a^{h_0}_{0,1}}\alpha_j + \sum_{s\in S} \frac{a^{h_0}_{s,2}}{a^{h_0}_{0,1}}\alpha_j$$
$$- \sum_{s\in S} \frac{a^{h_0}_{s,2}}{a^{h_0}_{0,1}\overline{p}_{s,2}} \overline{R}_{s,j}, \quad \forall j \in J.$$

It follows that the mean payment made to investors of the security j at state of nature s is lower than a *fixed proportion* of the value of collateral requirements at s,

$$\overline{R}_{s,j} < \frac{1}{\min_{s'\in S} a^{h_0}_{s',2}} \left(a^{h_0}_{0,2} + \sum_{s'\in S} a^{h_0}_{s',2}\right) \overline{p}_{s,2}\alpha_j.$$

Thus, the mean rate of default on asset j at state of nature s, denoted by $\tau_{s,j}$, satisfies,

$$\tau_{s,j} = \frac{\overline{p}_{s,1} d_{s,1} - \overline{R}_{s,j}}{\overline{p}_{s,1} d_{s,1}}$$
$$\geq \Upsilon(\alpha_j) := \left[1 - \frac{1}{\min_{s'\in S} a^{h_0}_{s',2}} \left(a^{h_0}_{0,2} + \sum_{s'\in S} a^{h_0}_{s',2}\right) \frac{a^{h_0}_{s,2}}{a^{h_0}_{s,1} d_{s,j}} \alpha_j\right]^+ .$$

Since $\Upsilon(\alpha_j)$ converges to one as α_j goes to zero, we conclude that the bankruptcy law has a limited effectiveness to reduce the mean rate of default when collateral requirements are low.

5 On garnishment rules

In our model, at any state of nature $s \in S$, for each vector of commodity prices $p_s \neq 0$, the continuous garnishment rule Φ_s satisfies

$$\Phi_s(p_s, R_s, w^h_s, z^h_0) = \left[1 - \frac{\Lambda_s\left(p_s, w^h_s, \mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)\right)}{\mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)}\right]^+ \mathcal{W}_s(p_s, R_s, w^h_s, z^h_0).$$

As exemptions are strictly positive, we can rewrite garnishment rules as a (variable) proportion of the amount of debt. Indeed, for any $s \in S$, there is a function $\zeta_s : \left(\mathbb{R}^L_+ \setminus \{0\}\right) \times \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to [0,1)$ such that, $\Phi_s(p_s, R_s, w^h_s, z^h_0) = \zeta_s\left(p_s, w^h_s, \mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)\right) \mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)$.

Note that functions $(\zeta_s; s \in S)$ are only required to be continuous, non-negative, and lower than one. Thus, we can have a variety of garnishment rules compatible with equilibrium existence. In particular, non-linear rules as those imposed in Araujo and Páscoa (2002).

Indeed, suppose that for some $s \in S$, the proportion $\zeta_s(p_s, w^h_s, \mathcal{W}_s)$ is equal to

$$\left(1 - \mathcal{B}\left(\left[\Psi_s(p_s, \varphi^h) - \mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s)\mathcal{W}_s\right]^+\right)\right) \mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s)$$
$$+ \mathcal{A}_u(p_s, w^h_s, \mathcal{W}_s)\mathcal{B}\left(\left[\Psi_s(p_s, \varphi^h) - \mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s)\mathcal{W}_s\right]^+\right),$$

where $\mathcal{W}_s = \mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)$ and functions $\mathcal{A}_u, \mathcal{A}_d : (\mathbb{R}^L_+ \setminus \{0\}) \times \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to [0,1]$ and $\mathcal{B} : \mathbb{R}_+ \to [0,1)$ are continuous and satisfy $\mathcal{B}(0) = 0 \leq \mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s) < \mathcal{A}_u(p_s, w^h_s, \mathcal{W}_s) \leq 1$.

With this specification, if the amount of unpaid debts $\Psi_s(p_s, \varphi^h)$ is lower than or equal to $\mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s)\mathcal{W}_s$, then the amount of resources that can be garnished in case of bankruptcy is equal to a proportion $\mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s)$ of $\mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)$. That is, when debts are lower than $\mathcal{A}_d(p_s, w^h_s, \mathcal{W}_s)\mathcal{W}_s$, garnished resources are sufficient to cover the whole amount of unpaid promises. Moreover, if the amount of unpaid debt increases, then the quantity of garnishable resources may increase asymptotically to a proportion $\mathcal{A}_u(p_s, w^h_s, \mathcal{W}_s)$ of the available wealth $\mathcal{W}_s(p_s, R_s, w^h_s, z^h_0)$. Thus, when $\mathcal{A}_u \equiv 1$, we have, as in Araujo and Páscoa (2002), non-linear garnishment rules that make exemptions go to zero as the amount of debt increases.

6 Concluding remarks

We introduce the possibility of bankruptcy into the general equilibrium model with collateralized credit markets of Dubey et al. (1995) and Geanakoplos and Zame (1997, 2002, 2007). In case of default, borrowers may lose more than collateral guarantees, as market regulations allow lenders to be reimbursed by the garnishment of debtors' wealth. Allowing for a continuum of agents, we show that equilibrium always exists in the economy, even when the garnishment of resources over collateral repossession could induce non-convexities on individuals' problems. The key assumptions of our model are the existence of positive exemptions in case of bankruptcy and the desirability of commodities used as collateral.

As a matter of future research, it might be interesting to extend our model to allow for more than two periods (or infinite horizon), to introduce additional payment enforcements over the garnishment of wealth, to include financial collaterals, or more complex securitization structures (as in Steinert and Torres-Martínez 2007). However, we want to highlight two natural questions that may be studied departing from our model.

First, it could be interesting to analyze the performance of the bankruptcy law (through the garnishment of wealth) in capturing resources over collateral values relative to another payment enforcement mechanisms, as those given by restrictions on future credit (see for instance Sabarwal 2003) or non-economic punishments that affect utility levels (as in Dubey et al. 1989, 2005). Second, although in our model garnished wealth can be reimbursed to lenders following different mechanisms, we could extend our results to price-dependent rules of distribution, for instance, to determine priorities over claims as a function of its sizes, as in Araujo and Páscoa (2002).

Appendix A: Proof of Theorem 1

To prove the existence of equilibrium, we introduce non-convex generalized games. In these games, there are fictitious players that choose prices and security payments, and each consumer maximizes his utility function, but is restricted to choose *bounded* budgetary feasible plans. First, we prove that those generalized games have equilibria. Second, by making upper bounds on admissible plans go to infinity, we find an equilibrium of our economy as a cluster point of the sequence of equilibria in generalized games.

For any $n \in \mathbb{N}$ consider the set

$$\mathbb{E}_n = \left\{(x,\theta,\varphi) \in \mathbb{E} : (x_{s,\ell},\theta_j,\varphi_j) \le (\alpha_{s,\ell}(n), n, n),\ \ \forall (s,\ell,j) \in S^* \times L \times J\right\},$$

where

$$\alpha_{s,\ell} = \begin{cases} n & \text{if } s = 0; \\ n + Y_{s,\ell}\left((n,\ldots,n) + n\sum_{j\in J} C_j\right) + 2n\overline{A}\#J & \text{if } s \neq 0. \end{cases}$$

and $\overline{A} := \max_{(s,j)\in S\times J} \sum_{\ell\in L} A_{s,j,\ell}$. Also, define

$$\Delta_0 = \left\{ z \in \mathbb{R}^L_+ \times \mathbb{R}^J_+ : \sum_{r\in L\cup J} z_r = 1 \right\},$$

$$\Delta_1 = \left\{ z \in \mathbb{R}^L_+ : \sum_{r\in L} z_r = 1 \right\}.$$

Given $s \in S$, we rewrite the unitary payments of a security $j \in J$ at this state of nature as $R_{s,j} = N_{s,j} + D_{s,j}(p_s)$, where $N_{s,j} \in [0, \overline{A}]$ denotes the contingent security payment over collateral values. Let $N = (N_{s,j}; (s,j) \in S \times J)$.

The generalized game $\mathcal{G}_n$ Given $n \in \mathbb{N}$, let $\mathcal{G}_n$ be a generalized game with a continuum of players, where only a finite number of them are atomic. In this game, the set of players is described as follows,
(a) Given a vector of prices and payments $(p, \pi, N) \in \widehat{\mathbb{V}} := \Delta_0 \times \Delta_1^S \times [0, \overline{A}]^{S\times J}$, each consumer $h \in [0,1]$ maximizes the function $v_n^h : \widehat{\mathbb{V}} \times \mathbb{E}_n \times \prod_{s\in S} \mathcal{R}_s \to \mathbb{R}$,

$$v_n^h((p,\pi,N), z_n^h, \beta_n^h)$$
$$= u^h\left(x_{n,0}^h + \sum_{j\in J} C_j \varphi_{n,j}^h, \left(x_{n,s}^h; s \in S\right)\right) - \sum_{s\in S} \Omega_s((p,\pi,N), z_n^h, \beta_n^h)$$

by choosing a plan

$$(z_n^h, \beta_n^h) = \left((x_n^h, \theta_n^h, \varphi_n^h), (\beta_{n,s,j}^h; j \in J)_{s\in S}\right) \in B_n^h(p,\pi,N) \times \prod_{s\in S} \mathcal{R}_s,$$

where $\Omega_s : \widehat{\mathbb{V}} \times \mathbb{E}_n \times \prod_{s\in S} \mathcal{R}_s \to \mathbb{R}_+$ is continuous, with $\Omega_s((p,\pi,N), z_n^h, \beta_n^h)$ given by

$$\left(\sum_{j\in J} \beta_{n,s,j}^h \left[p_s A_{s,j} - \; p_s Y_s(C_j)\right]^+ \varphi_{n,j}^h \right.$$
$$\left. - \min\left\{\Psi_s(p_s, \varphi_{n,j}^h), \Phi_s(p_s, N_s + D_s(p_s), w_0^h, z_{n,0}^h)\right\}\right)^2,$$

and $B_n^h(p,\pi,N) := B^h\left(p, \pi, (N_s + D_s(p_s))_{s\in S}\right) \bigcap \mathbb{E}_n$.

Let $\tau : \mathbb{E}_n \times \prod_{s\in S} \mathcal{R}_s \to \mathbb{E}_n \times [0,n]^{S\times J}$ be the continuous function given by $\tau((x,\theta,\varphi),\beta) = ((x,\theta,\varphi), \beta \odot \varphi)$, where $\beta \odot \varphi = (\beta_{s,j}\varphi_j; (s,j) \in S \times J)$. Denote by $\mathcal{F}_n$ the set of action profiles for players $h \in [0,1]$, that is, the set of functions $f : [0,1] \to \mathbb{E}_n \times \prod_{s\in S} \mathcal{R}_s$.

In addition to consumers $h \in [0,1]$, in the generalized game $\mathcal{G}_n$ there are players that take messages $m \in \mathrm{Mess}_n$ about the actions taken by the consumers as given, where

$$\mathrm{Mess}_n = \left\{ \int_{[0,1]} \tau(f(h))\,\mathrm{d}h : \; (f \in \mathcal{F}_n) \;\; \wedge \;\; (\tau \circ f \text{ is measurable}) \right\}.$$

(b) Given $m = \int_{[0,1]} ((x_n^h, \theta_n^h, \varphi_n^h), \beta_n^h \odot \varphi_n^h)\,\mathrm{d}h \in \mathrm{Mess}_n$, there exists a player a_0 that chooses a vector of prices $(p_0, \pi) \in \Delta_0$, in order to maximize the function

$$p_0 \int_{[0,1]} \left(x_{n,0}^h + \sum_{j\in J} C_j \varphi_{n,j}^h - w_0^h\right) \mathrm{d}h + \sum_{j\in J} \pi_j \int_{[0,1]} \left(\theta_{n,j}^h - \varphi_{n,j}^h\right) \mathrm{d}h.$$

(c) Given $m = \int_{[0,1]}((x_n^h, \theta_n^h, \varphi_n^h), \beta_n^h \odot \varphi_n^h)\,\mathrm{d}h \in \mathrm{Mess}_n$, for any $s \in S$, there is a player a_s that chooses a vector of prices $p_s \in \Delta_1$, in order to maximize the function

$$p_s \int\limits_{[0,1]} \left(x_{n,s}^h - w_s^h - Y_s \left(x_{n,0}^h + \sum_{j \in J} C_j \varphi_{n,j}^h \right) \right) \mathrm{d}h.$$

(d) For each pair $(s, j) \in S \times J$, there exists a player $c_{s,j}$ such that, given $(m, p_s) \in \mathrm{Mess}_n \times \Delta_1$, he chooses $N_{s,j} \in [0, \overline{A}]$ in order to maximize the function

$$-\left(N_{s,j} \int\limits_{[0,1]} \varphi_{n,j}^h\,\mathrm{d}h - [p_s A_{s,j} - p_s Y_s(C_j)]^+ \int\limits_{[0,1]} \beta_{n,s,j}^h \varphi_{n,j}^h\,\mathrm{d}h \right)^2,$$

where $m = \int_{[0,1]}((x_n^h, \theta_n^h, \varphi_n^h), \beta_n^h \odot \varphi_n^h)\,\mathrm{d}h$.

Definition 2 A Cournot–Nash equilibrium for the game $\mathcal{G}_n$ is given by a plan of strategies

$$\left(\left((\overline{p}_0^n, \overline{\pi}^n), \overline{p}_s^n, \overline{N}_{s,j}^n \right)_{(s,j) \in S \times J}; \left(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h, \overline{\beta}_n^h \right)_{h \in [0,1]} \right) \in \widehat{\mathbb{V}} \times \left(\mathbb{E}_n \prod_{s \in S} \mathcal{R}_s \right)^{[0,1]},$$

jointly with a message $\overline{m} \in \mathrm{Mess}_n$ such that, any player maximizes his objective function given $\overline{m}$ and the strategies chosen by the other players, where $\overline{m} = \int_{[0,1]}((\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h), \overline{\beta}_n^h \odot \overline{\varphi}_n^h)\,\mathrm{d}h$.

Lemma 1 *Under Assumptions (A1)–(A5), there exists $n^* \in \mathbb{N}$ such that, for any $n > n^*$, there is a Cournot–Nash equilibrium for $\mathcal{G}_n$.*

Proof In our game, a Cournot–Nash equilibrium is given as a consequence of Theorem 1 in Riascos and Torres-Martínez (2010) (see also Theorem 2.1 in Balder 1999).[14] The only requirement of this theorem that does not follow from direct verification is the lower-hemicontinuity of the correspondences of admissible strategies $\Gamma_n^h(p, \pi, N) = B_n^h(p, \pi, N) \times \prod_{s \in S} \mathcal{R}_s$, with $h \in [0, 1]$. However, as $\prod_{s \in S} \mathcal{R}_s$ is fixed and $B_n^h(p, \pi, N)$ is independent of the choice of β_n^h, it is sufficient to prove that $(B_n^h; h \in [0, 1])$ are lower-hemicontinuous correspondences.

Given $h \in [0, 1]$, consider the correspondence $\dot{B}_n^h$ that associates to each $(p, \pi, N) \in \widehat{\mathbb{V}}$ the collection of plans $(x_n^h, \theta_n^h, \varphi_n^h) \in \mathbb{E}_n$ that satisfy state-contingent constraints of $B_n^h(p, \pi, N)$ as strict inequalities. It follows from Assumption (A3)

[14] As an alternative approach to prove the existence of a Cournot–Nash equilibrium, we can use a purification of mixed strategy equilibria as in Balder (1999). This technique was used by Araujo et al. (2000), Araujo and Páscoa (2002), and Araujo et al. (2005).

that $\dot{B}_n^h$ has non-empty values. Also, since the constraints that define $\dot{B}_n^h(p,\pi,N)$ are given by inequalities that only include continuous functions, the correspondence $\dot{B}_n^h$ has an open graph. Therefore, for any $h \in [0,1]$, $\dot{B}_n^h$ is lower-hemicontinuous (see Hildenbrand 1974, Proposition 7, p. 27). Moreover, the correspondence that associates any vector $(p,\pi,N) \in \widehat{\mathbb{V}}$ to the closure of the set $\dot{B}_n^h(p,\pi,N)$ is also lower-hemicontinuous (see Hildenbrand 1974, p. 26).

We affirm that, closure$(\dot{B}_n^h) = B_n^h$. Since for any $(p,\pi,N) \in \widehat{\mathbb{V}}$, we have that closure$(\dot{B}_n^h(p,\pi,N)) \subset B_n^h(p,\pi,N)$, it is sufficient to ensure that $B_n^h(p,\pi,N) \subset$ closure$(\dot{B}_n^h(p,\pi,N))$.

Given $(x^h,\theta^h,\varphi^h) \in B_n^h(p,\pi,N) \subset \mathbb{E}_n$ and $(\epsilon,(\delta_s; s \in S^*), j) \in [0,1) \times [0,1)^{S^*} \times J$, let $\varphi_j^h(\epsilon,\delta_0) = (1-\delta_0)\varphi_j^h + \epsilon$. We want to prove that $(((1-\delta_s)x_s^h; s \in S^*), (1-\delta_0)\theta^h, (\varphi_j^h(\epsilon,\delta_0))_{j\in J}) \in \dot{B}_n^h(p,\pi,N)$. It is not difficult to verify that this last property effectively holds if $n > n^* := \max_{(s,\ell)\in S^*\times L} \overline{w}_{s,\ell}$,[15]

$$\epsilon \sum_{(\ell,j)\in L\times J} C_{j,\ell} < \delta_0 \min_{\ell\in L} w_{0,\ell}^h;$$

and, for any $s \in S$,

$$\delta_s = \begin{cases} \left[1-(1-\delta_0)\frac{G_s(p_s,R_s,w_s^h,z_0^h(\epsilon,\delta_0))}{p_s x_s^h}\right]^+, & \text{if } p_s x_s^h > 0; \\ 0, & \text{if } p_s x_s^h = 0; \end{cases}$$

where $z_0^h(\epsilon,\delta) := ((1-\delta_0)x_0^h, (1-\delta_0)\theta^h, (\varphi_j^h(\epsilon,\delta_0))_{j\in J})$ and

$$\begin{aligned} &G_s(p_s,R_s,w_s^h,z_0^h(\epsilon,\delta_0)) \\ &= p_s w_s^h + p_s Y_s\left((1-\delta_0)x_0^h + \sum_{j\in J} C_j\varphi_j^h(\epsilon,\delta_0)\right) \\ &\quad +(1-\delta_0)\sum_{j\in J} R_{s,j}\theta_j^h - M_s(p_s,R_s,w_s^h,z_0^h(\epsilon,\delta_0)).\end{aligned}$$ [16]

In fact, when $(x_0^h,\theta^h,\varphi^h)$ is changed to $((1-\delta_0)x_0^h, (1-\delta_0)\theta^h, (\varphi_j^h(\epsilon,\delta_0))_{j\in J})$, a quantity of resources $\delta_0 p_0 w_0^h$ becomes available at the first period. Thus, the condition that restricts ϵ ensures that a portion of these resources covers the cost of the emission of the new debt. Moreover, for any $s \in S$, the condition that defines δ_s ensures that, after the decision between payment or fill for bankruptcy at state of nature s, the agent h has resources to buy the bundle $(1-\delta_s)x_s^h$. Thus, the allocation $(((1-\delta_s)x_s^h; s \in S^*), (1-\delta_0)\theta^h, (\varphi_j^h(\epsilon,\delta_0))_{j\in J})$ belongs to $\dot{B}_n^h(p,\pi,N)$.

[15] The restriction over n is to ensure that, at any $s \in S^*$, agents can consume their entire physical endowment.

[16] Note that, $G_s(p_s,R_s,w_s^h,z_0^h(\epsilon,\delta_0))$ is strictly positive as a consequence of Assumption (A4), because there always exist exemptions in case of bankruptcy.

Making δ_0 go to zero (which implies that $(\epsilon; (\delta_s; s \in S))$ vanishes too), we conclude that $(x^h, \theta^h, \varphi^h)$ belongs to the closure of $\dot{B}_n^h(p, \pi, N)$. Thus, if $n > n^*$, the correspondence B_n^h is lower-hemicontinuous for each $h \in [0, 1]$. $\square$

Lemma 2 *Under Assumptions (A1)–(A5), for any $n > n^*$, given an equilibrium of $\mathcal{G}_n$,*

$$\left(\left((\overline{p}_0^n, \overline{\pi}^n), \overline{p}_s^n, \overline{N}_{s,j}^n\right)_{(s,j)\in S\times J}; \left(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h, \overline{\beta}_n^h\right)_{h\in[0,1]}, \overline{m}\right),$$

for each pair $(s, j) \in S \times J$ we have that,

$$\overline{N}_{s,j}^n \int_{[0,1]} \overline{\varphi}_{n,j}^h \,\mathrm{d}h = [\overline{p}_s^n A_{s,j} - \overline{p}_s^n Y_s(C_j)]^+ \int_{[0,1]} \overline{\beta}_{n,s,j}^h \overline{\varphi}_{n,j}^h \,\mathrm{d}h.$$

Proof Let $n > n^*$ and fix $(s, j) \in S \times J$. Since $\overline{N}_{s,j}^n \in [0, \overline{A}]$, it follows from the definition of the objective function of player $c_{s,j}$ that,

$$\overline{N}_{s,j}^n \int_{[0,1]} \overline{\varphi}_{n,j}^h \,\mathrm{d}h \leq [\overline{p}_s^n A_{s,j} - \overline{p}_s^n Y_s(C_j)]^+ \int_{[0,1]} \overline{\beta}_{n,s}^h \overline{\varphi}_{n,j}^h \,\mathrm{d}h,$$

where the strict inequality holds only when $\overline{N}_{s,j}^n = \overline{A}$ and $\int_{[0,1]} \overline{\varphi}_{n,j}^h \,\mathrm{d}h > 0$. However, this situation is not compatible with $\overline{p}_s^n \in \Delta_1$. $\square$

Definition 3 A vector $(\overline{p}^n, \overline{\pi}^n, \overline{R}^n) \in \Delta_0 \times \Delta_1^S \times [0, 2\overline{A}]^{S\times J}$, jointly with plans $\left((\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h); h \in [0, 1]\right) \in \mathbb{E}_n^{[0,1]}$, constitutes an n-equilibrium of $\mathcal{E}$ when market feasible conditions (2) and (3) of Definition 1 hold and, for each $h \in [0, 1]$ we have

$$(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h) \in argmax_{B^h(\overline{p}^n, \overline{\pi}^n, \overline{R}^n)\cap\mathbb{E}_n}\; u^h\left(x_0^h + \sum_{j\in J} C_j\varphi_j^h, \left(x_s^h; s \in S\right)\right).$$

Lemma 3 *Under Assumptions (A1)–(A6), the economy $\mathcal{E}$ has an n-equilibrium for any $n > n^*$.*

Proof Given $n > n^*$, let

$$\left(\left((\overline{p}_0^n, \overline{\pi}^n), \overline{p}_s^n, \overline{N}_{s,j}^n\right)_{(s,j)\in S\times J}; \left(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h, \overline{\beta}_n^h\right)_{h\in[0,1]}, \overline{m}\right)$$

be a Cournot–Nash equilibrium of $\mathcal{G}_n$. We want to prove that

$$\left(\left((\overline{p}_0^n, \overline{\pi}^n), \overline{p}_s^n, \overline{R}_{s,j}^n\right)_{(s,j)\in S\times J}; \left(\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h\right)_{h\in[0,1]}\right)$$

constitutes an n-equilibrium for the economy $\mathcal{E}$, where for each $(s, j) \in S \times J$ the unitary security payment is given by $\overline{R}^n_{s,j} = D_{s,j}(\overline{p}^n_s) + \overline{N}^n_{s,j}$.

Note that, in any Cournot–Nash equilibrium, each agent $h \in [0, 1]$ maximizes his utility function u^h. In fact, for any $h \in [0, 1]$ it is always feasible to choose parameters $(\overline{\beta}^h_{n,s,j}; j \in J)_{s\in S} \in \prod_{s\in S} \mathcal{R}_s$ in order to make $(\Omega^h_s)_{s\in S} = 0$. Therefore, to achieve our objective, it is sufficient to prove that market clearing conditions (2) and (3) of Definition 1 hold.

Step 1. There is no excess demand in physical and financial markets.

Integrating the first period budget constraint of $B^h_n(\overline{p}^n, \overline{\pi}^n, \overline{N}^n)$ through agents $h \in [0, 1]$, we have

$$\overline{p}^n_0 \int_{[0,1]} \left(\overline{x}^h_{n,0} + \sum_{j\in J} C_j \overline{\varphi}^h_{n,j} - w^h_0 \right) \mathrm{d}h + \sum_{j\in J} \overline{\pi}^n_j \int_{[0,1]} \left(\overline{\theta}^h_{n,j} - \overline{\varphi}^h_{n,j} \right) \mathrm{d}h \le 0.$$

Thus, the maximal value of the objective function of player a_0 is less than or equal to zero. Therefore, since $(\overline{p}^n_0, \overline{\pi}^n) \in \Delta_0$, we have that, for any $(\ell, j) \in L \times J$,

$$\int_{[0,1]} \left(\overline{x}^h_{n,0,\ell} + \sum_{j\in J} C_{j,\ell} \overline{\varphi}^h_{n,j} - w^h_{0,\ell} \right) \mathrm{d}h \le 0, \qquad \int_{[0,1]} \left(\overline{\theta}^h_{n,j} - \overline{\varphi}^h_{n,j} \right) \mathrm{d}h \le 0.$$

The last inequality, jointly with the result of Lemma 2, ensures that

$$\overline{N}^n_{s,j} \int_{[0,1]} \overline{\theta}^h_{n,j}\, \mathrm{d}h \le \overline{N}^n_{s,j} \int_{[0,1]} \overline{\varphi}^h_{n,j}\, \mathrm{d}h = [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+ \int_{[0,1]} \overline{\beta}^h_{n,s,j} \overline{\varphi}^h_{n,j}\, \mathrm{d}h.$$

As we argue above, it follows from Assumption (A5) that, for any $s \in S$,

$$\begin{aligned}
&\sum_{j\in J} \overline{\beta}^h_{n,s,j} [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+ \overline{\varphi}^h_{n,j} \\
&\quad = \min \left\{ \Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n), \Phi_s(\overline{p}^n_s, \overline{R}^n_s, w^h_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n) \right\},
\end{aligned}$$

which in turn implies that,

$$\begin{aligned}
&\sum_{j\in J} [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+ \int_{[0,1]} \overline{\beta}^h_{n,s,j} \overline{\varphi}^h_{n,j}\, \mathrm{d}h \\
&\quad = \int_{[0,1]} \min\{ \Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n), \Phi_s(\overline{p}^n_s, \overline{R}^n_s, w^h_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)\}\, \mathrm{d}h.
\end{aligned}$$

Therefore, aggregating budget constraints at $s \in S$ through agents $h \in [0,1]$, we obtain that

$$\overline{p}^n_s \left(\int\limits_{[0,1]} (\overline{x}^h_{n,s} - w^h_s)\, \mathrm{d}h - \int\limits_{[0,1]} Y_s \left(\overline{x}^h_{n,0} + \sum_{j \in J} C_j \overline{\varphi}^h_{n,j} \right) \mathrm{d}h \right) \leq 0.$$

which implies that the maximal value of the objective function of player a_s is less than or equal to zero. Since $\overline{p}^n_s \in \Delta_1$, it follows that, for any commodity $\ell \in L$

$$\int\limits_{[0,1]} (\overline{x}^h_{n,s,\ell} - w^h_{s,\ell})\, \mathrm{d}h \leq \int\limits_{[0,1]} Y_{s,\ell} \left(\overline{x}^h_{n,0} + \sum_{j \in J} C_j \overline{\varphi}^h_{n,j} \right) \mathrm{d}h.$$

Step 2. Commodity and asset prices are strictly positive. Securities have non-trivial payments.

For any commodity $\ell \in L$, we affirm that $\overline{p}^n_{0,\ell} > 0$. Otherwise, Assumption (A1) ensures that every agent would choose $\overline{x}^h_{n,0,\ell} = n$ and, therefore, $\int_{[0,1]} \overline{x}^h_{n,0,\ell}\, \mathrm{d}h = n > n^* = \max_{(s,\ell) \in S^* \times L} \overline{w}_{s,\ell} \geq \int_{[0,1]} w^h_{0,\ell}\, \mathrm{d}h$, which contradicts the results of Step 1. Analogously, for any pair $(\ell, s) \in L \times S$ we have that, $\overline{p}^n_{s,\ell} > 0$. In fact, in any other case, every agent would choose $\overline{x}^h_{n,s,\ell} = \alpha_{s,\ell}(n)$, which implies that,

$$\int\limits_{[0,1]} \overline{x}^h_{n,s,\ell}\, \mathrm{d}h = \alpha_{s,\ell}(n) > \int\limits_{[0,1]} \left(w^h_{s,\ell} + Y_s \left(\overline{x}^h_{n,0,\ell} + \sum_{j \in J} C_{j,\ell} \overline{\varphi}^h_{n,j} \right) \right) \mathrm{d}h,$$

a contradiction with the results of Step 1.

Thus, it follows from Assumption (A6) that, for any $j \in J$, there is a state of nature $s(j) \in S$ such that,

$$\overline{R}_{s(j),j} \geq \overline{D}_{s(j),j}(\overline{p}^n_{s(j)}) = \min \left\{ \overline{p}^n_{s(j)} A_{s(j),j}, \overline{p}^n_{s(j)} Y_{s(j)}(C_j) \right\} > 0.$$

Then, Assumption (A1) ensures that, for any $j \in J$, the unitary price $\overline{\pi}^n_j$ is strictly positive. Otherwise, every agent would choose $(\overline{\theta}^h_{n,j}, \overline{\varphi}^h_{n,j}) = (n, 0)$.[17] Thus, $\int_{[0,1]} (\overline{\theta}^h_{n,j} - \overline{\varphi}^h_{n,j})\, \mathrm{d}h = n > 0$, a contradiction with the fact that $\int_{[0,1]} (\overline{\theta}^h_{n,j} - \overline{\varphi}^h_{n,j})\, \mathrm{d}h \leq 0$.

That is, $((\overline{p}^n_0, \overline{\pi}^n), (\overline{p}^n_s; s \in S)) \gg 0$ and, for any $j \in J$, $(\overline{R}^n_{s,j}; s \in S) \neq 0$.

[17] On one hand, the investment on security j has no cost but delivers positive payments at $t = 1$. On the other hand, it follows from Assumption (A6) that agents do not have incentives to take short-positions in debt contract j, because it will induce a positive commitment at $s(j)$ without the right to receive any resources at $s = 0$.

Step 3. Market clearing conditions hold.

Since prices are strictly positive, it follows that for any agent $h \in [0,1]$ budget set constraints hold as equalities.[18] Therefore, since prices are strictly positive and there is no excess demand in physical and financial markets at $t = 0$, market clearing conditions hold at the first period. Thus, it follows that, at any $s \in S$,

$$\overline{R}^n_{s,j} \int\limits_{[0,1]} \overline{\theta}^h_{n,j}\,\mathrm{d}h = D_{s,j}(\overline{p}^n_s) \int\limits_{[0,1]} \overline{\varphi}^h_j\,\mathrm{d}h + [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+ \int\limits_{[0,1]} \overline{\beta}^h_{n,s,j}\overline{\varphi}^h_{n,j}\,\mathrm{d}h,$$

and parameters $(\overline{\beta}^h_{n,s,j}; (h,j) \in [0,1] \times J)_{s\in S}$ satisfy the requirements imposed at item (3) of Definition 1. Then, to finish the proof we need to ensure that physical market clearing conditions hold at any state of nature $s \in S$. But these properties are a direct consequence of the results proved at Steps 1 and 2, jointly with the fact that budget constraints hold as equalities.

Thus, $\left((\overline{p}^n_0, \overline{\pi}^n), \overline{p}^n_s, \overline{R}^n_{s,j})_{(s,j)\in S\times J}; (\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n)_{h\in[0,1]}\right)$ is an n-equilibrium of the economy $\mathcal{E}$.

Lemma 4 *Suppose that Assumptions (A1)–(A6) hold and let*

$$\left((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \left((\overline{x}^h_n, \overline{\theta}^h_n, \overline{\varphi}^h_n); h \in [0,1]\right)\right)$$

be an n-equilibrium of $\mathcal{E}$, with $n > n^*$. Consider the family of non-negative and integrable functions $\left\{g_n : [0,1] \to \mathbb{R}^{L\times S^*}_+ \times \mathbb{R}^J_+ \times \mathbb{R}^J_+ \times \mathbb{R}^{S\times J}_+\right\}_{n>n^*}$ given by,

$$g_n(h) = \left(\overline{x}^h_n, \overline{\theta}^h_n,\ \overline{\varphi}^h_n,\ (\overline{\beta}^h_{n,s,j}\ \overline{\varphi}^h_{n,j})_{(s,j)\in S\times J}\right), \quad \forall n > n^*.$$

[18] Actually, suppose that for some $h \in [0,1]$ the budgetary constraint at $s = 0$ holds as strict inequality at prices $(\overline{p}^n_0, \overline{\pi}^n)$. Then, $(\overline{x}^h_{n,0}, \overline{\theta}^h_n) = n(1,\ldots,1)$. Using the first period budget set constraint, we have that,

$$\begin{aligned} n &= \overline{p}^n_0 \overline{x}^h_{n,0} + \sum_{j\in J} \overline{\pi}^n_j \overline{\theta}^h_{n,j} < \overline{p}^n_0 w^h_0 + \sum_{j\in J} \overline{\pi}^n_j \overline{\varphi}^h_{n,j} \\ &< n^* \|\overline{p}^n_0\|_\Sigma + n\|\overline{\pi}^n\|_\Sigma < n, \end{aligned}$$

which is a contradiction. Analogously, assume that for agent h the budgetary constraint at $s \in S$ holds as a strict inequality. Then, $\overline{x}^h_{n,s} = (\alpha_{s,\ell}(n); \ell \in L)$, which is a contradiction, since

$$\begin{aligned} \sum_{\ell\in L} \overline{p}_{s,\ell}\alpha_{s,\ell}(n) &= \overline{p}^n_s \overline{x}^h_{n,s} \\ &< \overline{p}^n_s w^h_s + \overline{p}^n_s \left(Y_s\left((n,\ldots,n) + n\sum_{j\in J} C_j\right) + n\overline{A}(1,\ldots,1)\#J\right) \\ &< \sum_{\ell\in L} \overline{p}_{s,\ell}\alpha_{s,\ell}(n). \end{aligned}$$

Then, $\left\{\left(\overline{p}^n, \overline{\pi}^n, \overline{R}^n, \int\limits_{[0,1]} g_n(h)\,\mathrm{d}h\right)\right\}_{n>n^*}$ *is bounded.*[19]

Proof Since for any $n > n^*$, the vector $(\overline{p}^n, \overline{\pi}^n, \overline{R}^n) \in \Delta_0 \times \Delta_1^S \times [0, 2\overline{A}]^{S\times J}$, it follows that the sequence of equilibrium prices and payments is bounded. On the other hand, using the fact that $\left((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \left((\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h); h \in [0,1]\right)\right)$ is an n-equilibrium of $\mathcal{E}$ we have,

$$0 \leq \int\limits_{[0,1]} \overline{x}_{n,0}^h\,\mathrm{d}h \leq \int\limits_{[0,1]} w_0^h\,\mathrm{d}h,$$

$$0 \leq \sum_{j\in J} C_j \int\limits_{[0,1]} \overline{\varphi}_{n,j}^h\,\mathrm{d}h = \int\limits_{[0,1]} \sum_{j\in J} C_j \overline{\varphi}_{n,j}^h\,\mathrm{d}h \leq \int\limits_{[0,1]} w_0^h\,\mathrm{d}h,$$

$$0 \leq \int\limits_{[0,1]} \overline{\theta}_n^h\,\mathrm{d}h \leq \int\limits_{[0,1]} \overline{\varphi}_n^h\,\mathrm{d}h.$$

Moreover, for any $(s, j) \in S \times J$,

$$0 \leq \int\limits_{[0,1]} \overline{\beta}_{n,s,j}^h\,\overline{\varphi}_{n,j}^h\,\mathrm{d}h \leq \int\limits_{[0,1]} \overline{\varphi}_{n,j}^h\,\mathrm{d}h,$$

$$0 \leq \int\limits_{[0,1]} \overline{x}_{n,s}^h\,\mathrm{d}h \leq \int\limits_{[0,1]} \left(w_s^h + Y_s\left(w_0^h\right)\right)\mathrm{d}h,$$

where the last inequality is a consequence of the fact that $Y(x) \leq Y(y)$ if $x \leq y$. The result follows from Assumption (A3), since for any $j \in J$ there is $\ell \in L$ such that $C_{j,\ell} > 0$. □

It follows from the Lemma above that, given a sequence

$$\left\{\left((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \left((\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h); h \in [0,1]\right)\right)\right\}_{n>n^*}$$

of n-equilibria, there exists a convergent subsequence

$$\left\{\left(\overline{p}^{n_k}, \overline{\pi}^{n_k}, \overline{R}^{n_k}, \int\limits_{[0,1]} g_{n_k}(h)\,\mathrm{d}h\right)\right\}_{n_k>n^*} \subseteq \left\{\left(\overline{p}^n, \overline{\pi}^n, \overline{R}^n, \int\limits_{[0,1]} g_n(h)\,\mathrm{d}h\right)\right\}_{n>n^*}.$$

We denote by $(\overline{p}, \overline{\pi}, \overline{R})$ the associated limit of prices and payments. Also, applying the weak version of the multidimensional Fatou's Lemma to the sequence $\left\{g_{n_k}\right\}_{n_k>n^*}$

[19] Given $\left((\overline{p}^n, \overline{\pi}^n, \overline{R}^n); \left((\overline{x}_n^h, \overline{\theta}_n^h, \overline{\varphi}_n^h); h \in [0,1]\right)\right)$, $\left(\overline{\beta}_{n,s,j}^h; (h, s, j) \in [0,1] \times S \times J\right)$ are the delivery rates that satisfy condition (3) of Definition 1.

(see Hildenbrand (1974, page 69)), we can find a full measure set $\mathbb{P} \subset [0,1]$ and an integrable function $g : [0,1] \to \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}$, defined by $g(h) := (\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h, (\overline{\rho}^h_{s,j})_{(s,j)\in S\times J})$ such that, for each agent $h \in \mathbb{P}$, there is a subsequence of $\{g_{n_k}(h)\}_{n_k>n^*}$ that converges to $g(h)$, and

$$\int_{[0,1]} g(h)\,\mathrm{d}h \le \lim_{k\to\infty} \int_{[0,1]} g_{n_k}(h)\,\mathrm{d}h.$$

Thus, it follows that, for any $h \in \mathbb{P}$, the bundle $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)$ belongs to $B^h(\overline{p}, \overline{\pi}, \overline{R})$.

In addition, for any $(h, s) \in \mathbb{P} \times S$, there exists $(\overline{\beta}^h_{s,j};\, j \in J) \in \mathcal{R}_s$ such that, for any $j \in J$, $\overline{\rho}^h_{s,j} = \overline{\beta}^h_{s,j}\,\overline{\varphi}^h_j$ and

$$\begin{aligned}&\sum_{j\in J} \overline{\beta}^h_{s,j}\left[\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)\right]^+ \overline{\varphi}^h_j\\ &\quad = \min\left\{\Psi_s(\overline{p}_s, \overline{\varphi}^h),\, \Phi_s(\overline{p}_s, \overline{R}_s, w^h_s, \overline{x}^h_0, \overline{\theta}^h, \overline{\varphi}^h)\right\}.\end{aligned}$$ [20]

Lemma 5 *Under Assumptions (A1)–(A6), for each $h \in \mathbb{P}$, the plan $(\overline{x}^h, \overline{\theta}^h, \overline{\varphi}^h)$ is an optimal choice for agent h on $B^h(\overline{p}, \overline{\pi}, \overline{R})$.*

Proof Fix an agent $h \in \mathbb{P}$ and let $(\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h) \in B^h(\overline{p}, \overline{\pi}, \overline{R})$. It is clear that there exists $n^{**} > n^*$ such that, for any $n \ge n^{**}$, $(\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h)$ belong to $B^h_n(\overline{p}, \overline{\pi}, \overline{R})$.

Fix $n > n^{**}$. It follows from the sequential characterization of lower-hemicontinuity that there exists a sequence $\left\{(\tilde{x}^h_m, \tilde{\theta}^h_m, \tilde{\varphi}^h_m)\right\}_{m>n^{**}}$ such that, for any $m > n^{**}$, both $(\tilde{x}^h_m, \tilde{\theta}^h_m, \tilde{\varphi}^h_m) \in B^h_n(\overline{p}^m, \overline{\pi}^m, \overline{R}^m)$ and $\lim_{m\to\infty}(\tilde{x}^h_m, \tilde{\theta}^h_m, \tilde{\varphi}^h_m) = (\tilde{x}^h, \tilde{\theta}^h, \tilde{\varphi}^h)$. Since for m large enough $(\tilde{x}^h_m, \tilde{\theta}^h_m, \tilde{\varphi}^h_m) \in B^h_m(\overline{p}^m, \overline{\pi}^m, \overline{R}^m)$, it follows that,

$$u^h\left(\tilde{x}^h_{m,0} + \sum_{j\in J} C_j \tilde{\varphi}^h_{m,j},\, \left(\tilde{x}^h_{m,s};\, s \in S\right)\right) \le u^h\left(\overline{x}^h_{m,0} + \sum_{j\in J} C_j \overline{\varphi}^h_{m,j},\, \left(\overline{x}^h_{m,s};\, s \in S\right)\right).$$

[20] Given $h \in \mathbb{P}$, the convergence of a subsequence of $\{\overline{\varphi}^h_{n_k}, (\overline{\beta}^h_{n_k,s,j}\,\overline{\varphi}^h_{n_k,j})_{(s,j)\in S\times J}\}_{n_k>n^*}$ (those given by Fatou's Lemma), does not necessarily imply the convergence of the associated subsequence of $\left\{(\overline{\beta}^h_{n_k,s,j})_{(s,j)\in S\times J}\right\}_{n_k>n^*}$. However, the later sequence is bounded and, therefore, taking a subsequence if it is necessary, we can assume that it converges to a vector $(\overline{\beta}^h_{s,j};\, j \in J)_{s\in S} \in \prod_{s\in S} \mathcal{R}_s$. Thus, for any $(s, j) \in S \times J$, we have that $\overline{\rho}^h_{s,j} = \overline{\beta}^h_{s,j}\,\overline{\varphi}^h_j$. Finally, it follows from the continuity of functions Φ_s and Ψ_s that $\sum_{j\in J} \overline{\beta}^h_{s,j}\left[\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)\right]^+ \overline{\varphi}^h_j = \min\left\{\Psi_s(\overline{p}_s, \overline{\varphi}^h),\, \Phi_s(\overline{p}_s, \overline{R}_s, w^h_s, \overline{x}^h_0, \overline{\theta}^h, \overline{\varphi}^h)\right\}$.

Taking the limit as m goes to infinity, through the convergent subsequence of $\{g_{m'}(h)\}_{m'>n^*}$ given by Fatou's Lemma, we obtain that

$$u^h\left(\tilde{x}_0^h+\sum_{j\in J}C_j\tilde{\varphi}_j^h,\left(\tilde{x}_s^h;s\in S\right)\right)\leq u^h\left(\overline{x}_0^h+\sum_{j\in J}C_j\overline{\varphi}_j^h,\left(\overline{x}_s^h;s\in S\right)\right).$$

It follows that $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$ is an optimal choice for agent h on $B^h(\overline{p},\overline{\pi},\overline{R})$. □

By Lemma 5 and the monotonicity of utility functions, we have that $(\overline{p}_s;s\in S^*)\gg 0$. Therefore, for any $j\in J$, Assumption (A6) ensures that there is a state of nature $s(j)\in S$ such that $\overline{R}_{s(j),j}\geq\overline{D}_{s(j),j}(\overline{p}_{s(j)})=\min\left\{\overline{p}_{s(j)}A_{s(j),j},\overline{p}_{s(j)}Y_{s(j)}(C_j)\right\}>0$. Furthermore, this last property jointly with the monotonicity of preferences guarantees that, for any $j\in J$, the unitary price $\overline{\pi}_j$ is strictly positive.

Lemma 6 *Suppose that Assumptions (A1)–(A5) and (A7) hold. Then, for each* $j\in J$, $\overline{p}_0C_j>\overline{\pi}_j$.

Proof Since the set $\mathbb{P}$ has full measure, for any debt contract $j\in J$, the set $\mathbb{P}\cap H_j$ is non-empty. Thus, take as given $(h,j)\in(\mathbb{P}\cap H_j)\times J$ and suppose that $\overline{p}_0C_j\leq\overline{\pi}_j$.

In this context, agent h may sell any quantity $a>0$ of debt contract j to obtain resources at $t=0$ that allow him to consume the bundle $w_0^h+C_ja\gg 0$. This short-position on debt contract j has a limited commitment at any state of nature $s\in S$. In fact, if $\overline{p}_sA_{s,j}\leq\overline{p}_sY_s(C_j)$, agent h pays his debt and has resources to demand his initial endowment $w_s^h\gg 0$. Alternatively, if $\overline{p}_sA_{s,j}>\overline{p}_sY_s(C_j)$, even when agent h decides to not pay the whole amount of his debt, he has a positive amount of resources available for consumption, $\Lambda_s(\overline{p}_s,w_s^h,\mathcal{W}_s)>0$ [a consequence of Assumption (A4)]. Moreover, in this case, as the value of depreciated collateral is lower than the original promises, the amount of wealth $\mathcal{W}_s$ is (by definition) independent of a. Actually, $\mathcal{W}_s=\overline{p}_s(w_s^h+Y_s(w_0^h))$.

Therefore, regardless of a, he may consume (at least) at any state of nature $s\in S$ a bundle $\gamma_s(1,\ldots,1)$, where

$$0<\gamma_s=\overline{p}_s(\gamma_s,\ldots,\gamma_s)\leq\min\{\overline{p}_sw_s^h;\Lambda_s(\overline{p}_s,w_s^h,\overline{p}_s(w_s^h+Y_s(w_0^h)))\}.$$

Using this strategy, it follows from Assumption (A7) that, for a large enough, agent h could improve his utility function relative to the level that he obtains with the plan $(\overline{x}^h,\overline{\theta}^h,\overline{\varphi}^h)$, a contradiction. □

Lemma 7 *Suppose that Assumptions (A1)–(A7) hold. Then,* $\left\{g_{n_k}\right\}_{n_k\geq n^*}$ *is uniformly integrable and, for each* $h\in[0,1]$, $\left\{g_{n_k}(h)\right\}_{n_k\geq n^*}$ *is bounded.*

Proof For each $h\in[0,1]$, if the sequence $\left\{(\overline{x}_{n_k}^h,\overline{\theta}_{n_k}^h,\overline{\varphi}_{n_k}^h)\right\}_{n_k\geq n^*}$ is bounded, then $\left\{g_{n_k}(h)\right\}_{n_k\geq n^*}$ is bounded too. Since $\left(\overline{p},\overline{\pi},(\overline{p}_0C_j-\overline{\pi}_j)_{j\in J}\right)\gg 0$, there exists $\epsilon>0$ and $T^*\in\mathbb{N}$ such that $\left(\overline{p},\overline{\pi},(\overline{p}_0C_j-\overline{\pi}_j)_{j\in J}\right)\gg\epsilon(1,\ldots,1)$ and, for any $n_k>T^*$,

$$\| \left(\overline{p}^{n_k}, \overline{\pi}^{n_k}, (\overline{p}_0^{n_k} C_j - \overline{\pi}_j^{n_k})_{j\in J}\right) - \left(\overline{p}, \overline{\pi}, (\overline{p}_0 C_j - \overline{\pi}_j)_{j\in J}\right) \|_{\max} \leq \epsilon.$$

Therefore, for each $n_k > T^*$, $\| \left(\overline{p}^{n_k}, \overline{\pi}^{n_k}, (\overline{p}_0^{n_k} C_j - \overline{\pi}_j^{n_k})_{j\in J}\right) \|_{\max} \gg 0$. Using individuals' first period budget constraints, we have that, for any $(\ell, j) \in L \times J$ and for each $n_k > T^*$,

$$0 \leq \left(\overline{x}^h_{n_k,0,\ell}, \overline{\theta}^h_{n_k,j}, \overline{\varphi}^h_{n_k,j}\right) \leq \left(\frac{\overline{p}_0^{n_k} w_0^h}{\overline{p}_{0,\ell}^{n_k}}, \frac{\overline{p}_0^{n_k} w_0^h}{\overline{\pi}_j^{n_k}}, \frac{\overline{p}_0^{n_k} w_0^h}{\overline{p}_0^{n_k} C_j - \overline{\pi}_j^{n_k}} \right).$$

In addition, for any $(s, \ell) \in S \times L$,

$$0 \leq \overline{x}^h_{n_k,s,\ell} \leq \frac{\overline{p}_s^{n_k}\left(w_s^h + Y_s(\overline{x}^h_{n_k,0} + \sum_{j\in J} C_j \overline{\varphi}^h_{n_k,j})\right) + \sum_{j\in J} \overline{R}^{n_k}_{s,j} \overline{\theta}^h_{n_k,j}}{\overline{p}^{n_k}_{s,\ell}}.$$

Let $\zeta = \min_{(s,\ell,j)\in S^*\times L\times J} \left\{\overline{p}_{s,\ell}, \overline{\pi}_j, \overline{p}_0 C_j - \overline{\pi}_j\right\}$ and $\Pi_0 = \frac{1}{\zeta-\epsilon}\|\overline{w}\|_{\max}$ (which is well defined by the definition of ϵ). Then, for each $n_k > T^*$,

$$0 \leq \max_{(\ell,j)\in L\times J} \left\{\overline{x}^h_{n_k,0,\ell}, \overline{\theta}^h_{n_k,j}, \overline{\varphi}^h_{n_k,j}\right\} \leq \Pi_0,$$

and for any $s \in S$,

$$\begin{aligned} 0 &\leq \max_{\ell\in L} \overline{x}^h_{n_k,s,\ell} \\ &\leq \Pi_s := \Pi_0 \left(1 + \frac{1}{\zeta-\epsilon} \left\| Y_s \left((1,\ldots,1) + \sum_{j\in J} C_j\right) \right\|_{\max} + \frac{2\overline{A}}{\zeta-\epsilon} \# J \right). \end{aligned}$$

Therefore, for any $h \in [0,1]$, each component of the non-negative sequence $\left\{(\overline{x}^h_{n_k}, \overline{\theta}^h_{n_k}, \overline{\varphi}^h_{n_k})\right\}_{n_k\geq n^*}$ is bounded from above by $\Pi := \max_{s\in S^*} \Pi_s$. Since the upper bound of $\left\{g_{n_k}(h)\right\}_{n_k\geq n^*}$ is independent of $h \in [0,1]$, the family of functions $\left\{g_{n_k}\right\}_{n_k\geq n^*}$ is uniformly integrable (see Hildenbrand 1974, p. 52). □

It follows from Lemma 7 that the sequence of non-negative integrable functions $\left\{g_{n_k}\right\}_{n_k\geq n^*}$ satisfies the assumptions of the strong version of the multidimensional Fatou's Lemma (see Hildenbrand 1974, p. 69). Thus, we can find a full measure set $\widehat{\mathbb{P}} \subset [0,1]$ and an integrable function $\widehat{g} : [0,1] \to \mathbb{R}_+^{L\times S^*} \times \mathbb{R}_+^J \times \mathbb{R}_+^J \times \mathbb{R}_+^{S\times J}$, defined by $\widehat{g}(h) := (\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h, (\widehat{\rho}^h_{s,j})_{(s,j)\in S\times J})$ such that, for each agent $h \in \widehat{\mathbb{P}}$, there is a subsequence of $\left\{g_{n_k}(h)\right\}_{n_k\geq n^*}$ that converges to $\widehat{g}(h)$ and

$$\int_{[0,1]} \widehat{g}(h)\,\mathrm{d}h = \lim_{k\to\infty} \int_{[0,1]} g_{n_k}(h)\,\mathrm{d}h.$$[21]

In addition, for any state of nature $s \in S$, there exists $(\widehat{\beta}^h_{s,j}; j \in J) \in \mathcal{R}_s$ such that, for any $j \in J$, $\widehat{\rho}^h_{s,j} = \widehat{\beta}^h_{s,j}\,\widehat{\varphi}^h_j$ and $\sum_{j\in J} \widehat{\beta}^h_{s,j}\left[\overline{p}_s A_{s,j} - \overline{p}_s Y_s(C_j)\right]^+ \widehat{\varphi}^h_j = \min\left\{\Psi_s(\overline{p}_s, \widehat{\varphi}^h), \Phi_s(\overline{p}_s, \overline{R}_s, w^h_s, \widehat{x}^h_0, \widehat{\theta}^h, \widehat{\varphi}^h)\right\}$ (see footnote 20).

Therefore, it follows from the definition of $\widehat{g}$ that market clearing conditions of Definition (1) hold for the allocation $((\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h); h \in [0,1])$. Moreover, analogous arguments to those made at Lemma 5 ensure that, for any $h \in \widehat{\mathbb{P}}$, $(\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h)$ is an optimal allocation for agent h in $B^h(\overline{p}, \overline{\pi}, \overline{R})$.

Since $\left((\overline{p}_s)_{s\in S^*}, \overline{\pi}, (\overline{p}_0 C_j - \overline{\pi}_j)_{j\in J}\right) \gg 0$, each agent $h \in [0,1]$ has a compact budget set $B^h(\overline{p}, \overline{\pi}, \overline{R})$. Continuity of utility functions [Assumption (A1)] ensures that any agent $h \in [0,1] \setminus \widehat{\mathbb{P}}$ has an optimal allocation $(\breve{x}^h, \breve{\theta}^h, \breve{\varphi}^h) \in B^h(\overline{p}, \overline{\pi}, \overline{R})$. Thus, if we give to each $h \in [0,1] \setminus \widehat{\mathbb{P}}$ the allocation $(\breve{x}^h, \breve{\theta}^h, \breve{\varphi}^h)$ instead of $(\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h)$, we ensure that all consumers maximize their utility functions without changing the validity of market clearing conditions (because $[0,1] \setminus \widehat{\mathbb{P}}$ has zero measure).

Therefore, $\left((\overline{p}, \overline{\pi}, \overline{R}); ((\widehat{x}^h, \widehat{\theta}^h, \widehat{\varphi}^h); h \in \widehat{\mathbb{P}}); ((\breve{x}^h, \breve{\theta}^h, \breve{\varphi}^h); h \in [0,1] \setminus \widehat{\mathbb{P}})\right)$ is an equilibrium of $\mathcal{E}$. This concludes the proof of equilibrium existence in our economy.

Appendix B: Proof of Theorem 2

Since as now commodities can be perishable, to find an equilibrium with non-trivial security payments the key is to guarantee that, for each asset $j \in J$, there is $s \in S$ such that $N_{s,j} > 0$.[22] For this reason, we suppose that garnishable resources are bounded away from zero by a percentage of individuals' wealth [Assumption (B4)]. Actually, as initial endowments have a positive lower bound [Assumption (B1)], in case of bankruptcy the quantity of garnished resources is bounded away from zero, independent of the identity of the borrower. Thus, as claims are reimbursed in proportion to their sizes [Assumption (B2)], in case of default on a debt contract j at state of nature s, the mean payments that associated investors receive over the value of the collateral guarantees, $N_{s,j}$, are bounded away from zero too. The existence of these lower bounds on variables $(N_{s,j}; (s,j) \in S \times J)$ allows us to adapt the arguments made in the proof of Theorem 1, in order to ensure equilibrium existence without Assumption (A6).

We follow the same structure and notations used in the proof of Theorem 1. Thus, we consider a generalized game $\widehat{\mathcal{G}}_n$, which is obtained from $\mathcal{G}_n$ by changing two characteristics: the set where delivery rates $(\beta^h_{n,s,j}; (s,j) \in S \times J)$ are chosen, and the set of admissible payments $(N_{s,j}; (s,j) \in S \times J)$. More precisely, given

[21] Note that functions g and $\widehat{g}$ do not need to coincide.

[22] Note that debt contracts are only required to make non-trivial promises in at least one state of nature [Assumption (B3)] and, therefore, we cannot ensure that borrowers make payments over the minimum between the collateral value and the promise in more than one state of nature.

prices $(p_0, \pi) \in \Delta_0$, for each $(h, s, p_s) \in [0, 1] \times S \times \Delta_1$, we restrict delivery rates $(\beta^h_{n,s,j}; j \in J)$ to belong to $\mathcal{R}_s \cap [b_s(p_0, p_s, \pi), 1]^J$, where $\mathcal{R}_s$ satisfies Assumption (B2) and

$$b_s(p_0, p_s, \pi) = \min\left\{\max\left\{(1-\kappa_s)p_s\underline{w}\frac{\min_{j\in J}(p_0C_j - \pi_j)}{\#J\;\overline{A}\;\|\overline{w}\|_\Sigma}; \; 0\right\}; \; 1\right\},$$

where $\underline{w}$ and κ_s satisfy Assumptions (B1) and (B4), respectively.

Moreover, for any $(s, j) \in S \times J$, we restrict $N_{s,j}$ to belong to the set $[[p_sA_{s,j} - p_sY_s(C_j)]^+ b_s(p_0, p_s, \pi), \overline{A}]$.

This characterization of the generalized game $\widehat{\mathcal{G}}_n$ is well defined when Assumptions (A1)–(A4), (A7), and (B1)–(B4) hold. In addition, under these hypotheses, the following properties hold,

(a) *For each $n > n^*$, the generalized game $\widehat{\mathcal{G}}_n$ has a Cournot–Nash equilibrium.*

Proof Since $(b_s; s \in S)$ are continuous functions, the correspondence $\hat{\Gamma}^h_n : \mathbb{V} \twoheadrightarrow \mathbb{E}_n \times \prod_{s\in S}\mathcal{R}_s$ defined by $\hat{\Gamma}^h_n(p, \pi, N) = B^h_n(p, \pi, N) \times \prod_{s\in S}(\mathcal{R}_s \cap [b_s(p_0, p_s, \pi), 1]^J)$ is continuous and has non-empty and compact values. By analogous arguments to those made in Lemma 1, we ensure the existence of a Cournot–Nash equilibrium for $\widehat{\mathcal{G}}_n$. □

(b) *For any Cournot–Nash equilibrium of $\widehat{\mathcal{G}}_n$,*

$$\overline{N}^n_{s,j}\int_{[0,1]}\overline{\varphi}^h_{n,j}\,\mathrm{d}h = [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+\int_{[0,1]}\overline{\beta}^h_{n,s,j}\overline{\varphi}^h_{n,j}\,\mathrm{d}h.$$

Proof It follows from Lemma 2 that

$$\overline{N}^n_{s,j}\int_{[0,1]}\overline{\varphi}^h_{n,j}\,\mathrm{d}h \geq [\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+\int_{[0,1]}\overline{\beta}^h_{n,s,j}\overline{\varphi}^h_{n,j}\,\mathrm{d}h.$$

If the inequality is strict, we have that $\overline{N}^n_{s,j} > b_s(\overline{p}^n_0, \overline{p}^n_s, \overline{\pi}^n)[\overline{p}^n_s A_{s,j} - \overline{p}^n_s Y_s(C_j)]^+$. Thus, in the generalized game $\widehat{\mathcal{G}}_n$, player $c_{s,j}$ can improve the value of his objective function by decreasing the value of $\overline{N}^n_{s,j}$, a contradiction with the definition of Cournot–Nash equilibrium. □

(c) *In any equilibrium of $\widehat{\mathcal{G}}_n$, and for each $(h, s) \in [0, 1] \times S$, there exists $\widehat{\beta}^h_{n,s} \in [b_s(\overline{p}^n_0, \overline{p}^n_s, \overline{\pi}^n), 1]$ such that, equilibrium delivery rates $(\overline{\beta}^h_{n,s,j}; j \in J)$ satisfy $\widehat{\beta}^h_{n,s} = \overline{\beta}^h_{n,s,j}$, $\forall j \in J$. Moreover,*

$$\widehat{\beta}^h_{n,s}\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n) = \min\{\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n); \Phi_s(\overline{p}^n_s, \overline{R}^n_s, w^h_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)\},$$

where $\overline{R}^n_s = (\overline{R}^n_{s,j}; j \in J)$ and $\overline{R}^n_{s,j} = D_{s,j}(\overline{p}^n_s) + \overline{N}_{s,j}$.

Proof The existence of $(\widehat{\beta}^h_{n,s}; (h,s) \in [0,1] \times S)$ is a direct consequence of the definition of $(\mathcal{R}_s; s \in S)$ [Assumption (B2)]. On the other hand, it follows from the definition of the objective function of player $h \in [0,1]$ in the generalized game $\widehat{\mathcal{G}}_n$ that, if there is $s \in S$ such that

$$\widehat{\beta}^h_{n,s} \Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n) \neq \min\{\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n); \Phi_s(\overline{p}^n_s, \overline{R}^n_s, w^h_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)\},$$

then $\Phi_s(\overline{p}^n_s, \overline{R}^n_s, w^h_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n) < \Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n)$ and

$$b_s(\overline{p}^n_0, \overline{p}^n_s, \overline{\pi}^n) = \widehat{\beta}^h_{n,s} > \frac{\Phi_s(\overline{p}^n_s, \overline{R}^n_s, w^h_s, \overline{x}^h_{n,0}, \overline{\theta}^h_n, \overline{\varphi}^h_n)}{\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n)} \geq (1-\kappa_s)\frac{\overline{p}^n_s \underline{w}}{\overline{A}\sum_{j \in J} \overline{\varphi}^h_{n,j}},$$

where the last inequality follows from Assumptions (B1) and (B4), jointly with the definition of $\Psi_s(\overline{p}^n_s, \overline{\varphi}^h_n)$. Thus, $b_s(\overline{p}^n_0, \overline{p}^n_s, \overline{\pi}^n) > 0$, which implies that, for any $j \in J$, $\overline{p}^n_0 C_j - \overline{\pi}^n_j > 0$. From first period budget constraints, we obtain that $\sum_{j \in J} \overline{\varphi}^h_{n,j} \leq \frac{\#J\ \|\overline{w}\|_\Sigma}{\min_{k \in J}(\overline{p}^n_0 C_k - \overline{\pi}^n_k)}$. Therefore, $b_s(\overline{p}^n_0, \overline{p}^n_s, \overline{\pi}^n) > b_s(\overline{p}^n_0, \overline{p}^n_s, \overline{\pi}^n)$, a contradiction. □

As a consequence of properties (a)–(c), for any $n > n^*$, Step 1 of Lemma 3 holds. Thus, we can apply the weak version of the multidimensional Fatou's Lemma to obtain a cluster point of a sequence of Cournot–Nash equilibria. By the same arguments made in Lemma 5, for a generic set of consumers, individual allocations in the cluster point are optimal at limit prices $(\overline{p}, \overline{\pi}, \overline{R})$. Thus, by Assumption (A1), $(\overline{p}_{s,\ell}; (s,\ell) \in S^* \times L)$ are strictly positive. Moreover, as a consequence of Assumption (A4) and (A7), the same arguments of Lemma 6 can be applied to prove that $(\overline{p}_0 C_j - \overline{\pi}_j; j \in J) \gg 0$.

(d) *Even without Assumption (A6), asset prices $(\overline{\pi}_j; j \in J)$ are strictly positive.*

Proof Given an asset $j \in J$, if $D_{s,j}(\overline{p}_s) > 0$ for some $s \in S$, then $\overline{R}_{s,j} > 0$ and the monotonicity of preferences ensures that $\overline{\pi}_j > 0$. Alternatively, when $D_{s,j}(\overline{p}_s) = 0$ for any $s \in S$, it follows from Assumption (B3) that, there is $s_j \in S$ such that $[\overline{p}_{s_j} A_{s_j,j} - \overline{p}_{s_j} Y_{s_j}(C_j)]^+ > 0$. On the other hand, as $(\overline{p}_0 C_j - \overline{\pi}_j; j \in J) \gg 0$ and $(\overline{p}_s; s \in S^*) \gg 0$, we conclude that, for any $s \in S$, $b_s(\overline{p}_0, \overline{p}_s, \overline{\pi}) > 0$. Thus, at the state of nature s_j, the lower bound of the set of admissible payments $N_{s_j,j}$ is strictly positive, which implies that $\overline{N}_{s_j,j} > 0$. Then, $\overline{R}_{s_j,j} > 0$, which ensures the strict positivity of π_j. □

Since $(\overline{p}, \overline{\pi}, (\overline{p}_0 C_j - \overline{\pi}_j; j \in J)) \gg 0$, we can apply the same arguments made in Lemma 7 to bound individual allocations in a sequence of Cournot–Nash equilibria of $\widehat{\mathcal{G}}_n$. Thus, using the strong version of Fatou's Lemma, we obtain a cluster point of the sequence of Cournot–Nash equilibria in which: (i) prices are strictly positive; (ii) allocations are optimal for a generic set of agents; (iii) for these agents, budget constraints hold with equality; and (iv) there is no excess demand in physical or financial markets (a property that any element in the sequence of Cournot–Nash equilibria satisfies). Thus, market clearing conditions hold. Finally, with analogous arguments to those given at the end of the proof of Theorem 1, we conclude that there exists an equilibrium under the conditions of Theorem 2.

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