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# Controllability and Observability for a Linear Time Varying System with Piecewise Constant Delay

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**Abstract** In this note, we obtain some necessary and sufficient conditions for the controllability and observability in a linear time-varying control system with piecewise constant state variables. The controllability results can be understood in terms of classical control systems on intervals $[k, k+1]$ with $k \in \mathbb{Z}$. We have preferred to use a global treatment instead of the reduction to a discrete equation. We also derive some results for the linear time-invariant case. Illustrative examples are presented.

**Keywords** Differential equations · Piecewise constants arguments · Controllability · Observability

**Mathematics Subject Classification** 34A38 · 96B05 · 93B07

## 1 Introduction

Controllability and observability of linear systems described by differential equations [8, 9] and difference equations [6] are classical problems in control theory. As far as the authors know, there exists no results for linear systems with piecewise constant arguments and the purpose of this note is to extend some results to the system:

$$\dot{x}(t) = A(t)x(t) + A_0(t)x\big([t]\big) + B(t)u(t), \tag{1.1}$$

$$y(t) = C(t)x(t) + C_0(t)x\big([t]\big) + D(t)u(t), \tag{1.2}$$

where $[\cdot]$ denotes the ceiling or integer part function, $x(\cdot) \in \mathbb{R}^{n\times 1}$ is the state vector, the input $u(\cdot)$ is a $q$-dimensional real, bounded and measurable function, and the output $y(\cdot)$

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is an $r$-dimensional real function. In addition, $A(t)$, $A_0(t)$, $B(t)$, $C(t)$, $C_0(t)$ and $D(t)$ are continuous matrices with dimensions having compatible order.

The system (1.1) is a particular case of DEPCA (Differential Equation with Piecewise Constant Arguments) whose study was initiated in [16] and extensively developed in [1, 5, 25].

**Definition 1** A function $z(t)$ is a solution of (1.1) on an interval $(\alpha, \beta)$ if:

(i) $z(t)$ is continuous on $(\alpha, \beta)$,
(ii) The derivative exists at each point $t \in (\alpha, \beta)$ with the possible exception of the points $t = k \in \mathbb{Z} \cap (\alpha, \beta)$, where one-sided derivatives exist,
(iii) The equation is satisfied for $z(t)$ for any interval $(k, k+1) \subset (\alpha, \beta)$ and it holds for the right derivative of $z(t)$ at the points $t = k \in \mathbb{Z} \cap (\alpha, \beta)$.

## 1.1 Some Previous Results About DEPCA Systems

The system (1.1) can be viewed as a perturbation of the linear DEPCA system:

$$\dot{x}(t) = A(t)x(t) + A_0(t)x\big([t]\big), \tag{1.3}$$

which has been studied from stability and admissibility points of view: indeed, a first type of results are devoted to study the asymptotic stability of (1.3). The linear time invariant case (from now on, LTI) is studied in [25], where necessary and sufficient conditions are obtained in terms of spectral radius. The linear time varying case (from now on, LTV) has been studied in [1] and [20], where a fundamental matrix for (1.3) is obtained.

A second type of results are related to the $\mathcal{S}$-admissibility problem: if (1.3) is perturbed by a function $f(\cdot)$ of some space $\mathcal{S}$ leading to

$$\dot{x}(t) = A(t)x(t) + A_0(t)x\big([t]\big) + f(t), \tag{1.4}$$

the objective is to find a set of conditions on $A(\cdot)$ and $A_0(\cdot)$ ensuring the existence of a solution $x(\cdot) \in \mathcal{S}$ of (1.4). The case of periodic functions is studied in [4], the almost periodic case is studied in [26, 28, 29], the pseudo-almost periodic case is studied in [19, 31], the remotely almost periodic case is studied in [32] and the almost automorphic case in [3, 24, 29].

## 1.2 DEPCA as Delay Systems

It is interesting to note that (1.3) has an alternative formulation as a delayed differential equation

$$\dot{x}(t) = A(t)x(t) + A_0(t)x\big(t - \tau(t)\big), \tag{1.5}$$

with sawtooth delay $\tau(t)$ defined by

$$\tau(t) = t - [t]. \tag{1.6}$$

This fact suggest to consider (1.1)–(1.2) with $[t] = t - \tau(t)$ as a particular case of the delay control system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + A_0(t)x\big(t - \sigma(t)\big) + B(t)u(t) \\ y(t) = C(t)x(t) + C_0(t)x\big(t - \sigma(t)\big) + D(t)u(t), \end{cases} \tag{1.7}$$

which can be studied by considering the properties of the delay $\sigma(t)$:

- The study of (1.7) with constant delay (*i.e.*, $\sigma(t) = h > 0$) presents several differences with the classic theory. In particular, the study of controllability and observability become more complicated due to the infinite-dimensional framework (we refer to the reader to Sect. 2.3 from [21] and references therein for a deeper discussion) and several definitions of controllability and observability have been introduced: *e.g.*, absolute controllability [18], spectral controllability [13], Euclidean controllability [10]. The reader is refered to [22] for the observability case.
- The system (1.7) with non uniform sawtooth delay, *i.e.*, $\sigma(t) = \lambda t - \lambda s_k$ for any $t \in [s_k, s_{k+1})$, where $\lambda > 0$ and $\{s_k\}$ is an increasing and divergent sequence has been studied in stability theory and feedback stabilization of LTI systems with sampling data outputs in [11, 14, 23]. Nevertheless, as we pointed out before, to the best of our knowledge, there are no controllability and observability results for this type of systems. This make interesting to study (1.5)–(1.6) since is a particular case of control system with uniform sawtooth delay with $s_k = k$ and $\lambda = 1$.

The big difference between (1.5)–(1.6) and linear systems with constant delay is that—provided some additional assumptions—the variation of parameters formula can be deduced in a finite-dimensional framework for linear systems having sawtooth delays. This fact allows to study the controllability and observability of (1.1)–(1.2) in a way rather distant from the constant delay case and closer to the methods developed for the classical and impulsive control systems.

### 1.3 Description of Our Approach

The variation of parameters formula for (1.4) is a key tool in the study of controllability and observability for (1.1)–(1.2) and a careful revision shows several similarities with the impulsive formulation [7, 12, 17, 27, 30].

Provided that the matrices $A(t)$ and $A_0(t)$ satisfy some technical assumptions, we obtain necessary and sufficient controllability conditions. An interesting consequence is that our results can be reformulated in terms of the controllability conditions for the classic control system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

on intervals $[k, k+1]$ with $k \in \mathbb{Z}$. We also obtain necessary and sufficient conditions for observability, but—contrarily to the previous case—we cannot reformulate our result in a classical fashion.

When considering the LTI case and supposing again that $A$ and $A_0$ satisfy some technical assumptions, we deduce classical conditions for controllability. On the other hand, additional difficulties arise in the observability problem, obtaining new results.

In spite that DEPCA systems have a corresponding linear discrete system, which allows to deduce several properties, our methods will be based in a global treatment inspired in the variation of parameters formula for any $t$. The results obtained by a discrete way are less general. More details and comments will be given in the article.

The remainder of this note is organized as follows. Section 2 introduces some basic notation and recalls the variation of parameters formula for (1.1). Sections 3 and 4 study controllability and observability, respectively. The LTI case is considered separately in Sect. 5. Finally, Sect. 6 presents some numerical examples.

## 2 Preliminaries

### 2.1 Notation and Terminology

The Cauchy matrix of the unperturbed system:

$$\dot{x} = A(t)x \tag{2.1}$$

will be denoted by $\Phi(t)$, without loss of generality we will assume that $\Phi(0) = I$, the identity matrix. The transition matrix related to $A(t)$ will be denoted by:

$$\Phi(t,s) = \Phi(t)\Phi^{-1}(s). \tag{2.2}$$

In [1, p. 19] and [20], the following $n \times n$ matrices are introduced:

$$J(t,\tau) = I + \int_{\tau}^{t} \Phi(\tau,s)A_0(s)\,ds, \tag{2.3}$$

$$E(t,\tau) = \Phi(t,\tau) + \int_{\tau}^{t} \Phi(t,s)A_0(s)\,ds = \Phi(t,\tau)J(t,\tau). \tag{2.4}$$

Given a set of $n \times n$ matrices $Q_k$ $(k = 1,\ldots,m)$, we will consider the product in the backward sense:

$$\prod_{k=1}^{m} Q_k = \begin{cases} Q_m \cdots Q_2 Q_1 & \text{if } m \geq 1 \\ I & \text{if } m < 1. \end{cases}$$

Given a square matrix $Q$, its transpose and rank will be denoted respectively by $Q^T$ and $\mathrm{Rank}(Q)$. Finally, the euclidean norm of a vector $z \in \mathbb{R}^{n\times 1}$ will be denoted by $\|z\|_2 = \sqrt{z^T z}$.

### 2.2 Variation of Parameters

The variation of parameters plays a key role in the controllability and observability study. In order to make the article self-contained, we will recall a particular case of a result obtained by [1] and [20], where $[t]$ is replaced by a general step function.

From now on, the following assumption will be needed in our work:

**(A)** The matrix $J(t,\tau)$ is non singular for any $t,\tau \in [k,k+1]$ with $k \in \mathbb{Z}$.

*Remark 1* Assumption **(A)** will be fundamental to obtain controllability and observability conditions. Moreover, notice that:

(i) Non singularity of $\Phi(\cdot,\cdot)$ combined with **(A)**, imply that $E(t,k)$ is non singular for any fixed $k \in \mathbb{Z}$ and $t \in [k,k+1]$.
(ii) If $A_0(t)$ is "small" in some appropriate sense, the matrices $J(t,\tau)$ and $E(t,\tau)$ can be seen as a perturbation of the identity matrix and the transition matrix $\Phi(t,\tau)$ respectively. In consequence, **(A)** could be verified if $A_0$ is considered as a "small" perturbation.
(iii) An explicit condition for $A(t)$ and $A_0(t)$ can be obtained by defining

$$\rho_k(M) = \exp\left(\int_k^{k+1} \left|M(t)\right| dt\right),$$

where $M(t)$ is a matrix function and $|\cdot|$ denotes a matrix norm. Indeed, by considering a particular case of [20, Lemma 4.3], it is proved that if

$$\rho_k(A)\ln\rho_k(A_0) \le \nu < 1 \quad \text{for any } k\in\mathbb{Z},$$

then **(A)** is verified. In particular, observe that if $A(t)=0$, then **(A)** is satisfied if

$$\int_k^{k+1} \left|A_0(s)\right| ds \le \nu < 1 \quad \text{for any } k\in\mathbb{Z}.$$

In particular, if $A$ and $A_0$ are constant matrices, the condition is satisfied if $\int_0^1 |e^{-As}A_0|\,ds < 1$, see [3] and Sect. 5 for details.

**Lemma 1** *The solution of* (1.1) *with initial condition* $x(t_0)=x_0\in\mathbb{R}^{n\times 1}$ *is*:

(i) *If* $[t_0]\le t\le [t_0]+1$:

$$\begin{aligned} x(t) &= E\big(t,[t_0]\big)E^{-1}\big(t_0,[t_0]\big)x_0 \\ &\quad + \big\{\Phi(t,t_0) - E\big(t,[t_0]\big)E^{-1}\big(t_0,[t_0]\big)\big\}\int_{[t_0]}^{t_0}\Phi(t_0,s)B(s)u(s)\,ds \\ &\quad + \int_{t_0}^{t}\Phi(t,s)B(s)u(s)\,ds. \end{aligned} \tag{2.5}$$

(ii) *If* $t>[t_0]+1$:

$$\begin{aligned} x(t) &= E\big(t,[t]\big)\prod_{\ell=[t_0]}^{[t]-1}E(\ell+1,\ell)E^{-1}\big(t_0,[t_0]\big)x_0 \\ &\quad + E\big(t,[t]\big)\prod_{\ell=[t_0]+1}^{[t]-1}E(\ell+1,\ell)\Phi\big([t_0]+1,t_0\big)\int_{[t_0]}^{t_0}\Phi(t_0,s)B(s)u(s)\,ds \\ &\quad + E\big(t,[t]\big)\prod_{\ell=[t_0]}^{[t]-1}E(\ell+1,\ell)E^{-1}\big(t_0,[t_0]\big)\int_{[t_0]}^{t_0}\Phi(t_0,s)B(s)u(s)\,ds \\ &\quad + E\big(t,[t]\big)\prod_{\ell=[t_0]+1}^{[t]-1}E(\ell+1,\ell)\int_{t_0}^{[t_0]+1}\Phi\big([t_0]+1,s\big)B(s)u(s)\,ds \\ &\quad + E\big(t,[t]\big)\sum_{r=[t_0]+1}^{[t]-1}\left\{\prod_{\ell=r+1}^{[t]-1}E(\ell+1,\ell)\right\}\int_r^{r+1}\Phi(r+1,s)B(s)u(s)\,ds \\ &\quad + \int_{[t]}^{t}\Phi(t,s)B(s)u(s)\,ds. \end{aligned} \tag{2.6}$$

*Proof* Firstly, let us assume that $[t_0]\le t<[t_0]+1$. Hence, $[t]=[t_0]$ and study the system:

$$\dot{x}(t) = A(t)x(t) + A_0(t)x\big([t_0]\big) + B(t)u(t).$$

By using (2.2) combined with (2.3)–(2.4) and integrating between $[t_0]$ and $t$, it can be verified that its solutions are defined by:

$$\begin{aligned} x(t) &= \Phi\big(t,[t_0]\big)x\big([t_0]\big) + \int_{[t_0]}^{t} \Phi(t,s)\big\{A_0(s)x\big([t_0]\big) + B(s)u(s)\big\}\,ds \\ &= \left\{\Phi\big(t,[t_0]\big) + \int_{[t_0]}^{t} \Phi(t,s)A_0(s)\,ds\right\}x\big([t_0]\big) + \int_{[t_0]}^{t} \Phi(t,s)B(s)u(s)\,ds \\ &= E\big(t,[t_0]\big)x\big([t_0]\big) + \int_{[t_0]}^{t} \Phi(t,s)B(s)u(s)\,ds \end{aligned}$$

and, by evaluating at $t = t_0$ together with **(A)**, we can deduce that:

$$x\big([t_0]\big) = E^{-1}\big(t_0,[t_0]\big)\left\{x_0 - \int_{[t_0]}^{t_0} \Phi(t_0,s)B(s)u(s)\,ds\right\}. \tag{2.7}$$

By using (2.7) together with the identity above, we can deduce that if $t_0 \leq t \leq [t_0+1]$, the solution is:

$$\begin{aligned} x(t) = {} & E(t,t_0)E^{-1}\big(t_0,[t_0]\big)x_0 - E\big(t,[t_0]\big)E^{-1}\big(t_0,[t_0]\big)\int_{[t_0]}^{t_0} \Phi(t_0,s)B(s)u(s)\,ds \\ & + \int_{[t_0]}^{t} \Phi(t,s)B(s)u(s)\,ds \end{aligned}$$

and (2.5) follows by noticing that:

$$\int_{[t_0]}^{t} \Phi(t,s)B(s)u(s)\,ds = \Phi(t,t_0)\int_{[t_0]}^{t_0} \Phi(t_0,s)B(s)u(s)\,ds + \int_{t_0}^{t} \Phi(t,s)B(s)u(s)\,ds.$$

Similarly, if $t \in (k,k+1]$ and $k \geq [t_0]+1$, we study the system

$$\dot{x}(t) = A(t)x(t) + A_0(t)x(k) + B(t)u(t), \quad \text{with } k = [t],$$

and it is easy to verify that its solution is given by:

$$x(t) = E\big(t,[t]\big)x(k) + \int_{[t]}^{t} \Phi(t,s)B(s)u(s)\,ds \quad \text{with } k = [t]. \tag{2.8}$$

The continuity of $x(\cdot)$ implies that, by letting $t \to k+1$, we can deduce that $x_k = x(k)$ is the solution of the difference equation:

$$\begin{cases} x_{k+1} = E(k+1,k)x_k + \displaystyle\int_{k}^{k+1} \Phi(k+1,s)B(s)u(s)\,ds \\ x_{[t_0]+1} = x\big([t_0]+1\big). \end{cases} \tag{2.9}$$

By using the variation of parameters for linear difference equations (see *e.g.*, [6]) it follows that:

$$x_k = \prod_{\ell=[t_0]+1}^{k-1} E(\ell+1,\ell)x\big([t_0]+1\big)$$

$$+\sum_{r=[t_0]+1}^{k-1}\left\{\prod_{\ell=r+1}^{k-1}E(\ell+1,\ell)\right\}\int_r^{r+1}\Phi(r+1,s)B(s)u(s)\,ds. \tag{2.10}$$

By inserting this identity in (2.8), it follows that

$$\begin{aligned}x(t)&=E\bigl(t,[t]\bigr)\prod_{\ell=[t_0]+1}^{[t]-1}E(\ell+1,\ell)x_{[t_0]+1}\\&\quad+E\bigl(t,[t]\bigr)\sum_{r=[t_0]+1}^{[t]-1}\left\{\prod_{\ell=r+1}^{[t]-1}E(\ell+1,\ell)\right\}\int_r^{r+1}\Phi(r+1,s)B(s)u(s)\,ds\\&\quad+\int_{[t]}^{t}\Phi(t,s)B(s)u(s)\,ds\end{aligned}$$

and (2.6) is obtained by the continuity of the solution together with

$$\begin{aligned}x_{[t_0]+1}&=E\bigl([t_0]+1,[t_0]\bigr)E^{-1}\bigl(t_0,[t_0]\bigr)x(t_0)\\&\quad+\left\{\Phi\bigl([t_0]+1,t_0\bigr)-E\bigl([t_0]+1,[t_0]\bigr)E^{-1}\bigl(t_0,[t_0]\bigr)\right\}\int_{[t_0]}^{t_0}\Phi(t_0,s)B(s)u(s)\,ds\\&\quad+\int_{t_0}^{[t_0]+1}\Phi\bigl([t_0]+1,s\bigr)B(s)u(s)\,ds,\end{aligned}$$

which is obtained by letting $t\to[t_0]+1$ in (2.5). □

*Remark 2* In the special case $t_0=[t_0]$, the formula (2.6) becomes:

$$\begin{aligned}x(t)&=E\bigl(t,[t]\bigr)\prod_{\ell=t_0}^{[t]-1}E(\ell+1,\ell)x_0\\&\quad+E\bigl(t,[t]\bigr)\prod_{\ell=t_0+1}^{[t]-1}E(\ell+1,\ell)\int_{t_0}^{[t_0]+1}\Phi(t_0+1,s)B(s)u(s)\,ds\\&\quad+E\bigl(t,[t]\bigr)\sum_{r=t_0+1}^{[t]-1}\left\{\prod_{\ell=r+1}^{[t]-1}E(\ell+1,\ell)\right\}\int_r^{r+1}\Phi(r+1,s)B(s)u(s)\,ds\\&\quad+\int_{[t]}^{t}\Phi(t,s)B(s)u(s)\,ds,\end{aligned}$$

which was previously deduced by Wiener (see *e.g.*, [25, Th. 1.45]).

*Remark 3* The assumption **(A)** plays a key role in order to ensure the existence of a solution. Indeed, for any bounded and measurable function $u\colon\mathbb{R}\to\mathbb{R}^q$, there exists a one to one correspondence between $x(t_0)=x_0$ and $x([t_0])$ defined by (2.7).

*Remark 4* Equation (2.5) provides a backward continuation for $[t_0]\le t<t_0$.

## 3 Controllability

**Definition 2** The linear system (1.1) is called **controllable on** $[t_0, t_f]$ ($t_0 < t_f$) if for any couple of vectors $x_0 \in \mathbb{R}^{n\times 1}$ and $x_f \in \mathbb{R}^{n\times 1}$, there exists a piecewise continuous function $u\colon [t_0, t_f] \to \mathbb{R}^q$ such that the corresponding solution of (1.1) with initial condition $x(t_0) = x_0$ satisfies $x(t_f) = x_f$. Moreover, the system (1.1) is called **completely controllable** if is controllable on any interval.

In order to study the controllability problem, let us assume that $t_f > [t_0]+1$ and introduce the $n \times n$ matrices:

$$\Lambda_k(t_f) := \Lambda_k = \begin{cases} E(t_f, [t_f]) \prod_{\ell=k+1}^{[t_f]-1} E(\ell+1, \ell) & \text{if } [t_0] \le k < [t_f], \\ I & \text{if } k = [t_f], \end{cases} \tag{3.1}$$

which are non singular since **(A)**. In addition, let us define the $n \times q$ matrices:

$$Z_k(t) = \begin{cases} \Lambda_k \Phi(k+1, t)B(t) & \text{if } k \in \{[t_0], [t_0]+1, \ldots, [t_f]-1\}, \\ \Phi(t_f, t)B(t) & \text{if } k = [t_f], \end{cases} \tag{3.2}$$

the $n \times n$ matrices:

$$\begin{aligned} W_{[t_0]} = W\big(t_0, [t_0]+1\big) &= \int_{t_0}^{[t_0]+1} Z_{[t_0]}(s) Z_{[t_0]}^T(s)\, ds \\ &=: \Lambda_{[t_0]} G\big(t_0, [t_0]+1\big) \Lambda_{[t_0]}^T, \end{aligned} \tag{3.3}$$

$$W_{[t_f]} =: W\big([t_f], t_f\big) = \int_{[t_f]}^{t_f} Z_{[t_f]}(s) Z_{[t_f]}^T(s)\, ds = G\big([t_f], t_f\big) \tag{3.4}$$

and, for any integer $k \in \{[t_0]+1, \ldots, [t_f]-1\}$:

$$W_k = W(k, k+1) = \int_k^{k+1} Z_k(s) Z_k^T(s)\, ds =: \Lambda_k G(k, k+1) \Lambda_k^T, \tag{3.5}$$

where the $n \times n$ matrices $G(\alpha, \beta)$ are defined by:

$$G(\alpha, \beta) = \int_\alpha^\beta \Phi(\beta, s) B(s) B^T(s) \Phi^T(\beta, s)\, ds. \tag{3.6}$$

The special case $A(t) = 0$ leads to:

$$J(t, \tau) = E(t, \tau) = I + \int_\tau^t A_0(s)\, ds, \tag{3.7}$$

which allow us to define

$$W_{[t_0]}^0 = \Lambda_{[t_0]} G_0\big(t_0, [t_0]+1\big) \Lambda_{[t_0]}^T, \qquad W_{[t_f]}^0 = G_0\big([t_f], t_f\big)$$

and

$$W_k^0 = \Lambda_k G_0(k, k+1) \Lambda_k^T \quad \text{for any } k \in \big\{[t_0]+1, \ldots, [t_f]-1\big\},$$

where $\Lambda_k$ are defined by (3.1) with (3.7) and $G_0(\alpha,\beta)$ is defined by:

$$G_0(\alpha,\beta)=\int_\alpha^\beta B(t)B^T(t)\,dt.$$

Finally, in this section it will be assumed that $u(t)=0$ for any $t<t_0$.

**Theorem 1** *Assume that* **(A)** *is satisfied,* $t_f>[t_0]+1$. *The system* (1.1) *is controllable on* $[t_0,t_f]$ *if and only if at least one of the matrices* $G(t_0,[t_0]+1)$, $G([t_f],t_f)$ *or* $G(k,k+1)$ *with* $k\in\{[t_0]+1,\dots,[t_f]-1\}$ *has rank* $n$.

*Proof* Firstly, we will assume that there exists $k\in\{[t_0]+1,\dots,[t_f]-1\}$ such that $\operatorname{Rank} G(k,k+1)=n$. In addition, we will consider $x_0$ as initial state and $x_f$ as final one. Note that the variation of parameters formula (2.6) implies:

$$\begin{aligned}x(t_f)&=\Lambda_{[t_0]}E\big([t_0]+1,[t_0]\big)E^{-1}\big(t_0,[t_0]\big)x_0+\int_{t_0}^{[t_0]+1}Z_{[t_0]}(s)u(s)\,ds\\&\quad+\sum_{k=[t_0]+1}^{[t_f]-1}\int_k^{k+1}Z_k(s)u(s)\,ds+\int_{[t_f]}^{t_f}Z_{[t_f]}(s)u(s)\,ds.\end{aligned}\tag{3.8}$$

As the matrices $\Lambda_k$ and $G(k,k+1)$ are non singular, the Theorem 2 from [2, Appendix A] says that $\operatorname{Rank}(W_k)=n$, which allow to define the control:

$$u(t)=\begin{cases}Z_k^T(t)u_k^* & \text{if } t\in[k,k+1)\\ 0 & \text{otherwise,}\end{cases}$$

where $u_k^*$ is defined by:

$$u_k^*=W_k^{-1}\big\{x_f-\Lambda_{[t_0]}E\big([t_0]+1,t_0\big)E^{-1}\big(t_0,[t_0]\big)x_0\big\}.$$

Upon inserting $u(\cdot)$ in the parameter's variation formula (3.8), we obtain that:

$$\begin{aligned}x(t_f)&=\Lambda_{[t_0]}E\big([t_0]+1,[t_0]\big)E^{-1}\big(t_0,[t_0]\big)+\bigg(\int_k^{k+1}Z_k(s)Z_k^T(s)\,ds\bigg)u_k^*\\&=\Lambda_{[t_0]}E\big([t_0]+1,[t_0]\big)E^{-1}\big(t_0,[t_0]\big)+W_ku_k^*=x_f.\end{aligned}$$

and the controllability follows. A similar result can be obtained by an identical approach when $\operatorname{Rank} G(t_0,[t_0]+1)=n$ or $\operatorname{Rank} G([t_f],t_f)=n$. The details are left to the reader.

Secondly, we will assume that (1.1) is controllable on $[t_0,t_f]$ and, as a first step, we will verify the following property:

$$\forall\alpha\in\mathbb{R}^{n\times1}\setminus\{0\}\quad\text{exists some } k\in\big\{[t_0],\dots,[t_f]\big\}\text{ such that }\alpha^TW_k\alpha>0,\tag{3.9}$$

indeed, otherwise:

$$\exists\alpha\in\mathbb{R}^{n\times1}\setminus\{0\}\quad\text{such that }\alpha^TW_k\alpha=0\text{ for any }k=[t_0],\dots,[t_f],$$

which implies by (3.5) that:

$$\begin{cases} \alpha^T Z_{[t_0]}(t) = 0 & \text{for any } t \in \big[t_0, [t_0]+1\big), \\ \alpha^T Z_{[t_f]}(t) = 0 & \text{for any } t \in \big[t_f, [t_f]\big), \\ \alpha^T Z_{[t]}(t) = 0 & \text{for any } t \in [k, k+1) \text{ with } k \in \big\{[t_0]+1, \ldots, [t_f]-1\big\}. \end{cases} \tag{3.10}$$

On the other hand, the controllability on $[t_0, t_f]$ implies that given a couple $(x_0, 0)$, there exists a control function $u\colon [t_0, t_f] \to \mathbb{R}$ such that $x(t_0) = x_0$ and $x(t_f) = 0$.

Now, we choose the initial condition

$$x_0 = E\big(t_0, [t_0]\big)E^{-1}\big([t_0]+1, [t_0]\big)\Lambda_{[t_0]}^{-1}\alpha,$$

which is inserted in (3.8). By using the fact that $x(t_f) = 0$, we can conclude that.

$$0 = \alpha + \int_{t_0}^{[t_0]+1} Z_{[t_0]}(s)u(s)\,ds + \sum_{k=[t_0]+1}^{[t_f]-1} \int_k^{k+1} Z_k(s)u(s)\,ds + \int_{[t_f]}^{t_f} Z_{[t_f]}(s)u(s)\,ds.$$

By multiplying by $\alpha^T$ and using (3.10), we can conclude that $0 = \alpha^T\alpha = \|\alpha\|_2^2$, obtaining a contradiction with $\alpha \neq 0$ and (3.9) follows. Now, as (3.9) is verified and $W_k$ is positive definite, we can conclude that $\text{Rank}(W_k) = n$. Finally, the invertibility of $\Lambda_k$ implies that $G(k, k+1)$ has rank $n$ and the Theorem follows. □

A careful lecture of the proof shows that Eq. (3.9) can be interpreted in several ways, leading to the alternative formulation of Theorem 1:

**Proposition 1** *The following sufficient and necessary controllability conditions hold when* $t_f > [t_0]+1$ *and* **(A)** *is satisfied*:

(i) *If at least one of the matrices* $G(t_0, [t_0]+1)$, $G([t_f], t_f)$ *or* $G(k, k+1)$ *with* $k \in \{[t_0]+1, \ldots, [t_f]-1\}$ *has rank* $n$, *then* (1.1) *is controllable on* $[t_0, t_f]$.
(ii) *If the system* (1.1) *is controllable on* $[t_0, t_f]$, *then*

$$\text{Rank}[W_{[t_0]} \quad W_{[t_0]+1} \quad \cdots \quad W_{[t_f]}] = n,$$

which is usual in the impulsive literature [7, 30]. We point out the remarkable simplicity of our first formulation.

In addition, the controllability result can be reformulated in terms of the control system

$$\dot{x} = A(t)x + B(t)u(t). \tag{3.11}$$

**Proposition 2** *Provided that* **(A)** *is verified and* $t_f > [t_0]+1$, *the system* (1.1) *is controllable on* $[t_0, t_f]$ *if and only if the system* (3.11) *is controllable in at least one of the intervals* $[t_0, [t_0]+1]$, $[k, k+1]$ (*for some* $k \in \{[t_0]+1, \ldots, [t_f]-1\}$) *or* $[[t_f], t_f]$.

*Proof* The proof is a consequence of a classical controllability result for LTV systems (see *e.g.*, Theorem 1 from [9]). □

If $t_f < [t_0]+1$, a similar consequence (the proof is given for the reader) is the following result:

**Corollary 1** *Assume that* **(A)** *is satisfied and* $t_f \in (t_0, [t_0]+1)$. *Then* (1.1) *is controllable on* $[t_0, t_f]$ *if and only if the matrix*:

$$G(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, s) B(s) B^T(s) \Phi^T(t_f, s)\, ds$$

*has rank* $n$.

By following the lines of the proof of the Theorem 1, we can easily obtain the corresponding result:

**Corollary 2** *Assume that* $I + \int_\tau^t A_0(s)\, ds$ *is non singular for any* $t, \tau \in [j, j+1)$ *with* $j \in \mathbb{Z}$ *and* $t_f > [t_0]+1$. *The system* (1.1) *with* $A(t) = 0$ *is controllable on* $[t_0, t_f]$ *if and only if at least one of the matrices* $G_0(k, k+1)$, $G_0([t_f], t_f)$ *or* $G_0(t_0, [t_0]+1)$ *has rank* $n$.

## 4 Observability

**Definition 3** The unforced linear system (1.1)–(1.2):

$$\dot{x}(t) = A(t)x(t) + A_0(t)x\big([t]\big), \tag{4.1}$$

$$y(t) = C(t)x(t) + C_0(t)x\big([t]\big), \tag{4.2}$$

is called **observable on** $[t_0, t_f]$ ($t_0 < t_f$) if every initial condition $x_0 \in \mathbb{R}^{n\times 1}$ can be determined from the knowledge of the output $y(t) \in \mathbb{R}^{r\times 1}$ on $[t_0, t_f]$.

Let us introduce the $r \times n$ matrix function:

$$H(t, \tau) = C(t)E(t, \tau) + C_0(t), \tag{4.3}$$

the $n \times n$ matrices:

$$X_{[t]} = \begin{cases} I & \text{if } t_0 \le t < [t_0]+1 < t_f, \\ \prod_{\ell=[t_0]}^{[t]-1} E(\ell+1, \ell) & \text{if } [t_0]+1 \le t, \end{cases} \tag{4.4}$$

which are non singular since **(A)**. Moreover, we define the $r \times n$ matrices,

$$R_{[t_0]}(t) = H(t, t_0) \quad \text{and} \quad R_k(t) = H(t, k)X_k \quad \text{with } k \in \big\{[t_0]+1, \ldots, [t_f]\big\}, \tag{4.5}$$

the $n \times n$ matrices:

$$M_{[t_0]} = M\big(t_0, [t_0]+1\big) = \int_{t_0}^{[t_0]+1} R_{[t_0]}^T(s) R_{[t_0]}(s)\, ds =: V\big(t_0, [t_0]+1\big), \tag{4.6}$$

$$M_{[t_f]} = M\big([t_f], t_f\big) = \int_{[t_f]}^{t_f} R_{[t_f]}^T(s) R_{[t_f]}(s)\, ds =: X_{[t_f]}^T V\big([t_f], t_f\big) X_{[t_f]} \tag{4.7}$$

and, for any integer $k \in \{[t_0]+1, \ldots, [t_f]-1\}$:

$$M_k = M(k, k+1) = \int_k^{k+1} R_k(s) R_k^T(s)\, ds =: X_k^T V(k, k+1) X_k, \tag{4.8}$$

where the $n \times n$ matrices $V(\alpha, \beta)$ are defined by:

$$V(\alpha, \beta) = \int_{\alpha}^{\beta} H^T\big(s, [s]\big) H\big(s, [s]\big)\, ds. \tag{4.9}$$

The special case $A(t) = 0$ leads to the matrices $E(t, \tau) = J(t, \tau)$ defined by (3.7). Now, we define the matrices

$$M_{[t_f]}^0 = X_{[t_f]}^T V_0\big([t_f], t_f\big) X_{[t_f]}, \qquad M_{[t_0]}^0 = V_0\big(t_0, [t_0] + 1\big)$$

and for any $k \in \{[t_0] + 1, \ldots, [t_f] - 1\}$:

$$M_k^0 = X_k^T V_0(k, k+1) X_k,$$

where $X_k$ is defined by (4.4) with (3.7) and

$$V_0(\alpha, \beta) = \int_k^{k+1} \big\{C(t) E\big(t, [t]\big) + C_0(t)\big\}^T \big\{C(t) E\big(t, [t]\big) + C_0(t)\big\}\, dt$$

with $E(\cdot, \cdot)$ defined by (3.7).

**Theorem 2** *Assume that* **(A)** *is satisfied and* $t_f > [t_0] + 1$, *the system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$ *if and only if at least one of the matrices* $V(t_0, [t_0] + 1)$, $V([t_f], t_f)$ *or* $V(k, k+1)$ *with* $k \in \{[t_0] + 1, \ldots, [t_f] - 1\}$ *has rank* $n$.

*Proof* Firstly, we assume that there exists some integer $k \in \{[t_0] + 1, \ldots, [t_f] - 1\}$ such that $\operatorname{Rank} V(k, k+1) = n$. By using (2.6) with $u(t) = 0$, it follows that:

$$y(t) = H\big(t, [t]\big) X_{[t]} E^{-1}\big(t_0, [t_0]\big) x_0 = R_{[t]}(t) E^{-1}\big(t_0, [t_0]\big) x_0. \tag{4.10}$$

Now, let us multiply the left of (4.10) by the function:

$$\phi_k(t) = \begin{cases} R_k^T(t) & \text{if } k \le t < k+1 \\ 0 & \text{otherwise,} \end{cases}$$

and integrate over $[t_0, t_f]$, obtaining:

$$M(k, k+1) E^{-1}\big(t_0, [t_0]\big) x(t_0) = \int_k^{k+1} X_k^T H^T\big(s, [s]\big) y(s)\, ds.$$

By using the fact that the matrices $X_k$ and $V(k, k+1)$ are non singular, the Theorem 2 from [2, Appendix A] says that $\operatorname{Rank} M(k, k+1) = n$, which implies that $M(k, k+1) E^{-1}(t_0, [t_0])$ is non singular. In consequence, we have that

$$x(t_0) = E\big(t_0, [t_0]\big) M^{-1}(k, k+1) \int_k^{k+1} R_{[k]}^T(t) y(t)\, dt$$

and the observability follows. The cases $\operatorname{Rank} V(t_0, [t_0] + 1) = n$ and $\operatorname{Rank} V([t_f], t_f) = n$ can be proved in a similar way and are given for the reader.

Secondly, we assume that (4.1)–(4.2) is observable on $[t_0, t_f]$ and, as an intermediate step, we will prove that

$$\forall \alpha \in \mathbb{R}^{n \times 1} \setminus \{0\} \quad \text{exists some } k \in \big\{[t_0], \ldots, [t_f]\big\} \text{ such that } \alpha^T M_k \alpha > 0, \tag{4.11}$$

indeed, otherwise

$$\exists \alpha \in \mathbb{R}^{n\times 1} \setminus \{0\} \quad \text{such that } \alpha^T M_k \alpha = 0 \text{ for any } k = [t_0], \dots, [t_f].$$

Now, if we consider the initial condition $x(t_0) = x_0 = E(t_0, [t_0])\alpha$ and use (4.10), it follows that:

$$\int_{t_0}^{t_f} \|y(s)\|_2^2 \, ds = \int_{t_0}^{t_f} y^T(s) y(s) \, ds = \sum_{j=[t_0]}^{[t_f]} \alpha^T M_j \alpha = 0,$$

thus $y(t) = R_{[t]}(t)\alpha = 0$ almost everywhere for $t \in [t_0, t_f]$, which implies that the initial condition $x_0 = E(t_0, [t_0])\alpha$ is unobservable, obtaining a contradiction and (4.11) is verified. Now, we can deduce that some matrix $M_k$ has rank $n$ and the Theorem follows by using the invertibility of $X_k$, which implies that $V(k, k+1)$ has rank $n$. □

As in the controllability proof, the property (4.11) can be reformulated in several ways, leading to an equivalent "impulsive type" result:

**Proposition 3** *Assume that* **(A)** *is satisfied and* $t_f > [t_0] + 1$,

(i) *If at least one of the matrices* $V(t_0, [t_0]+1)$, $V([t_f], t_f)$ *or* $V(k, k+1)$ *with* $k \in \{[t_0]+1, \dots, [t_f]-1\}$ *has rank* $n$, *then the system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$.

(ii) *If the system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$, *then the matrix*

$$M(t_0, t_f) = M\big(t_0, [t_0]+1\big) + \sum_{k=[t_0]+1}^{[t_f]-1} M(k, k+1) + M\big([t_f], t_f\big). \tag{4.12}$$

*is non singular.*

By following the lines of the proof of Theorem 2, we can deduce the byproduct:

**Corollary 3** *Assume that* **(A)** *is satisfied and* $t_0 < t_f \le [t_0] + 1$. *The system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$ *if and only if the matrix*:

$$V(t_0, t_f) = \int_{t_0}^{t_f} \big\{C(t)E\big(t, [t_0]\big) + C_0(t)\big\}^T \big\{C(t)E\big(t, [t_0]\big) + C_0(t)\big\} \, dt,$$

*is non singular.*

By following the lines of the proof of Theorem 2, we obtain:

**Corollary 4** *Assume that* $J(t, \tau) = I + \int_\tau^t A_0(s)\, ds$ *is non singular for any* $t, \tau \in [k, k+1]$ *with* $k \in \mathbb{Z}$ *and* $t_f > [t_0] + 1$, *the system* (4.1)–(4.2) *with* $A(t) = 0$ *is observable on* $[t_0, t_f]$ *if and only if at least one of the matrices* $V_0(t_0, [t_0]+1)$, $V_0([t_f], t_f)$ *or* $V_0(k, k+1)$ *with* $k \in \{[t_0]+1, \dots, [t_f]-1\}$ *has rank* $n$.

## 5 The Linear Time Invariant Case

When $A(t) = A$, $A_0(t) = A_0$ and $B(t) = B$, it follows that

$$J(t,\tau) = I + \int_{\tau}^{t} e^{A(\tau-s)} A_0\,ds = I + \int_{0}^{t-\tau} e^{-As} A_0\,ds, \tag{5.1}$$

and

$$E(t,\tau) = e^{A(t-\tau)} + \int_{\tau}^{t} e^{A(t-s)} A_0\,ds = e^{A(t-\tau)} + \int_{0}^{t-\tau} e^{As} A_0\,ds. \tag{5.2}$$

This notation has been introduced in [25, p. 46]. In addition, the special case $E(k+1,k)$ (with $k \in \mathbb{Z}$) will be denoted as follows:

$$E(k+1,k) = e^{A} + \int_{0}^{1} e^{As} A_0\,ds = \mathcal{M}(A, A_0). \tag{5.3}$$

In order to study the above expressions and without loss of generality, we will assume that the characteristic polynomial of the matrix $A$ is

$$p(\lambda) = \sum_{i=0}^{n-1} d_i \lambda^i + \lambda^n \quad \text{with } d_i \in \mathbb{R}. \tag{5.4}$$

*Remark 5* The Cayley–Hamilton Theorem allows to prove (see *e.g.*, [15]) the existence of $n$ scalar analytic functions $\{\beta_i(t)\}_{i=0}^{n-1}$ such that

$$e^{tA} = \sum_{i=0}^{n-1} \beta_i(t) A^i, \tag{5.5}$$

where the series expansion of $t \mapsto \beta_i(t)$ has coefficients defined in terms of $d_i$.

**Lemma 2** *The function $E(t,\tau)$ defined by (5.2) can be represented as*:

$$E(t,\tau) = I + \sum_{i=1}^{n} \left( \int_{0}^{t-\tau} \beta_{i-1}(s)\,ds \right) A^{i-1}(A + A_0), \tag{5.6}$$

*where the terms $\beta_i(\cdot)$ are defined in (5.5).*

*Proof* Notice that (5.4) and (5.5) imply that:

$$\begin{aligned} e^{At}A &= \beta_0(t)A + \beta_1(t)A^2 + \cdots + \beta_{n-2}(t)A^{n-1} + \beta_{n-1}(t)A^n \\ &= \sum_{i=1}^{n-1} \beta_{i-1}(t)A^i - d_0\beta_{n-1}(t)I - \beta_{n-1}(t)\sum_{i=1}^{n-1} d_i A^i \\ &= -d_0\beta_{n-1}(t)I + \sum_{i=1}^{n-1} \big\{\beta_{i-1}(t) - d_i\beta_{n-1}(t)\big\} A^i. \end{aligned}$$

This identity combined with (5.4) and

$$e^{At} = I + \int_0^t e^{As} A\, ds, \tag{5.7}$$

will allow us to verify that:

$$\beta_0(t) = 1 - d_0 \int_0^t \beta_{n-1}(s)\, ds, \tag{5.8}$$

$$\int_0^t \beta_{i-1}(s)\, ds = \beta_i(t) + d_i \int_0^t \beta_{n-1}(s)\, ds \quad \text{for } i = 1, \dots, n-1. \tag{5.9}$$

By using (5.5) and (5.9), we can write (5.2) as follows:

$$\begin{aligned}
E(t,\tau) &= \sum_{i=0}^{n-1} \beta_i(t-\tau) A^i + \sum_{i=0}^{n-2} \int_0^{t-\tau} \beta_i(s)\, ds\, A^i A_0 + \int_0^{t-\tau} \beta_{n-1}(s)\, ds\, A^{n-1} A_0 \\
&= \sum_{i=0}^{n-1} \beta_i(t-\tau) A^i + \sum_{i=1}^{n-1} \int_0^{t-\tau} \beta_{i-1}(s)\, ds\, A^{i-1} A_0 + \int_0^{t-\tau} \beta_{n-1}(s)\, ds\, A^{n-1} A_0 \\
&= \sum_{i=0}^{n-1} \beta_i(t-\tau) A^i + \sum_{i=1}^{n-1} \left\{ \beta_i(t-\tau) + d_i \int_0^{t-\tau} \beta_{n-1}(s)\, ds \right\} A^{i-1} A_0 \\
&\quad + \int_0^{t-\tau} \beta_{n-1}(s)\, ds\, A^{n-1} A_0 \\
&= \beta_0(t-\tau) I + \sum_{i=1}^{n-1} \beta_i(t-\tau) A^{i-1}(A + A_0) \\
&\quad + \left( \int_0^{t-\tau} \beta_{n-1}(s)\, ds \right) \sum_{i=1}^{n-1} \left\{ d_i A^{i-1} + A^{n-1} \right\} A_0.
\end{aligned}$$

By using (5.8) and Cayley–Hamilton's Theorem, we obtain:

$$\begin{aligned}
E(t,\tau) &= I + \sum_{i=1}^{n-1} \beta_i(t-\tau) A^{i-1}(A + A_0) \\
&\quad + \left( \int_0^{t-\tau} \beta_{n-1}(s)\, ds \right) \left( \sum_{i=1}^{n-1} \left\{ d_i A^{i-1} + A^{n-1} \right\} A_0 - d_0 I \right) \\
&= I + \sum_{i=1}^{n-1} \left( \beta_i(t-\tau) + d_i \int_0^{t-\tau} \beta_{n-1}(s)\, ds \right) A^{i-1}(A + A_0) \\
&\quad + \int_0^{t-\tau} \beta_{n-1}(s)\, ds\, A^{n-1}(A + A_0).
\end{aligned}$$

and the identity (5.6) follows by using (5.9). □

*Remark 6* As expected by (5.2), the reader can verify that (5.6) becomes $E(t,\tau) = e^{A(t-\tau)}$ when $A_0 = 0$.

As before, we will assume throughout this section that:

(a) For any $t < t_0$, we have $u(t) = 0$.
(b) The matrix defined by (5.1) is nonsingular for any couple $(t,\tau)$ such that $t - \tau \in [0,1]$, *e.g.*, **(A)** is satisfied. See Remark 1 for details

## 5.1 Controllability Results

In this framework, the matrices $\Lambda_k$ defined in (3.1) becomes:

$$\Lambda_k = \begin{cases} E(t_f,[t_f])\mathcal{M}(A,A_0)^{[t_f]-k-1} & \text{if } k \in \{[t_0],\dots,[t_f]-1\}, \\ I & \text{if } k = [t_f], \end{cases}$$

which are non singular since **(A)**. As before, let us define the $n \times q$ matrices:

$$Z_k(s) = \begin{cases} \Lambda_k e^{A(k+1-s)}B & \text{if } k \in \{[t_0],\dots,[t_f]-1\} \\ e^{A(t_f-s)}B & \text{if } k = [t_f]. \end{cases}$$

For any $k \in \{[t_0]+1,\dots,[t_f]-1\}$, the matrices $W_k$ defined by (3.5) becomes:

$$\begin{aligned} W_k &= \Lambda_k\left(\int_k^{k+1} e^{A(k+1-s)}BB^T e^{A^T(k+1-s)}\,ds\right)\Lambda_k^T \\ &= \Lambda_k\left(\int_0^1 e^{As}BB^T e^{A^T s}\,ds\right)\Lambda_k^T. \end{aligned}$$

Finally, we define:

$$\begin{aligned} W_{[t_0]} &= \Lambda_{[t_0]}\left(\int_{t_0}^{[t_0]+1} e^{A([t_0]+1-s)}BB^T e^{A^T([t_0]+1-s)}\,ds\right)\Lambda_{[t_0]}^T \\ &= \Lambda_{[t_0]}\left(\int_0^{1-(t_0-[t_0])} e^{As}BB^T e^{A^T s}\,ds\right)\Lambda_{[t_0]}^T. \end{aligned}$$

and

$$W_{[t_f]} = \int_{[t_f]}^{t_f} e^{A(t_f-s)}BB^T e^{A^T(t_f-s)}\,ds = \int_0^{t_f-[t_f]} e^{As}BB^T e^{A^T s}.$$

In addition, if $t_f < [t_0]+1$, we can define:

$$W(t_0,t_f) = E(t_0,t_f)\left(\int_0^{t_f-t_0} e^{As}BB^T e^{A^T s}\,ds\right)E^T(t_0,t_f).$$

*Remark 7* By using (5.5), it can be proved (see *e.g.*, Th. 3.1 from [33]) that

$$\text{Rank}\left(\int_0^t e^{As}BB^T e^{A^T s}\,ds\right) = n \quad \text{for any } t > 0$$

if and only if the classic controllability matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{5.10}$$

has rank $n$.

An important consequence of Remark 7 is that $\text{Rank}(\mathcal{C}) = n$ if and only if all the matrices $W_k$ defined above have rank $n$. This fact combined with Theorem 1 implies the following results:

**Theorem 3** *The system* (1.1) *is completely controllable if and only if* $\text{Rank}(\mathcal{C}) = n$.

**Corollary 5** *If* $A = 0$, *then the system* (1.1) *is completely controllable if and only if* $\text{Rank}(B) = n$.

As in the LTV case, Theorem 3 can be reformulated as follows:

**Proposition 4** *The system* (1.1) *is completely controllable if and only if the system*

$$\dot{x} = Ax + Bu$$

*is completely controllable.*

It is interesting to emphasize that the controllability problem can be also addressed by using constant control functions $u(t) = u_k$ for any $t \in [k, k+1)$ and studying the linear discrete system:

$$\begin{cases} x_{k+1} = \mathcal{M}(A, A_0)x_k + \tilde{B}u_k \\ x_{[t_0]+1} = x\big([t_0]+1\big), \end{cases} \quad \text{with } \tilde{B} = \int_0^1 e^{As}B\,ds, \tag{5.11}$$

obtained by the variation of parameters. It is well known that (5.11) is controllable if and only if the matrix

$$\begin{bmatrix} \tilde{B} & \mathcal{M}(A, A_0)\tilde{B} & \cdots & \mathcal{M}(A, A_0)^{n-1}\tilde{B} \end{bmatrix}$$

has rank $n$. Nevertheless, a careful lecture of this result (see *e.g.*, [6, Theorem 10.4]) show us that it is necessary to consider intervals $[t_0, t_f]$ of length bigger than $n$, while our results do not have this restriction.

## 5.2 Observability Results

By (5.3), we can deduce that

$$X_{[t]} = \mathcal{M}(A, A_0)^{[t]-[t_0]} \quad \text{for } t_0 < t < [t_f].$$

A direct consequence of Lemma 2 allows to write (4.3) as:

$$H\big(t, [t]\big) = C + C_0 + \sum_{i=1}^{n} \left( \int_0^{t-[t]} \beta_{i-1}(s)\,ds \right) CA^{i-1}(A + A_0). \tag{5.12}$$

**Theorem 4** *The system (4.1)–(4.2) is observable on* $[t_0, t_f]$ *with* $t_f > [t_0] + 1$ *if and only if the* $r(n+1) \times n$ *matrix*:

$$\mathcal{O} = \begin{bmatrix} C + C_0 \\ C(A + A_0) \\ \vdots \\ CA^{n-1}(A + A_0) \end{bmatrix} \tag{5.13}$$

*has rank* $n$.

*Proof* Firstly, we will assume that $\text{Rank}(\mathcal{O}) = n$ and prove that any matrix $V(k, k+1)$ (with $k \in \{[t_0] + 1, \ldots, [t_f] - 1\}$) defined in Sect. 4 has rank $n$. Indeed, otherwise if Rank $V(k, k+1) < n$, there exists a non zero vector $\alpha \in \mathbb{R}^{n\times 1}$ such that:

$$\int_k^{k+1} \left\| \left\{ CE(t,k) + C_0 \right\} \alpha \right\|_2^2 dt = 0,$$

which implies that the continuous map $t \mapsto \psi(t) = (CE(t,k) + C_0)\alpha$ is zero on $[k, k+1]$. By letting $t \to k^+$ leads to:

$$\psi\left(k^+\right) = (C + C_0)\alpha = 0.$$

In addition, it can be verified that the $j$-th right derivatives of $\psi(t)$ evaluated at $t = k^+$ are equal to:

$$\psi^{(j)}\left(k^+\right) = CA^{j-1}(A + A_0)\alpha = 0, \quad \text{with } j \in \mathbb{N}.$$

By using (5.4) combined with Cayley–Hamilton's Theorem, we can see that:

$$CA^n(A + A_0) = -\sum_{i=1}^{n} d_{i-1} CA^{i-1}(A + A_0)$$

and we can conclude that the $j$-th (with $j \geq n + 1$) derivatives $\psi^{(j)}(k^+)$ can be deduced from the previous ones.

Now, we obtain the system:

$$\mathcal{O}\alpha = 0.$$

As $\text{Rank}(\mathcal{O}) = n$, we conclude that $\alpha = 0$, obtaining a contradiction. In consequence, any matrix $V(k, k+1)$ with $k \geq [t_0] + 1$ has rank $n$ and the observability follows from Theorem 2. A similar observability result can be obtained with the matrix $V([t_f], t_f)$ and the details are left to the reader. Nevertheless, note that this computation is not direct for $V(t_0, [t_0] + 1)$.

Secondly, we will assume that the system (4.1)–(4.2) is observable and prove that $\text{Rank}(\mathcal{O}) = n$. Indeed, otherwise if $\text{Rank}(\mathcal{O}) < n$, then there exists a nonzero vector $\alpha \in \mathbb{R}^{n\times 1}$ such that:

$$(C + C_0)\alpha = 0 \quad \text{and} \quad CA^{i-1}(A + A_0)\alpha = 0 \quad \text{for } i = 1, \ldots, n. \tag{5.14}$$

On the other hand, as the system is observable, Theorem 2 says that there exists $j \in \{[t_0], \ldots, [t_f]\}$ such that $M_j$ is positive definite. Now, we will consider the non zero vector

$v_\alpha = X_{[j]}^{-1}\alpha$ with $\alpha$ satisfying (5.14). Now, we use (5.12) combined with (5.14) to observe that

$$\alpha^T V(j, j+1)\alpha$$
$$= \alpha^T \int_j^{j+1} H(t, j)^T \left\{ C + C_0 + \sum_{i=1}^{n} \int_0^{t-[t]} \beta_{i-1}(s) C A^{i-1}(A + A_0)\, ds \right\} \alpha\, dt = 0,$$

obtaining a contradiction with the positive definiteness of $M_j$ and the result follows. □

*Remark 8* For the authors, it is surprising to verify the lack of the duality between controllability and observability conditions, which is verified in the classical theory.

A careful lecture of the previous proof shows that the computations for the rank for $V(t_0, [t_0] + 1)$ are more complicated but are useful to study the observability when $t_f < [t_0] + 1$.

**Corollary 6** *Assume that* $[t_0] \le t_0 < t_f \le [t_0] + 1$. *The following sufficient and necessary observability conditions for* (4.1)–(4.2) *hold*:

(i) *If the* $r(n+1) \times n$ *matrix*:

$$\mathcal{O}_{\tau(t_0)} = \begin{bmatrix} Ce^{A\tau(t_0)} + C\int_0^{\tau(t_0)} e^{As} A_0\, ds + C_0 \\ Ce^{A\tau(t_0)}(A + A_0) \\ \vdots \\ Ce^{A\tau(t_0)} A^{n-1}(A + A_0) \end{bmatrix} \quad \text{with } \tau(t_0) = t_0 - [t_0] \tag{5.15}$$

*has rank* $n$, *then* (4.1)–(4.2) *is observable in* $[t_0, t_f]$.

(ii) *If* (4.1)–(4.2) *is observable in* $[t_0, t_f]$, *then the observability matrix* $\mathcal{O}$ *defined by* (5.13) *has rank* $n$.

*Proof* By using Corollary 3, we know that the system is observable on $[t_0, t_f]$ if and only if

$$\operatorname{Rank} V(t_0, t_f) = \int_{t_0}^{t_f} \left\{C(t)E\big(t, [t_0]\big) + C_0(t)\right\}^T \left\{C(t)E\big(t, [t_0]\big) + C_0(t)\right\} dt = n.$$

Now, by following the lines of the previous proof, we can deduce that $\operatorname{Rank} \mathcal{O}_{\tau(t_0)} = n$ implies that $\operatorname{Rank} V(t_0, t_f) = n$ and the observability follows. Finally, the proof of statement (ii) is similar. □

*Remark 9* This result is weaker than the previous one. The main obstacle is that $[t_0, t_f] \cap \mathbb{Z} = \emptyset$ and the derivatives of $\psi(s)$ cannot be evaluated at integer numbers as before. This induces an additional difficulty to the observability problem. Nevertheless, it is interesting to point out that if $t_0 = [t_0]$, then $\mathcal{O}_{\tau(t_0)} = \mathcal{O}$ and our previous result is recovered.

Some particular cases can be studied and proved in a similar way but with less restrictions. The following result shows that if $A_0 = 0$, it is not correct to evaluate $A_0 = 0$ in Corollary 4 (*i.e.*, the observability conditions are not continuous with respect to $A_0$):

**Corollary 7** *If* $A_0 = 0$, *then the system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$ *with* $t_f > [t_0] + 1$ *if and only if the* $rn \times n$ *matrix*:

$$\begin{bmatrix} C + C_0 \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

*has rank* $n$.

*Proof* If the rank is $n$, the observability can be deduced by noticing that the $j$-th right derivatives of $t \mapsto \psi(t) = (Ce^{At} + C_0)\alpha$ evaluated at $t = k^+$ are equal to:

$$\psi^{(j)}\big(k^+\big) = CA^j \alpha = 0, \quad \text{with } j \in \mathbb{N},$$

and by Cayley–Hamilton's Theorem, we can see that the $j$-th with $j \geq n$ right derivatives $\psi^{(j)}(k^+)$ can be deduced from the previous ones (observe that it was $j \geq n + 1$ when $A_0 \neq 0$). Conversely, if the system is observable, the property $\operatorname{Rank}(\mathcal{O})$ can be deduced as in the previous proof by noticing that (4.3) has the simpler form:

$$H\big(t, [t]\big) = C + C_0 + \sum_{j=0}^{n-1} \beta_j(t) CA^j.$$

□

Notice that if $C_0 = 0$, we obtain the necessary and sufficient observability condition for classical LTI control systems. The last results can be proved in a similar way and its proof is given for the reader:

**Corollary 8** *If* $C + C_0 = 0$, *then the system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$ *with* $t_f > [t_0] + 1$ *if and only if the* $rn \times n$ *matrix*:

$$\begin{bmatrix} C(A + A_0) \\ \vdots \\ CA^{n-1}(A + A_0) \end{bmatrix}$$

*has rank* $n$.

**Corollary 9** *If* $A + A_0 = 0$, *then the system* (4.1)–(4.2) *is observable on* $[t_0, t_f]$ *with* $t_f > [t_0] + 1$ *if and only if* $\operatorname{Rank}(C + C_0) = n$.

**Corollary 10** *Assume that* $A = 0$ *and* $J(t, \tau) = I + (t - \tau)A_0$ *is non singular for any* $t, \tau \in [j, j + 1]$ *with* $j \in \mathbb{Z}$ *and* $t_f > [t_0] + 1$, *then* (4.1)–(4.2) *is observable if and only if the matrix*

$$\begin{bmatrix} C + C_0 \\ CA_0 \end{bmatrix}$$

*has rank* $n$.

As before, the observability problem can be addressed by studying the linear discrete system:

$$\begin{cases} x_{k+1} = \mathcal{M}(A, A_0)x_k \\ y_k = \{C + C_0\}x_k. \end{cases} \tag{5.16}$$

Indeed, it is well known that (5.16) is observable if and only if

$$\begin{bmatrix} C + C_0 \\ (C + C_0)\mathcal{M}(A, A_0) \\ \vdots \\ (C + C_0)\mathcal{M}(A, A_0)^{n-1} \end{bmatrix}$$

has rank $n$. Nevertheless, a careful lecture of this criterion (see *e.g.*, [6, Theorem 10.13]) show us that it is necessary to consider intervals $[t_0, t_f]$ of length bigger than $n$ while our results do not have this restriction.

## 6 Examples

*Example 1* Let us consider the classic control system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t),$$

which is neither controllable nor observable since the controllability and observability matrices have rank one. This system (when $\varepsilon = \nu = 0$) can be submitted to a discontinuous delay combined with a perturbation:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \varepsilon^2 & 0 \end{bmatrix} x(t) + \nu \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x\big([t]\big) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) - \nu \begin{bmatrix} 1 & 0 \end{bmatrix} x\big([t]\big).$$

Now, we can verify that

$$e^{At} = \begin{bmatrix} \cosh(\varepsilon t) & \sinh(\varepsilon t) \\ \varepsilon \sinh(\varepsilon t) & \varepsilon \cosh(\varepsilon t) \end{bmatrix} \quad \text{and} \quad J(t, \tau) = \begin{bmatrix} 1 + \nu\{1 - \cosh(\varepsilon(t - \tau))\} & 0 \\ * & 1 \end{bmatrix}.$$

For any couple $(t, \tau)$ satisfying $t - \tau \in [0, 1]$, it can proved that $J(t, \tau)$ is non singular either if $\nu \leq 0$ or $\nu > 0$ and $(\varepsilon, t - \tau) \in \mathbb{R} \times [0, 1]$ with $\nu|1 - \cosh(\varepsilon(t - \tau))| < 1$. The controllability and observability matrices:

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{bmatrix} \quad \text{and} \quad \mathcal{O} = \begin{bmatrix} C + C_0 \\ C(A + A_0) \\ CA(A + A_0) \end{bmatrix} = \begin{bmatrix} -\nu & 1 \\ \varepsilon^2 + \nu & 0 \\ 0 & \varepsilon^2 \end{bmatrix}$$

have rank 2. Finally, Theorem 3 says that the DEPCA system is completely controllable and Theorem 4 says that is observable on any interval $[t_0, t_f]$ with $t_f > [t_0] + 1$.

*Example 2* Let us consider the system (1.1) with the matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad C_0 = \begin{bmatrix} -1 & -1.1 \end{bmatrix}.$$

We can verify that

$$e^{At} = \begin{bmatrix} 1 & 10(1 - e^{-0.1t}) \\ 0 & e^{-0.1t} \end{bmatrix}$$

and

$$J(t,\tau) = \begin{bmatrix} 1 + 10(t-\tau) - 100(e^{0.1(t-\tau)} - 1) & 0 \\ * & 1 \end{bmatrix}$$

and we can verify that $J(t,\tau)$ is non singular for any couple $(t,\tau)$ such that $t - \tau \in [0,1)$ since the function $\eta \mapsto 1 + 10\eta - 100(e^{0.1\eta} - 1)$ is positive for any $\eta \in [0,1]$. Now, observe that:

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

has rank 1 and Corollary 3 says that the DEPCA system cannot be controllable. On the other hand, the matrix

$$\mathcal{O} = \begin{bmatrix} C + C_0 \\ C(A + A_0) \\ CA(A + A_0) \end{bmatrix} = \begin{bmatrix} 0 & -0.1 \\ 1 & 0.9 \\ 0.9 & -0.09 \end{bmatrix}$$

has rank 2 and Theorem 4 says that the DEPCA system is observable on any interval $[t_0, t_f]$ with $t_f > [t_0] + 1$.

## 7 Concluding Remarks

In this paper the DEPCA control systems (1.1) and (4.1)–(4.2) have been considered under the fundamental assumption **(A)**, which imposes some restrictions to the matrices $A(t)$ and $A_0(t)$. Necessary and sufficient conditions for the controllability of (1.1) and observability of (4.1)–(4.2) on $[t_0, t_f]$ have been derived (see Theorems 1 and 2 respectively). It is interesting to point out that Theorem 1 says that the controllability can be studied by considering the classic system (3.11) on the intervals $[k, k+1] \subset [t_0, t_f]$ with $k \in \mathbb{Z}$.

On the other hand, when considering the LTI case, we obtain classical controllability conditions (see Theorem 3) but deduce some new ones (see Theorem 4 and Corollary 8) for observability, this is due to our Cayley–Hamiltons's characterization of the matrix $E(t,\tau)$ (see Lemma 2). In addition, new difficulties arise when $t_f \leq [t_0] + 1$ (see Corollary 6) and "classical" results (see Corollary 7) can be deduced when $A_0 = 0$.

In spite of controllability and observability in an LTI framework can be also considered by studying the linear discrete systems (5.11) and (5.16), an important limitation is the assumption that $[t_0, t_f]$ has a length bigger than $n$. However, discrete systems arising from DEPCA equations are interesting on itself.

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