Harmonic Analysis of Radon Filtrations for S_n and $GL_n(q)$

M. Francisca Yáñez*

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Abstract

We present a unified approach to the study of Radon transforms related to symmetric groups and to general linear groups $GL_n(q)$, regarded as $q-$analogues of the former. In both cases, we define a sequence of generalized Radon transforms which are intertwining operators for natural representations associated to Gel'fand spaces for our groups. This sequence enables us to decompose in a recursive way these natural representations and to compute explicitly the associated spherical functions. Our methods and results are related by $q-$analogy.

Keywords: Radon Transform, Spherical Function, q-analogue, Gelfand Space, Natural Representation

1 Introduction

In 1917 Radon [7] showed that a function on Euclidean space could be recovered from its integrals over affine hyperplanes, solving in this way the inversion problem for the nowadays called "Radon Transform" which associates to a point function f the hyperplane function $Rf : H \mapsto \int_H f$. Later, Gel'fand [2] and Helgason [3, 4] studied the Radon Transform in the more general setting of homogeneous spaces for a Lie group G.

In 1975 Soto-Andrade [8] used finite Radon Transforms associated to the geometry of finite symplectic spaces to study the intertwining algebra of natural representations of the finite similitude symplectic group $GSp(4, q)$ in function spaces on isotropic flags and so obtained by decomposition the principal series representations of $GSp(4, q)$.

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In 1977 Dunkl [1] used finite Radon Transforms to obtain a convenient basis of the intertwining algebra of the natural representation of $GL_n(q)$ associated to the action of this group on the lattice of subspaces in $n-$dimensional space, to decompose this representation and to calculate the corresponding spherical functions.

In 1983 Grinberg [5] independently took up this viewpoint in the real case to analyze the Radon Transform on compact two point homogeneous spaces, considering it as an intertwining operator between representations of the isometry group $G = U(n+1)$.

Later, in 2002, Marco and Parcet [6] took advantage of finite Radon transforms to study the interwining algebra of the natural representation $L^2(\mathcal{P}(\Omega))$ of the symmetric group $S(\Omega)$ on a finite set Ω, associated to the power set $\mathcal{P}(\Omega)$ of Ω and to decompose this representation.

On the other, it should be noticed that Stanton [8] also considered the Gel'fand spaces (S_n, P_s) and $(GL_n(q), V_s)$ and proved that the spherical functions correspond to discrete orthogonal polynomials, without using Radon Transforms.

We describe now the results in this paper, which were already announced in [11]. For G being the symmetric group or its $q-$analogue, the finite general linear group, the Radon transforms that we consider are G -intertwining operators between natural representations $L^2(X)$ associated to the action of G on Gel'fand spaces X (i.e., G -spaces whose associated natural representation is multiplicity free).

We introduce here Gel'fand spaces (S_n, P_s) for the symmetric group S_n and their "q-analogues" $(GL_n(q), V_s)$, for the finite general linear group $GL_n(q)$, where P_s is the set of all subsets of $N = \{1, 2, ...n\}$ with just s elements, and V_s is the set of all s-dimensional vector subspaces of $V = \mathbf{F}_q^n$, where $\mathbf{F}_q$ is the finite field with q elements, where $q = p^n$, p being a prime number.

We define for each positive integer s a "generalized Radon Transform" R_s, between the natural unitary representations $L^2(P_s)$ and $L^2(P_{s+1})$ of S_n. This is done in an analogous way to the construction of the classical Radon transform.

In theorems 1 and 2 we prove, applying the harmonic analysis techniques developped in [6], that the sequence of these operators and the sequence of their adjoints give a "resolution" of the natural representations $L^2(P_s)$ and $L^2(P_{n-s})$, respectively. That is, we show that the subrepresentations $KerR_t^*$ (resp. $KerR_{n-t}$) of S_n give all the irreducible components of $L^2(P_s)$ (resp. $L^2(P_{n-s})$) . Simultaneously we construct the corresponding spherical functions. Furthermore, we prove in theorem 3 that the multiplicity-free natural representations $L^2(P_s)$ and $L^2(P_{n-s})$ are isomorphic via the operator $\mathcal{C}_s^*$ induced by taking complement s.

In Theorem 4 we prove mutatis mutandis for the $q-$analogue $(GL_n(q), V_s)$ of (S_n, P_s), the results corresponding to those obtained in Theorems 1 and 2 for (S_n, P_s). Finally in Theorem 5 we prove the $q-$analogue of Theorem 3. However the $q-$analogue $\mathcal{C}_s^*(q)$ of the operator $\mathcal{C}_s^*$ and the proof are more involved in the case of $GL_n(q)$. Indeed, $\mathcal{C}_s^*(q)$ is given by a summation over all possible supplementaries to a given subspace.

2 The Gel'fand space (S_n, P_s).

Let $G = S_n$ and $N = \{1, 2, ...n\}$. We denote by P_s the set of all subsets of N with just s elements. We consider the natural, transitive, action of S_n on P_s.

We endow (S_n, P_s) with the S_n-invariant distance d defined by $d(X, X') = |X - X'| = |X| - |X \cap X'| = \frac{1}{2}|X \triangle X'|)$.

Proposition 1 *The distance d is an S_n-invariant function on $P_s \times P_s$ which classifies the $G-$orbits in $P_s \times P_s$. The number of orbits is $s+1$.*

Proof: It is enough to prove that (Y, Y') and (Z, Z') belong to the same orbit iff $|Y \cap Y'| = |Z \cap Z'|$. Let $\sigma \in S_n$ such that $\sigma(Y) = Z$ and $\sigma(Y') = Z'$. Since σ is bijective, we have $\sigma(Y \cap Y') = Z \cap Z'$. On the other side, if $|Y \cap Y'| = |Z \cap Z'|$ then we can find a biyection $\sigma : N \to N$ such that

$$\sigma(Y-(Y\cap Y')) = Z-(Z\cap Z'); \quad \sigma(Y\cap Y') = Z\cap Z'; \quad \sigma(Y'-(Y\cap Y')) = Z'-(Z\cap Z')$$

and $\sigma(N - (Y \cap Y')) = N - (Z \cap Z')$.

Then $\sigma(Y) = Z$ and $\sigma(Y') = Z'$.

$\square$

Definition 1 *Let $\Omega_i(X_0)$ be the set of all elements X of P_s such that $d(X, X_0) = i$.*

Furthermore, for $0 \leq t \leq s \leq n$ we define the pseudo-distance ℓ on $P_t \times P_s$ by: $\ell(Y, X) = |Y - X|$.

Remark 1 *The number of elements of $\Omega_i(X_0)$ is* $\left(\begin{array}{c} s \\ s-i \end{array}\right)\left(\begin{array}{c} n-s \\ i \end{array}\right)$.

Corollary 1 *The natural unitary representation* $(L^2(P_s),\tau)$ *associated to the geometric space* (S_n, P_s) *is a Gel'fand geometric space, i.e., the unitary representation* $(L^2(P_s),\tau)$ *, (where* τ *is defined by* $(\tau_\sigma f)(X) = f(\sigma^{-1}(X))$ *for* $X \in P_s, f \in L^2(P_s), \sigma \in S_n$ *), is multiplicity free. The number of irreducible components of* $L^2(P_s)$ *is* $s+1$.

Our purpose here is to decompose the natural representation $(L^2(P_s),\tau)$ with the help of certain operators which we denote by R_s and R_s^*. Their definition is completely analogous to the classical Radon Transform in the continuous case.

Definition 2 *For* $0 \le s \le n$, $Z \in P_s$ *and* $X \in P_{s+1}$, *let*

a) $R_s : L^2(P_s) \to L^2(P_{s+1})$, *be given by*
$R_s(f)(X) = \sum_{Z \subset X} f(Z);\;\; f \in L^2(P_s)$
and

b) $R_s^* : L^2(P_{s+1}) \to L^2(P_s)$, *be given by*
$R_s^*(f)(Z) = \sum_{Z \subset X} f(X);\;\; f \in L^2(P_{s+1})$.

Remark 2 *A straightforward calculation shows that* R_s *and* R_s^* *are intertwining operators and*

$$R_s^* \circ R_s = (n-s)Id + M_1$$

where $M_k = M_{\Omega_k}$ *is the averaging operator associated to the orbit* Ω_k *of* P_s^2/S_n *defined by*

$$M_k(f)(X) = \sum_{d(X,X')=k} f(X'), \quad (X \in P_s).$$

Proposition 2 *For* $1 \le s \le n$ *, we have*

i) $L^2(P_s) = ImR_{s-1} \perp KerR_{s-1}^*$,

ii) $L^2(P_{n-s}) = ImR_{n-s}^* \perp KerR_{n-s}$.

Proof: By computing the Fourier coefficients of a function $f \in L^2(P_s)$ such that $\langle f, R_{s-1}\delta_Z\rangle\rangle = 0$, where $\delta_Z(Z') = \delta_{Z,Z'}$ we obtain

$$(ImR_{s-1})^\perp = KerR_{s-1}^*.$$

Case ii) is completely analogous .

$\square$

The following Theorem leads us towards the decomposition of the multiplicity-free natural representation $(L^2(P_m), \tau)$ into its irreducible components H^i $(0 \leq i \leq m)$ and gives us the spherical functions Φ_i associated to each H^i.

Theorem 1 *Let* $0 \leq m \leq \frac{n}{2}$. *The sequence:*

$$L^2(P_0) \overset{R_0}{\to} L^2(P_1) \to \cdots L^2(P_{m-1}) \overset{R_{m-1}}{\to} L^2(P_m)$$

is an inductive system of monomorphisms which provides all irreducible components of $L^2(P_m)$ *as successive orthogonal supplements.*

Proof: In order to prove that R_s is injective for $0 \leq s < \frac{n}{2}$, we will apply induction. Let $s = 1$. Since $KerR_0$ is trivial we have the decomposition $L^2(P_1) = H^o \perp H^1$, where H^o is the trivial one dimensional representation and $H^1 = KerR_0^*$ is the Steinberg representation of dimension $n-1$ of S_n. We construct next the spherical functions Φ_0, Φ_1 associated to H^o and H^1 , respectively. As we know, Φ_i will be the spherical function associated to H^i if and only if Φ_i is invariant under the action of the isotropy group K_1 of the origin $X_0 = \{1\}$. In our case this means that the values of Φ_i at $X \in P_1$ depend only on the distance $d(X_0, X)$. Furthermore Φ_i should take the values 1 at X_0 and Φ_i must belong to H^i. These conditions allow us to get:

$$\Phi_0 \equiv 1$$

$$\Phi_1(X) = \begin{cases} 1 & X = X_0 \\ -\frac{1}{n-1} & d(X, X_0) = 1 \end{cases}$$

Next we have to prove that R_1 is an injective operator. This is however equivalent to proving that $R_1^* \circ R_1$ is an automorphism of $L^2(P_1)$. Since $R_1^* \circ R_1 = (n-1)Id + M_1$, it is enough to prove that the eigenvalues of M_1 are different from $-(n-1)$. We recall that the eigenvalues λ_i of M_1 are given by $M_1(\Phi_i)(X_0)$. In this way, we obtain: $\lambda_0 = n-1$ and $\lambda_1 = 1$. Then R_1 is injective.

Let us suppose that R_t is injective for $0 \leq t < s$; $s < \frac{n}{2}$. We have to prove that R_s injective.

Due to the induction hypothesis and using proposition 1 we have:

$$L^2(P_s) = H^0 \oplus H^1 \oplus \cdots \oplus H^s \qquad \text{(orthogonal sum)}$$

where H^0 is the one dimensional trivial representation and

$$H^t = (R_{s-1} \circ \cdots \circ R_t)\,(KerR^*_{t-1}),\ (1 \le t \le s),$$

with dimension $\begin{pmatrix} n \\ t \end{pmatrix} - \begin{pmatrix} n \\ t-1 \end{pmatrix}$.

We have decomposed $L^2(P_s)$ in $(s+1)-$non isomorphic subrepresentations of S_n. Since the number of S_n-orbits in P_s^2/S_n is just $s+1$, we have proved that the subspaces H^t, $0 \le t \le s$, are all the irreducible components of S_n appearing in $L^2(P_s)$. We are now able to construct the spherical functions Φ_t associated to each H^t.

We observe that $f \in H^t$ if and only if there exists a function $h_t \in KerR^*_{t-1}$ such that $f(X) = \sum_{Y\in P_t, Y\subset X} h_t(Y)$, $X \in P_s$. In order to obtain the spherical function $\Phi_t \in H^t$, we need to define a function h_t on P_t such that $\sum_{Y\in P_t, Z\subset Y} h_t(Y) = 0$ for each $Z \in P_{t-1}$ and

$$\Phi_t(X) = \sum_{Y\subset X} h_t(Y) = \sum_{k=k_0}^{min(j,t)} \left(\sum_{Y\subset X, \ell(Y,X_0)=k} h_t(Y) \right),$$

where $X_0 = \{1, 2, \cdots, s\}$, $d(X_0, X) = j$ and $\ell(Y, X_0) = k_0$ where $k_0 = 0$ if $j \le s-t$ or $k_0 = t+j-s$ if $j \ge s-t$. The values of Φ_t at $X \in P_s$ must depend only on the distance $d(X_0, X)$. We impose the following conditions to the function h_t :

i) $\sum_{Y\in P_t, Z\subset Y} h_t(Y) = 0$, $(Z \in P^{t-1})$

ii) $h_t(Y) = \alpha_t^k$ if $\ell(Y, X_0) = k$

Now let us suppose that we have found one such function h_t and let us define the following functions Φ_t on P_t by:

$$\Phi_t(X) = \sum_{Y\subset X} h_t(Y) = \sum_{k=k_0}^{min(j,t)} |A_t^{k,j}|\alpha_t^k,$$

where

$$A_t^{k,j} = \{Y \in P_t : \ell(Y, X_0) = k, Y \subset X, d(X, X_0) = j.$$

In order to get an element $Y \in A_t^{k,j}$, we need to choose $t-k$ elements among the $s-j$ elements of $Y \cap X_0$ and k elements among the j elements of $Y - X_0$. Therefore $|A_t^{k,j}| = \begin{pmatrix} s-j \\ t-k \end{pmatrix} \begin{pmatrix} j \\ k \end{pmatrix}$.

Since for all σ belonging to the isotropy group K_t, we have $d(\sigma X, \sigma X_0) = d(X, X_0)$ and $|A_t^{k,j}|$ depends only on the number of elements of X and

Y, we obtain that Φ_t is invariant under the action of K_t. Then, for Φ_t to be a spherical function, we need still only to fulfill the normalization condition $\Phi_t(X_0) = 1$ or equivalently:

iii) $A_t^{0,0}\alpha_t^0 = 1$.

So we obtain the function h_t by solving the following system of linear equations arising from i), ii) and iii)

$$\begin{cases} |C_t^{k,k}|\alpha_t^k + |C_t^{k+1,k}|\alpha_t^{k+1} = 0, & 0 \leq k \leq t-1 \\ \alpha_0 = \begin{pmatrix} s \\ t \end{pmatrix}^{-1}, & \end{cases}$$

where

$$C_t^{k',k} = \{Y \in P_t : Z \subset Y,\ \ell(Y, X_0) = k',\ \ell(Z, X_0) = k\},\ Z \in P_{t-1}.$$

We have $Y = Z \cup \{y\}$. Then if $y \in X_0$ we get $k' = k$, and if $y \notin X_0$ then $k' = k+1$. A combinatorial computation gives us that $|C_t^{k,k}| = \begin{pmatrix} s-(t-1-k) \\ 1 \end{pmatrix}$ and $|C_t^{k+1,k}| = \begin{pmatrix} n-(s+k) \\ 1 \end{pmatrix}$.

Solving this system we get

$$\alpha_t^0 = \begin{pmatrix} s \\ t \end{pmatrix}^{-1} \quad \text{and} \quad \alpha_t^k = \frac{(-1)^k (s-t+k)!(n-s-k)!}{\begin{pmatrix} s \\ t \end{pmatrix}(s-t)!(n-s)!}, \quad 1 \leq k \leq t.$$

Then the spherical function Φ_t associated to H^t is given by

$$\Phi_t(X) = \sum_{k=k_0}^{min(j,t)} (-1)^k \begin{pmatrix} s-j \\ t-k \end{pmatrix} \begin{pmatrix} j \\ k \end{pmatrix} \gamma_t^k$$

where $j = d(X_0, X)$ and $\gamma_t^k = \begin{pmatrix} s \\ t \end{pmatrix}^{-1} \frac{(s-t+k)!(n-s-k)!}{(s-t)!(n-s)!}$.

We have to compute now the eigenvalues of M_1 with respect to our decomposition of $L^2(P_s)$. As we know, the eigenvalues λ_t associated with H^t are given by $\lambda_t = M_1(\Phi_t)(X_0)$. Then we have:

$$\lambda_t = (n-s)(s-t) - (s-t+1)t, \quad 0 \leq t \leq s$$

We want to show that $\lambda_t \neq -(n-s)$. Let us suppose that for some t the eigenvalue $\lambda_t = -(n-s)$. By replacing $\lambda_t = -(n-s)$ in the last equation and dividing by $s-t+1 \neq 0$, we obtain that $t+s=n$. But since by hypothesis $t+s<n$, we conclude that $\lambda_t \neq -(n-s)$; $0 \leq t \leq s$ and $R_s^* \circ R_s$ is an automorphism. Then R_s is injective for all $0 \leq s < m$ and

$$L^2(P_m) \cong (ImR_0) \oplus (ImR_0)^{\perp} \oplus \cdots \oplus (ImR_{m-1})^{\perp} \quad (orthogonal\ sum)$$

In this way, for each $0 \leq m \leq \frac{n}{2}$, we have constructed all the irreducible components for the natural representation $L^2(P_m)$ of S_n and the associated spherical functions.

$\square$

In order to prove that for $0 \leq m \leq \frac{n}{2}$

$$L^2(N) \overset{R^*_{n-1}}{\to} L^2(P_{n-1}) \to \cdots \overset{R^*_{n-m}}{\to} L^2(P_{n-m}),$$

is a resolution for the Gel'fand space $L^2(P_{n-m})$, we consider the following preliminaries

Proposition 3 *Let us define the S_n-invariant application $\mathcal{C}_s$ from P_s to P_{n-s}, $(0 \leq s \leq \frac{n}{2})$ by $\mathcal{C}_s(X) = N - X$. The following combinatorial properties of $\mathcal{C}_s$ hold:*

Let $Y \in P_t$, $X_0, X \in P_s$ and $Z \in P_{t-1}$

1. *If $\ell(Y, X_0) = k$ then $\ell(\mathcal{C}_s(X_0), \mathcal{C}_s(Y)) = k$*
2. *Let $B_t^{k,j} = \{Y' \in P_{n-t} : Y' \supset X',\ d(X', X_0') = j,\ \ell(X_0', Y') = k\}$ where $X', X_0' \in P_{n-s}$, then $|A_t^{k,j}| = |B_t^{k,j}|$.*
3. *Let $D_t^{k',k} = \{Y' \in P_{n-t} : Y' \subset Z',\ \ell(X_0', Z') = k,\ \ell(X_0', Y') = k'\}$ where $Z' \in P_{n-(t-1)}$, $X_0' \in P_{n-s})$, then $|C_t^{k',k}| = |D_t^{k',k}|$, $k' = k, k+1$.*
4. *The map $\mathcal{C}_s$ is an S_n-invariant bijection and $\mathcal{C}_s^{-1} = \mathcal{C}_{n-s}$.*

Proof: Since $(N-X_0)-(N-Y)=Y-X_0$ we get the first statement. In connection with the second statement, we note that, in order to find $Y' \in B_t^{k,j}$, we need to choose $j-k$ elements of the j elements of $X_0' - X'$ and we need to select the $(n-t)-(n-s+j-k)$ remaining elements of the $n-(n-s+j)$ elements of $N-(X_0' \cup (X'-X_0'))$. Then $|B_t^{k,j}| = \begin{pmatrix} s-j \\ s-j-(t-k) \end{pmatrix} \begin{pmatrix} j \\ j-k \end{pmatrix} = |A_t^{k,j}|$.

Finally, if $Y' \in D_t^{k',k}$ then $Y' = Z' - \{z\}$ and $(X_0' - Y') = (X_0' - Z') \cup (X_0' \cap \{z\})$. Now, if $z \in X_0'$ then $|X_0' - Y'| = k+1$ and if $z \notin X_0'$ then $|X_0' - Y'| = |X_0' - Z'| = k$. Therefore $|D_t^{k,k}| = n-(t-1)-(n-s-j) = s-t+j+1$ and $|D_t^{k+1,k}| = (n-s-j)$.

□

Using these properties and following the same procedure employed in the proof of Theorem 1 we get the following Theorem:

Theorem 2 *Let* $0 \leq s \leq \frac{n}{2}$

1. R^*_{n-s} *is an injective intertwining operator.*
2. $L^2(P_{n-s}) \cong ImR^*_{n-1} \oplus KerR_{n-1} \oplus \cdots \oplus KerR_{n-s}$. *(orthogonal sum)*
3. *The spherical function* φ_t *associated to each irreducible sub-representation* $L^t \cong KerR_{n-t}$ *of* $(L^2(P_{n-s}(N), S_n)$ *is given by* $\varphi_t(X') = \Phi_t(\mathcal{C}_{n-s}(X'),\ X' \in P_{n-s}$.

□

Theorem 3 *Let* $0 \leq s \leq \frac{n}{2}$ $L^2(P_s) \cong L^2(P_{n-s})$ *and* $KerR_{n-s} \cong KerR^*_{s-1}$.

Proof: We define $\mathcal{C}_s^* : L^2(P_{n-s}) \to L^2(P_s)$, $0 \leq s \leq \frac{n}{2}$ by $\mathcal{C}_s^*(f) = f \circ \mathcal{C}_s$, $(f \in L^2(P_{n-s})$. Let $h \in KerR_{n-s}$ and $Z \in P_{s-1}$ then

$$\sum_{Z \subset Y} \mathcal{C}_s^*(h)(Y) = \sum_{Z \subset Y} h(N-Y) = \sum_{Y' \subset N-Z} h(Y') = 0.$$

Therefore $\mathcal{C}_s^*(KerR_{n-s}) = KerR^*_{s-1}$. From this and from Theorems 1 and 2 we get this Theorem.

□

3 The q-analogue Gel'fand space $(GL_n(q), V_m)$.

Let $V = \mathbf{F}_q^n$ and V_s the set of all $s-$dimensional subspaces of V, $0 \leq s \leq n$. We fix a basis $\{e_1, e_2, ..., e_n\}$ of V and let $W_0 \in V_s$, be the subspace of V generated by $\{e_1, e_2, ..., e_s\}$ and $W_0' \in V_{n-s}$ the subspace generated by $\{e_{s+1}, ..., e_n\}$.

We construct a resolution for the Gel'fand space $L^2(V_m, GL_n(q))$, through the generalized Radon transform following the same procedure used for $(L^2(P_m), S_n), 0 \leq m \leq \frac{n}{2}$. The combinatorial calculations are obtained by taking the q-analogue of $\begin{pmatrix} n \\ m \end{pmatrix}$.

Definition 3 *1. We define the q-analogue of* $n!$ *by* $n!_q = n_q(n-1)_q \cdots 1_q$ *where* $n_q = \frac{q^n-1}{q-1}$ *and the q-analogue of* $\begin{pmatrix} n \\ m \end{pmatrix}$ *by*

$$\begin{pmatrix} n \\ m \end{pmatrix}_q = \frac{n!_q}{(n-s)!_q s!_q}$$

2. Let $(0 \leq s \leq m)$. *We define the q-analogue pseudo distances* ℓ_q *on* $V_s \times V_m$, *by* $\ell_q(U, W) = dimU - dim(U \cap W)$, $(U \in V_s, W \in V_m)$.

Remark 3 *1. The gaussian binomial coefficient* $\begin{pmatrix} n \\ s \end{pmatrix}_q$ *gives the number of* $s-$*dimensional subspaces of* V.

2. If $s = m$ *then* $\ell_q = d_q$ *defines a distance on* $V_m \times V_m$ *which classifies the orbits of the action on* $V_m \times V_m$ *of* $GL_n(q)$.

3. The natural unitary representation $(L^2(V_s), \tau)$ *associated to the geometric space* $(GL_n(q), V_s)$ *is a Gel'fand geometric space, i.e., the unitary representation* $(L^2(V_s), \tau)$ *, (where* τ *is defined by* $(\tau_g f)(W) = f(g^{-1}(W))$ *for* $W \in V_s, f \in L^2(V_s), g \in GL_n(q)$ *), is multiplicity free. The number of irreducible components of* $L^2(V_s)$ *is* $s+1$.

Let us consider the following Lemma.

Lemma 1 *Let* $W \in V_s$. *The number of* $l-$*dimensional subspaces* Z *of* V *such that the dimension of* $Z \cap W$ *is* t *is given by:*

$$q^{(s-t)(l-t)} \begin{pmatrix} n-s \\ l-t \end{pmatrix}_q \begin{pmatrix} s \\ t \end{pmatrix}_q .$$

Proof:

Consider a $t-$dimensional subspace U of W. There are $\left(\begin{array}{c} s \\ t \end{array}\right)_q$ such subspaces. Let us choose the basis $\{u_1,..,u_t,w_1,..,w_{s-t},v_1,..,v_{n-s}\}$ of V, where $U=<u_1,..,u_t>$ and $W=U\oplus<w_1,..,w_{s-t}>$. In order to construct $l-$dimensional subspaces Z so that $U=Z\cap W$, we need to add $l-t$ linearly independent vectors, $z_1,..,z_{l-t}$, to $\{u_1,..,u_t\}$ satisfying $dim(<z_1,..,z_{l-t}>\cap W)=0$.

Let $z_i=\sum_{k=1}^{t}\alpha_k^i u_k+\sum_{k=1}^{s-t}\beta_k^i w_k+\sum_{k=1}^{n-s}\gamma_k^i v_k$. Since $z_i\notin<w_1,..,w_{s-t}>$, then $(\alpha_1^i,..,\alpha_t^i,\gamma_1^i,..,\gamma_{n-s}^i)\neq\vec{0}, 1\le i\le(l-t)$. Moreover, if we redefine $z_i'=z_i-\sum_{k=1}^{t}\alpha_k^i u_k$ it is verified that $<u_1,..,u_t,z_1,..,z_{l-t}>=<u_1,..,u_t,z_1',..,z_{l-t}'>$. Thus there are

$$\frac{q^{s-t}(q^{n-s}-1)q^{s-t}(q^{n-s}-q)\cdots,q^{s-t}(q^{n-s}-q^{l-t-1})}{(q^{l-t}-1)\cdots(q^{l-t}-q^{l-t-1})}$$

ways to complete U, from which the lemma follows.

$\square$

Corollary 2 *Let $W\in V_s$, $W=W_1\oplus W_2$. The number of $k-$dimensional subspaces Z of W such that $\ell_q(Z,W_1)=k$ is given by*

$$q^{(dimW_1)k}\left(\begin{array}{c} dimW_2 \\ k \end{array}\right)_q .$$

Definition 4 *Next we consider the following sets, (which are q-analogue to the sets $\Omega_i(X_0)$, $A_t^{k,j}$, $C_t^{k,k'}$, $B_t^{k,j}$, $D_t^{k,k'}$ defined in the above section).*

1. $\Omega_i^q(W_0)=\{W\in V_s|d_q(W,W_0)=i\}$
2. $A(q)_t^{k,j}=\{U\in V_t|\ell_q(U,W_0)=k,\ U\subset W,\ d_q(W,W_0)=j\}$
   where $W\in V_s$.
3. $C(q)_t^{k',k}=\{U\in V_t|Z<U,\ell_q(U,W_0)=k',\ \ell_q(Z,W_0)=k\}$
   where $Z\in V_{t-1}$ *and* $k'=k,\ k+1$
4. $B(q)_t^{k,j}=|\{U'\in V_{n-t}|W'<U',d_q(W',W_0')=j,\ell_q(W_0',U')=k\}$
   where $W'\in V_{n-s}$.

5. $D(q)_t^{k',k} = \{U' \in V_{n-t} | U' < Z', \ell_q(W_0', Z') = k, \ell_q(W_0', U') = k'\}$ *where* $Z' \in V^{n-(t+1)}$ *and* $k' = k+1,\ k.$

The following proposition gives us the number of elements of the $q-$analogue sets defined above, getting similar results to those contained in proposition 3.

Proposition 4 *1.* $|\Omega_j^q(W_0)| = \left(\begin{array}{c} s \\ s-j \end{array}\right)_q \left(\begin{array}{c} n-s \\ j \end{array}\right)_q q^{j^2}$

2. $|A(q)_t^{k,j}| = \left(\begin{array}{c} s-j \\ t-k \end{array}\right)_q \left(\begin{array}{c} j \\ k \end{array}\right)_q q^{((s-j)-(t-k))k}$

3. $|C(q)_t^{k,k}| = \left(\begin{array}{c} s-(t-1-k) \\ 1 \end{array}\right)_q$

4. $|C(q)_t^{k+1,k}| = \left(\begin{array}{c} n-(s+k) \\ 1 \end{array}\right)_q q^{s-(t-(k+1))}$

5. If $0 \le s \le n/2$, *then* $|A(q)_t^{k,j}| = |B(q)_k^j|$, $|D(q)_t^{k,k}| = |C(q)_t^{k,k}|$ *and* $|D(q)_t^{k+1,k}| = |C(q)_t^{k+1,k}|$.

Proof: Let $V = W_0 \oplus W_0'$. We have that $dim(W \cap W_0) = s-j$. Applying the Lemma, for $l = s$ and $t = s-j$, we get the first item.

If $U \in A(q)_t^{k,j}$ then $U \cap W_0 < W \cap W_0$. Let X be a $k-$dimensional subspace of $W \cap W_0$. In order to get a subspace U of $A(q)_t^{k,j}$, we have to complete each subspace $X = U \cap W_0$ with a supplement X' so that $X' < W$ and $X' \cap W_0 = \{\vec{0}\}$. Let $W \cap W_0 = X \oplus Y$, then $W = X \oplus Y \oplus Z$. So, we have that X' is a subspace of $Y \oplus Z$, such that $X' \cap Y = \{\vec{0}\}$. Applying the above lemma we get the second item.

To prove item 3, we remark that in order to construct a subspace $U \in C(q)_t^{k,k}$ we need to complete Z with a one dimensional subspace $< v >$ such that $v \in W_0$ but $v \notin (Z \cap W_0)$. Let $W_0 = (Z \cap W_0) \oplus Y$. Since $v \notin Z$, in order to complete Z. it is enough to consider all the one dimensional subspaces $< v >$ of Y. From this item 3 follows.

To prove item 4 let $Z = (Z \cap W_0) \oplus X$, and $W_0 = (Z \cap W_0) \oplus Y$. Since $X \cap W_0 = \{\vec{0}\}$ we have that $V = Z \oplus Y \oplus Y'$. To get U we need that

$v \notin (Z \oplus Y)$. Moreover, since $Z < U$ it is enough that $v = y + y'$ where $y \in Y$, $y' \in Y'$ and $y' \neq \vec{0}$. Then item 4 follows.

analogous ly, we calculate that:

$$|B(q)_t^{k,j}| = \left(\begin{array}{c} s-j \\ s-j-(t-k) \end{array} \right)_q \left(\begin{array}{c} j \\ j-k \end{array} \right)_q q^{((s-j)-(t-k))k},$$

$$|D(q)_t^{k,k}| = \left(\begin{array}{c} s-(t-1-k) \\ s-t-k \end{array} \right)_q$$

and

$$|D(q)_t^{k+1,k}| = \left(\begin{array}{c} n-(s+k) \\ 1 \end{array} \right)_q q^{s-(t-(k+1))}.$$

getting item 5.

$\square$

2.1.3. q-Radon Transforms $R(q)_s$ and $R(q)_s^*$.

We define $R(q)_s$ and $R(q)_s^*$ in an analogous way to R_s and R_s^*. We replace P_s by V_s and $X \subset Y$ by $U < W$ (subspace of W). As we can expect, similar properties to those contained in Propositions 1 and 2 are satisfied for $R(q)_s$ and $R(q)_s^*$. Following the same procedure used to prove Theorem 1 and using Proposition 4 and Corollary 2 we prove the following Theorem:

Theorem 4 *Let* $0 \leq s \leq n/2$;

a) If $H^t(q) = (R(q)_{s-1} \circ \cdots \circ R(q)_t)(KerR(q)_{t-1}^*$ *for* $1 \leq t \leq s$ *then*

$$L^2(V_s) \cong ImR(q)_0 \oplus KerR(q)_0^* \oplus \cdots \oplus KerR(q)_{s-1}^* \quad (\textit{orthogonal sum})$$

and the spherical funtion $\Phi(q)_t$ *asociated with* $H^t(q) \cong KerR(q)_{t-1}^*$ *is given by*

$$\Phi(q)_t(W) = \sum_{k=k_0}^{min(t,j)} (-1)^k q^{\frac{k}{2}(k-1-2j)} \left(\begin{array}{c} s-j \\ t-k \end{array} \right)_q \left(\begin{array}{c} j \\ k \end{array} \right)_q \gamma(q)_t^k$$

where $W \in V_s, d(W_0, W) = j, \quad \gamma(q)_t^k = \begin{pmatrix} s \\ t \end{pmatrix}_q^{-1} \frac{(s-t+k)!_q (n-s-k)!_q}{(s-t)!_q (n-s)!_q}$
and $k_0 = 0$ *if* $t \leq (s-j)$ *and* $k_0 = t + j - s$ *if* $t \geq (s-j)$.

b) If $H^{n-t}(q) = (R(q)^*_{n-s} \circ \cdots \circ R(q)^*_{n-t-1})(KerR(q)_{n-t})$ *for* $1 \leq t \leq s$ *then*

$$L^2(V_{n-s}) \cong ImR(q)^*_{n-1} \oplus KerR(q)_{n-1} \oplus \cdots \oplus KerR(q)_{n-s},$$

and the spherical function $\varphi(q)_t$ *asociated with* $H^{n-t}(q) \cong KerR(q)_{n-t}$ *, is given by*

$$\varphi(q)_t(W') = \Phi(q)_t(W)$$

where: $W \in V^s$ *and* $d_q(W', W'_0) = d_q(W, W_0) = j$.

Proof: Following the same procedure used in theorem 1 to get that if R_t is injective for $1 \leq t < s$ then $L^2(P_s) = H^0 \oplus H^1 \oplus ... \oplus H^s$ we obtain for $0 \leq s < \frac{n}{2}$ that

$$L^2(V_s) = H^0(q) \oplus H^1(q) \oplus \cdots \oplus H^s(q) \qquad \text{(orthogonal sum)}$$

where $H^0(q)$ is the one dimensional trivial representation and $H^t(q) = (R(q)_{s-1} \circ \cdots \circ R(q)_t)\ (KerR(q)^*_{t-1}),\ (1 \leq t \leq s),$ with dimension $\begin{pmatrix} n \\ t \end{pmatrix}_q - \begin{pmatrix} n \\ t-1 \end{pmatrix}_q$.

As in theorem 1, these subspaces afford all the irreducible components of the multiplicity-free natural representation of $GL_n(q)$ in $L^2(V_s)$, because they have different dimensions and their number equals the number of orbits of $GL_n(q)$ in $V_s \times V_s$.

Next we proceed to construct the spherical function $\Phi(q)_t$ associated to $H^t(q)$. We have to find a set of functions $h(q)_t \in KerR(q)^*_{t-1}$ such that

$$\Phi(q)_t(W) = \sum_{Y<W} h(q)_t(Y) = \sum_{k=k_0}^{min(j,t)} \left(\sum_{Y<W, \ell_q(Y,W_0)=k} h(q)_t(Y) \right)$$

where $d_q(W_0, W) = j$ and $\ell_q(Y, W_0) = k_0$ where $k_0 = 0$ if $j \leq s-t$ or $k_0 = t + j - s$ if $j \geq s - t$. We require that the function $h(q)_t$ satisfies the same kind of conditions we imposed on h_t, obtaining the following expression:

$$\Phi(q)_t(W) = \sum_{k=k_0}^{\min(j,t)} |A(q)_t^{k,j}| \alpha(q)_t^k$$

where $\alpha(q)_t^k = h(q)_t(Y)$ if $\ell_q(Y, W_0) = k$, and

$$\alpha(q)_t^0 = \left(\begin{array}{c} s \\ t \end{array} \right)_q^{-1} \quad \text{and} \quad \alpha(q)_t^k = \frac{C(q)_t^{0,0} C(q)_t^{1,1} \cdots C(q)_t^{k-1,k-1}}{C(q)_t^{0,1} C(q)_t^{1,2} \cdots C(q)_t^{k-1,k}} \alpha(q)_t^0.$$

Using the combinatorial results of proposition 4 we get finally the expression for the spherical function $\Phi(q)_t$ mentioned above.

Next we get the eigenvalues $\lambda(q)_t$ of the average operator $M(q)_1^s$ by computing $M(q)_1^s(\Phi(q)_t)(W_0)$

$$\lambda(q)_t = q(n-s)_q \left((s-t)_q - \frac{(s-t+1)_q t_q}{q(n-s)_q} \right), \ 0 \leq t \leq s$$

We need that $\lambda(q)_t \neq -(n-s)_q$. If we compute for which values of t the equality holds, we obtain that $\lambda(q)_t = -(n-s)_q$ if and only if $t = n-s$. Since by hiphotesis $t+s < n$, we conclude that $\lambda(q)_t \neq -(n-s)_q$; $0 \leq t \leq s$ and $R(q)_s^* \circ R(q)_s$ is an automorphism. Then $R(q)_s$ is injective for all $0 \leq s < \frac{n}{2}$ and

$$L^2(V_s) \cong (ImR(q)_0) \oplus (ImR(q)_0)^{\perp} \oplus \cdots \oplus (ImR(q)_{s-1})^{\perp} \quad (\textit{orthogonal sums})$$

In order to prove b) we follow the same procedure used in a). We get:

$$\varphi(q)_t(W') = \sum_{k=k_0}^{\min(j,t)} |B(q)_t^{k,j}| \beta(q)_t^k$$

where

$$\beta(q)_t^0 = \left(\begin{array}{c} s \\ s-t \end{array} \right)_q^{-1} \quad \text{and} \quad \beta(q)_t^k = \frac{D(q)_t^{0,0} D(q)_t^{1,1} \cdots D(q)_t^{k-1,k-1}}{D(q)_t^{0,1} D(q)_t^{1,2} \cdots D(q)_t^{k-1,k}} \beta(q)_t^0$$

and by applying proposition 4 we prove b).

$\square$

The following definition corresponds to the operator $q-$analogue to $\mathcal{C}_s^*$ defined in theorem 3 of the last section.

Definition 5 *Let* $0 \leq s \leq \frac{n}{2}$. *We define* $\mathcal{C}(q)_s^* : L^2(V_{n-s}) \to L^2(V_s)$, *by* $\mathcal{C}(q)_s^*(f)(W) = \sum_{W' \oplus W = V} f(W')$, *for* $f \in L^2(V_{n-s}), W \in V_s$.

Proposition 5 *a)* $\mathcal{C}(q)_s^*$ *is a non trivial intertwining operator between* $(L^2(V_{n-s}), \tau')$ *and* $(L^2(V_s), \tau)$,

b) $\mathcal{C}(q)_s^*(KerR(q)_{n-s}) = KerR(q)_{s-1}^*$,

c) $\mathcal{C}(q)_s^* \circ R(q)_{n-s}^* = q^{((n-s)-(s-1))}(R(q)_{s-1} \circ \mathcal{C}(q)_{s-1}^*)$.

Proof:

a) Let $g \in GL_n(q)$, $f \in L^2(V_{n-s})$ and $W \in V_s$.

We remark that $(\mathcal{C}(q)_s^* \circ \tau_g')(f)(W) = \sum_{W' \oplus W = V} f(g^{-1}W')$ may be expressed as

$$\sum_{W' \oplus W = V} f(g^{-1}W') = \sum_{gW' \oplus W = V} f(W').$$

It is easy to prove that the sets $A = \{W' \in V_S : gW' \oplus W = V\}$ and $B = \{W' \in V_s : W' \oplus g^{-1}W = V\}$ are equal. If g is an automorphism we have that

$$\sum_{W' \oplus g^{-1}W = V} f(W') = \sum_{gW' \oplus W = V} f(W'),$$

from which we obtain a).

b) Let $h \in (KerR(q)_{n-s}), Z \in V_{s-1}$. We have to prove that for each $Z \in V_{s-1}$,

$$\sum_{Z < Y} \sum_{Y' \oplus Y = V} h(Y') = 0$$

is fulfilled.

We will prove first that the condition $Z < Y, \quad Y' \oplus Y = V$ is equivalent to the condition $Y' \in V_{n-s}, dim(Y' \cap Z) = 0$. If we let Y' be a subspace of V_{n-s} such that $Y' \oplus Y = V$ for some Y which contains Z then $dim(Y' \cap Z) = 0$.

Conversely, if $Y' \in V_{n-s}$ and $dim(Y' \cap Z) = 0$ then we can find one dimensional subspaces $< w >$ such that $V = Y' \oplus Z \oplus < w >$. So, for each $< w >$ we find $Y = Z \oplus < w >$ such that $Z < Y$ and $Y \oplus Y' = V$.

Moreover, applying Lemma 1 we compute:

$$|Y : Z < Y||Y' : Y \oplus Y' = V| = q^{s(n-s)}(n-(s-1))_q,$$

$$|Y' \in V_{n-s} : dim(Y' \cap Z) = 0| = q^{(s-1)(n-s)} \left(\begin{array}{c} n-(s-1) \\ n-s \end{array} \right)_q,$$

and

$$|Y : Z < Y, \;\; Y \oplus Y' = V| = q^{n-s}.$$

Therefore we get:

$$\sum_{Z<Y} \sum_{Y' \oplus Y = V} h(y) = q^{n-s} \sum_{dim(Y' \cap Z)=0} h(Y')$$

On the other hand,in a similar way we obtain

$$\sum_{Z' \oplus Z = V} \sum_{Y' < Z'} h(Y') = \sum_{dim(Y' \cap Z)=0} |Z' : Y' < Z', Z' \oplus Z = V| h(Y'),$$

and we compute that:

$$|Z' : Z' \oplus Z = V||Y' < Z'| = q^{(s-1)(n-(s-1)))} \left(\begin{array}{c} n-(s-1) \\ n-s \end{array} \right)_q,$$

$$|Y' \in V_(n-s) : dim(Y' \cap Z) = 0| = q^{(s-1)(n-s)} \left(\begin{array}{c} n-(s-1) \\ n-s \end{array} \right)_q,$$

and

$$|Z' : Y' < Z', Z' \oplus Z = V| = q^{(s-1)},$$

obtaining

$$\sum_{Z' \oplus Z = V} \sum_{Y' < Z'} h(Y') = q^{s-1} \sum_{dim(Y' \cap Z) = 0} h(Y').$$

So , for each $Z \in V_{s-1}$ we have that

$$q^{n-s} \sum_{Z' \oplus Z = V} \sum_{Y' < Z'} h(Y') = q^{s-1} \sum_{Z < Y} \sum_{Y' \oplus Y = V} h(Y').$$

But due to the fact that $h \in Ker(q)_{n-s}$ we have that $\sum_{Y' < Z'} h(Y') = 0$ for each $Z' \in V_{n-(s-1)}$.

Therefore the left side of the last equation is equal to zero and then $\sum_{Z<Y} \sum_{Y' \oplus Y = V} h(Y') = 0$.

c) Let $g \in L^2(V_{n-(s-1)})$ and $Y \in L^2(V_s)$. We have that

$$(\mathcal{C}(q)^*_s \circ R(q)^*_{n-s})(g)(Y) = \sum_{Y' \oplus Y = V} (\sum_{Y' < Z'} g(Z')).$$

First, we note that:

$$\sum_{Y' \oplus Y = V} (\sum_{Y' < Z'} g(Z')) = \sum_{dim(Z' \cap Y) = 1} K_{Z'} g(Z').$$

where $K_{Z'} = |Y' : Y' < Z', dim(Y' \cap Y) = 0, dim(Z' \cap Y) = 1|$.

Indeed, if Z' is a $n - (s-1)$ -subspace of V and it satisfies that $Y' < Z'$ for some supplementary subspace Y' of Y then we can find a one dimensional subspace $< u >$ such that $Z' = Y' \oplus < u >$. Since $V = Y \oplus Y'$ there exist unique vectors $u_1 \in Y$ and $u_2 \in Y'$, such that $u = u_1 + u_2$. So we have that $Z' \cap Y = < u_1 >$ and therefore $dim(Z' \cap Y) = 1$.

Conversely, if $Z' \in V_{n-(s-1)}$ and $Z' \cap Y = < u >$, then we can find an $(n-s)-$ dimensional subspace Y' such that $Z' = < u > \oplus Y'$.

So, if there is a vector $w \in Y' \cap Y$ such that $w \neq \vec{0}$ then the subspace $< u > \oplus < w >$ should be a two dimensional subspace of $Z' \cap Y$ and this would contradict the condition $dim(Z' \cap Y) = 1$.

Therefore, since $dim(Y' \cap Y) = 0$ and $dimY + dimY' = n$ we get that Y' is a supplementary subset of Y for which we have that $Z' < Y'$.

Moreover, applying Lemma 1 we compute:

$$|Y' \in V_{n-s} : Y' \oplus Y = V| \quad |Z' \in V_{n-(s-1)} : Y' < Z', Y' \oplus Y = V|$$
$$= q^{s(n-s)} \left(\begin{array}{c} s \\ 1 \end{array} \right)_q ,$$

$$|Z' \in V_{n-(s-1)} : dim(Z' \cap Y) = 1| = q^{(s-1)(n-s)} \left(\begin{array}{c} s \\ 1 \end{array} \right)_q ,$$

and

$$|Y' \in V_{n-s} : Y' < Z', dim(Z' \cap Y) = 1, Y' \oplus Y = V| = q^{1(n-s)} \left(\begin{array}{c} n-s \\ n-s \end{array} \right)_q .$$

In this way we get

$$(\mathcal{C}(q)^*_s \circ R(q)^*_{n-s})(g)(Y) = q^{n-s} \sum_{dim(Z' \cap Y)=1} g(Z').$$

On the other hand, we note that

$$(R(q)_{s-1} \circ \mathcal{C}(q)^*_{s-1})(g)(Y) = \sum_{Z<Y} (\sum_{Z' \oplus Z = V} g(Z')).$$

We claim that

$$\sum_{Z<Y} (\sum_{Z' \oplus Z = V} g(Z')) = \sum_{dim(Z' \cap Y)=1} J_Z g(Z').$$

where $J_Z = |Z \in V_{s-1} : Z \oplus Z' = V, Z < Y, dim(Z' \cap Y) = 1|$.

If we consider an $n-(s-1)-$dimensional subspace Z', such that $Z' \oplus Z = V$ for some subspace Z, which is a subspace of Y, then we can find a vector v such that $Y = Z \oplus < v >$.

Then there are unique vectors $z_1 \in Z$ and $z_2 \in Z'$ such that $v = z_1 + z_2$ and since we have found a non zero vector $z_2 \in (Y \cap Z')$ and $dimZ = dimY - 1$ then the dimension of the subspace $Y \cap Z'$ must be one. In the same way if $Z' \in V_{n-(s-1)}$ and $dim(Z' \cap Y) = 1$ there exists an $(s-1)-$ dimensional subspace Z such that $Y = (y \cap Z') \oplus Z$ and $dim(Z \cap Z') = 0$. Then we have found an $(s-1)-$ dimensional subspace Z of Y such that $Z' \oplus Z = V$.

Moreover we compute:

$$\begin{aligned}
|Z \in V_{s-1} : Z < Y| \quad & |Z' \in V_{n-(s-1)} : Z' \oplus Z = V, Z < Y| = \\
& = \begin{pmatrix} s \\ s-1 \end{pmatrix}_q q^{(s-1)(n-(s-1))} \begin{pmatrix} n-(s-1) \\ n-(s-1) \end{pmatrix}_q = \\
& = q^{(s-1)(n-(s-1))} \begin{pmatrix} s \\ 1 \end{pmatrix}_q ,
\end{aligned}$$

$$|Z' \in V_{n-(s-1)} : dim(Z' \cap Y) = 1| = q^{(s-1)(n-s)} \begin{pmatrix} s \\ 1 \end{pmatrix}_q ,$$

and

$$|Z \in V_{s-1} : Z \oplus Z' = V, Z < Y, dim(Z' \cap Y) = 1| = q^{1(s-1)} \begin{pmatrix} s-1 \\ s-1 \end{pmatrix}_q ,$$

getting in this way that

$$(R(q)_{s-1} \circ \mathcal{C}(q)^*_{s-1})(g)(Y) = q^{s-1} \sum_{dim(Z' \cap Y)=1} g(Z').$$

Therefore we have

$$\frac{1}{q^{n-s}} (\mathcal{C}(q)^*_s \circ R(q)^*_{n-s}) = \frac{1}{q^{s-1}} (R(q)_{s-1} \circ \mathcal{C}(q)^*_{s-1})$$

from which we get item c).

$\square$

To prove that $\mathcal{C}(q)^*_s$ is an isomorphism between the unitary natural representations $(L^2(V_{n-s}), \tau')$ and $(L^2(V_s), \tau)$ for $0 \leq s \leq \frac{n}{2}$, we need the following lema.

Lemma 2 *The number of* $(n-s)-$ *dimensional subspaces* W' *such that* $W' \oplus W_0 = V$ *and* $d(W', W_0') = j$ *is given by*

$$|N_j| = \left(\begin{array}{c} n-s \\ n-s-j \end{array} \right)_q \frac{((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=0}^{j-1}(q^j - q^i)}{(q^j - 1)}.$$

Proof:

Let U be an $(n-s-j)-$ dimensional subspace of W' and $\{u_1, ..., u_{n-s-j}\}$ a basis of U. We complete this basis with vectors $\{z_1, .., z_j\}$ to obtain a basis of W' . So, we have that $\{e_1, .., e_s, u_1, .., u_{n-s-j}, z_1, .., z_j\}$ is a basis of V.

To construct subspaces W', for each subspace U, we have to complete U with vectors $w_k = \sum_{i=1}^{s} a_i^k e_i + \sum_{t=1}^{j} b_t^k z_t$, where $v_k = (a_1^k, ..a_s^k) \neq 0$ and $p_k = (b_1^k, ..b_j^k) \neq 0$ for $1 \leq k \leq j$.

Since $dim(W' \cap W_0) = 0$ and $W' \cap W_0' = U$, we need to choose $\{v_1, .., v_j\}$ and $\{p_1, .., p_j\}$ linear independent, since, if these sets are linear dependent, we can find scalars $c_i, d_i, 1 \leq i \leq j$, not all zeros, such that $\sum_{k=1}^{j} c_k v_k = 0$ or $\sum_{k=1}^{j} d_k p_k = 0$. Then $\sum_{k=1}^{j} a_i^k c_k = 0$, for $1 \leq i \leq s$ or $\sum_{k=1}^{j} b_t^k d_k = 0$, for $1 \leq t \leq s$.

In this way

$$\sum_{i=0}^{s}(\sum_{k=1}^{j} a_i^k c_k)e_i = 0$$

or

$$\sum_{i=0}^{s}(\sum_{k=1}^{j} b_t^k d_k)z_k = 0.$$

Therefore

$$\sum_{k=1}^{j} c_k(\sum_{i=1}^{s} a_i^k e_i) = 0$$

or

$$\sum_{k=1}^{j} d_k(\sum_{t=1}^{j} b_t^k z_t) = 0.$$

So, we obtain that $\sum_{k=1}^{j} c_k w_k \in W_0'$ or $\sum_{k=1}^{j} d_k w_k \in W_0$ and also $dim(W' \cap W_0) \neq 0$ or $W' \cap W_0 \neq U$.

Suppose now we have chosen $W' =< u_1, .., u_{n-s-j}, w_1, .., w_j$ and $W'' =< u_1, .., u_{n-s-j}, w_1', .., w_j' >$ and $w_k' = \sum_{i=1}^{s} x_i^k e_i + \sum_{t=1}^{j} y_t^k z_t$.

We have that $W' = W''$ if and only if there are scalars $l_1^k, l_2^k, .., l_j^k, 1 \leq k \leq j$ such that

$$(x_1^k, x_2^k, .., x_s^k) = \sum_{i=1}^{j} l_i^k v_i,$$

$$(y_1^k, y_2^k, .., y_j^k) = \sum_{i=1}^{j} l_i^k p_i,$$

for $1 \leq k \leq j$.

Also, among all choices of $(v_1, v_2, .., v_j, p_1, p_2, .., p_j)$ there are $q^j - 1$ which span the same subspace. Now, since there are $(q^s - 1)..(q^s - q^{j-1})$ ways to choose the vectors $v_1, v_2, .., v_j$ and $(q^j - 1)..(q^j - q^{j-1})$ ways to choose the vectors $p_1, p_2, .., p_j$, we have

$$\frac{((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=0}^{j-1}(q^j - q^i)}{(q^j - 1)}$$

different ways to complete the subspace . U Since there are $\left(\begin{array}{c} n-s \\ n-s-j \end{array}\right)_q$ ways to choose subspaces U of W_0', we finally get

$$|N_j| = \left(\begin{array}{c} n-s \\ n-s-j \end{array}\right)_q \frac{((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=0}^{j-1}(q^j - q^i)}{(q^j - 1)},$$

$\square$

Theorem 5 *Let* $0 \leq s \leq \frac{n}{2}$. *The unitary natural representations* $(L^2(V_{n-s}), \tau')$ *and* $(L^2(V_s), \tau)$ *of the finite linear group* $GL_n(q)$ *are isomorphic.*

Proof: Using the previous propositions and theorems, it is enough to prove that $\mathcal{C}(q)_s^*$ is injective.

We will prove this by induction on s.

If $s = 0$, we have that $\mathcal{C}(q)_0^*(f)(\{\vec{0}\}) = f(V)$, for all $f \in L^2(V_n)$, then $\mathcal{C}(q)_0^*$ is non trivial and thus injective.

If $s = 1$, we have that

$$L^2(V_{n-1}) = R(q)_{n-1}^*(L^2(V_n)) \perp Ker R(q)_{n-1}.$$

Let $f \in L^2(V_{n-1})$, we have that $f = f_1 + f_2$ where $f_1 \in R(q)_{n-1}^*(L^2(V_n))$ and $f_2 \in Ker R(q)_{n-1}$.

Since $f_1 = R(q)_{n-1}^*(g)$, for some $g \in L^2(V_n)$ we obtain that $\mathcal{C}(q)_1^*(f_1) = \mathcal{C}(q)_1^*(R(q)_{n-1}^*(g))$.

By using the last proposition, we get

$$\mathcal{C}(q)_1^*(f_1) = q^{n-1}(R(q)_0^* \circ \mathcal{C}(q)_0^*)(g).$$

If $\mathcal{C}(q)_1^*(f_1) = 0$ then $(R(q)_0^* \circ \mathcal{C}(q)_0^*)(g) = 0$.

Since $(R(q)_0^*$ is injective, we deduce $\mathcal{C}(q)_0^*(g) = 0$. Then $g = 0$ and therefore $f_1 = 0$.

In this way we have that $\mathcal{C}(q)_1^*$ restricted to the subspace $R(q)_{n-1}^*(L^2(V_n))$ of $L^2(V_{n-1})$ is injective.

From the previous proposition $\mathcal{C}(q)_1^*(f_2) \in Ker R(q)_0^*$. If we compute $\mathcal{C}(q)_1^*(\varphi(q)_1)(W_0)$ where $\varphi(q)_1$ is the spherical function of $Ker R(q)_{n-1}$ we obtain:

$$\mathcal{C}(q)_1^*(\varphi(q)_1)(W_0) = \frac{1}{q}$$

Then $\mathcal{C}(q)_1^* \neq 0$ and then $\mathcal{C}(q)_1^*$ is injective.

We suppose now that $\mathcal{C}(q)_t^*$ is injective for all $t < s$.

As in the previous case we have the decomposition

$$L^2(V_{n-s}) = R(q)_{n-s}^*(L^2(V_{n-(s-1)})) \perp Ker R(q)_{n-s}.$$

We notice first that to prove that $\mathcal{C}(q)_s^*$ is injective it is enough to prove that the restriction of $\mathcal{C}(q)_s^*$ to the components $R(q)_{n-s}^*(L^2(V_{n-(s-1)}))$ and $Ker R(q)_{n-s}$ is injective.

Indeed, if $f_1 \in R(q)^*_{n-s}(L^2(V_{n-(s-1)}))$, then $f_1 = R(q)^*_{n-s}(g)$ for some $g \in L^2(V_{n-(s-1)})$. By using item c) of last proposition we find that

$$\mathcal{C}(q)^*_s(R(q)^*_{n-s}(g)) = \frac{q^{n-s}}{q^{s-1}}(R(q)_{s-1} \circ \mathcal{C}(q)^*_{s-1})(g).$$

Therefore $\mathcal{C}(q)^*_s(f_1)$ lies in $R(q)_{s-1}((L^2(V_{n-(s-1)})$.

Moreover, since $f_2 \in KerR(q)_{n-s}$, we have that $\mathcal{C}(q)^*_s(f_2)$ lies in $R(q)^*_{s-1}$ since we have proved that $\mathcal{C}(q)^*_s(R(q)_{n-s}) = KerR(q)^*_{s-1}$

Therefore, if we have $\mathcal{C}(q)^*_s(f) = 0$, then writing $f = f_1 + f_2$, where $f_1 \in R(q)^*_{n-s}(L^2(V_{n-(s-1)}))$ and $f_2 \in KerR(q)_{n-s}$, we see that necessarily $\mathcal{C}(q)^*_s(f_1) = 0$ and $\mathcal{C}(q)^*_s(f_2) = 0$. So, if if we know that the restrictions of $\mathcal{C}(q)^*_s$ to $R(q)^*_{n-s}(L^2(V_{n-(s-1)}))$ and $KerR(q)_{n-s}$. are injective, we conclude that $\mathcal{C}(q)^*_s$ is injective.

Now, we prove that the restriction of $\mathcal{C}(q)^*_s$ to the subspace $R(q)^*_{n-s}(L^2(V_{n-(s-1)}))$ is injective.

Write $f_1 \in R(q)^*_{n-s}(L^2(V_{n-(s-1)}))$ as $f_1 \in R(q)^*_{n-s}(L^2(V_{n-(s-1)}))$ for some $g \in L^2(V_{n-(s-1)})$. So, if $\mathcal{C}(q)^*_s(f_1) = 0$ then $\mathcal{C}(q)^*_{s-1}(g) = 0$.

Since $R(q)_{s-1}$ is injective we deduce that $\mathcal{C}(q)^*_{s-1}(g) = 0$. Therefore by the induction hypothesis we obtain that $g = 0$, and therefore $f_1 = 0$.

To conclude the proof, we prove now that $\mathcal{C}(q)^*_s$ restricted to the irreducible subrepresentation $KerR(q)_{n-s}$ of $L^2(V_s)$ is injective.

For this we consider the spherical function $\varphi(q)_s$ of the irreducible representation $KerR^*_{n-s}$ and we compute $\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0)$.

We obtain that

$$\begin{aligned}\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0) &= \textstyle\sum_{W' \oplus W_0 = V}(\varphi(q)_s)(W') \\ &= \textstyle\sum_{j=0}^{s} |N_j|(\varphi(q)_s)(W') \\ &= 1 + \textstyle\sum_{j=1}^{s} |N_j|(\varphi(q)_s)(W'),\end{aligned}$$

where $N_j = \{W' : W' \oplus W_0 = V, d(W', W'_0) = j\}$

We recall that :

$$\varphi(q)_s)(W') = \sum_{k=k_0}^{min(j,s)} (-1)^k q^{\frac{k}{2}(k-2j-1)} \left(\begin{array}{c} j \\ k \end{array}\right)_q \left(\begin{array}{c} s-j \\ s-k \end{array}\right)_q, k!_q \frac{(n-s-k)!_q}{(n-s)!_q},$$

where $j = d(W', W'_0)$ and $k_0 = j$.

Then for $1 \leq j \leq s$ we have

$$\varphi(q)_s)(W') = (-1)^{\frac{-j}{2}(j+1)} \frac{j!_q(n-s-j)!_q}{(n-s)!_q}.$$

In this way

$$\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0) = 1 + \sum_{j=1}^{s} |N_j|(-1)^j (q)^{\frac{-j}{2}(j+1)} \frac{j!_q (n-s-j)!_q}{(n-s)!_q}$$

From lema 2 we have that

$$|N_j| = \left(\begin{array}{c} n-s \\ n-s-j \end{array} \right)_q \frac{((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=0}^{j-1}(q^j - q^i)}{(q^j - 1)},$$

Replacing $|N_j|$ in the last equation we get,

$$\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0) = 1 + \sum_{j=1}^{s} (-1)^j q^{\frac{j(j-3)}{2}} ((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=1}^{j-1}(q^i - 1).$$

If we put $\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0) = S + T$ where

$$S = 1 + \sum_{j=1}^{2} (-1)^j q^{\frac{j(j-3)}{2}} ((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=1}^{j-1}(q^i - 1)$$

and

$$T = \sum_{j=3}^{s} (-1)^j q^{\frac{j(j-3)}{2}} ((\Pi_{i=0}^{j-1}(q^s - q^i)\Pi_{i=1}^{j-1}(q^i - 1),$$

we have

$$S = 1 + (q^s - 1)(q^{s-1} - q^{s-2} - 1).$$

and

$$T = (q^s - 1)\sum_{j=3}^{s} (-1)^j q^{\frac{j(j-3)}{2}} ((\Pi_{i=0}^{j-1}(q^{s-1} - q^i)\Pi_{i=1}^{j-1}(q^i - 1).$$

In this way , if $\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0) = 0$ then $T = -S$ i.e.

$$(q^s - 1)\sum_{j=3}^{s} (-1)^j q^{\frac{j(j-3)}{2}} ((\Pi_{i=0}^{j-1}(q^{s-1} - q^i)\Pi_{i=1}^{j-1}(q^i - 1) = -1 - (q^s - 1)(q^{s-1} - q^{s-2} - 1).$$

Therefore $(q^s - 1)$ would be a factor of $-1 - (q^s - 1)(q^{s-1} - q^{s-2} - 1)$.
But

$$\frac{-1 - (q^s - 1)(q^{s-1} - q^{s-2} - 1)}{(q^s - 1)} = -q^{s-1} + q^{s-2} + 1 + \frac{-1}{(q^s - 1)}.$$

Then $(q^s - 1)$ is not a factor of $-1 - (q^s - 1)(q^{s-1} - q^{s-2} - 1)$ Therefore $\mathcal{C}(q)^*_s(\varphi(q)_s)(W_0) \neq 0.$. Then $\mathcal{C}(q)^*_s$ restricted to $KerR(q)_{n-s}$ is injective. The injectivity of $\mathcal{C}(q)^*_s$ follows.

$\square$

Remark 4 *For* $0 \leq s \leq \frac{n}{2}$, *we define*

$$\mathcal{C}(q)^*_{n-s} : L^2(V_s) \to L^2(V_{n-s}),$$

by

$$\mathcal{C}(q)^*_{n-s}(f)(W') = \sum_{W \oplus W' = V} f(W),$$

for $f \in L^2(V_s), W' \in V_{n-s}$.

By proceeding in a similar way as we did in the proof of proposition 5 and theorem 5, we get the following results:

a) $\mathcal{C}(q)^*_{n-s}$ *is a non trivial intertwining operator between* $(L^2(V_s), \tau)$ *and* $(L^2(V_{n-s}), \tau')$,

b) $\mathcal{C}(q)^*_{n-s}(KerR(q)^*_{s-1}) = KerR(q)_{n-s}$,

c) $(\mathcal{C}(q)^*_{n-s} \circ R(q)_{s-1}) = q^{((n-s)-(s-1))}(R(q)^*_{n-s} \circ \mathcal{C}(q)^*_{n-(s-1)})$.

d) $\mathcal{C}(q)^*_{n-s}$ *is injective. (The q-analogue to the set* N_j *is the set* $S_j = \{W : W \oplus W'_0 = V, d(W, W_0) = j\}$ *and* $|N_j| = |S_j|$. *)*

e)

$$(\mathcal{C}(q)^*_{n-s} \circ \mathcal{C}(q)^*_s) \circ R(q)^*_{n-s} = R(q)^*_{n-s} \circ (\mathcal{C}(q)^*_{n-(s-1)} \circ \mathcal{C}(q)^*_{s-1})$$

and

$$R(q)_{s-1} \circ (\mathcal{C}(q)^*_{s-1} \circ \mathcal{C}(q)^*_{n-(s-1)}) = (\mathcal{C}(q)^*_s \circ \mathcal{C}(q)^*_{n-s}) \circ R(q)_{s-1}.$$

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Maria Francisca Yañez
mfyanez@uchile.cl
Escuela de Pregrado
Facultad de Ciencias Químicas y Farmacéuticas
Universidad de Chile
Olivos 1007
Santiago, Chile