Split scheduling with uniform setup times

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Abstract We study a scheduling problem in which jobs may be split into parts, where the parts of a split job may be processed simultaneously on more than one machine. Each part of a job requires a setup time, however, on the machine where the job part is processed. During setup, a machine cannot process or set up any other job. We concentrate on the basic case in which setup times are job-, machine- and sequence-independent. Problems of this kind were encountered when modelling practical problems in planning dis-

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aster relief operations. Our main algorithmic result is a polynomial-time algorithm for minimising total completion time on two parallel identical machines. We argue, why the same problem with three machines is not an easy extension of the two-machine case, leaving the complexity of this case as a tantalising open problem. We give a constant-factor approximation algorithm for the general case with any number of machines and a polynomial-time approximation scheme for a fixed number of machines. For the version with the objective to minimise total *weighted* completion time, we prove NP-hardness. Finally, we conclude with an overview of the state of the art for other split scheduling problems with job-, machine- and sequence-independent setup times.

1 Introduction

We consider a scheduling problem with *setup times* and *job splitting*. Given a set of identical parallel machines and a set of jobs with processing times, the goal of the scheduling problem is to schedule the jobs on the machines such that a given objective, for example, the makespan or the sum of completion times, is minimised. With ordinary pre-emption, feasible schedules do not allow multiple machines to work on the same job simultaneously. In *job splitting*, this constraint is dropped. Without setup times, allowing job splitting makes many scheduling problems trivial: both for minimising makespan and for minimising total (weighted) completion time, an optimal schedule is obtained by splitting the processing time of each job equally over all machines, and processing the jobs in an arbitrary order on each machine in case of makespan, and in (weighted) shortest processing time

first [(W)SPT] order in case of total (weighted) completion time. See Xing and Zhang (2000) for an overview of several classical scheduling problems which become polynomially solvable if job splitting is allowed.

In the presence of release times, minimising total completion time with ordinary pre-emption is NP-hard (Du et al. 1990), whereas it is easy to see that if we allow job splitting, then splitting all jobs equally over all machines and applying the shortest remaining processing time first (SRPT) rule gives an optimal schedule.

Triviality disappears, when setup times are present, i.e., when each machine requires a setup time before it can start processing the next job (part). During setup, a machine cannot process any job nor it can set up the processing of any other job (part). Problems for which the setup times are allowed to be sequence-dependent are usually NP-hard, as such problems tend to exhibit routinglike features. For example, the Hamiltonian path problem in a graph can be reduced to the problem of minimising the makespan on a single machine, where each job corresponds to a node in the graph, the processing times are 1, and the setup time between job i and j is 0 if the graph contains an edge between i and j, and 1 otherwise. However, as we will see, adding setup times leads to challenging algorithmic problems, already if the setup times are assumed to be job-, machine- and sequence-independent.

We encountered such problems in studying disaster relief operations (Van der Ster 2010). For example, in modelling flood relief operations, the machines are pumps and the jobs are locations to be drained. Or in the case of earthquake relief operations, the machines are teams of relief workers, and the jobs are locations to be cleared. The setup is the time required to install the team on the new location. Although, in principle, these setup times consist partly of travel time, which is sequence-dependent, the travel time is negligible compared to the time required to equip the teams with instructions and tools for the new location. Hence, considering the setup times as being location- and sequence-independent was in this case an acceptable approximation of reality.

In this paper, we concentrate on a basic scheduling problem and consider the variation where we allow job splitting with setup times that are job-, machine- and sequence-independent, to which we will refer here as *uniform setup times*, i.e., we assume a *uniform* setup time s. There exists little literature on this type of scheduling problem. The problem of minimising makespan on parallel identical machines is in the standard scheduling notation of Graham et al. (1979) denoted as $P||C_{\max}$ (see Sect. 8 for an instruction on this notation). This problem $P||C_{\max}$, but then with job splitting and setup times that are job-dependent, but sequence- and machine-independent, is considered by Xing and Zhang (2000), and Chen et al. (2006). Chen et al. (2006) mention that this problem is NP-hard in the strong sense, and only

weakly NP-hard if the number of machines is assumed constant. Straightforward reductions from the 3- PARTITION and SUBSET SUM problem show that these hardness results continue to hold, if setup times are uniform. Chen et al. (2006) provide a 5/3-approximation algorithm for this problem and an FPTAS for the case of a fixed number of machines. A PTAS for the version of $P||C_{\max}$ with pre-emption and job-dependent, but sequence- and machine-independent setup times was given by Schuurman and Woeginger (1999). It remains open whether a PTAS exists with job splitting rather than preemption, even if the setup times are uniform. See Liu and Edwin Cheng (2004) and Potts and Wassenhove (1992) for a more extensive literature on problems with pre-emption and setup times.

Our problem is related to scheduling problems with malleable tasks. A malleable task may be scheduled on multiple machines, and a function $f_j(k)$ is given that denotes the processing speed if j is processed on k machines. If k machines process task j for L time, then $f_j(k)L$ units of task j are completed. What we call, job splitting is referred to as malleable tasks with linear speedups, i.e., the processing time required on k machines is $1/k$ times the processing time required on a single machine. We remark that job splitting with setup times is not a special case of scheduling malleable tasks, because of the discontinuity caused by the setup times. We refer the reader to Drozdowski (2009) for an extensive overview of the literature on scheduling malleable tasks.

The main algorithmic result of our paper considers the job splitting variant of the problem of minimising the sum of completion times on identical machines, with uniform setup times: given a set of m identical machines, n jobs with processing times $p_1, \ldots, p_n$, and a setup time s, the objective is to schedule the jobs on the machines to minimise total completion time ($\sum C_j$) (where the chosen objective is inspired by the disaster relief application). The version of this problem with ordinary pre-emption and fixed setup time s is solved by the shortest processing time first rule (SPT); the option of preemption is not used by the optimum. However, the situation is much less straightforward for job splitting. If s is very large, then an optimal schedule minimises the contribution of the setup times to the objective, and a job will only be split over several machines if no other job is scheduled after the job on these machines. It is not hard to see that the jobs that are not split are scheduled in SPT order.

If s is very small (say 0), then each job is split over all machines and the jobs are scheduled in SPT order. However, for other values of s, it appears to be a non-trivial problem to decide how to schedule the jobs, as splitting a job over multiple machines decreases the completion time of the job itself, but it increases the total load on the machines, and hence the completion times of later jobs.

Consider the following instance as an example. There are 3 machines and 6 jobs, numbered $1, 2, \ldots, 6$, with process-
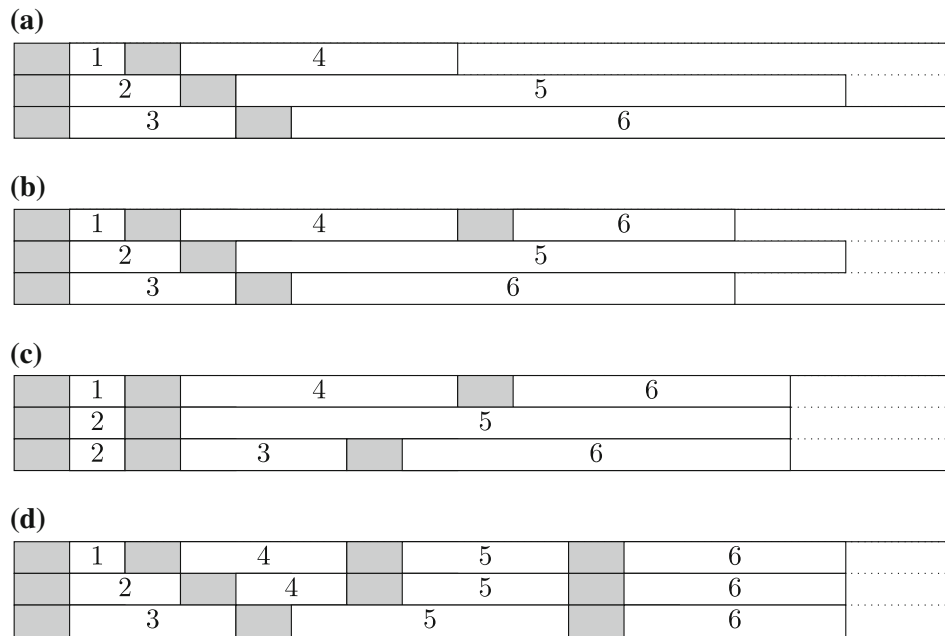
(a)

(b)

(c)

(d)

Fig. 1 Gantt charts depicting the schedules for the instance described in Sect. 2. The *grey blocks* indicate the setup times, the *numbered blocks* are scheduled job parts. Each row of blocks gives the schedule for a machine

ing times 1, 2, 3, 5, 11 and 12, respectively, and setting up a machine takes 1 time unit. One could consider filling up a schedule in round-robin style, assigning the jobs to machine 1, 2, 3, 1, 2, 3, respectively. This schedule is given in the Gantt chart in Fig. 1a. The schedule has objective value 49.

By splitting job 6 over machines 1 and 3, instead of processing it on machine 3 only, we can lower the completion time of job 6, and this improves the objective value since there are no jobs scheduled after job 6. In fact, to get the best improvement in objective value, we make sure that both job parts of job 6 finish at the same time, see Fig. 1b. The objective value of the schedule is 45.

Splitting jobs early in the schedule, may increase the objective value, as (many) later jobs may experience delays. For example, if we choose to split job 2 over machines 2 and 3, we will cause delays for jobs 3 and 6, while improving the completion times of jobs 2 and 5. If we require that job parts of the same job end at the same time, we get the schedule pictured in Fig. 1c with objective value 46. Finally, Fig. 1d depicts the optimal schedule with objective value 40.

This example illustrates the inherent trade-off in this problem mentioned earlier: splitting jobs will decrease the completion times of some jobs, but it also may increase the completion times of other jobs.

In Sect. 3 we present a polynomial-time algorithm for the case in which there are two machines. The algorithm is based on a careful analysis of the structure of optimal solutions to this problem. Properties of optimal solutions that hold under any number of machines are presented in a preliminary section. Though a first guess might be that the problem would

remain easy on any fixed number of machines, we will show by some examples in Sect. 4 that nice properties, which make the algorithm work for the 2-machine case, fail to hold for three machines already. The authors are split between thinking that we have encountered another instance of Lawler's "mystical power of twoness" (Lenstra 1998), a phrase signifying the surprisingly common occurance that problems are easy, when a problem parameter (here the number of machines) is two, but NP-hard when it is three, or that we just lacked the necessary flash of insight to find a polynomial-time algorithm. We present a constant-factor approximation algorithm for the general case with any number of machines in Sect. 5, and in Sect. 6 we give a polynomial-time approximation scheme for the case of a fixed number of machines. We leave the complexity of the problem (even for only three machines) as a tantalising open problem for the scheduling research community. We show in Sect. 7 that introducing weights for the jobs makes the problem NP-hard, already on 2 machines. We finish the paper by giving a table with the state of the art for other split scheduling problems with uniform setup times. We summarise whether they are known to be NP-hard or in P, and present the best known approximation ratios.

2 Preliminaries

An instance is given by m parallel identical machines and n jobs. Job j has processing time p_j, for $j = 1, \ldots, n$. Each job may be split into *parts* and multiple parts of the same job may be processed simultaneously. Before a machine can start

processing a part of a job, a fixed setup time s is required. During setup of a job (part), the machine cannot simultaneously process or setup another job (part). The objective is to minimise the sum of the completion times of the jobs (total completion time), which is equivalent to minimising the average completion time.

Here, we derive some properties of an optimal schedule, which are valid for any number of machines. Some additional properties for the special case of two machines, presented in Sect. 3, will lead us to a polynomial-time algorithm for this special case. We show in Sect. 4 that the additional properties that make the 2-machine case tractable do not hold for the case of three machines.

Claim Let σ be a feasible schedule with job completion times $C_1 \leq C_2 \leq \cdots \leq C_n$. Let σ' be obtained from σ by rescheduling the job parts on each machine in order $1, 2, \ldots, n$. Then, $C'_j \leq C_j$ for $j = 1, \ldots, n$.

Proof Let q_{ij} be the time that j is processed on machine i in σ and let C_{ij} be the time that j finishes on machine i. Let $y_{ij} = s + q_{ij}$ if $q_{ij} > 0$ an let $y_{ij} = 0$ otherwise. Fix some job j and machine i. Let $k = \arg\max\{C_{ik} \mid 1 \leq k \leq j\}$. Then $C_j \geq C_k \geq C_{ik} \geq \sum_{h=1}^{j} y_{ih} = C'_{ij}$, where the first inequality is by assumption and the last one by the fact that all work on jobs smaller than or equal to j has been done on machine i at time C_{ik}. Since $C_j \geq C'_{ij}$ for any machine i on which j is scheduled the proof follows.   □

The claim above has several nice corollaries. First, note that if in an optimal schedule $C_1 \leq C_2 \leq \cdots \leq C_n$, then we maintain an optimal schedule with the same completion time for each job by scheduling the job parts on each machine in the order $1, 2, \ldots, n$. This allows to characterise an optimal schedule by a permutation of the jobs and the times that job j is processed on each machine i. The optimal schedule is then obtained by adding a setup time s for each non-zero job part and processing them in the order of the permutation on each machine. Consequently, in the optimal schedule obtained, each machine contains at most one part of each job.

We thus have the following lemma, which we will use throughout this work.

Lemma 1 *There exists an optimal schedule such that each machine contains at most one part of each job.*

In the sequel, given a schedule, we use M_j to denote the set of machines on which parts of job j are processed. We will sometimes say that a machine processes job j, if it processes a part of job j.

Lemma 2 *There exists an optimal schedule that satisfies the property of Lemma 1 such that on each machine the job parts are processed (started and completed) in SPT order of the corresponding jobs.*

Proof Among the optimal schedules that satisfy Lemma 1, we choose the schedule that minimises $\sum_h p_h C_h$. By the observations preceding Lemma 1, we may assume the jobs are numbered $1, \ldots, n$ so that $C_1 \leq C_2 \leq \ldots \leq C_n$, and each machine processes the job parts in the order given by the numbering of the jobs. Suppose by contradiction that there exist jobs j, k such that $p_j < p_k$ and there exists some machine that processes job k before j, i.e., $C_k \leq C_j$. Choose among such pairs of jobs j, k a pair that minimises $j - k$. Note that any machine that processes both j and k must process k immediately before j, since if there is some job ℓ that is processed between them, then $C_k \leq C_\ell \leq C_j$, and either $p_\ell > p_j$ or $p_\ell \leq p_j < p_k$, so either j, ℓ or ℓ, k should have been chosen instead of the pair j, k. We now show how to define a new optimal schedule for which $\sum_h p_h C_h$ is strictly less than for the original schedule, thus contradicting the choice of our schedule.

Note that $M_j \cap M_k \neq \emptyset$. We define a new schedule by rescheduling both jobs within the time slots these jobs occupy in the current schedule (including the slots for the setup times). First remove both jobs. Then consider the machines in M_k one by one, starting with the machines in $M_k \setminus M_j$ and fill up the slots previously used by job k, until we have completely scheduled job j including the setup times. This is possible since $p_j < p_k$. We consider the remaining slots, which we note are single time intervals for each machine by our choice of j and k. We will show that they provide sufficient time for the processing and set up of job k, by showing that the combined number of setups for j and k does not increase.

Let M'_j and M'_k denote the sets of machines occupied by j and k, respectively, in the new schedule. We distinguish two cases. If job j cannot be rescheduled completely in the slots used by k in $M_k \setminus M_j$ then we have $M'_k \subseteq M_j$. Together with $M'_j \subseteq M_k$ it follows that $(M'_j \cap M'_k) \subseteq (M_j \cap M_k)$. Hence, any machine containing both j and k in the new schedule did also contain both jobs in the old schedule, and therefore, there are no extra setups on any machine needed.

Now consider the case that job j is rescheduled completely in the slots used by k in $M_k \setminus M_j$. Then, after adding job k, the total number of setups needed for j and k does not increase since there is at most one machine of $M_k \setminus M_j$ containing both jobs in the schedule, but none of the machines in $M_j \cap M_k$ is used by j in the new schedule.

We conclude that the remaining slots after scheduling job j provide sufficient room to feasibly schedule both the processing of job k and the required setups. Note that, if there is some machine on which the time allotted to k is at most s, then we can simply leave the machine idle for that time interval.

Let C' denote the new completion times. We have $C'_j \leq C_k$ and $C'_k \leq \max\{C_j, C_k\}$, since in the new schedule j is processed only where job k was processed in the old schedule, and job k is processed in the new schedule only where

either job j or job k was processed in the old schedule. For all other jobs, the completion time remains the same. Now, by assumption, we have that $C_j > C_k$, and hence $C'_k \leq C_j$. Therefore, the sum of completion times did not increase, and $\sum_h p_h C'_h < \sum_h p_h C_h$, which contradicts the choice of the original schedule.   □

From now on, assume that jobs are numbered in SPT order, i.e., $p_1 \leq \cdots \leq p_n$. Given a schedule, we call a job *balanced* if it completes at the same time on all machines on which it is processed.

Lemma 3 *There exists an optimal schedule that satisfies the properties of Lemma 1 and Lemma 2 in which all jobs are balanced.*

Proof Consider an optimal schedule of the form of Lemma 1 and Lemma 2 with a minimum number of job parts. Let C_j be the completion time of j in this schedule and define M_j for this schedule as before. Consider the following linear program in which there is a variable x_{ij} for all pairs i, j with $i \in M_j$, indicating the amount of processing time of job j assigned to machine i:

$$\min \sum_j C_j$$
$$\text{s.t.} \sum_{i \in M_j} x_{ij} = p_j \ \forall j = 1, \ldots, n,$$
$$\sum_{k \leq j:\ M_k \ni i} (s + x_{ik}) \leq C_j \ \forall j = 1, \ldots, n, \ \forall i \in M_j,$$
$$x_{ij} \geq 0, \ C_j \geq 0 \quad \forall j = 1, \ldots, n, \ \forall i \in M_j.$$

Note that a schedule that satisfies Lemmas 1 and 2 gives a feasible solution to the LP, and on the other hand that any feasible solution to the LP gives a schedule with total completion time at most the objective value of the LP: if there exist some j and $i \in M_j$ such that $x_{ij} = 0$, then the LP objective value is at least the total completion time of the corresponding schedule, as there is no need to set up for job j on machine i if $x_{ij} = 0$. We know that a solution is a basic solution to this LP, only if the number of variables that are non-zero is at most the number of linearly independent tight constraints (not including the non-negativity constraints). By the minimality assumption on the optimal schedule, in any optimal solution to the LP all C_j and x_{ij} variables are non-zero, which gives a total of $n + \sum_j |M_j|$ variables. Since there are only $n + \sum_j |M_j|$ constraints, all constraints must be tight, which proves the lemma.   □

3 An $O(n \log n)$-time algorithm for two machines

Given a feasible schedule, we call a job j a d-job, if $|M_j| = d$. In this section we assume that the number of machines is two.

Lemma 4 *Let σ be an optimal schedule for a 2-machine instance that satisfies the properties of Lemmas 1, 2 and 3. Let $j < k$ be two consecutive 2-jobs. If there are 1-jobs between j and k, then there is at least one 1-job on each machine. Also, the last 2-job is either not followed by any job or is followed by at least one 1-job on each machine.*

Proof Let j and k be two consecutive 2-jobs and assume there is at least one in-between 1-job on machine 1 and none on machine 2. Let s_1, s_2 be the start time of job j on, respectively, machine 1 and 2. We may assume without loss of generality that $s_1 \geq s_2$: otherwise we just swap the schedules of the two machines for the interval $[0, C_j]$ and get the inequality. We change the schedule of j and k and the in-between 1-jobs as follows. Job j is completely processed on machine 2, starting from time s_2, and the in-between 1-jobs are moved forward such that the first starts at time s_1. Let Δ be the amount of processing on job j that was previously assigned to machine 1, where we note that $\Delta \leq \frac{1}{2} p_j$. We increase the part of job k on machine 1 by Δ, and decrease the part of job k on machine 2 by Δ. This is possible, since the part of job k that was previously on machine 2 is at least $\frac{1}{2} p_k \geq \frac{1}{2} p_j \geq \Delta$.

The completion time of each of the in-between 1-jobs decreases by $\Delta + s$, the completion time of job j increases by Δ and the completion time of job k remains unchanged. The total completion time is thus reduced by at least s. If j is the last 2-job then, we can make the same adjustment.   □

Lemma 5 *In the case of two machines, there are no 1-jobs after a 2-job in an optimal schedule satisfying the properties of Lemmas 1, 2 and 3.*

Proof Suppose the lemma is not true. Then, there must be a 2-job j that is directly followed by a 1-job. By Lemma 4, there must be at least one such 1-job on each machine, say jobs h and k. Assume without loss of generality that $p_h \leq p_k$. Let x_{1j}, x_{2j} be the processing time of j on machine 1 and 2, respectively. As argued before, without loss of generality we assume that $x_{1j} \geq x_{2j}$. Let us define the starting time of j as zero, and let $\Delta = x_{1j} - x_{2j}$. Note that $C_j = \frac{1}{2}(\Delta + p_j + 2s)$. Then, the sum of the three completion times is

$$C_j + C_h + C_k = C_j + (C_j + p_h + s) + C_k$$
$$= \Delta + p_j + 2s + p_h + s + C_k. \quad (1)$$

We reschedule the jobs j, h, and k as follows, while the remaining schedule stays the same. Place job j, the shortest among j, h, and k, on machine 1 (unsplit), job h on machine 2 (unsplit), and behind these two, job k is split on machine 1 and 2, in such a way that it completes on one machine at time C_h and at time C_k on the other. The sum of the completion times of the three jobs becomes

$$(p_j + s) + (\Delta + p_h + s) + C_k,$$

which is exactly s less than the sum of the three completion times in (1) from before the switch. □

Given the previous lemmas, we see that the 2-jobs are scheduled in SPT order at the end. By Lemma 2, the first 2-job, say job k, is not shorter than the preceding 1-jobs. But this implies that the 1-jobs can be scheduled in SPT order without increasing the completion time of job k and the following jobs. By considering each of the n jobs as the first 2-job, we immediately obtain a $O(n^2)$-time algorithm to solve the problem. Carefully updating consecutive solutions leads to a faster method.

Theorem 1 *There exists an $O(n \log n)$ algorithm for minimising the total completion time of jobs on two identical parallel machines with job splitting and uniform setup times.*

Proof Suppose we schedule the first k jobs (for any $1 \leq k \leq n$) in SPT order as 1-jobs and the other jobs in SPT order as 2-jobs. We would like to compute the change in objective value that results from changing job k from a 1-job to a 2-job. However, this happens to give a rather complicated formula. It is much easier to consider the change for job $k - 1$ and k simultaneously.

The schedule for the 1-jobs $j < k - 1$ does not change. To facilitate the exposition, suppose that job $k - 1$ starts at time zero and job k starts at time a. Then, $C_{k-1} + C_k = p_{k-1} + s + a + p_k + s$. After turning the jobs into 2-jobs, the new completion times become $C'_{k-1} = (a + p_{k-1} + 2s)/2$ and $C'_k = (a + p_{k-1} + p_k + 4s)/2$. Hence,

$$C'_{k-1} + C'_k - C_{k-1} - C_k = s - p_k/2.$$

In addition, each job $j > k$ completes s time units later. Hence, the total increase in objective value due to turning both job $k - 1$ and k from a 1-job into a 2-job is

$$f(k) := (n - k + 1)s - p_k/2.$$

Notice that $f(k)$ is decreasing in k, since $s > 0$ and p_k is non-decreasing in k. Hence, either there exists some $k \in \{2, \ldots, n\}$ such that $f(k) < 0$ and $f(k - 1) \geq 0$, or either $f(n) \geq 0$, or $f(2) < 0$.

Suppose there exists some $k \in \{2, \ldots, n\}$ such that $f(k) < 0$ and $f(k - 1) \geq 0$. The optimal schedule is to have either $k - 1$ or $k - 2$ unsplit jobs, since the first inequality and monotonicity implies that a schedule with $k - 2$ unsplit jobs has a better objective value than a schedule with k or more unsplit jobs, and the second inequality and monotonicity implies that a schedule with $k - 1$ unsplit jobs has a better objective value than a schedule with $k - 3$ or fewer unsplit jobs.

If $f(n) \geq 0$ then the optimal solution is either to have only 1-jobs or have only job n as a 2-job. If $f(2) < 0$ then the optimal solution is either to have only 2-jobs or have only job 1 as a 1-job.

Straightforward implementation of the above gives the desired algorithm, the running time of which is dominated by sorting the jobs in SPT order. □

4 Troubles on more machines

The properties exposed in Sect. 2 have been proven to hold for any number of machines. The properties presented in Sect. 3 were shown specifically for two machines only. In this section, we investigate their analogues for three and more machines. We will present some examples of instances that show that the extension is far from trivial. It keeps the complexity of the problem on three and more machines as an intriguing open problem.

Lemma 5 shows that for two machines, there always exists an optimal schedule in which $|M_j|$ is monotonically non-decreasing in j. The following lemma shows that this does not hold for an arbitrary number of machines.

Lemma 6 *There exist instances for which there is no optimal schedule in which $|M_j|$ is monotonically non-decreasing in j.*

Proof Consider the instance on three machines having 10 jobs with their vector of processing times $p = (3, 10, 10, 10, 10, 50, 50, 50, 50, 50)$ (1 small job, 4 medium-sized jobs and 5 large jobs) and $s = 0.7$. We slightly perturb the processing times if necessary, obtaining $p_j < p_{j+1}$ for all $j = 1, 2, \ldots, n - 1$.

We found all optimal solutions for this instance by exhaustive search. An optimal solution is depicted in Fig. 2a. As we see, job 2 is split over machines 2 and 3, but job 3 starting later than job 2 is not split. Jobs 4 and 5 are again what we call 2-jobs and are split over machines 2 and 3. The large jobs are all split over all three machines.

Below, we will describe all other optimal solutions to this instance. We will consider two solutions to be the same, if one solution can be obtained from the other by a relabelling of machines, and/or (repeatedly) swapping the schedule of two machines from some time t till the end of the schedule, if these two machines both complete processing of some job at time t.

The second optimal schedule, in Fig. 2b, is obtained by scheduling job 1 on machine 1, job 2 split on machines 2 and 3, job 3 on machine 2 (or 3) and jobs 4 and 5 as split jobs on the machines not used by job 3. The remaining jobs are again all split on all three machines. It is easily verified that the objective of this schedule is the same as the objective of the schedule in Fig. 2a: the completion time of job 3 increases by 2, and the completion times of jobs 4 and 5 each decrease by 1, and all other completion times remain the same. The remaining two optimal schedules, in Fig. 2c, d are obtained by switching jobs 3 and 4 in the first two optimal schedules.
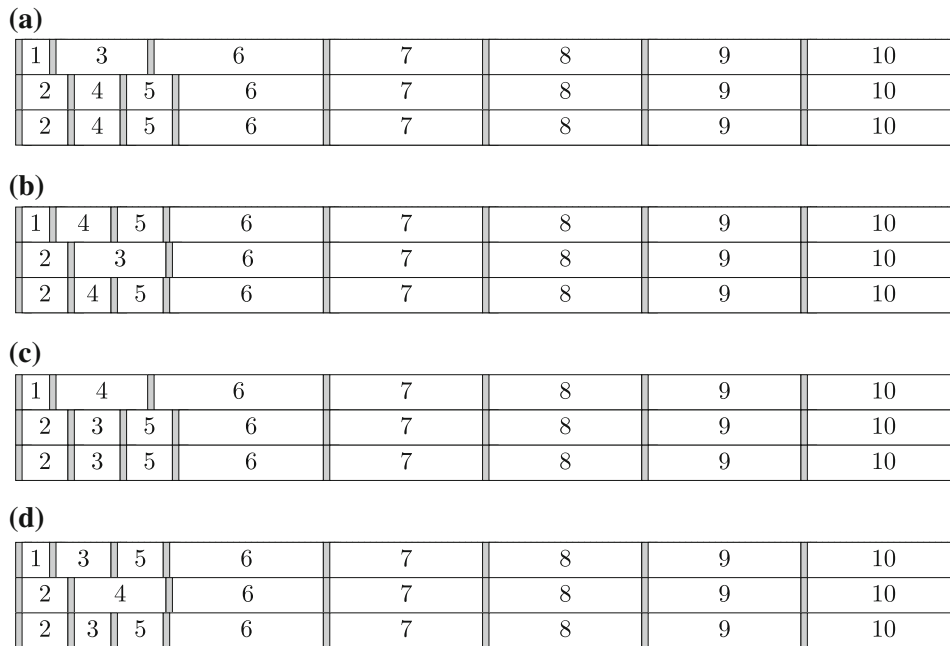
(a)

1	3		6		7		8		9		10
2	4	5	6		7		8		9		10
2	4	5	6		7		8		9		10

(b)

1	4	5	6		7		8		9		10
2	3		6		7		8		9		10
2	4	5	6		7		8		9		10

(c)

1	4		6		7		8		9		10
2	3	5	6		7		8		9		10
2	3	5	6		7		8		9		10

(d)

1	3	5	6		7		8		9		10
2	4		6		7		8		9		10
2	3	5	6		7		8		9		10

Fig. 2 Gantt charts depicting the optimal solutions to the 3-machine instance with processing times $p = (3, 10, 10, 10, 10, 50, 50, 50, 50, 50)$ (1 small job, 4 medium-sized jobs and 5 large jobs) and $s = 0.7$. The *grey blocks* indicate the setup times, the *numbered blocks* are scheduled job parts. Each row of blocks gives the schedule for a machine
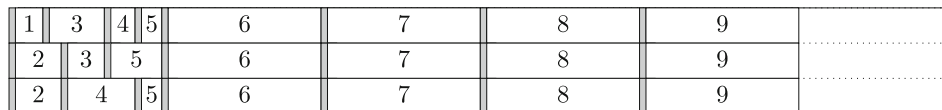
1	3	4	5	6		7		8		9	
2	3	5	6		7		8		9		
2	4	5	6		7		8		9		

Fig. 3 Gantt chart depicting the unique optimal solutions to the 3-machine instance with processing times $p = (3, 10, 10, 10, 10, 50, 50, 50, 50)$ (1 small job, 4 medium-sized jobs and 4 large jobs) and $s = 0.7$

We note that these schedules continue to be optimal if the processing times are slightly perturbed, as mentioned earlier.

All optimal solutions for this instance share the property that job 2 is a 2-job, and either job 3 or job 4 is a 1-job, which proves the lemma. □

If we slightly change the instance from the proof of Lemma 6 by deleting one of the large jobs, then there is a unique optimal solution, which splits job 3 over machines 1 and 2 and continues with splitting job 4 over machines 1 and 3. Job 5 and the four large jobs are split over all three machines, see Fig. 3.

Lemma 6 and the fact that a subtle change in the problem instance causes such a substantial change in the optimal schedule bodes ill for an algorithmic approach like the one in Sect. 3.

5 Approximation algorithm

We will now show a constant-factor approximation algorithm for our problem, for an arbitrary number of machines. We remark that we do not know whether this problem is NP-hard, but the examples in the previous section do show that the way a job is scheduled in an optimal schedule may depend on jobs that occur later in the schedule. Our approximation algorithm, on the other hand, is remarkably simple, and only uses a job's processing time and the setup time to determine how to schedule the job.

We schedule the jobs in order of non-decreasing processing time. Let $s > 0$ and let α be some constant that will be determined later. Job j will be scheduled such that it completes as early as possible under the restriction that it uses at most $\ell_j := \min\{\lceil \alpha p_j / s \rceil, m\}$ machines. Thus, the job will be scheduled on the at most ℓ_j machines that have minimum load in the schedule so far. It is easy to see that a job is always balanced this way.

Theorem 2 *The algorithm described above is a* $(2 + \alpha)$-*approximation algorithm for minimising the total completion time with job splitting and uniform setup times, provided that* $\alpha \geq \frac{1}{4}(\sqrt{17} - 1)$.

Proof Let σ be the schedule produced by the described algorithm. Note that the total load (processing times plus setup times) of all jobs in σ up to, but not including, job j is upper

bounded by $L_j = \sum_{k<j}(p_k + \ell_k s)$, since job k introduced at most ℓ_k setups. Therefore, the average load on the ℓ_j least-loaded machines is upper bounded by L_j/m. Since job j is balanced, we can thus upper bound the completion time $\tilde{C}_j$ of job j in the schedule by $L_j/m + p_j/\ell_j + s$. Note that this is an upper bound on the completion time of job j when we try to schedule it on at most ℓ_j machines.

Noting that

$$p_j/\ell_j = p_j/\min\{\lceil \alpha p_j/s \rceil, m\}$$
$$\leq p_j/\lceil \alpha p_j/s \rceil + p_j/m$$
$$\leq (1/\alpha)s + p_j/m,$$

and

$$\ell_k s = \min\{\lceil \alpha p_k/s \rceil, m\}s < \alpha p_k + s,$$

we obtain

$$\tilde{C}_j \leq L_j/m + p_j/\ell_j + s$$
$$\leq \frac{1}{m}\sum_{k<j}(p_k + \ell_k s) + p_j/\ell_j + s$$
$$< \frac{1}{m}\sum_{k<j}\big((1+\alpha)p_k + s\big) + p_j/m + (1 + 1/\alpha)s$$
$$\leq \frac{1+\alpha}{m}\sum_{k\leq j}p_k + \left(\frac{j-1}{m} + 1 + \frac{1}{\alpha}\right)s.$$

We can lower bound the sum of completion times in an *optimal* schedule by $\sum_j(s + \frac{1}{m}\sum_{k\leq j}p_k)$: suppose we only needed a setup time for the first job to be processed on a machine, for any machine. Clearly, the optimal sum of completion times for this problem gives a lower bound on the optimum for the original problem. Now, the optimal schedule when we only need a setup time for the first job on a machine processes the jobs in SPT order and splits each job over all machines, which gives a sum of completion times of $\sum_j(s + \frac{1}{m}\sum_{k\leq j}p_k)$.

Also, in any schedule, at most m jobs are preceded by only one setup, at most another m by two setups, etc., giving a lower bound of $\sum_j \lceil j/m \rceil s$ on the sum of completion times: this is exactly the optimal value when all processing times are 0. We will show below that $\sum_j \lceil \frac{j}{m} \rceil s \geq \sum_j \frac{j-1}{m}s + \frac{1}{2}ns$.

Hence, by using $1 + \alpha$ times the first bound, and 1 time the second bound, we get

$$(2+\alpha)\sum_j C_j$$
$$\geq (1+\alpha)\sum_j\left(s + \frac{1}{m}\sum_{k\leq j}p_k\right) + \left(\sum_j\frac{j-1}{m}s + \frac{1}{2}ns\right)$$
$$= \sum_j\left(\frac{1+\alpha}{m}\sum_{k\leq j}p_k + (1+\alpha)s + \frac{j-1}{m}s + \frac{1}{2}s\right),$$

which is at least as large as $\sum_j \tilde{C}_j$ provided $\alpha > 0$ and $\frac{3}{2} + \alpha \geq 1 + \frac{1}{\alpha}$, which is equivalent to $\alpha \geq \frac{1}{4}(\sqrt{17} - 1)$.

Next, we show that $\sum_j \lceil \frac{j}{m} \rceil s \geq \sum_j \frac{j-1}{m}s + \frac{1}{2}ns$. Let $j = qm + a$ for some $q \geq 0$ and $a \in \{1, \ldots, m\}$. Then

$$\left\lceil \frac{j}{m} \right\rceil - \frac{j-1}{m} = (q+1) - (qm + a - 1)/m$$
$$= 1 - (a-1)/m.$$

Now assume that $n = rm + b$, for some integer $r \geq 0$ and $b \in \{1, \ldots, m\}$. Then

$$\sum_{j=1}^n\left\lceil \frac{j}{m} \right\rceil - \sum_{j=1}^n\frac{j-1}{m}$$
$$= r\sum_{a=1}^m\left(1 - \frac{a-1}{m}\right) + \sum_{a=1}^b\left(1 - \frac{a-1}{m}\right)$$
$$= rm + b - r\sum_{a=1}^m\frac{a-1}{m} - \sum_{a=1}^b\frac{a-1}{m}$$
$$= n - r(m-1)/2 - \frac{1}{2}(b-1)b/m$$
$$\geq n - r(m-1)/2 - \frac{1}{2}(b-1)$$
$$= n - (rm + b)/2 + r/2 + 1/2$$
$$= n/2 + r/2 + 1/2 \geq n/2.$$

Hence multiplying both sides with s yields

$$\sum_{j=1}^n\left\lceil \frac{j}{m} \right\rceil s \geq \sum_{j=1}^n\frac{j-1}{m}s + \frac{1}{2}ns.$$

$\square$

Corollary 1 *There exists a $2 + \frac{1}{4}(\sqrt{17} - 1) < 2.781$-approximation algorithm for minimising total completion time with job splitting and uniform setup times.*

6 A polynomial-time approximation scheme

We give an approximation scheme which runs in polynomial time if the number of machines is assumed constant. The idea is simple: by splitting a job j, at most p_j on its completion time can be saved. Denote by OPT the sum of completion times of an optimal schedule. It is easy to show that the value of a non-preemptive SPT schedule is no more than $\sum_j p_j$ larger than OPT. In particular, if we schedule the first $K = n - \lceil m/\epsilon \rceil$ jobs by non-preemptive SPT then the extra cost is at most $\sum_{j=1}^K p_j$. But, as we will see, this is only an ϵ-fraction of the total completion time of the last $\lceil m/\epsilon \rceil$ jobs. These last jobs we schedule optimally given the schedule of the first K jobs.

Now, we define the algorithm and its running time in more detail. Let, as before, $p_1 \leq \cdots \leq p_n$. Let $K = n - \lceil m/\epsilon \rceil$. (If $K \leq 0$ then $n \leq \lceil m/\epsilon \rceil$ and the optimal solution can be found in constant time.) Let ρ be an optimal schedule and

let $\rho(K)$ be the schedule ρ restricted to the jobs $1, 2, \ldots, K$. By Lemma 2 we may assume that $\rho(K)$ has no idle time. Let $t_i(\rho)$ be the completion time of machine i in $\rho(K)$. The algorithm makes an approximate guess about the values $t_i(\rho)$. That means, it finds values t_i such that

$$t_i(\rho) \le t_i \le t_i(\rho) + s + p_K. \tag{2}$$

Note that for any i, we have $t_i(\rho) \le K(s + p_K)$. Hence, we need to try only K^m guesses for $(t_1, \ldots, t_m)$. Assume from now that we guessed $(t_1, \ldots, t_m)$ correctly, i.e., (2) is satisfied.

We apply SPT to the jobs $1, 2, \ldots, K$ such that no machine i is loaded more than $t_i + s + p_K$. This can easily be done as follows: apply list scheduling in SPT order and *close* a machine once its load becomes t_i or more. Let T_i be the completion time of machine i in the resulting schedule. Then, $T_i \le t_i + s + p_K \le t_i(\rho) + 2(s + p_K)$. Next, we find a near-optimal completion of the schedule by guessing for each job $j > K$ a set M_j and apply linear programming. There are $2^{m(n-K)}$ possibilities for choosing such sets, which is a constant. The linear program works as follows. Note that the LP of Sect. 2 can be extended to do the following. Given a set M_j for each job j and a time T_i for each machine, we can find the optimal schedule among all schedules for which: (i) job parts are in SPT order on each machine, (ii) machine i does not start before T_i, (iii) job j can only be scheduled on machines in M_j and (iv) job j has a setup time s for each machine in M_j even when its processing time x_{ij} is zero. Note that it is not clear if the LP gives us the real optimal completion since, we have not proved that the SPT properties hold also for optimal schedules if an initial part is fixed, as we do here. However, we can show that the solution given by the LP is close to optimal.

Approximation ratio Let σ be the final schedule and let $\tilde{C}_j$ be the completion time of job j. Here, we use OPT to denote the objective value of optimal schedule ρ. For any $h \in \{1, \ldots, n\}$ define $\mu_h = \sum_{k=1}^{h} (s + p_k)/m$. Then, for any schedule, the h − th completion time is at least μ_h. Hence,

$$\text{OPT} \ge \sum_{h=1}^{n} \mu_h \ge \sum_{h=K+1}^{n} \mu_h \ge \sum_{h=K+1}^{n} \mu_K$$

$$= \lceil m/\epsilon \rceil \mu_K \ge \frac{1}{\epsilon} \sum_{k=1}^{K} (s + p_k).$$

For the rest the proof, we will denote by C_h the completion time of job h in the optimal schedule and by $\tilde{C}_h$ the completion time of job h in the schedule produced by our algorithm. Further, we will use the notation $C^{(h)}$ for the h − th completion time of ρ, $(h = 1, \ldots, n)$. Notice that $C^{(h)}$ is not necessarily equal to C_h. For $h \le K$ it is easy to see that $\tilde{C}_h \le C^{(h)} + s + p_h$. This implies

$$\sum_{h=1}^{K} \tilde{C}_h \le \sum_{h=1}^{K} (C^{(h)} + s + p_h)$$

$$= \sum_{h=1}^{K} C^{(h)} + \sum_{h=1}^{K} (s + p_h)$$

$$\le \sum_{h=1}^{K} C^{(h)} + \epsilon \text{OPT}. \tag{3}$$

So for the first K jobs we are doing fine. Next, we give a bound on the total completion time of the other jobs.

Let M_j^* be the set of machines used by job j in the optimal schedule ρ. One of the guesses of the algorithm will be $M_j = M_j^*$ for $j > K$. We show that the corresponding LP-solution gives a near-optimal completion of the schedule.

A feasible LP solution is to take for x_{ij}, $j > K$, the values that correspond to ρ and choose values $C_j^{LP} = C_j + 2(s + p_K)$, where we we remind that C_j is the completion time of job j in the optimal schedule ρ. The latter is feasible since $T_i \le t_i(\rho) + 2(s + p_K)$. Hence, we can bound the total completion times of jobs $K + 1, \ldots, n$ by

$$\sum_{h=K+1}^{n} \tilde{C}_h \le \sum_{h=K+1}^{n} C_h^{LP}$$

$$\le \sum_{h=K+1}^{n} (C_h + 2(s + p_K))$$

$$= \left(\sum_{h=K+1}^{n} C_h \right) + 2(n - K)(s + p_K). \tag{4}$$

To bound the second term in the right-hand side of (4) we derive another bound on OPT:

$$\text{OPT} \ge \sum_{h=1}^{n} \mu_h \ge \sum_{h=K+1}^{n} \mu_h$$

$$= \sum_{h=K+1}^{n} \sum_{k=1}^{h} (s + p_k)/m$$

$$\ge \sum_{h=K+1}^{n} \sum_{k=K+1}^{h} (s + p_k)/m$$

$$\ge \sum_{h=K+1}^{n} (h - K)(s + p_K)/m$$

$$> \frac{1}{2}(n - K)^2 (s + p_K)/m$$

$$\ge \frac{1}{2}(n - K)(s + p_K)/\epsilon.$$

Combining this with (4) we get

$$\sum_{h=K+1}^{n} \tilde{C}_h \le \sum_{h=K+1}^{n} C_h + 4\epsilon \text{OPT} \le \sum_{h=K+1}^{n} C^{(h)} + 4\epsilon \text{OPT}.$$

Adding (3) we can bound the total completion time by $(1 + 5\epsilon)$OPT.

7 Hardness for weighted completion times

We prove that introducing weights for the jobs in our problem makes it strongly NP-hard for any number of machines and weakly NP-hard for 2 machines.

Theorem 3 *The problem of minimising total weighted completion time with job splitting and uniform setup times on parallel identical machines ($P|s, split| \sum w_j C_j$) is strongly NP-hard.*

Proof We reduce from 3-PARTITION: given $3n$ positive numbers $a_1, \ldots, a_{3n}$ and a number A such that $a_1 + \cdots + a_{3n} = nA$, does there exist a partition $A_1, \ldots, A_n$ of $\{1, \ldots, 3n\}$ such that $|A_i| = 3$ and $\sum_{j \in A_i} a_j = A$ for all i?

Given an instance of 3-PARTITION, we construct the following instance of our scheduling problem: We have n machines and $3n$ jobs. We set $p_j = a_j$ and $w_j = a_j + s$ for all $j = 1, \ldots, 3n$, where the setup time s is some large enough number, to be defined later.

The idea behind the reduction is the following: the large setup time will make sure that exactly three jobs are scheduled (unsplit) per machine. The weights are chosen such that a schedule where all machines complete at exactly the same time is optimal, if such a schedule is feasible.

Suppose we schedule the jobs unsplit where A_i is the set of jobs processed on machine i. Then, the cost of the schedule is:

$$\sum_{j=1}^{3n} w_j C_j = \sum_{i=1}^{n} \sum_{j \in A_i} w_j C_j$$
$$= \sum_{i=1}^{n} \sum_{j \in A_i} (s + a_j) \sum_{k \leq j} (s + a_k)$$
$$= \sum_{i=1}^{n} \sum_{j \in A_i} \sum_{k \in A_i : k \leq j} (s + a_j)(s + a_k)$$
$$= \frac{1}{2} \sum_{i=1}^{n} \left[\left(\sum_{j \in A_i} (s + a_j) \right)^2 + \sum_{j \in A_i} (s + a_j)^2 \right]$$
$$= \frac{1}{2} \sum_{i=1}^{n} l_i^2 + \frac{1}{2} \sum_{j=1}^{3n} (s + a_j)^2,$$

where l_i is the total load on machine i. Note that the second term is independent of the schedule. This cost is minimised when $l_i = l_h$ for all i and h and this can be realised if a perfect 3-partition exists. Let us denote this minimum by OPT_{3P}.

If no perfect 3-partition exists, then any schedule where no jobs are split has strictly higher cost than OPT_{3P}. It remains to prove that also any schedule with at least one split job has a strictly higher cost than OPT_{3P}.

First observe that

$$\text{OPT}_{3P} = \frac{1}{2} \sum_{i=1}^{n} (3s + A)^2 + \frac{1}{2} \sum_{j=1}^{3n} (s + a_j)^2$$
$$= 6ns^2 + O(ns).$$

Now assume that at least one job is split, then there are at least $3n + 1$ setup times of s each. Consider the extreme case where all $3n$ values a_j are zero. In this case, it is easy to see that the weighted sum of the $3n$ completion times is at least $(6n + 1)s^2$. Clearly, this bound holds as well for arbitrary value a_j. For large enough s we have $(6n + 1)s^2 > \text{OPT}_{3P}$. □

Theorem 4 *The problem $P2|s, split| \sum w_j C_j$ is weakly NP-hard.*

Proof We now reduce from a restricted form of the SUBSET SUM problem: Given $2n$ positive integers $a_1, \ldots, a_{2n}$ such that $a_1 + \cdots + a_{2n} = 2A$, is there a set $I \subset \{1, \ldots, 2n\}$ such that $|I| = n$ and $\sum_{i \in I} a_i = A$? Given an instance of SUBSET SUM, we construct the following instance of our scheduling problem. We have 2 machines and $2n$ jobs. We set $p_j = a_j$ and $w_j = a_j + s$ for $j = 1, \ldots, 2n$, where the setup time s is some large enough number, to be defined later. The proof follows the same reasoning as the previous proof: the large setup time will now make sure that exactly n jobs are scheduled (unsplit) per machine, and the weights will make sure that a schedule where the two machines complete at exactly the same time is optimal, if such a schedule is feasible.

Suppose we schedule the jobs unsplit. Then, just as in the proof above for an arbitrary number of machines we have that the cost of the schedule is:

$$\sum_{j=1}^{2n} w_j C_j = \frac{1}{2}(l_1^2 + l_2^2) + \frac{1}{2} \sum_{j=1}^{2n} (s + a_j)^2,$$

where l_i is the total load on machine i. Note that the second term is independent of the schedule. This cost is minimised when $l_1 = l_2$ and this can be realised if a perfect subset I exists. Let us denote this minimum by OPT_S.

If no perfect subset exists, any unsplit schedule has strictly higher cost. It remains to prove that also any schedule with at least one split has a strictly higher cost than OPT_S.

First observe that

$$\text{OPT}_S = (ns + A)^2 + \frac{1}{2} \sum_{j=1}^{2n} (s + a_j)^2$$
$$= (n^2 + n)s^2 + O(ns).$$

Now assume that at least one job is split, then there are at least $2n + 1$ setup times of s each. Consider the extreme case where all $2n$ values a_j are zero. In this case, it is easy to see that the weighted sum of the $2n$ completion times is at least

Table 1 Minimising total (weighted) completion time and makespan with job splitting and setup times

Problem	Complexity	Algorithm
$P \mid \text{split} \mid \sum C_j$	In P	Divide jobs equally over the machines in SPT order
$P2 \mid s, \text{split} \mid \sum C_j$	In P	Algorithm of Sect. 3
$Pm \mid s, \text{split} \mid \sum C_j$	?	PTAS of Sect. 6
$P \mid s, \text{split} \mid \sum C_j$	?	2.781-approx. of Sect. 5
cf. $P \mid s, \text{pmtn} \mid \sum C_j$	In P	SPT
$P \mid \text{split} \mid \sum w_j C_j$	In P	Divide jobs equally over the machines in WSPT order
cf. $P \mid \text{pmtn} \mid \sum w_j C_j$	NP-hard (Bruno et al. 1974)	PTAS (Afrati et al. 1999)
$P \mid s, \text{split} \mid \sum w_j C_j$	NP-hard	-
cf. $P \mid s, \text{pmtn} \mid \sum w_j C_j$	NP-hard	-
$P \mid s, \text{split} \mid C_{\max}$	NP-hard (Van der Ster 2010) [cf. Chen et al. (2006)]	$\frac{5}{3}$-approximate split/assignment (Chen et al. 2006)
$P \mid s, \text{split} \mid C_{\max}$ if $p_j \geq s \; \forall j$	NP-hard	$\frac{3}{2}$-approximate wrap-around (Van der Ster 2010) algorithm
cf. $P \mid s, \text{pmtn} \mid C_{\max}$	NP-hard (Schuurman and Woeginger 1999)	PTAS (Schuurman and Woeginger 1999)

$(n^2 + n + 1)s^2$. Clearly, this bound holds as well for arbitrary values a_j. For large enough s we have $(n^2 + n + 1)s^2 > \text{OPT}_S$. □

8 Epilogue

In the following table, we gather the state of the art on scheduling problems with job splitting and uniform setup times. For describing the problems in the first column of the table we use the standard three-field scheduling notation (Graham et al. 1979). In the first field, expressing the processor environment, we only consider parallel identical machines, denoted by P, possibly with the number of parallel machines mentioned additionally. In the second field, expressing job characteristics, the term 'pmtn' denotes ordinary preemption, 'split' denotes job splitting as we consider in this paper and s denotes the presence of uniform setup times. Though this paper is mainly concerned with problems with a total completion time objective, indicated by $\sum C_j$ in the third field, expressing the objective, we will also show the state of the art on the total weighted completion time (indicated by $\sum w_j C_j$) and on the makespan (indicated by $C_{\max}$) (Table 1).

In the second column, we summarise the complexity status of these problems. A question mark indicates that the complexity of the problem is unknown. In the third column, we give the best approximation guarantee known, where a '-' indicates that no algorithm with a performance guarantee is known. If we consider it relevant, we also present, as a footnote, the knowledge on the comparable version with preemption instead of splitting.

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