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ORIGINAL PAPER

Characterization of Extremal Antipodal Polygons

O. Aichholzer · L. E. Caraballo · J. M. Díaz-Báñez · R. Fabila-Monroy · C. Ochoa · P. Nigsch

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Abstract Let S be a set of $2n$ points on a circle such that for each point $p \in S$ also its antipodal (mirrored with respect to the circle center) point p' belongs to S. A polygon P of size n is called *antipodal* if it consists of precisely one point of each antipodal pair (p, p') of S. We provide a complete characterization of antipodal polygons which

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O. Aichholzer
Institute for Software Technology, University of Technology Graz, Graz, Austria
e-mail: oaich@ist.tugraz.at

L. E. Caraballo · J. M. Díaz-Báñez (✉)
Departamento de Matemática Aplicada II, Universidad de Sevilla, Sevilla, Spain
e-mail: dbanez@us.es

L. E. Caraballo
e-mail: luisevaristocaraballo-ext@us.es

R. Fabila-Monroy
Departamento de Matemáticas, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Mexico City, Mexico
e-mail: ruyfabila@math.cinvestav.edu.mx

C. Ochoa
Departamento de Ciencias de la Computación (DCC), Universidad de Chile, Santiago, Chile
e-mail: cochoa@dcc.uchile.cl

P. Nigsch
University of Technology Graz, Graz, Austria
e-mail: paul.nigsch@student.tugraz.at

maximize (minimize, respectively) the area among all antipodal polygons of S. Based on this characterization, a simple linear time algorithm is presented for computing extremal antipodal polygons. Moreover, for the generalization of antipodal polygons to higher dimensions we show that a similar characterization does not exist.

Keywords Antipodal points · Extremal area polygons · Discrete and computational geometry

1 Introduction

For a point $p = (x_1, x_2) \in I\!R^2$, let $p' := (-x_1, -x_2)$ be the antipodal point of p. Consider a set S of points on a circle centered at the origin such that for each point $p \in S$ also its antipodal point p' belongs to S. We choose one point from each antipodal pair of S such that their convex hull is as large or as small (with respect to its area) as possible. It is easy to see that, with this selection, the largest polygon will have to contain the origin, but the smallest polygon does not. In Fig. 1 an example of a thin (the smallest) and a thick (the largest) polygon is shown. An interesting question, which immediately suggests itself, is whether any polygon of S containing the center has larger area than any polygon that does not. We will formalize the mentioned concepts of thin and thick polygons and answer this question for sets in the plane as well as for sets in higher dimensions.

We start by introducing the problem formally in the plane. A set of $2n$ ($n \geq 3$) points on the unit circle centered at the origin is called an *antipodal point set* if for every point p it also contains its antipodal point p'. Let $S := \{p_1, p'_1, p_2, p'_2, \ldots, p_n, p'_n\}$ be such a set. An *antipodal* polygon on S is a convex polygon having as vertices precisely one point from each antipodal pair (p_i, p'_i) of S. A *thin* antipodal polygon P is an antipodal polygon whose vertices are consecutive points on the circle. For n odd, a *thick* antipodal polygon P is an antipodal polygon such that its vertices are every other point of S along the circle. For n even, we add the exception that exactly one pair of vertices of P are consecutive points on the circle. See Fig. 1. Note that an

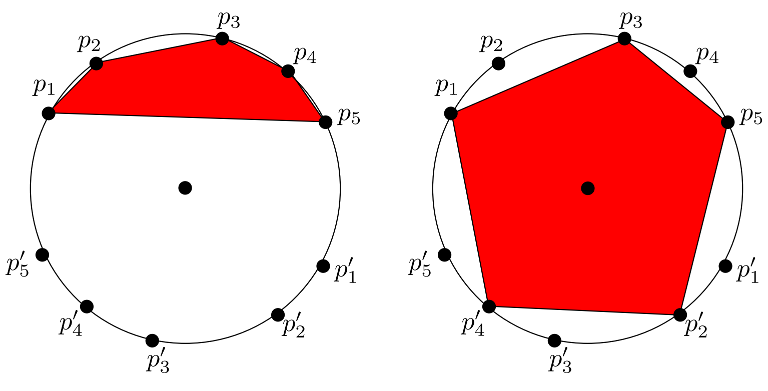

Fig. 1 A thin (*left*) and a thick (*right*) antipodal polygon

antipodal polygon P could be thin, thick, or neither, but P cannot both be thin and thick at the same time. Moreover, a thin antipodal polygon does not contain the center of the circle and a non-thin antipodal polygon always contains it.
We will consider the following questions:

- Does a thick antipodal polygon always have larger area than a thin antipodal polygon?
- How efficiently can one compute an antipodal polygon with minimal (maximal) area?
- What can be said about antipodal polygons in higher dimensions?

1.1 Related Work

The questions studied here are related to and motivated by several other (geometric) problems, some of which we mention below.

Extremal problems: Plane geometry is rich of extremal problems, often dating back till the ancient Greeks. During the centuries many of these problems have been solved by geometrical reasoning. Specifically, extremal problems on convex polygons have attracted the attention of both fields, geometry and optimization. In computational geometry, efficient algorithms have been proposed for computing extremal polygons with respect to several different properties [4]. In operations research, global optimization techniques have been extensively studied to find convex polygons maximizing a given parameter [3]. A geometric extremal problem similar to the one studied in this paper was solved by Fejes Tóth [12] almost fifty years ago. He showed that the sum of pairwise distances determined by n points contained in a circle is maximized when the points are the vertices of a regular n-gon inscribed in the circle. Recently, the discrete version of this problem has been reviewed in [14] and problems considering maximal area instead of the sum of inter-point distances have been solved in [10]. The last two mentioned references are also related to music theory, see below.

Stabbing problems: The problem of stabbing a set of objects by a polygon (transversal problems in the mathematics literature) has been widely studied. For example, in computational geometry, Arkin et al. [1] considered the following problem: a set S of segments is *stabbable* if there exists a convex polygon whose boundary C intersects every segment in S; the closed convex chain C is then called a (convex) transversal or *stabber* of S. Arkin et al. [1] proved that deciding whether S is stabbable is an NP-hard problem. In a recent paper [7], the problem of stabbing the set S of line segments by a simple polygon but with a different criterion has been considered. A segment s is stabbed by a simple polygon P if at least one of the two endpoints of s is contained in P. The task is to find a simple polygon P that stabs S and has minimum(maximum) area among those that stab S. In [7] it is shown that if S is a set of n pairwise disjoint segments, the problem of computing the minimum and maximum area (perimeter) polygon stabbing S can be solved in polynomial time. However, for general (crossing) segments the problem is NP-hard. Note that our problem can be seen as a constrained version of the problem studied in [7] in which each segment joins two antipodal points on a circle.

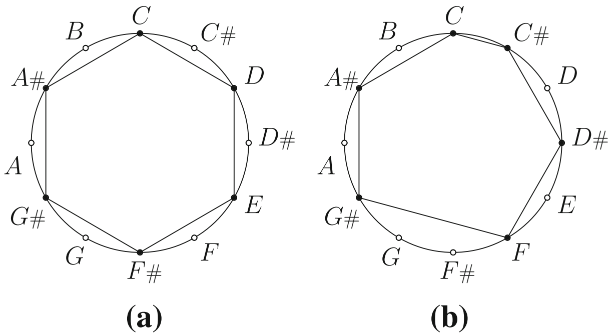

Fig. 2 The subsets in **a** and **b** represent maximally even scales with and without tritones, respectively

Music Theory: There exists a surprisingly high number of applications of mathematics to music theory. Questions about variation, similarity, enumeration, and classification of musical structures have long intrigued both musicians and mathematicians. In some cases, these problems inspired mathematical discoveries. The research in music theory has illuminated problems that are appealing, nontrivial, and, in some cases, connected to deep mathematical questions. The problem introduced in this paper comes from a question related to geometric measures of musical scales and rhythms [14]. An antipodal polygon is related with the *tritone* concept in music theory. Typically, the notes of a scale are represented by a polygon in a clock diagram. In a chromatic scale, each whole tone can be further divided into two semitones. Thus, we can think on a clock diagram with twelve points representing the twelve equally spaced pitches that represent the chromatic universe (using an equal tempered tuning). The pitch class diagram is illustrated in Fig. 2. A tritone is traditionally defined as a musical interval composed of three whole tones. Thus, it is any interval spanning six semitones. In Fig. 2a, the polygon represents a scale containing the tritones $CF\#$, $DG\#$, $EA\#$. The tritone is defined as a restless interval or dissonance in Western music from the early Middle Ages. This interval was mostly avoided in medieval ecclesiastical singing because of its dissonant sound. The name *diabolus in musica* (the Devil in music) has been applied to the interval from at least the early 18th century [11]. In this context, an antipodal polygon corresponds to a subset of notes or harmonic scale avoiding the tritone. On the other hand, one of the properties that musicologists and mathematicians have observed in music in various oral traditions is what is called regularity of the rhythm (or of the musical scale). Regular rhythms have been defined as those which maximize a particular geometric measure [14,15]. Thus, a maximal antipodal polygon represents a maximally even set that avoids the tritone. A relationship between extremal polygons and musical scales has been shown in [10].

Inscribed polygons with no antipodal vertices have also been considered in the analysis of musical rhythms [2,5]. A rhythm has the *rhythmic oddity* property if, when represented on a circle, it does not contain two onsets (the black points in Fig. 3) that lie diametrically opposite of each other. Note that the property ensures that one cannot split the circle into two parts of equal length whatever the chosen breaking onsets is. Thus, these patterns possess a particular type of asymmetry. Many musical

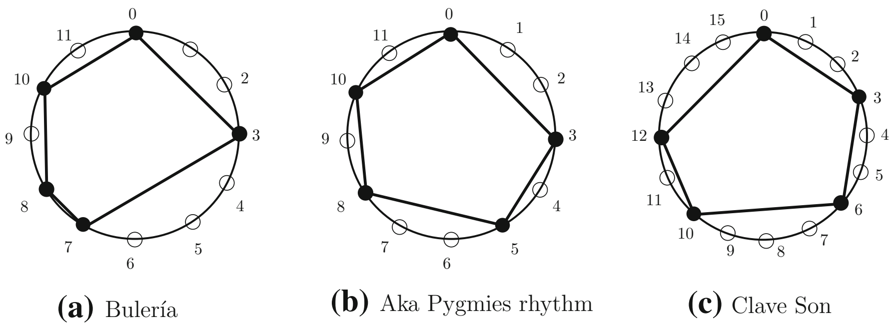

Fig. 3 **a** The bulería rhythm used in Spain, **b** a rhythm used by the Aka Pygmies of Central Africa, **c** the clave son of Cuba

traditions all over the world have asymmetric rhythmic patterns. Indeed, rhythms with the oddity property (antipodal polygons have such a property) play a fundamental role in traditional music as, for instance, flamenco music, african music or cuban music. The *bulería pattern* used in the flamenco music of Spain (Fig. 3a)), the Aka Pygmies pattern of Central Africa (Fig. 3b)) and the *clave son* in Cuba (Fig. 3c)) can all be considered as antipodal polygons with $k < n$ vertices. See [2,6,13] respectively, for a detailed study on the preference of theses rhythms in their cultural contexts.

1.2 Our Results

In this paper we prove the following general result:

Claim Given an antipodal point set $S \subset I\!R^2$, every thin antipodal polygon on S has less area than any non-thin antipodal polygon on S.

In addition we show that the 2-dimensional case is special in the sense that the above result can not be generalized to higher dimensions.

The analogue result holds for thick antipodal polygons when n is odd but surprisingly turns out to be wrong when n is even; for n even we provide an example of an antipodal non-thick polygon having larger area than a thick antipodal polygon. However, we are able to prove the following result:

Claim Given an antipodal point set $S \subset I\!R^2$, for every non-thick antipodal polygon on S there exists a thick antipodal polygon on S with larger area.

Note that above claims imply that an antipodal polygon with minimum (resp. maximum) area is thin (resp. thick). As a consequence, we will show that the extremal problems for antipodal polygons can be solved in linear time.

2 Thin Antipodal Polygons

Assume that the clockwise circular order of S around the origin is $p_1, p_2, \ldots, p_n$, $p'_1, p'_2, \ldots, p'_n$. For every point q in S, let P_q be the thin antipodal polygon that contains as vertices q and the next $n-1$ clockwise consecutive points of S. Note that all thin antipodal polygons are of this form and that P_q and $P_{q'}$ are congruent, that is, P_q and $P_{q'}$ are the same polygon in different positions.

The next lemma characterizes triangles that contain a given point of S and maximize the area.

Lemma 1 *For a point $p \in S$ let ℓ be the line containing p and p'. Let τ be the triangle determined by p, and its two neighbors in S. Among all triangles that have as vertices p and one point of S in each of the two half-planes defined by ℓ, τ has strictly the smallest area.*

Proof Let τ' be a triangle with vertices in S, containing p as a vertex and with a vertex in each of the two half-planes defined by ℓ. Assume that τ' is different from τ. Let b be the side opposite to p in τ and b' be the side opposite to p in τ'. Note that b' is at least as large as b, because S is an antipodal point set and ℓ contains the origin. The height of τ' with respect to p is larger than the height of τ with respect to p, as otherwise b' would have to intersect b, which is not possible by construction. Thus the area of τ' is larger than the area of τ.

We split the proof of Claim 1.2 into the three cases $n = 3$, $n = 4$, and $n \geq 5$.

Lemma 2 *For $n = 3$, every thin antipodal polygon on S has area strictly less than that of any non-thin antipodal polygon on S.*

Proof In this case the only non-thin polygons are the two triangles τ and τ' with vertex sets $\{p_1, p_2', p_3\}$ and $\{p_1', p_2, p_3'\}$, respectively. Note that τ has the same area as τ'. In addition, by Lemma 1, τ has larger area than P_{p_2} and τ' has larger area than P_{p_1} and P_{p_3}.

Lemma 3 *For $n = 4$, every thin antipodal polygon on S has area strictly less than that of any non-thin antipodal polygon on S.*

Proof In this case a non-thin antipodal polygon P has exactly two consecutive points; without loss of generality assume that they are p_1 and p_2. Thus, P is the convex quadrilateral p_1, p_2, p_4, p_3'. We show that P has larger area than P_{p_1}, P_{p_2}, $P_{p_3'}$ and $P_{p_4'}$.

By Lemma 1 the triangle $p_4' p_1 p_2$ has less area than the triangle $p_3' p_1 p_2$. By Lemma 2 the triangle $p_3' p_2 p_4$ has larger area than the triangle $p_3' p_4' p_2$ and also larger than the triangle $p_4' p_2 p_3$. Thus P has larger area than $P_{p_3'}$ and also larger than $P_{p_4'}$. By Lemma 1 the triangle $p_1 p_2 p_3$ has less area than the triangle $p_1 p_2 p_4$. By Lemma 2 the triangle $p_3' p_1 p_4$ has larger area than the triangle $p_1 p_3 p_4$. Thus P has larger area than P_{p_1}.

It remains to show that P has larger area than P_{p_2}. Let ℓ be the line passing through p_1 and p_1'. Rotate ℓ continuously clockwise around the origin and translate p_1 and p_1' until p_1 meets p_2 and p_1' meets p_2'. See Fig. 4. Throughout the motion the area of P_{p_2} is strictly increasing. To see this, observe that the height of the triangle with vertices p_2, p_4 and p_1' is strictly increasing, as otherwise during the rotation p_1' must

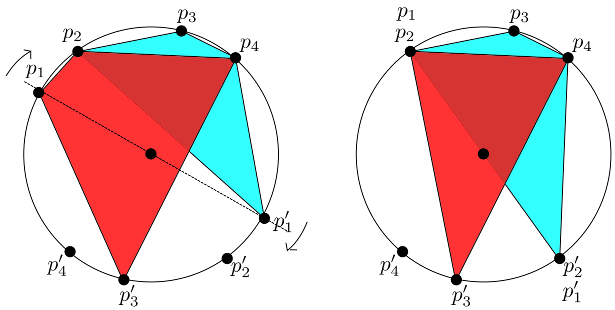

Fig. 4 The rotation in the proof of Lemma 3 and its limit case

eventually cross the perpendicular bisector of the segment p_2p_4. However, this cannot happen since p_1' reaches p_2' before it reaches this line.

On the other hand, the area of P might at first be strictly increasing and then at some point be strictly decreasing. Moreover, if this is the case, there is a point in time at which P has the same area as in the beginning of the motion (and will strictly decrease afterwards) and the area of P_{p_2} has increased. Thus, we can assume that during the whole motion the area of P is strictly decreasing and the area of P_{p_2} is strictly increasing. We will show that at the end of the motion P and P_{p_2} have equal area. This implies that at the beginning of the motion the area of P is larger than the area of P_{p_2}.

At the end of the motion P coincides with the triangle $p_2p_4p_3'$ and P_{p_2} with the quadrilateral $p_2p_3p_4p_2'$. We split the quadrilateral $p_2p_3p_4p_2'$ into the triangles $p_2p_3p_4$ and $p_2'p_2p_4$, sharing the side $\overline{p_2p_4}$. The height of the triangle $p_2p_4p_3'$ with respect to $\overline{p_2p_4}$ has the same length as the sum of the heights of the triangles $p_2p_3p_4$ and $p_2'p_2p_4$ with respect to $\overline{p_2p_4}$ (This can bee seen by using the triangle $p_4'p_3'p_2'$). Hence, the area of the triangle $p_2p_4p_3'$ equals the area of the quadrilateral $p_2p_3p_4p_2'$.

We are now ready to prove our first claim.

Theorem 1 *Every thin antipodal polygon on S has less area than any non-thin antipodal polygon on S.*

Proof We proceed by induction on n where Lemmas 2 and 3 cover the induction base. For $n \geq 5$ let P be a non-thin antipodal polygon on S and let T be a triangulation of P. Let p be a vertex of degree two in T such that the triangle τ of T having p as a vertex does not contain the origin in its interior (as any triangulation has two ears, there is at least one ear that does not contain the origin). Let q and r be the two neighbors of p in S and τ' the triangle with vertices p, q and r. By Lemma 1 the area of τ' is equal to or less than the area of τ.

The polygon P' with vertices of P without p is a non-thin antipodal polygon for $S \backslash \{p, p'\}$. By induction, P' has larger area than any thin antipodal polygon on $S \backslash \{p, p'\}$. Some of these thin polygons (specifically, any polygon that includes both r and q) together with τ' form antipodal polygons on S. Using this observation and the fact that the area of P_{p_i} is the same as the area of $P_{p_i'}$, it follows that, except for

P_p and P_q ($P_{p'}$ and $P_{q'}$), all antipodal thin polygons on S have area strictly less than P. However, for $n \geq 5$, P can be triangulated so that p is not the middle nor the last vertex (clockwise) of an ear. As any triangulation has two ears, there is an ear that does not contain the origin. Using the previous arguments for this new triangulation and ear it can be shown that the area of P is strictly larger than the area of P_p ($P_{p'}$), and similarly for P_q ($P_{q'}$).

3 Thick Antipodal Polygons

In this section we present two area increasing operations on antipodal polygons. Using a sequence of these operations a non-thick antipodal polygon can be transformed into a thick antipodal polygon. We will use this to prove Claim 1.2.

We start with an (arbitrary) antipodal polygon P. By *flipping* a point q of S we denote the following operation: if q is a vertex of P, then choose q' instead of q; if q is not a vertex of P then choose q instead of q'. The two operations described in the following Lemmas 4 and 5 are sequences of such flips.

Lemma 4 *If P has three consecutive points q_1, q_2 and q_3 of S as vertices, then flipping q_2 provides a polygon P' of larger area.*

Proof Let q_4' be the point after q_3' in P and q_0' be the point before q_1' in P. Moreover, τ_1 is the triangle with vertex set $\{q_1, q_2, q_3\}$ and τ_2 the triangle with vertex set $\{q_0', q_2', q_4'\}$. The difference of the areas of P and P' is equal to the difference in the areas of τ_1 and τ_2. However, τ_1 has the same area as the triangle with vertex set $\{q_1', q_2', q_3'\}$; by Lemma 1 the area of this triangle is less than that of τ_2.

From now on, we assume that P does not contain three consecutive points of S as vertices. Otherwise we apply the operation described in Lemma 4.

Lemma 5 *Let $q_1, q_2, \ldots, q_m$ $(4 \leq m \leq n)$ be consecutive points of S and let P be an antipodal polygon with vertices from S such that:*

- *P does not contain three consecutive points of S,*
- *P contains q_1 and q_2,*
- *P contains either both q_{m-1} and q_m, or neither of them, and*
- *every other point from q_3 to q_{m-1} belongs to P.*

Let P' be the antipodal polygon obtained from P, by flipping each point q_i $(2 \leq i \leq m-1)$. Then P' has larger area than P.

Proof We proceed by comparing the area of $P \backslash P'$ with the area of $P' \backslash P$. Note that these areas are formed by disjoint triangles. If m is odd then q_m and q_{m-1} are vertices of P (see Fig. 5, left) and if m is even then P does not contain any of them (see Fig. 5, right). In Fig. 5 (left) the points a and b can match with q_1 and q_m, respectively and, in Fig. 5 (right) the points c and d exist and can be coincident. Let $\mathcal{T} = (P \backslash P') \cup (P' \backslash P)$ (light-shaded and dark-shaded triangles in Fig. 5). For each p in $\{q_2, q_2', \ldots, q_{m-1}, q_{m-1}'\}$ let $\tau(p)$ be the triangle in $\mathcal{T}$ that contains p as a vertex. The difference in the area of P and the area of P' equals the difference in the areas of

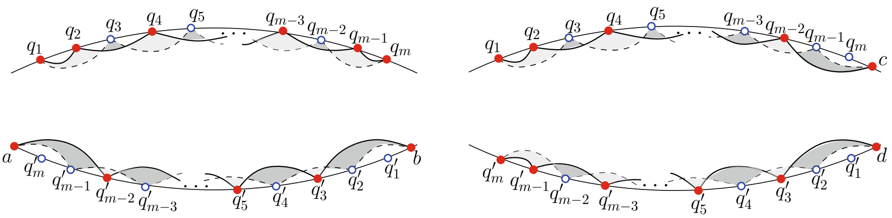

Fig. 5 Schematic diagram of the flip operations described in Lemma 5 to transform P into P'. The *light-shaded triangles* are $P \backslash P'$ and the *dark-shaded triangles* are $P' \backslash P$

those triangles contained in P and those contained in P'. For $4 \leq i \leq m-3$, the area of $\tau(q_i)$ equals the area of $\tau(q'_i)$ and one of them is contained in P while the other is contained in P', see Fig. 5. Thus, the difference in the areas of P and P' depends only on the areas of $\tau(q_2)$, $\tau(q'_2)$, $\tau(q_3)$, $\tau(q'_3)$, $\tau(q_{m-2})$, $\tau(q'_{m-2})$, $\tau(q_{m-1})$, and $\tau(q'_{m-1})$ Note that the area of $\tau(q_2)$ is smaller than the area of $\tau(q'_2)$ and that P contains $\tau(q_2)$ while P' contains $\tau(q'_2)$. The same holds for $\tau(q_3)$ and $\tau(q'_3)$, see again Fig. 5.

If P contains both q_{m-1} and q_m, then $\tau(q_{m-1})$ is contained in P and $\tau(q'_{m-1})$ is contained in P'. In this case the area of $\tau(q_{m-1})$ is smaller than the area of $\tau(q'_{m-1})$, see Fig. 5 (left).

If P does not contain q_{m-1} and q_m, then $\tau(q'_{m-1})$ is contained in P and $\tau(q_{m-1})$ is contained in P'. In this case the area of $\tau(q'_{m-1})$ is smaller than the area of $\tau(q_{m-1})$, see Fig. 5 (right). The same argument can be applied to $\tau(q_{m-2})$ and $\tau(q'_{m-2})$. As a consequence, in all cases the area of P is smaller than the area of P'.

We are now ready to prove Claim 1.2.

Theorem 2 *For every non-thick antipodal polygon on S, there exists a thick antipodal polygon on S with larger area.*

Proof The idea is to transform any non-thick antipodal polygon into a thick antipodal polygon by using flipping transformations. Let Q be a non-thick antipodal polygon. Then S contains three consecutive points as vertices of Q or S contains a sequence $q_1, q_2, \ldots, q_m$ $(4 \leq m \leq n)$ fulfilling that Q contains q_1 and q_2, Q contains either both q_{m-1} and q_m, or neither of them, and every other point from q_3 to q_{m-1} belongs to Q. Using Lemmas 4 and 5 we obtain an antipodal polygon with larger area than Q. Repeating these operations until they can no longer be applied we obtain a thick polygon. The process terminates as in the operations described in Lemmas 4 and 5 the number of consecutive points in S as vertices of the non-thick antipodal polygon decreases.

Corollary 1 *For n odd, every thick antipodal polygon on S has larger area than a non-thick antipodal polygon on S.*

Proof For n odd there are only two antipodal thick polygons and they have the same area.

We now provide an example of a set of points and a non-thick antipodal polygon that has larger area than a thick antipodal polygon of this set.

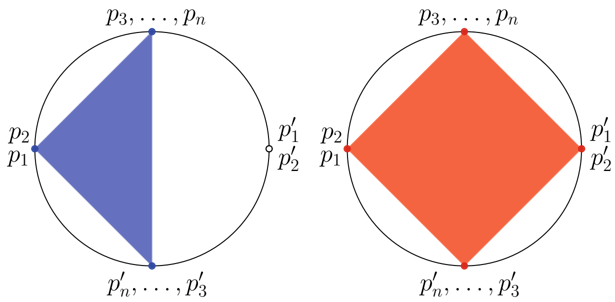

Fig. 6 Construction for Theorem 3

Theorem 3 *For all $n \geq 6$ even, there exist sets of n points with a non-thick antipodal polygon and a thick antipodal polygon such that the non-thick antipodal polygon has larger area than the thick antipodal polygon.*

Proof Place p_1 and p_2 arbitrarily close to $(-1, 0)$; thus p'_1 and p'_2 are arbitrarily close to $(1, 0)$. Place $p_3, \ldots, p_n$ arbitrarily close to $(0, 1)$; thus $p'_3, \ldots, p'_n$ are arbitrarily close to $(0, -1)$ as illustrated in Fig. 6. Let P be the thick antipodal polygon that contains both p_1 and p_2 as vertices. Let Q be any non-thick antipodal polygon that contains p_1, p'_2, p_3 and p'_4 as vertices. Note that P is arbitrarily close to the triangle with vertices $(0, 1)$, $(0, -1)$ and $(1, 0)$; Q is arbitrarily close to the quadrilateral with vertices $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(0, -1)$. Thus the area of P is arbitrarily close to 1, while the area of Q is arbitrarily close to 2.

It is worth mentioning that the algorithmic version of the optimization problem in which the input is a set of line segments, each connecting two (non necessarily antipodal) points on the circle, has been shown to be NP-hard [7]. Surprisingly, the antipodal version can be easily solved in linear time by using the above characterizations. According to Theorem 1, an antipodal polygon with minimum area is a thin antipodal polygon. Thus, since there exist $O(n)$ thin antipodal polygons, we can sweep in a linear number of steps around the circle and update in constant time the area of two consecutive thin antipodal polygons. On the other hand, according to Theorem 2, if n is odd, there are only two thick antipodal polygons (the alternating polygons). For n even, there exists a linear number of thick antipodal polygons (having two consecutive points and the rest in alternating position). In the last case, a linear sweep around the circle can also be used to compute in linear time a thick antipodal polygon that maximizes the area.

4 Higher Dimensions: Antipodal Polytopes

In this section we consider a generalization of the problem to higher dimensions. Assume therefore that all points are now placed on the unit d-dimensional sphere. Instead of antipodal polygons we thus have antipodal polytopes.

The natural generalization for higher dimensions is as follows. Let $S := \{p_1, p'_1, p_2, p'_2, \ldots, p_n, p'_n\}$ be a set of n antipodal pairs on the unit d-dimensional

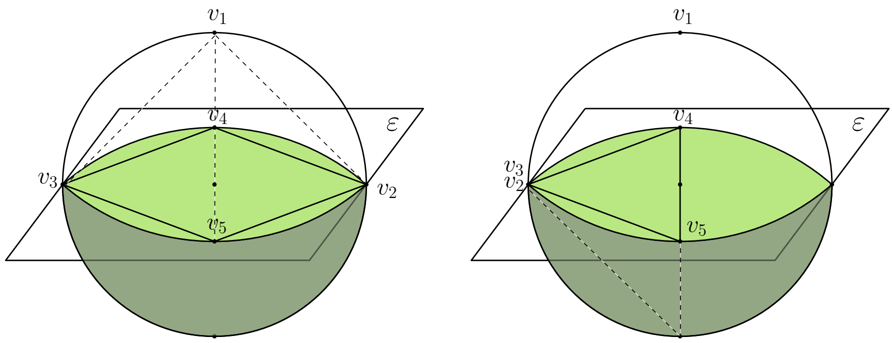

Fig. 7 An example showing that Theorem 1 can not be generalized to three-dimensions

sphere centered at the origin. A *thin* antipodal polytope is an antipodal polytope whose vertices all lie on one side of some hyperplane passing through the origin. Note that this definition generalizes the thin antipodal polygon's notion.

In dimension 3 or greater Theorem 1 does not hold. There are antipodal point sets $S \subset I\!R^d$ such that there exists an antipodal thin polytope with larger d-dimensional volume than a non-thin antipodal polytope on S. We start by providing a three dimensional example and then argue how to generalize it to higher dimensions.

For some small $\varepsilon > 0$, let $\delta = \sqrt{1 - 2\varepsilon^2}$ and consider the set S_1 of the five points $v_1 := (0, 0, 1)$, $v_2 := (\delta, \varepsilon, \varepsilon)$, $v_3 := (-\delta, \varepsilon, \varepsilon)$, $v_4 := (\varepsilon, \delta, \varepsilon)$, and $v_5 := (\varepsilon, -\delta, \varepsilon)$. Let S be the antipodal point set consisting of S_1 and all its antipodal points. The convex hull of S_1 is a pyramid with a square base (with corners $v_2, \ldots, v_5$) which lies in the horizontal plane just ε above the origin, see Fig. 7. The top of the pyramid is at height 1. Thus, this pyramid does not contain the origin in its interior, and for $\varepsilon \to 0$ the volume of the pyramid converges to $2/3$.

To obtain our second polyhedron first flip the vertex v_1 to $v_1' := (0, 0, -1)$. This gives a similar upside-down pyramid, which contains the origin in its interior. By also flipping v_2 to $v_2' := (-\delta, -\varepsilon, -\varepsilon)$, we essentially halve the base of the pyramid to be a triangle. We denote the resulting point set by $S_2 = \{v_1', v_2', v_3, v_4, v_5\} \subset S$. Note that v_2' and v_3 are rather close together. As the triangle v_3, v_4, v_5 lies above the origin, the convex hull of S_2 still contains the origin in its interior, see again Fig. 7. Moreover, the volume of the convex hull of S_2 converges to $1/3$ for $\varepsilon \to 0$, and thus towards half of the volume of the convex hull of S_1.

So together these two polyhedra constitute an example which shows that Theorem 1 can not be generalized to higher dimensions: S is a set of five antipodal pairs of points on the surface of the 3-dimensional unit sphere such that the convex hull of S_1 does not contain the origin, while the convex hull of S_2 does. But in the limit the volume of the convex hull of S_1 becomes twice as large as the volume of the convex hull of S_2.

It is straight forward to observe that this example can be generalized to any dimension $d \geq 4$. There we have $2d - 1$ antipodal pairs of points, where we set $\delta = \sqrt{1 - (d-1)\varepsilon^2}$ and every point has one coordinate at $\pm\delta$ and the remaining coordinates at $\pm\varepsilon$, analogous to the 3-dimensional case. For $d - 1$ of the coordinate axes two such pairs are "aligned" as in the 3-dimensional example, and for the last axis there is only one such pair. The resulting polytope does not contain the origin. Flipping the vertex of the singular pair and one vertex for all but one aligned pairs results in a polytope which contains the origin, but has a volume of only $1/2^{d-2}$ of the first polytope.

We call a d-dimensional antipodal polytope *thick* if the number of vertices in any half-space defined by a hyperplane through the origin contains at least $\lceil \frac{n-d}{2} \rceil$ points of the polytope. Note that in the two dimensional case, a thick antipodal polygon satisfies that at least $\lceil \frac{n-2}{2} \rceil$ of its vertices lie in both open half-planes defined by any given line through the origin.[1]

It is not clear that for a given antipodal set in $\mathbb{R}^d$ an antipodal thick polytope should exist. However, for every $n \geq d$, there exist antipodal sets in $\mathbb{R}^d$ that admit an antipodal thick polytope. For the proof of this remark we use the following Lemma.

Lemma 6 (Gale's Lemma [8]). *For every $d \geq 0$ and every $k \geq 1$, there exists a set $X \subset S^d$ of $2k + d$ points such that every open hemisphere of S^d contains at least k points of X.*

From the proof of Gale's Lemma in [9] (page 64), it follows that the provided set does not contain an antipodal pair of points. Recall that $S^{d-1} \subset \mathbb{R}^d$; let X be the subset of S^{d-1} provided by Gale's Lemma for $k = \lceil \frac{n-d+1}{2} \rceil$. If necessary remove a point from X so that X consists of exactly n points. Let X' be the set of antipodal points of X and set $S := X \cup X'$. Let P be the antipodal polytope on S with X as a vertex set. It follows from Gale's Lemma that P is thick.

5 Open Problems

Let us assume that we are given a circular lattice with an antipodal set of $2n$ points (evenly spaced) and we would like to compute an extremal antipodal k-polygon with $k < n$ vertices. This problem is significantly different to the considered case $k = n$. Recall that, for $k = n$, the linear algorithms proposed in this paper are strongly based on the simple characterization for the extremal antipodal polygons. Namely, the minimal thin antipodal polygon has consecutive vertices and the thick antipodal polygon has an alternating configuration. It is not difficult to come up with examples for which that characterization does not hold in the general case $k < n$. On the other hand, finding the extremal antipodal $(n-1)$-polygon, called $(2n, n-1)$-problem for short, can be easily reduced to solve $O(n)$ times the $(2(n-1), n-1)$-problem. To see this, observe that in the $(2n, n-1)$-problem an antipodal pair is not selected and can thus be removed from the input. This approach gives a simple $O(n^{n-k+1})$ time algorithm for solving the general $(2n, k)$-problem. This leaves the open question if the $(2n, k)$-problem can be solve in $o(n^k)$ time.

Instead of area, it is also interesting to consider other extremal measures, like perimeter or the sum of inter-point distances. Finally, for higher dimensions, we leave the existence of thick polytopes for arbitrary antipodal point sets as an open problem.

[1] This property is not "if and only if" because there also exist non-thick polygons fulfilling the property.

For future research it is also interesting to consider alternative definitions of thin and thick antipodal polytopes for higher dimensions.

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