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Tails of the endpoint distribution of directed polymers

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Abstract. We prove that the random variable $\mathcal{T} = \arg\max_{t\in\mathbb{R}}\{\mathcal{A}_2(t) - t^2\}$, where $\mathcal{A}_2$ is the Airy$_2$ process, has tails which decay like e^{-ct^3}. The distribution of $\mathcal{T}$ is a universal distribution which governs the rescaled endpoint of directed polymers in $1+1$ dimensions for large time or temperature.

Résumé. Nous prouvons qu'une variable aléatoire $\mathcal{T} = \arg\max_{t\in\mathbb{R}}\{\mathcal{A}_2(t) - t^2\}$, où $\mathcal{A}_2$ est un processus Airy$_2$ a une queue qui décroît comme e^{-ct^3}. La distribution de $\mathcal{T}$ est une distribution universelle qui gouverne la position du point final d'un polymère dirigé en dimension $1+1$ à temps grand ou à grande température.

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1. Introduction

Consider the following model of a *directed polymer in a random environment*. A *polymer path* is a nearest-neighbor random walk path $\pi = (\pi_0, \pi_1, \ldots)$ in $\mathbb{Z}$ started at the origin, that is, $\pi_0 = 0$ and $\pi_k - \pi_{k-1} = \pm 1$. On $\mathbb{Z}_+ \times \mathbb{Z}$ we place a collection of independent random weights $\{\omega_{i,j}\}_{i\geq 0, j\in\mathbb{Z}}$. The *weight* of a polymer path segment π of length N is defined as

$$W_N(\pi) = \mathrm{e}^{\beta\sum_{k=0}^N \omega_{i,\pi_i}}$$

for some fixed $\beta > 0$ which is known as the *inverse temperature*. If we restrict our attention to paths of length N which go from the origin to some given $x \in \mathbb{Z}$ then we talk about a *point-to-point polymer*, defined through the path measure

$$Q^{\mathrm{point}}_{N,x}(\pi) = \frac{1}{Z^{\mathrm{point}}(N,x)} W_N(\pi)$$

for π of length N going from the origin to x and $Q^{\mathrm{point}}_{N,x}(\pi) = 0$ otherwise. The normalizing constant $Z^{\mathrm{point}}(N,x) = \sum_{\pi:\pi(0)=0,\pi(N)=x} W_N(\pi)$ is known as the *point-to-point partition function*. Similarly, if we consider all possible paths of length N then we talk about a *point-to-line polymer*, defined through the path measure

$$Q^{\mathrm{line}}_{N}(\pi) = \frac{1}{Z^{\mathrm{line}}(N)} W_N(\pi)$$

for π of length N and $Q_N^{\mathrm{line}}(\pi)=0$ otherwise, with the *point-to-line partition function* $Z^{\mathrm{line}}(N)=\sum_{k=-N}^{N} Z^{\mathrm{point}}(N,k)$.

Our main interest will be the point-to-line case. The main quantities of interest in this case are the partition function and the position of the endpoint of the randomly chosen path, which we will denote by $\mathcal{T}_N$. It is widely believed that these quantities should satisfy the scalings

$$\log\bigl(Z^{\mathrm{line}}(N)\bigr)\sim aN+bN^{1/3}\chi \quad \text{and} \quad \mathcal{T}_N\sim N^{2/3}\mathcal{T},$$

where the constants a and b may depend on the distribution of the $\omega_{i,j}$ and β, but χ and $\mathcal{T}$ should be universal (up to some moment assumptions on the $\omega_{i,j}$'s).

While there are few results available in the general case described above, the zero-temperature limit $\beta\to\infty$, known as *last passage percolation*, is very well understood, at least for some specific choices of the environment variables $\omega_{i,j}$. We will restrict the discussion to *geometric last passage percolation*, where one considers a family $\{\omega_{i,j}\}_{i\in\mathbb{Z}^+,j\in\mathbb{Z}}$ of independent geometric random variables with parameter q (i.e., $\mathbb{P}(\omega_{i,j}=k)=q(1-q)^k$ for $k\geq 0$) and defines the *point-to-point last passage time* by

$$L(N,y)=\max_{\pi:\pi(0)=0,\pi(N)=y}\sum_{i=0}^{N}\omega_{i,\pi(i)}$$

and the *point-to-line last passage time* by

$$L(N)=\max_{y=-N,\dots,N}L(N,y).$$

We remark that this model is usually defined on $(\mathbb{Z}^+)^2$, which corresponds to rotating our picture by 45 degrees and working on the dual lattice. Although the exact results we will describe next have been proved for that case, the picture in our situation is morally the same, and hence for simplicity we present the results for last passage percolation on $\mathbb{Z}_+\times\mathbb{Z}$.

We define the rescaled process $t\mapsto H_N(t)$ by linearly interpolating the values given by scaling $L(N,y)$ through the relation

$$L(N,y)=c_1N+c_2N^{1/3}H_N\bigl(c_3N^{-2/3}y\bigr),$$

where the constants c_i have explicit expressions which depend only on q and can be found in [12]. The point-to-line rescaled process is then given by

$$G(N)=\sup_{t\in[-c_3N^{1/3},c_3N^{1/3}]}H_N(t),$$

and it is known in this case [4] that

$$G(N)\sim aN+bN^{1/3}\chi \tag{1.1}$$

with χ having the Tracy–Widom largest eigenvalue distribution for the Gaussian Orthogonal Ensemble (GOE) from random matrix theory [24] (the analogous result holds in the point-to-point case with χ now having the Tracy–Widom largest eigenvalue distribution for the Gaussian Unitary Ensemble (GUE) [23]). On the other hand, [12] showed that

$$H_N(t)\to\mathcal{A}_2(t)-t^2$$

in distribution as $N\to\infty$, in the topology of uniform convergence on compact sets. Here $\mathcal{A}_2$ is the Airy$_2$ process, which we describe below, and which is a universal limiting spatial fluctuation process in such models. As a consequence of Johansson's result (see also [6]), (1.1) translates into

$$\mathbb{P}\Bigl(\sup_{t\in\mathbb{R}}\bigl(\mathcal{A}_2(t)-t^2\bigr)\leq m\Bigr)=F_{\mathrm{GOE}}\bigl(4^{1/3}m\bigr) \tag{1.2}$$

(the $4^{1/3}$ arises from scaling considerations, or alternatively from the direct proof given in [6]).

The Airy$_2$ process was introduced by [16], and is defined through its finite-dimensional distributions, which are given by a Fredholm determinant formula: given $x_0,\dots,x_n\in\mathbb{R}$ and $t_0<\cdots<t_n$ in $\mathbb{R}$,

$$\mathbb{P}\big(\mathcal{A}_2(t_0)\le x_0,\dots,\mathcal{A}_2(t_n)\le x_n\big)=\det\big(I-\mathrm{f}^{1/2}K_{\mathrm{ext}}\mathrm{f}^{1/2}\big)_{L^2(\{t_0,\dots,t_n\}\times\mathbb{R})}, \tag{1.3}$$

where we have counting measure on $\{t_0,\dots,t_n\}$ and Lebesgue measure on $\mathbb{R}$, f is defined on $\{t_0,\dots,t_n\}\times\mathbb{R}$ by $\mathrm{f}(t_j,x)=\mathbf{1}_{x\in(x_j,\infty)}$, and the *extended Airy kernel* [9,13,16] is defined by

$$K_{\mathrm{ext}}\big(t,\xi;t',\xi'\big)=\begin{cases}\int_0^\infty \mathrm{d}\lambda\,\mathrm{e}^{-\lambda(t-t')}\,\mathrm{Ai}(\xi+\lambda)\,\mathrm{Ai}(\xi'+\lambda), & \text{if } t\ge t'\\ -\int_{-\infty}^0 \mathrm{d}\lambda\,\mathrm{e}^{-\lambda(t-t')}\,\mathrm{Ai}(\xi+\lambda)\,\mathrm{Ai}(\xi'+\lambda), & \text{if } t<t',\end{cases}$$

where $\mathrm{Ai}(\cdot)$ is the Airy function. In particular, the one point distribution of $\mathcal{A}_2$ is given by the Tracy–Widom GUE distribution. An alternative formula for $\mathcal{A}_2$ due to [16], which is the starting of the proofs given in [6] of (1.2) and (1.8) below, and also of the main result of this paper, is given by

$$\begin{aligned}&\mathbb{P}\big(\mathcal{A}_2(t_0)\le x_0,\dots,\mathcal{A}_2(t_n)\le x_n\big)\\&\quad=\det\big(I-K_{\mathrm{Ai}}+\bar{P}_{x_0}\mathrm{e}^{(t_0-t_1)H}\bar{P}_{x_1}\mathrm{e}^{(t_1-t_2)H}\cdots\bar{P}_{x_n}\mathrm{e}^{(t_n-t_0)H}K_{\mathrm{Ai}}\big),\end{aligned} \tag{1.4}$$

where K_{Ai} is the *Airy kernel*

$$K_{\mathrm{Ai}}(x,y)=\int_{-\infty}^0 \mathrm{d}\lambda\,\mathrm{Ai}(x-\lambda)\,\mathrm{Ai}(y-\lambda),$$

H is the *Airy Hamiltonian* $H=-\partial_x^2+x$ and $\bar{P}_a$ denotes the projection onto the interval $(-\infty,a]$. Here, and in everything that follows, the determinant means the Fredholm determinant on the Hilbert space $L^2(\mathbb{R})$, unless a different Hilbert space is indicated in the subscript (the last formula (1.4) should be compared with (1.3), where the Fredholm determinant is computed in an extended space). The equivalence of (1.3) and (1.4) was derived in [16] and [17]. We refer the reader to [6,18] for more details.

Coming back to geometric last passage percolation, we turn to the random variables

$$\mathcal{T}_N=\inf\Big\{t:\sup_{s\le t}H_N(s)=\sup_{s\in\mathbb{R}}H_N(s)\Big\},$$

which correspond to the location of the endpoint of the maximizing path with unconstrained endpoint (that is, the zero-temperature point-to-line polymer). From the above discussion one expects the following:

Theorem 1. *Let $\mathcal{T}=\arg\max_{t\in\mathbb{R}}\{\mathcal{A}_2(t)-t^2\}$. Then, as $N\to\infty$, $\mathcal{T}_N\to\mathcal{T}$ in distribution.*

This result was proved by [12] under the additional hypothesis that the supremum of $\mathcal{A}_2(t)-t^2$ is attained at a unique point. The uniqueness was proved, using two different methods, by [7] and by Moreno Flores and us [15].

Although the result of Theorem 1 has only been proved in the case of geometric (or exponential) last passage percolation, the key point is that the *polymer endpoint distribution* is expected to be *universal* for directed polymers in random environments in $1+1$ dimensions, and even more broadly in the KPZ universality class, for example in particle models such as asymmetric attractive interacting particle systems (e.g. the asymmetric exclusion process), where second class particles play the role of polymer paths. This problem has received quite a bit of recent interest in the physics literature, see [15] and references therein for more details.

In [15] we obtained an explicit expression for the distribution of $\mathcal{T}$. More precisely, we obtained an explicit expression for the joint density of

$$\mathcal{T}=\arg\max_{t\in\mathbb{R}}\big\{\mathcal{A}_2(t)-t^2\big\}\quad\text{and}\quad\mathcal{M}=\max_{t\in\mathbb{R}}\big\{\mathcal{A}_2(t)-t^2\big\},$$

which we will denote as $f(t,m)$. To state the formula we need some definitions. Let B_m be the integral operator with kernel

$$B_m(x,y)=\mathrm{Ai}(x+y+m).$$

Recall that [8] showed that F_{GOE} can be expressed as the determinant

$$F_{\mathrm{GOE}}(m) = \det(I - P_0 B_m P_0), \tag{1.5}$$

where P_a denotes the projection onto the interval $[a, \infty)$ (the formula essentially goes back to [20]). In particular, note that since $F_{\mathrm{GOE}}(m) > 0$ for all $m \in \mathbb{R}$, (1.5) implies that $I - P_0 B_m P_0$ is invertible. For $t, m \in \mathbb{R}$ define the function

$$\psi_{t,m}(x) = 2\mathrm{e}^{xt}\big[t\,\mathrm{Ai}\big(x + m + t^2\big) + \mathrm{Ai}'\big(x + m + t^2\big)\big]$$

and the kernel

$$\Psi_{t,m}(x, y) = 2^{1/3}\psi_{t,m}\big(2^{1/3}x\big)\psi_{-t,m}\big(2^{1/3}y\big).$$

Then the joint density of $\mathcal{T}$ and $\mathcal{M}$ is given by

$$\begin{aligned} f(t,m) &= \det(I - P_0 B_{4^{1/3}m} P_0 + P_0 \Psi_{t,m} P_0) - F_{\mathrm{GOE}}\big(4^{1/3}m\big) \\ &= \mathrm{tr}\big[(I - P_0 B_{4^{1/3}m} P_0)^{-1} P_0 \Psi_{t,m} P_0\big] F_{\mathrm{GOE}}\big(4^{1/3}m\big). \end{aligned} \tag{1.6}$$

Integrating over m one obtains a formula for the probability density $f_{\mathrm{end}}(t)$ of $\mathcal{T}$, although it does not appear that the resulting integral can be computed explicitly. One can readily check nevertheless that $f_{\mathrm{end}}(t)$ is symmetric in t. Figure 1, taken from [15], shows a plot of the marginal $\mathcal{T}$ density.

The goal of this paper is to study the decay of the tails of $\mathcal{T}$. We will prove:

Theorem 2. *There is a $c > 0$ such that for every $\kappa > \frac{32}{3}$ and large enough t,*

$$\mathrm{e}^{-\kappa t^3} \le \mathbb{P}\big(|\mathcal{T}| > t\big) \le c\mathrm{e}^{(-4/3)t^3 + 2t^2 + \mathcal{O}(t^{3/2})}.$$

We believe that the correct exponent is the $-\frac{4}{3}$ obtained in the upper bound (we remark that we have not attempted to get better estimates on the lower order terms in the upper bound). The tail decay of order e^{-ct^3} confirms a prediction made in the physics literature in [11], see also [14]. Their idea is to argue by analogy with the argmax of Brownian motion minus a parabola. In that case one has a complete analytical solution [10].

We will give two proofs of the upper bound, both in Section 2. The first one is based on a direct application of the formula (1.6) for the joint density of $\mathcal{T}$ and $\mathcal{M}$. The second proof will start from a probabilistic argument and then use the continuum statistics formula for the Airy_2 process obtained in [6] to estimate the probability that the maximum is attained very far from the origin. This formula corresponds to the continuum limit of (1.4) and is given

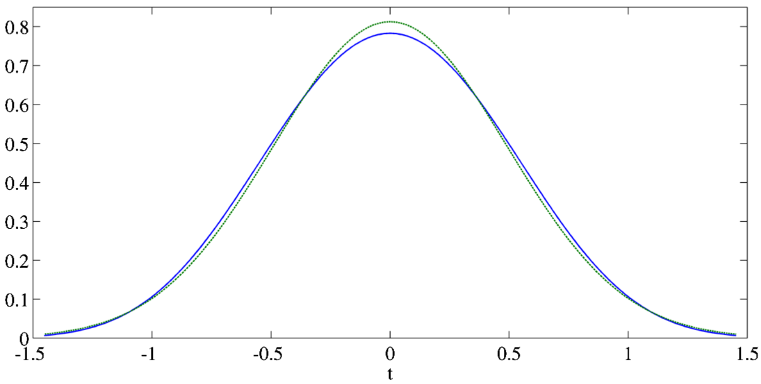

Fig. 1. Plot of the density of $\mathcal{T}$ compared with a Gaussian density with the same variance 0.2409 (dashed line). The excess kurtosis $\mathbb{E}(\mathcal{T}^4)/\mathbb{E}(\mathcal{T}^2)^2 - 3$ is -0.2374.

as follows (see [6] for more details). Fix $\ell < r$ and $g \in H^1([\ell, r])$ and define an operator $\Theta^g_{[\ell,r]}$ acting on $L^2(\mathbb{R})$ by $\Theta^g_{[\ell,r]} f(\cdot) = u(r,\cdot)$, where $u(r,\cdot)$ is the solution at time r of the boundary value problem

$$\begin{aligned} &\partial_t u + Hu = 0 \quad \text{for } x < g(t), t \in (\ell, r), \\ &u(\ell, x) = f(x)\mathbf{1}_{x<g(\ell)}, \\ &u(t,x) = 0 \quad \text{for } x \geq g(t). \end{aligned} \tag{1.7}$$

Then

$$\mathbb{P}\big(\mathcal{A}_2(t) \leq g(t) \text{ for } t \in [\ell, r]\big) = \det\big(I - K_{\mathrm{Ai}} + \Theta^g_{[\ell,r]} e^{(r-\ell)H} K_{\mathrm{Ai}}\big). \tag{1.8}$$

We remark that in the second proof actually get an upper bound with a larger $\mathcal{O}(t^2)$ correction in the exponent.

Not surprisingly, the lower bound turns out to be more difficult (in fact, for the upper bound we can basically use the estimate $|\det(I+A) - \det(I+B)| \leq \|A-B\|_1 e^{1+\|A\|_1+\|B\|_1}$ for trace class operators A and B directly to estimate the decay by computing the trace norm of two operators; no such estimate is available for the lower bound). In this case we will have to use a probabilistic argument to extract the lower bound from the well-known exact asymptotics for the tails of the GUE distribution, and then show that the remaining terms are of lower order. For this last task we will use again (1.8), but the argument is much more complicated than for the upper bound. Interestingly, it will involve turning an instance of (1.8) which mixes continuum and discrete statistics for $\mathcal{A}_2$ back into an extended kernel formula.

Remark 1.1.

1. *A few days before submitting this article, we became aware of the very recent work of* [21], *where he obtains, using non-rigorous arguments, an alternative formula for the joint distribution function of $\mathcal{M}$ and $\mathcal{T}$. His formula is obtained by taking the limit in N of a known formula for the joint distribution of the maximum and location of the maximum for the top line of N non-intersecting Brownian excursions, which is expected to converge to the* Airy$_2$ *process. The resulting formula is expressed in terms of quantities associated to the Hastings–McLeod solution of the Painlevé II equation, and has tails decaying like* $e^{(-4/3)t^3}$.
2. *During the refereeing process,* [3] *proved the equivalence of the formula of* [21] *and* (1.6). *Hence the rigorous validity of the formula of* [21] *is established based on* [15], *as well as the tail decay.*

2. Upper bound

Throughout the paper c and C will denote positive constants whose values may change from line to line. We will denote by $\|\cdot\|_{\mathrm{op}}$, $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively the operator, trace class and Hilbert–Schmidt norms of operators on $L^2(\mathbb{R})$ (see Section 3 of [6] for the definitions or [22] for a complete treatment). We will use the following facts repeatedly (they can all be found in [22]): if A and B are bounded linear operators on $L^2(\mathbb{R})$, then

$$\begin{aligned} &\|AB\|_1 \leq \|A\|_2\|B\|_2, \qquad \|AB\|_2 \leq \|A\|_{\mathrm{op}}\|B\|_2, \qquad \|AB\|_2 \leq \|A\|_2\|B\|_{\mathrm{op}}, \\ &\|A\|_{\mathrm{op}} \leq \|A\|_2 \leq \|A\|_1, \\ &\|A\|_2^2 = \int_{\mathbb{R}^2} dx\, dy\, A(x,y)^2, \end{aligned} \tag{2.1}$$

where in the last one we are assuming that A has integral kernel $A(x,y)$. We will also use the bound

$$\big|\det(I+A) - \det(I+B)\big| \leq \|A-B\|_1 e^{\|A\|_1+\|B\|_1+1} \leq \|A-B\|_1 e^{\|A-B\|_1+2\|B\|_1+1} \tag{2.2}$$

for any two trace class operators A and B.

We recall that the shifted Airy functions $\phi_\lambda(x) = \mathrm{Ai}(x-\lambda)$ are the generalized eigenfunctions of the Airy Hamiltonian, as $H\phi_\lambda = \lambda\phi_\lambda$, and the Airy kernel K_{Ai} is the projection of H onto its negative generalized eigenspace (see Remark 1.1 of [6]). This implies that $\mathrm{e}^{sH}K_{\mathrm{Ai}}$ has integral kernel

$$\mathrm{e}^{sH}K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-s\lambda}\,\mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda). \tag{2.3}$$

It also implies that $\mathrm{e}^{aH}K_{\mathrm{Ai}}\mathrm{e}^{bH}K_{\mathrm{Ai}} = \mathrm{e}^{(a+b)H}K_{\mathrm{Ai}}$. We will use this fact several times in this and the next section.

2.1. *First proof*

We start by writing

$$\mathbb{P}(\mathcal{T} > t) \le \mathbb{P}(\mathcal{T} > t, \mathcal{M} > -2t) + \mathbb{P}(\mathcal{M} \le -2t).$$

By (1.2) the second probability on the right side equals $F_{\mathrm{GOE}}(-2^{5/3}t) \le c\mathrm{e}^{-(4/3)t^3}$, where the tail bound can be found in [2]. Thus it will be enough to prove that

$$\mathbb{P}(\mathcal{T} > t, \mathcal{M} > -2t) \le c\mathrm{e}^{(-4/3)t^3 + 2t^2 + \mathcal{O}(t^{3/2})}. \tag{2.4}$$

Let $s \ge t$. We will assume for the rest of the proof that $m > -2t$. Using (1.5) and the first formula in (1.6) we get from (2.2) that

$$f(s,m) \le \|P_0\Psi_{s,m}P_0\|_1 \mathrm{e}^{1+2\|P_0 B_{4^{1/3}m} P_0\|_1 + \|P_0\Psi_{s,m}P_0\|_1}. \tag{2.5}$$

Using the identity

$$\int_{-\infty}^\infty \mathrm{d}u\,\mathrm{Ai}(a+u)\,\mathrm{Ai}(b-u) = 2^{-1/3}\,\mathrm{Ai}\big(2^{-1/3}(a+b)\big)$$

and letting $\varepsilon = t^{-1}$ we may write

$$\begin{aligned} &P_0 B_{4^{1/3}m} P_0 = 2^{1/3} Q_1 Q_2 \\ &\quad \text{with } Q_1(x,\lambda) = \mathbf{1}_{x\ge 0}\,\mathrm{Ai}\big(2^{1/3}x + m + \lambda\big)\mathrm{e}^{(1/2)\varepsilon\lambda},\ Q_2(\lambda,y) = \mathrm{e}^{(-1/2)\varepsilon\lambda}\,\mathrm{Ai}\big(2^{1/3}y + m - \lambda\big)\mathbf{1}_{y\ge 0}. \end{aligned} \tag{2.6}$$

Lemma 4.2 now gives

$$\|P_0 B_{4^{1/3}m} P_0\|_1 \le ct^{3/2}. \tag{2.7}$$

On the other hand, recall that the trace norm of an operator Ψ acting on $L^2(\mathbb{R})$ is defined as

$$\|\Psi\|_1 = \sum_{n=1}^\infty \langle e_n, |\Psi| e_n\rangle,$$

where $\{e_n\}_{n\ge 1}$ is any orthonormal basis of $L^2(\mathbb{R})$ and $|\Psi| = \sqrt{\Psi^*\Psi}$ is the unique positive square root of the operator $\Psi^*\Psi$. For the case $\Psi = P_0\Psi_{s,m}P_0$, since Ψ is a rank one operator it is easy to check that $\Psi^*\Psi$ has only one eigenvector, and in fact it is given by $\mathbf{1}_{x\ge 0}\psi_{-s,m}(2^{1/3}x)$ with associated eigenvalue $\lambda_{s,m} = 2^{1/3}\|P_0\psi_{s,m}(2^{1/3}\cdot)\|_2^2\|P_0\psi_{-s,m}(2^{1/3}\cdot)\|_2^2$. We deduce that $\|P_0\Psi_{s,m}P_0\|_1 = \sqrt{\lambda_{s,m}}$, and then by (2.5) and (2.7) we get

$$\int_{-2t}^\infty \mathrm{d}m f(s,m) \le \int_{-2t}^\infty \mathrm{d}m\sqrt{\lambda_{s,m}}\mathrm{e}^{1+ct^{3/2}+\sqrt{\lambda_{s,m}}}. \tag{2.8}$$

Now Lemma 4.3 gives

$$\begin{aligned}\int_{-2t}^{\infty} \mathrm{d}m\sqrt{\lambda_{s,m}} &= 2^{1/6}\int_{-2t}^{\infty} \mathrm{d}m \big\| P_0\psi_{s,m}\big(2^{1/3}\cdot\big)\big\|_2 \big\| P_0\psi_{-s,m}\big(2^{1/3}\cdot\big)\big\|_2 \\ &\le 2^{1/6}\bigg[\int_{-2t}^{\infty}\mathrm{d}m \big\| P_0\psi_{s,m}\big(2^{1/3}\cdot\big)\big\|_2^2\bigg]^{1/2}\bigg[\int_{-2t}^{\infty}\mathrm{d}m \big\| P_0\psi_{-s,m}\big(2^{1/3}\cdot\big)\big\|_2^2\bigg]^{1/2} \\ &\le c\mathrm{e}^{(-4/3)s^3+2st},\end{aligned}$$

and it is not hard to see from the proof of Lemma 4.3 that $\lambda_{s,m}$ is bounded uniformly for $m \ge -2t$, $s > t$ and large enough t. We deduce then from (2.8) that $\int_{-2t}^{\infty}\mathrm{d}m f(s,m) \le c\mathrm{e}^{(-4/3)s^3+2st+\mathcal{O}(t^{3/2})}$, and hence

$$\begin{aligned}\mathbb{P}(\mathcal{T} > t, \mathcal{M} > -2t) &= \int_t^{\infty}\mathrm{d}s\int_{-2t}^{\infty}\mathrm{d}m f(s,m) \le c\int_t^{\infty}\mathrm{d}s \mathrm{e}^{(-4/3)s^3+2st+\mathcal{O}(t^2)} \\ &\le c\mathrm{e}^{(-4/3)t^3+2t^2+\mathcal{O}(t^{3/2})},\end{aligned}$$

where the last estimate can be easily obtained from an application of Laplace's method, see the proof of Lemma 4.1 for a similar estimate. This gives (2.4) and the upper bound of Theorem 2.

2.2. *Second proof*

Since we already have a full proof of the upper bound, we will skip some details. The key result for this proof is the following:

Proposition 2.1. *Fix* $L \ge 1$. *Then there is a* $c > 0$ *such that for every* $m > 0$

$$\mathbb{P}\Big(\sup_{x\in[-L,L]}\mathcal{A}_2(x) > m+1\Big) \le c\mathrm{e}^{(-4/3)m^{3/2}}.$$

Proof. By (1.8) we have, writing $g_m(s) = s^2 + m$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{x\in[-L,L]}\mathcal{A}_2(x) \le m+1\Big) &\ge \mathbb{P}\Big(\sup_{x\in[-L,L]}\big(\mathcal{A}_2(x) - x^2\big) \le m\Big) \\ &= \det\big(I - K_{\mathrm{Ai}} + \mathrm{e}^{LH}K_{\mathrm{Ai}}\Theta^{g_m}_{[-L,L]}\mathrm{e}^{LH}K_{\mathrm{Ai}}\big),\end{aligned} \tag{2.9}$$

where we have used the cyclic property of the determinant together with the identity $\mathrm{e}^{2LH}K_{\mathrm{Ai}} = \mathrm{e}^{LH}K_{\mathrm{Ai}}\mathrm{e}^{LH}K_{\mathrm{Ai}}$ (see the remark after (2.3)). Now recall from Theorem 1.3 of [6] that

$$F_{\mathrm{GOE}}\big(4^{1/3}m\big) = \det\big(I - \mathrm{e}^{LH}K_{\mathrm{Ai}}R_L\mathrm{e}^{LH}K_{\mathrm{Ai}}\big),$$

where R_L is defined in (1.5) in [6]. Therefore by (2.2) we deduce that

$$\det\big(I - K_{\mathrm{Ai}} + \mathrm{e}^{LH}K_{\mathrm{Ai}}\Theta^{g_m}_{[-L,L]}\mathrm{e}^{LH}K_{\mathrm{Ai}}\big) \ge F_{\mathrm{GOE}}\big(4^{1/3}m\big) - \|A-B\|_1\mathrm{e}^{1+\|A-B\|_1+2\|B\|_1}, \tag{2.10}$$

where

$$A = K_{\mathrm{Ai}} - \mathrm{e}^{LH}K_{\mathrm{Ai}}\Theta^{g_m}_{[-L,L]}\mathrm{e}^{LH}K_{\mathrm{Ai}} \quad \text{and} \quad B = \mathrm{e}^{LH}K_{\mathrm{Ai}}R_L\mathrm{e}^{LH}K_{\mathrm{Ai}}$$

and we have used the triangle inequality in the exponent. Now $\|B\|_1$ can easily be bounded by some constant uniformly in $m > 0$ by an argument similar to the one used to obtain (2.7). On the other hand, using the decomposition of $\Theta^{g_m}_{[-L,L]}$ given in (3.5) in [6] we have

$$A - B = \mathrm{e}^{LH}K_{\mathrm{Ai}}\Omega_L\mathrm{e}^{LH}K_{\mathrm{Ai}},$$

where $\Omega_L = (R_L - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2}) - (\mathrm{e}^{-2LH} - \bar{P}_{m+L^2}\mathrm{e}^{-2LH}\bar{P}_{m+L^2})$. By Lemma 4.4 we get $\|A - B\|_1 \le c\mathrm{e}^{-\eta m^{3/2}}$ for some $\eta > \frac{4}{3}$, and then using this and (2.10) in (2.9) we obtain

$$\mathbb{P}\Big(\sup_{x\in[-1,1]} \mathcal{A}_2(x) \le m+1\Big) \ge F_{\mathrm{GOE}}\big(4^{1/3}m\big) - c\mathrm{e}^{-\eta m^{3/2}}.$$

The result follows from this and the asymptotics [2] $F_{\mathrm{GOE}}(4^{1/3}m) \ge 1 - cm^{-3/2}\mathrm{e}^{(-4/3)m^{3/2}}$. □

Using Proposition 2.1 we can derive the upper bound. Start by observing that, for fixed $\sigma \in (0,1)$,

$$\begin{aligned}\mathbb{P}\big(\mathcal{T}\in[s,s+2]\big) &\le \mathbb{P}\Big(\sup_{x\in[s,s+2]}\big(\mathcal{A}_2(x)-x^2\big) > \mathcal{A}_2(0)\Big) \le \mathbb{P}\Big(\sup_{x\in[s,s+2]}\mathcal{A}_2(x) > \mathcal{A}_2(0)+s^2\Big)\\ &\le \mathbb{P}\Big(\sup_{x\in[s,s+2]}\mathcal{A}_2(x) > (1-\sigma)s^2\Big) + \mathbb{P}\big(\mathcal{A}_2(0) < -\sigma s^2\big).\end{aligned} \tag{2.11}$$

The last probability equals $F_{\mathrm{GUE}}(-\sigma s^2)$, and then the asymptotics obtained in [2] give $\mathbb{P}(\mathcal{A}_2(0) < -\sigma s^2) \le c\mathrm{e}^{(-1/12)\sigma^3 s^6}$. For the other term on the right side of (2.11) we use Proposition 2.1 and the stationarity of the Airy$_2$ process to write, for $s \ge t$,

$$\mathbb{P}\Big(\sup_{x\in[s,s+2]}\mathcal{A}_2(x) > (1-\sigma)s^2\Big) \le c\mathrm{e}^{(-4/3)[(1-\sigma)s^2-1]^{3/2}}.$$

Therefore

$$\mathbb{P}\big(\mathcal{T}\in[s,s+2]\big) \le c\mathrm{e}^{-\min\{(1/12)\sigma^3 s^6,(4/3)[(1-\sigma)s^2-1]^{3/2}\}}.$$

Take $\sigma = 4^{2/3}s^{-1}$ so that the minimum in the above exponent equals $\frac{4}{3}[s^2 - 4^{2/3}s - 1]^{3/2} = \frac{4}{3}s^3 + \mathcal{O}(s^2)$ for large enough s. We deduce that

$$\mathbb{P}\big(\mathcal{T}\in[s,s+2]\big) \le c\mathrm{e}^{(-4/3)s^3+\mathcal{O}(s^2)}.$$

Summing this inequality over intervals of the form $[t+2k, t+2(k+1)]$ for $k \ge 0$ gives

$$\mathbb{P}(\mathcal{T} > t) \le c\mathrm{e}^{(-4/3)t^3+\mathcal{O}(t^2)}$$

for some $c > 0$ and large enough t, and now the upper bound in Theorem 2 (with a worse $\mathcal{O}(t^2)$ correction) follows from the symmetry of $\mathcal{T}$.

3. Lower bound

As we mentioned, the lower bound turns out to be more delicate, because in this case a simple bound like (2.2) is not available. The main idea of the proof is to compare the probability we are interested in with an expression involving the one-dimensional marginal of the Airy$_2$ process, and then extract the lower bound from known asymptotics for the Tracy–Widom GUE distribution. This comparison will introduce an error term, and most of the work in the proof will be to show that this error term is of lower order.

The first step in the comparison is to write, for $t > 0$, $\beta \ge 0$ and $s > 0$,

$$\mathbb{P}\big(|\mathcal{T}| > t\big) \ge \mathbb{P}\big(\mathcal{A}_2(x) - x^2 \le \beta t^2\ \forall x \le t, \mathcal{A}_2(t+s) - (t+s)^2 > \beta t^2\big). \tag{3.1}$$

The idea is the following. If s is now taken to be reasonably large, then $\mathcal{A}_2(t+s)$ and $\mathcal{A}_2(x)$, $x \le t$, should have decorrelated somewhat. Assuming they have completely decorrelated, we would have that the right side is bounded below by $\mathbb{P}(\mathcal{A}_2(t+s) - (t+s)^2 > \beta t^2)$, which has the correct decay if we choose $s = \alpha t$. So the whole proof comes down to estimating the correction coming from the correlation.

Of course, from (3.1) we have

$$\mathbb{P}\big(|\mathcal{T}|>t\big)=\mathbb{P}\big(\mathcal{A}_2(x)-x^2\leq\beta t^2\ \forall x\leq t\big)\\ -\mathbb{P}\big(\mathcal{A}_2(x)-x^2\leq\beta t^2\ \forall x\leq t,\mathcal{A}_2(t+s)-(t+s)^2\leq\beta t^2\big). \tag{3.2}$$

The bound will then follow from the following lemma.

Lemma 3.1. *Let $\beta>3$. There is an $\alpha_0>0$ (which depends on β) such that if $\alpha\in(0,\alpha_0)$ and $s=\alpha t$, then for large enough t we have*

$$\mathbb{P}\big(\mathcal{A}_2(x)-x^2\leq\beta t^2\ \forall x\leq t,\mathcal{A}_2(t+s)-(t+s)^2\leq\beta t^2\big)\\ \leq\mathbb{P}\big(\mathcal{A}_2(x)-x^2\leq\beta t^2\ \forall x\leq t\big)\mathbb{P}\big(\mathcal{A}_2(t+s)-(t+s)^2\leq\beta t^2\big)\\ \cdot\left[1+\frac{1}{2}a_2(t+s)^{-3}\mathrm{e}^{(-4/3)(\beta+1)^{3/2}(t+s)^3}\right],$$

where a_2 is defined implicitly in (3.3).

To see how the lower bound follows from this, let $\beta>3$ and choose α as in the lemma. Then letting $s=\alpha t$ and using the lemma and (3.2) we get

$$\mathbb{P}\big(|\mathcal{T}|>t\big)\geq\mathbb{P}\big(\mathcal{A}_2(x)-x^2\leq\beta t^2\ \forall x\leq t\big)\\ \cdot\left(1-\mathbb{P}\big(\mathcal{A}_2(t+s)-(t+s)^2\leq\beta t^2\big)\left[1+\frac{1}{2}a_2(t+s)^{-3}\mathrm{e}^{(-4/3)(\beta+1)^{3/2}(t+s)^3}\right]\right).$$

Now if we let $p_0=\mathbb{P}(\mathcal{A}_2(x)-x^2\leq 0\ \forall x\in\mathbb{R})=F_{\mathrm{GOE}}(0)>0$ then, since $t>0$, the first probability on the right side above is larger than p_0. On the other hand,

$$\mathbb{P}\big(\mathcal{A}_2(t+s)-(t+s)^2\leq\beta t^2\big)=F_{\mathrm{GUE}}\big((\beta+1)t^2+2st+s^2\big)\leq F_{\mathrm{GUE}}\big((\beta+1)(t+s)^2\big)\\ \leq 1-a_2(t+s)^{-3}\mathrm{e}^{(-4/3)(\beta+1)^{3/2}(t+s)^3} \tag{3.3}$$

for some explicit constant $a_2>0$ and large enough t, see for instance [2] for the precise bounds on the tails of the GUE distribution. This implies that

$$\mathbb{P}\big(|\mathcal{T}|>t\big)\geq p_0\left(\frac{1}{2}a_2(t+s)^{-3}\mathrm{e}^{(-4/3)(\beta+1)^{3/2}(t+s)^3}+\frac{1}{2}a_2^2(t+s)^{-6}\mathrm{e}^{(-8/3)(\beta+1)^{3/2}(t+s)^3}\right)\\ \geq c(1+\alpha)^{-3}t^{-3}\mathrm{e}^{(-4/3)(1+\beta)^{3/2}(1+\alpha)^3t^3},$$

and now the lower bound in Theorem 2 follows from choosing $\beta>3$ and a small enough $\alpha>0$ so that $\frac{4}{3}(1+\beta)^{3/2}(1+\alpha)^3<\kappa$.

Our goal then is to prove Lemma 3.1. For this we need an expression for the probability we want to bound. The answer follows from a simple extension of the result in [19], where we obtained an explicit expression for probabilities of the form

$$\mathbb{P}\Big(\sup_{x\leq t}\big(\mathcal{A}_2(x)-x^2\big)\leq m\Big)$$

and showed that they to correspond (after a suitable shift) to the one-dimensional marginals of the Airy$_{2\to 1}$ process [5]. To state the extension of that formula, define for $a,t\in\mathbb{R}$ the operators

$$\varrho_{a,t}f(x)=f\big(2\big(a+t^2\big)-x\big)\quad\text{and}\quad M_{a,t}f(x)=\mathrm{e}^{2t(x-a-t^2)}f(x)$$

acting on $f\in L^2(\mathbb{R})$. We will say that an operator acting on $L^2(\mathbb{R})$ is *identity plus trace class* if it can be written in the form $I+A$ with A a trace class operator.

Lemma 3.2. *With the above definitions, and for any $a, b \in \mathbb{R}$ and $s > 0$,*

$$\begin{aligned}&\mathbb{P}\big(\mathcal{A}_2(x) - x^2 \le a \ \forall x \le t, \mathcal{A}_2(t+s) - (t+s)^2 \le b\big)\\&\quad = \det\big(I - K_{\mathrm{Ai}} + K_{\mathrm{Ai}}(I - M_{a,t}\varrho_{a,t})\bar{P}_{a+t^2}\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}\mathrm{e}^{sH}K_{\mathrm{Ai}}\big).\end{aligned} \tag{3.4}$$

Moreover, the operator inside this determinant is identity plus trace class.

Proof. For $L > 0$ it is straighforward to adapt the proof given in [6] of the continuum statistics formula (1.8) to deduce that

$$\begin{aligned}&\mathbb{P}\big(\mathcal{A}_2(x) - x^2 \le a \ \forall x \in [-L, t], \mathcal{A}_2(t+s) - (t+s)^2 \le b\big)\\&\quad = \det\big(I - K_{\mathrm{Ai}} + \Theta_{[-L,t]}\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}\mathrm{e}^{(L+t+s)H}K_{\mathrm{Ai}}\big),\end{aligned}$$

where $\Theta_{[-L,t]}$ is defined as $\Theta^{g}_{[-L,t]}$ (see (1.7)) for $g(x) = x^2 + a$. Since $\mathrm{e}^{(L+t+s)H}K_{\mathrm{Ai}} = \mathrm{e}^{sH}K_{\mathrm{Ai}}\mathrm{e}^{(L+t)H}K_{\mathrm{Ai}}$ and $K_{\mathrm{Ai}} = \mathrm{e}^{sH}K_{\mathrm{Ai}}\mathrm{e}^{-sH}K_{\mathrm{Ai}} = \mathrm{e}^{-sH}K_{\mathrm{Ai}}\mathrm{e}^{sH}K_{\mathrm{Ai}}$ (see the remark after (2.3)), we can use the cyclic property of the determinant to turn the last determinant into

$$\det\big(I - K_{\mathrm{Ai}} + \mathrm{e}^{(L+t)H}K_{\mathrm{Ai}}\Theta_{[-L,t]}\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}\mathrm{e}^{sH}K_{\mathrm{Ai}}\big). \tag{3.5}$$

Rewriting the above operator as

$$I - \mathrm{e}^{-sH}K_{\mathrm{Ai}}P_{b+(t+s)^2}\mathrm{e}^{sH}K_{\mathrm{Ai}} - \mathrm{e}^{(L+t)H}K_{\mathrm{Ai}}\big(\mathrm{e}^{-(L+t)H} - \Theta_{[-L,t]}\big)\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}\mathrm{e}^{sH}K_{\mathrm{Ai}}, \tag{3.6}$$

it follows from an easy adaptation of the proof of Proposition 3.2 of [6] that the operator is identity plus trace class. On the other hand, in the proof of Theorem 1 of [19] it was shown that

$$\mathrm{e}^{(L+t)H}K_{\mathrm{Ai}}\Theta_{[-L,t]} \xrightarrow[L\to\infty]{} K_{\mathrm{Ai}}(I - M_{a,t}\varrho_{a,t})\bar{P}_{a+t^2}$$

in Hilbert–Schmidt norm, and a straightforward extension of the proof shows that the same holds if we post-multiply both sides by $\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}N$, where $Nf(x) = (1+x^2)^{1/2}f(x)$. Thus, since $N^{-1}\mathrm{e}^{sH}K_{\mathrm{Ai}}$ is Hilbert–Schmidt by (3.4) in [6] deduce by (2.1) that

$$\mathrm{e}^{(L+t)H}K_{\mathrm{Ai}}\Theta_{[-L,t]}\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}\mathrm{e}^{sH}K_{\mathrm{Ai}} \xrightarrow[L\to\infty]{} K_{\mathrm{Ai}}(I - M_{a,t}\varrho_{a,t})\bar{P}_{a+t^2}\mathrm{e}^{-sH}\bar{P}_{b+(t+s)^2}\mathrm{e}^{sH}K_{\mathrm{Ai}}$$

in trace class norm. This together with (3.5) and (3.6) yields (3.4) and, in particular, the fact that the operator inside this determinant is identity plus trace class. □

The key to obtain Lemma 3.1 from Lemma 3.2 is to turn the last determinant into the determinant of a 2×2 matrix kernel. Observe that, since $M_{a,t}$ and P_{a+t^2} commute and $P_{a+t^2}\varrho_{a,t} = \varrho_{a,t}\bar{P}_{a+t^2}$, the formula (3.4) can be rewritten as

$$\begin{aligned}&\mathbb{P}\big(\mathcal{A}_2(x) - x^2 \le a \ \forall x \le t, \mathcal{A}_2(t+s) - (t+s)^2 \le b\big)\\&\quad = \det\big(I - K_{\mathrm{Ai}} + K_{\mathrm{Ai}}(I - Q)\mathrm{e}^{-sH}(I - P_2)\mathrm{e}^{sH}K_{\mathrm{Ai}}\big),\end{aligned} \tag{3.7}$$

where

$$P_1 = P_{a+t^2}, \qquad P_2 = P_{b+(t+s)^2} \quad \text{and} \quad Q = P_1(I + M_{a,t}\varrho_{a,t}) = P_{a+t^2} + M_{a,t}\varrho_{a,t}\bar{P}_{a+t^2}. \tag{3.8}$$

Note that $Q^2 = Q$ (although Q is not a projection in $L^2(\mathbb{R})$, as it is an unbounded operator). This formula has exactly the same structure as the formula (1.4) for the finite-dimensional distributions of the Airy$_2$ process (in the case $n = 2$) which, we recall, is equivalent to the extended kernel formula (1.3). The equivalence of the two types of formulas was developed by [16] and [17], and later made rigorous and extended to the Airy$_1$ case by us in [18]. The same method will work for (3.7), and the result is the following:

Proposition 3.3. *With the notation introduced above,*

$$\begin{aligned}&\mathbb{P}\big(\mathcal{A}_2(x)-x^2\le a\ \forall x\le t, \mathcal{A}_2(t+s)-(t+s)^2\le b\big)\\&\quad=\det\left(I-\Gamma\begin{bmatrix}QK_{\rm Ai}P_1 & Q\mathrm{e}^{-sH}(K_{\rm Ai}-I)P_2\\ P_2\mathrm{e}^{sH}K_{\rm Ai}P_1 & P_2K_{\rm Ai}P_2\end{bmatrix}\Gamma^{-1}\right)_{L^2(\mathbb{R})^2},\end{aligned}\tag{3.9}$$

where

$$\Gamma=\begin{bmatrix}G&0\\0&G\end{bmatrix}\quad \textit{with } Gf(x)=\mathrm{e}^{-2tx}\varphi^{-1}(x)f(x) \textit{ and } \varphi(x)=\big(1+x^2\big)^{1/2}.$$

In particular, the operator above is identity plus trace class.

The conjugation by Γ is needed to make the operator trace class. Note that there is a slight difference between this formula and the ones for the Airy$_1$ and Airy$_2$ processes: the formula is not written in the most symmetric way, as the first column in the brackets is post-multiplied by P_1 instead of Q. Formally there is no difference, because since $Q^2=Q$ one can pre-multiply the whole matrix by $\left[\begin{smallmatrix}Q&0\\0&Q\end{smallmatrix}\right]$ and then use the cyclic property of the determinant to turn this into a post-factor of $P_1Q=Q$ for the first column. But this form of the formula will turn out to be better for obtaining the desired bounds in Lemma 3.4.

The proof of Proposition 3.3 follows the steps of the proofs in [17,18], but given the slight difference noted above, and since it is short and easy to present in the two-dimensional case, we include the details.

Proof of Proposition 3.3. We remark that all the manipulations performed on Fredholm determinants below rely on knowing that the operators inside each of them are identity plus trace class (see the proof of Theorem 1 of [18] for more details), but this can be seen in each case from Lemma 3.2 and Lemma 3.4 and similar estimates.

Let D denote the determinant in (3.9). The operator inside it can be factored as

$$\begin{aligned}&\begin{bmatrix}I & GQ\mathrm{e}^{-sH}P_2G^{-1}\\0&I\end{bmatrix}\\&\quad\cdot\left(I-\Gamma\begin{bmatrix}QK_{\rm Ai}P_1-Q\mathrm{e}^{-sH}P_2\mathrm{e}^{sH}K_{\rm Ai}P_1 & Q\mathrm{e}^{-sH}K_{\rm Ai}P_2-Q\mathrm{e}^{-sH}P_2K_{\rm Ai}P_2\\ P_2\mathrm{e}^{sH}K_{\rm Ai}P_1 & P_2K_{\rm Ai}P_2\end{bmatrix}\Gamma^{-1}\right).\end{aligned}$$

Note that the determinant of the first matrix on the right side above is 1, so D equals

$$\begin{aligned}&\det\left(I-\Gamma\begin{bmatrix}QK_{\rm Ai}P_1-Q\mathrm{e}^{-sH}P_2\mathrm{e}^{sH}K_{\rm Ai}P_1 & Q\mathrm{e}^{-sH}K_{\rm Ai}P_2-Q\mathrm{e}^{-sH}P_2K_{\rm Ai}P_2\\ P_2\mathrm{e}^{sH}K_{\rm Ai}P_1 & P_2K_{\rm Ai}P_2\end{bmatrix}\Gamma^{-1}\right)_{L^2(\mathbb{R})^2}\\&\quad=\det\left(I-\Gamma\begin{bmatrix}QK_{\rm Ai}-Q\mathrm{e}^{-sH}P_2\mathrm{e}^{sH}K_{\rm Ai} & 0\\ P_2\mathrm{e}^{sH}K_{\rm Ai} & 0\end{bmatrix}\begin{bmatrix}P_1 & \mathrm{e}^{-sH}P_2\\0&0\end{bmatrix}\Gamma^{-1}\right)_{L^2(\mathbb{R})^2}.\end{aligned}$$

Since $K_{\rm Ai}$ is a projection we may pre-multiply each entry in the second bracket by $K_{\rm Ai}$ and then use the cyclic property of the determinant to get

$$\begin{aligned}D&=\det\left(I-\begin{bmatrix}K_{\rm Ai}P_1 & K_{\rm Ai}\mathrm{e}^{-sH}P_2\\0&0\end{bmatrix}\begin{bmatrix}QK_{\rm Ai}-Q\mathrm{e}^{-sH}P_2\mathrm{e}^{sH}K_{\rm Ai} & 0\\ P_2\mathrm{e}^{sH}K_{\rm Ai} & 0\end{bmatrix}\right)_{L^2(\mathbb{R})^2}\\&=\det\left(I-\begin{bmatrix}K_{\rm Ai}QK_{\rm Ai}-K_{\rm Ai}Q\mathrm{e}^{-sH}P_2\mathrm{e}^{sH}K_{\rm Ai}+\mathrm{e}^{-sH}K_{\rm Ai}P_2\mathrm{e}^{sH}K_{\rm Ai} & 0\\ 0 & 0\end{bmatrix}\right)_{L^2(\mathbb{R})^2}\\&=\det\big(I-K_{\rm Ai}QK_{\rm Ai}+K_{\rm Ai}Q\mathrm{e}^{-sH}P_2\mathrm{e}^{sH}K_{\rm Ai}-\mathrm{e}^{-sH}K_{\rm Ai}P_2\mathrm{e}^{sH}K_{\rm Ai}\big)_{L^2(\mathbb{R})}.\end{aligned}$$

This last determinant equals the one on the right side of (3.7), and the result follows. ☐

The 2×2 matrix kernel formula is useful because it will allow us to extract easily the first two factors in the bound given in Lemma 3.1. This idea was introduced by [25], where he studied the asymptotics in t of $\mathbb{P}(\mathcal{A}_2(0) \le s_1, \mathcal{A}_2(t) \le s_2)$.

Proof of Lemma 3.1. We start with the formula in Proposition 3.3, with $a = b = \beta t^2$, and use the idea introduced in [25]: factor out the two diagonal terms in the determinant and then estimate the remainder. More precisely, we write

$$\begin{aligned} I - \Gamma &\begin{bmatrix} QK_{\mathrm{Ai}}P_1 & Q\mathrm{e}^{-sH}(K_{\mathrm{Ai}} - I)P_2 \\ P_2\mathrm{e}^{sH}K_{\mathrm{Ai}}P_1 & P_2K_{\mathrm{Ai}}P_2 \end{bmatrix}\Gamma^{-1} \\ &= \left(I - \Gamma\begin{bmatrix} QK_{\mathrm{Ai}}P_1 & 0 \\ 0 & P_2K_{\mathrm{Ai}}P_2 \end{bmatrix}\Gamma^{-1}\right) \\ &\quad \cdot \left(I - \Gamma\begin{bmatrix} 0 & (I - QK_{\mathrm{Ai}}P_1)^{-1}Q\mathrm{e}^{-sH}(K_{\mathrm{Ai}} - I)P_2 \\ (I - P_2K_{\mathrm{Ai}}P_2)^{-1}P_2\mathrm{e}^{sH}K_{\mathrm{Ai}}P_1 & 0 \end{bmatrix}\Gamma^{-1}\right) \end{aligned}$$

and then recognize that the determinant of the first factor on the right side equals

$$\begin{aligned} &\det\bigl(I - GQK_{\mathrm{Ai}}P_1G^{-1}\bigr)\det\bigl(I - GP_2K_{\mathrm{Ai}}P_2G^{-1}\bigr) \\ &\quad = \mathbb{P}\bigl(\mathcal{A}_2(x) - x^2 \le 0\ \forall x \le t\bigr)\mathbb{P}\bigl(\mathcal{A}_2(t+s) - (t+s)^2 \le 0\bigr). \end{aligned}$$

To get the last equality, observe first that $\det(I - GP_2K_{\mathrm{Ai}}P_2G^{-1}) = F_{\mathrm{GUE}}((t+s)^2)$ which is the second factor on the right side. For the first one we note that, by the cyclic property of determinants and the facts that $K_{\mathrm{Ai}}^2 = K_{\mathrm{Ai}}$ and $P_1Q = Q$,

$$\det\bigl(I - GQK_{\mathrm{Ai}}P_1G^{-1}\bigr) = \det(I - K_{\mathrm{Ai}}QK_{\mathrm{Ai}}) = \mathbb{P}\bigl(\mathcal{A}_2(x) - x^2 \le 0\ \forall x \le t\bigr)$$

by (2.9) in [19]. So we are left with estimating

$$\det\left(I - \Gamma\begin{bmatrix} 0 & (I - QK_{\mathrm{Ai}}P_1)^{-1}Q\mathrm{e}^{-sH}(K_{\mathrm{Ai}} - I)P_2 \\ (I - P_2K_{\mathrm{Ai}}P_2)^{-1}P_2\mathrm{e}^{sH}K_{\mathrm{Ai}}P_1 & 0 \end{bmatrix}\Gamma^{-1}\right)_{L^2(\mathbb{R})^2}.$$

The last determinant equals $\det(I - \widetilde{K})$, with $\widetilde{K} = R_{1,1}R_{1,2}R_{2,2}R_{2,1}$ and

$$\begin{aligned} R_{1,1} &= G(I - QK_{\mathrm{Ai}}P_1)^{-1}G^{-1}, \qquad R_{1,2} = GQ\mathrm{e}^{-sH}(K_{\mathrm{Ai}} - I)P_2, \\ R_{2,2} &= (I - P_2K_{\mathrm{Ai}}P_2)^{-1}, \qquad R_{2,1} = P_2\mathrm{e}^{sH}K_{\mathrm{Ai}}P_1G^{-1}. \end{aligned} \tag{3.10}$$

Now $|\det(I - \widetilde{K}) - \det(I)| \le \|\widetilde{K}\|_1\mathrm{e}^{1+\|\widetilde{K}\|_1} \le \|\widetilde{K}\|_1\mathrm{e}^2$ for $\|\widetilde{K}\|_1 \le 1$, so the proof will be complete once we show that

$$\|\widetilde{K}\|_1 \le \frac{1}{2}\mathrm{e}^{-2}a_2(t+s)^{-3}\mathrm{e}^{(-4/3)(\beta+1)^{3/2}(t+s)^3}. \tag{3.11}$$

To get this estimate we use (2.1) to write

$$\|\widetilde{K}\|_1 \le \|R_{1,1}\|_1\|R_{1,2}\|_1\|R_{2,2}\|_1\|R_{2,1}\|_1$$

and use Lemma 3.4, which gives, writing $\sigma = 1 + \beta$,

$$\|\widetilde{K}\|_1 \le cs^{-3/2}t^{1/2}\bigl(t^2 + s^2\bigr)^{3/2}\mathrm{e}^{(-2/3)(\sigma t^2+2ts+s^2)^{3/2}-(2/3)\sigma^{3/2}t^3-s(\sigma t^2+2ts+s^2)}.$$

Now taking $s = \alpha t$ we get

$$\|\widetilde{K}\|_1 \le c\alpha^{-3/2}\bigl(1 + \alpha^2\bigr)^{3/2}t^2\mathrm{e}^{(-4/3)(1+\alpha)^3\sigma^{3/2}t^3-h_\sigma(\alpha)t^3},$$

where $h_\sigma(\alpha) = \frac{2}{3}\sigma^{3/2} + \alpha(\sigma + 2\alpha + \alpha^2) - \frac{2}{3}(\sigma + 2\alpha + \alpha^2)^{3/2}$. Observe that for fixed $\sigma = \beta + 1 > 4$ we have $h_\sigma(0) = 0$ and $h'_\sigma(0) > 0$, which implies that $h_\sigma(\alpha) > 0$ for small enough α. Therefore (3.11) holds for small enough α and large enough t, and the result follows. □

Lemma 3.4. *Let $R_{1,1}$, $R_{1,2}$, $R_{2,2}$ and $R_{2,1}$ be defined as in* (3.10). *Then there is a $c > 0$ such that if t and s are large enough and $\sigma = \beta + 1 \geq 4$,*

$$\|R_{1,1}\|_1 \leq 2, \tag{3.12a}$$

$$\|R_{1,2}\|_1 \leq cs^{-1/2}t^{-1}\big(t^2 + s^2\big)^2 e^{-2\sigma t^3 - s(\sigma t^2 + 2ts + s^2)}, \tag{3.12b}$$

$$\|R_{2,2}\|_1 \leq 2, \tag{3.12c}$$

$$\|R_{2,1}\|_1 \leq cs^{-1}t^{3/2}\big(t^2 + s^2\big)^{-1/2} e^{(-2/3)(\sigma t^2 + 2ts + s^2)^{3/2} - (2/3)\sigma^{3/2}t^3 + 2\sigma t^3}. \tag{3.12d}$$

The proof of this result is postponed to Section 4.2.

4. Estimates of operator norms

We will use below the following well-known estimates for the Airy function (see (10.4.59-60) in [1]):

$$\big|\mathrm{Ai}(x)\big| \leq C e^{(-2/3)x^{3/2}} \quad \text{for } x > 0, \qquad \big|\mathrm{Ai}(x)\big| \leq C \quad \text{for } x \leq 0. \tag{4.1}$$

We start some with some basic integral estimates involving the Airy function:

Lemma 4.1. *There is a $c > 0$ such that for any $m > 0$ and $\alpha, \alpha', k, t \in \mathbb{R}$ such that if $\alpha > 0$ or $m \geq \frac{1}{4}\alpha^2 t^2$ we have, for large enough t,*

$$\int_m^\infty dx\, x^k e^{-\alpha t x}\,\mathrm{Ai}(x)^2 \leq ct^{2k-1} e^{-\alpha t m - (4/3)m^{3/2}}$$

and

$$\int_m^\infty dx \int_0^\infty dy\, x^k e^{-\alpha t x}\,\mathrm{Ai}(x+y)^2 e^{\alpha' y} \leq ct^{2k-1} e^{-\alpha t m - (4/3)m^{3/2}}.$$

Proof. Using (4.1) the first integral is bounded by

$$c\int_m^\infty dx\, x^k e^{-\alpha t x - (4/3)x^{3/2}} = ct^{2(k+1)} \int_{mt^{-2}}^\infty dx\, x^k e^{-(\alpha x + (4/3)x^{3/2})t^3}.$$

The exponent is maximized for $x \geq 0$ at $x = \frac{1}{4}\alpha^2 \leq m$ if $\alpha < 0$ and at 0 otherwise, so the first estimate follows from a simple application of Laplace's method, see Lemma 5.1 of [6]. The second integral is bounded in the same way after noting that $\mathrm{Ai}(x+y) \leq ce^{(-2/3)x^{3/2} - (2/3)y^{3/2}}$ for $x, y \geq 0$. □

4.1. *Estimates used for the upper bound*

Lemma 4.2. *Let Q_1 and Q_2 be defined as in* (2.6). *Then there is a $c > 0$ such that for $m \geq -2t$ and $t \geq 1$ we have*

$$\|Q_1\|_2 \leq ct^{3/4} \quad \textit{and} \quad \|Q_2\|_2 \leq ct^{3/4}.$$

Proof. Writing $\tilde{x} = 2^{1/3}x$ we have

$$\|Q_1\|_2^2 = \int_0^\infty dx \int_{-\infty}^\infty d\lambda\, e^{\varepsilon\lambda}\,\mathrm{Ai}(\tilde{x} + m + \lambda)^2 = \int_0^\infty dx\, e^{-\varepsilon\tilde{x} - \varepsilon m} \int_{-\infty}^\infty d\lambda\, e^{\varepsilon\lambda}\,\mathrm{Ai}(\lambda)^2$$

$$\leq ct \int_{-\infty}^\infty d\lambda\, e^{\lambda/t}\,\mathrm{Ai}(\lambda)^2$$

by our assumption $m \geq -2t$ and the facts that $\varepsilon = t^{-1}$ and $t \geq 1$. Using the estimate $|\operatorname{Ai}(\lambda)| \leq c|\lambda|^{-1/4}$ as $\lambda \to -\infty$ (see (10.4.60) in [1]) and (4.1) we deduce that $\|Q_1\|_2^2 \leq ct^{3/2}$. The bound for Q_2 is proved in exactly the same way. □

Lemma 4.3. *For large enough $t > 0$ and $s > t$ we have*

$$\int_{-2t}^{\infty} \mathrm{d}m \, \| P_0 \psi_{s,m}(2^{1/3} \cdot) \|_2^2 \leq \mathrm{e}^{(-4/3)s^3 + 4st}$$

and

$$\int_{-2|t|}^{\infty} \mathrm{d}m \, \| P_0 \psi_{-s,m}(2^{1/3} \cdot) \|_2^2 \leq \mathrm{e}^{(-4/3)s^3}.$$

Proof. Since $|\operatorname{Ai}'(x)|$ satisfies the same bound (4.1) as $\operatorname{Ai}(x)$ for $x > 0$ (see (10.4.61) in [1]) we have for $s \geq t > 2$ and $m \geq -2t$ that

$$\begin{aligned} \| P_0 \psi_{s,m}(2^{1/3} \cdot) \|_2^2 &= \int_0^{\infty} \mathrm{d}x 4 \mathrm{e}^{2xs} \big[s \operatorname{Ai}\big(x + m + s^2\big) + \operatorname{Ai}'\big(x + m + s^2\big) \big]^2 \\ &\leq c\big(1 + s^2\big) \int_0^{\infty} \mathrm{d}x \mathrm{e}^{2xs - (4/3)(x+m+s^2)^{3/2}}. \end{aligned}$$

Integrating over $m \geq -2t$ and scaling m and x by s^2 we get

$$\begin{aligned} \int_{-2t}^{\infty} \mathrm{d}m \, \| P_0 \psi_{s,m}(2^{1/3} \cdot) \|_2^2 &\leq cs^6 \int_{-2ts^{-2}}^{\infty} \mathrm{d}m \int_0^{\infty} \mathrm{d}x \mathrm{e}^{2xs^3 - (4/3)(x+m+1)^{3/2} s^3} \\ &= cs^6 \int_0^{\infty} \mathrm{d}m \int_0^{\infty} \mathrm{d}x \mathrm{e}^{2xs^3 - (4/3)(x+m+1-2ts^{-2})^{3/2} s^3} \\ &\leq cs^6 \int_0^{\infty} \mathrm{d}m \int_0^{\infty} \mathrm{d}x \mathrm{e}^{[2x - (4/3)(x+m+1)^{3/2}] s^3} \mathrm{e}^{4ts\sqrt{x+m+1}}, \end{aligned}$$

where in the last line we used the inequality $\frac{4}{3}(x + m + 1 - 2ts^{-2})^{3/2} \geq \frac{4}{3}(x + m + 1)^{3/2} - 4ts^{-2}\sqrt{x + m + 1}$ for $x > 0$, $m \geq -2t$ and $s \geq t > 2$. The term in brackets in the first exponential in the last integral is maximized at $x = m = 0$, and then applying Laplace's method as in the proof of Lemma 4.1 gives

$$\int_0^{\infty} \mathrm{d}m \, \| P_0 \psi_{s,m}(2^{1/3} \cdot) \|_2^2 \leq cs^3 \mathrm{e}^{-(4/3)s^3 + 4ts}.$$

This gives the first bound. The second bound is similar (and slightly simpler). □

Lemma 4.4. *Let Ω_L^m be the operator defined in* (1.6) *in* [6] *for $m, L > 0$* (*here we are making the dependence on m explicit in the notation*). *Then there is an $\eta > \frac{4}{3}$ satisfying the following*: *for fixed $L > 0$ there is a $c > 0$ such that for all $m > 0$*

$$\big\| \mathrm{e}^{LH} K_{\mathrm{Ai}} \Omega_L^m \mathrm{e}^{LH} K_{\mathrm{Ai}} \big\|_1 \leq c \mathrm{e}^{-\eta m^{3/2}}.$$

Proof. The proof of this result can be adapted from the proof of Lemmas 5.2 and 5.3 of [6]. In that result it is only proved that the above norm is finite, but one can get the above estimate by carefully keeping track of the dependence in m. We leave the details to the reader. □

4.2. *Proof of Lemma* 3.4

We will use Lemma 4.1 repeatedly without reference. The assumption $\sigma = \beta + 1 \geq 4$ enters crucially as it ensures in each case that the hypothesis of the lemma holds. Throughout the proof we will assume that $t \geq t_0$, where $t_0 > 1$ should be taken as large as needed to make the estimates work.

Recall the notation introduced in (3.8). In the present case we have $a = b = \beta t^2$, and thus recalling that $\sigma = 1 + \beta$ and writing $r^2 = \sigma t^2 + 2ts + s^2$, $M = M_{\beta t^2,t}$ and $\varrho = \varrho_{\beta t^2,t}$ to simplify the notation, we have

$$P_1 = P_{\sigma t^2}, \qquad P_2 = P_{r^2} \quad \text{and} \quad Q = P_1(I + M\varrho).$$

We also define the multiplication operator

$$Nf(x) = \varphi(x)^{-1} f(x),$$

where, as before, $\varphi(x) = (1 + x^2)^{1/2}$. Finally, we will use repeatedly the decomposition

$$K_{\mathrm{Ai}} = B_0 P_0 B_0,$$

where, we recall, $B_0(x, y) = \mathrm{Ai}(x + y)$.

Let us start with the first estimate. Since $Q = P_1 + P_1 M\varrho$, we have

$$\left\| GQKP_1G^{-1}\right\|_1 \leq \|GP_1B_0P_0\|_2 \left\| P_0B_0P_1G^{-1}\right\|_2 + \|GP_1M\varrho B_0P_0N\|_2 \left\|N^{-1}P_0B_0P_1G^{-1}\right\|_2. \tag{4.2}$$

Now

$$\|GP_1B_0P_0\|_2^2 = \int_{\sigma t^2}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y \varphi(x)^{-2}\mathrm{e}^{-4tx}\,\mathrm{Ai}(x+y)^2 \leq ct^{-5}\mathrm{e}^{-4\sigma t^3-(4/3)\sigma^{3/2}t^3},$$

while, recalling that $M\varrho f(x) = \mathrm{e}^{2t(x-\sigma t^2)} f(2\sigma t^2 - x)$, we have

$$\begin{aligned}\|GP_1M\varrho B_0P_0N\|_2^2 &= \int_{\sigma t^2}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y \varphi(x)^{-2}\mathrm{e}^{-4\sigma t^3}\,\mathrm{Ai}\big(2\sigma t^2 - x + y\big)^2\varphi(y)^{-2}\\ &\leq \mathrm{e}^{-4\sigma t^3}\|\mathrm{Ai}\|_\infty^2 \left\|P_1\varphi^{-1}\right\|_2^2 \left\|\varphi^{-1}\right\|_2^2 \leq ct^{-2}\mathrm{e}^{-4\sigma t^3}.\end{aligned}$$

Similarly

$$\left\|P_0B_0P_1G^{-1}\right\|_2^2 = \int_0^{\infty} \mathrm{d}x \int_{\sigma t^2}^{\infty} \mathrm{d}y \varphi(y)^2\mathrm{e}^{4ty}\,\mathrm{Ai}(x+y)^2 \leq ct^3\mathrm{e}^{4\sigma t^3-(4/3)\sigma^{3/2}t^3},$$

and one can easily see that the same estimate holds with a possibly larger constant for $\|N^{-1}P_0B_0P_1G^{-1}\|_2^2$. Putting these estimates together with (4.2) we deduce that

$$\left\|GQKP_1G^{-1}\right\|_1 \leq ct\mathrm{e}^{-(4/3)\sigma^{3/2}t^3} < \frac{1}{2}$$

for large enough t, and then

$$\|R_{1,1}\|_1 \leq \sum_{k\geq 0} \left\|\big(GQK_{\mathrm{Ai}}P_1G^{-1}\big)^k\right\|_1 \leq \sum_{k\geq 0} \left\|GQK_{\mathrm{Ai}}P_1G^{-1}\right\|_1^k < 2,$$

which gives (3.12a).

We turn now to $R_{1,2}$. Since $\mathrm{e}^{-sH}(K_{\mathrm{Ai}} - I)$ has integral kernel (in x, y) given by $\int_{-\infty}^0 \mathrm{d}\lambda \mathrm{e}^{s\lambda}\,\mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)$ (see (2.3) and the paragraph around it), we may use the decomposition $Q\mathrm{e}^{-sH}(K_{\mathrm{Ai}} - I)P_2 = QB_0\bar{P}_0\mathrm{e}^{s\xi}B_0P_2$, where $\mathrm{e}^{a\xi}$ is the multiplication operator defined by $(\mathrm{e}^{a\xi} f)(x) = \mathrm{e}^{ax} f(x)$, so that

$$\|R_{1,2}\|_1 \leq \|GQB_0\bar{P}_0N\|_2 \left\|N^{-1}\bar{P}_0\mathrm{e}^{s\xi}B_0P_2\right\|_2. \tag{4.3}$$

Now

$$\|GP_1B_0\bar{P}_0N\|_2^2 = \int_{\sigma t^2}^{\infty} dx \int_{-\infty}^{0} dy e^{-4tx}\varphi(x)^{-2}\operatorname{Ai}(x+y)^2\varphi(y)^{-2}$$
$$\leq \|\operatorname{Ai}\|_\infty^2 \|P_1\varphi^{-1}\|_2^2\|\varphi^{-1}\|_2^2 e^{-4\sigma t^3} \leq ct^{-2}e^{-4\sigma t^3},$$

while

$$\|GP_1M\varrho B_0\bar{P}_0N\|_2^2 = \int_{\sigma t^2}^{\infty} dx \int_{-\infty}^{0} dy e^{-4\sigma t^3}\varphi(x)^{-2}\operatorname{Ai}\bigl(2\sigma t^2 - x + y\bigr)^2\varphi(y)^{-2} \leq ct^{-2}e^{-4\sigma t^3}$$

in a similar way. On the other hand

$$\bigl\|N^{-1}\bar{P}_0e^{s\xi}B_0P_2\bigr\|_2^2 = \int_{-\infty}^{0} dx \int_{r^2}^{\infty} dy \varphi(x)^2 e^{2sx}\operatorname{Ai}(x+y)^2$$
$$= \int_{r^2}^{\infty} dy e^{-2sy}\int_{-\infty}^{y} dx\bigl(1+(x-y)^2\bigr)e^{2sx}\operatorname{Ai}(x)^2.$$

We split the x integral into the regions $(-\infty, 0]$ and $(0, y]$. On the first one we can estimate the integral by

$$\|\operatorname{Ai}\|_\infty^2 \int_{r^2}^{\infty} dy e^{-2sy}\int_{-\infty}^{0} dx\bigl(1+(x-y)^2\bigr)e^{2sx} \leq cr^4s^{-2}e^{-2sr^2},$$

while on the second one we estimate by

$$c\int_{r^2}^{\infty} dy e^{-2sy}\int_{0}^{y} dx\bigl(1+(x-y)^2\bigr)e^{-(4/3)x^{3/2}+2sx} \leq cr^4s^{-1}e^{-2sr^2},$$

giving

$$\bigl\|N^{-1}\bar{P}_0e^{s\xi}B_0P_2\bigr\|_2^2 \leq cr^4s^{-1}e^{-2sr^2}.$$

Putting the three bounds in (4.3) gives (3.12b).

For $R_{2,2}$ we observe that $\|P_2K_{\mathrm{Ai}}P_2\|_1 \leq \|P_2B_0P_0\|_2\|P_0B_0P_2\|_2$, which can easily be seen to be bounded by $\frac{1}{2}$ for large enough t by bounds similar to (and simpler than) those used to prove (3.12a), and thus we get (3.12c) in exactly the same way.

Finally, for $R_{2,1}$ we use a similar decomposition as for $R_{1,2}$: using (2.3) we may write

$$\bigl\|P_2e^{sH}K_{\mathrm{Ai}}P_1G^{-1}\bigr\|_1 \leq \bigl\|P_2B_0e^{-s\xi/2}P_0\bigr\|_2\bigl\|P_0e^{-s\xi/2}B_0P_1G^{-1}\bigr\|_2.$$

Now

$$\bigl\|P_2B_0e^{-s\xi/2}P_0\bigr\|_2^2 = \int_{r^2}^{\infty} dx \int_{0}^{\infty} dy e^{-sy}\operatorname{Ai}(x+y)^2 \leq cs^{-1}r^{-1}e^{(-4/3)r^3}$$

and

$$\bigl\|P_0e^{-s\xi/2}B_0P_1G^{-1}\bigr\|_2^2 = \int_{0}^{\infty} dx \int_{\sigma t^2}^{\infty} dy e^{-sx}\operatorname{Ai}(x+y)^2\varphi(y)^2e^{4ty} \leq cs^{-1}t^3e^{4\sigma t^3-(4/3)\sigma^{3/2}t^3},$$

which gives (3.12d).

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