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Clique-perfectness of complements of line graphs

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ABSTRACT

A graph is clique-perfect if the maximum number of pairwise disjoint maximal cliques equals the minimum number of vertices intersecting all maximal cliques for each induced subgraph. In this work, we give necessary and sufficient conditions for the complement of a line graph to be clique-perfect and an $O(n^2)$-time algorithm to recognize such graphs. These results follow from a characterization and a linear-time recognition algorithm for matching-perfect graphs, which we introduce as graphs where the maximum number of pairwise edge-disjoint maximal matchings equals the minimum number of edges intersecting all maximal matchings for each subgraph. Thereby, we completely describe the linear and circular structure of the graphs containing no bipartite claw, from which we derive a structure theorem for all those graphs containing no bipartite claw that are Class 2 with respect to edge-coloring.

1. Introduction

Numerous major theorems in combinatorics are formulated in terms of min–max relations of dual graph parameters.

Perfect graphs were defined by Berge in terms of a min–max inequality involving clique and chromatic number. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to assign different colors to adjacent vertices of G. The maximum size of a clique in G is its *clique number* $\omega(G)$. Clearly, the min–max type inequality $\omega(G) \leq \chi(G)$ holds for every graph G. Berge [3] called a graph G *perfect* if and only if the equality $\omega(H) = \chi(H)$ holds for each induced subgraph H of G.

An important result about perfect graphs is the *Perfect Graph Theorem* which states that the complement of a perfect graph is also perfect [29,40]. Thus, a graph G is perfect if and only if clique and chromatic number coincide for each induced subgraph of its complement $\overline{G}$. The clique number of $\overline{G}$ is the *stability number* $\alpha(G)$, which is the maximum number of pairwise nonadjacent vertices of G. The chromatic number of $\overline{G}$ is the *clique covering number* $\theta(G)$, which is the minimum number of cliques of G covering all its vertices. Hence, the min–max type inequality $\alpha(G) \leq \theta(G)$ holds for every graph G and, by virtue of the Perfect Graph Theorem, a graph G is perfect if and only if $\alpha(H) = \theta(H)$ holds for each induced subgraph H of G.

A *hole* or *antihole* in a graph G is an induced subgraph isomorphic to the chordless cycle on k vertices C_k or its complement $\overline{C_k}$, respectively, for some $k \geq 5$. If k is odd, then the hole or antihole is *odd*; otherwise it is *even*. Berge [3] conjectured that a graph is perfect if and only if it has no odd holes and no odd antiholes. This conjecture was proved to be true about 40 years later and is now known as the *Strong Perfect Graph Theorem* [19].

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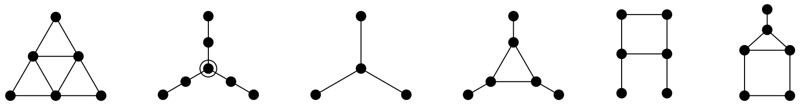

Fig. 1. Some small graphs. The circled vertex is the *center* of the bipartite claw.

Theorem 1.1 (*Strong Perfect Graph Theorem [19]*). *A graph is perfect if and only if it has no odd holes and no odd antiholes.*

A polynomial-time recognition algorithm for perfect graphs was given in [18].

The class of clique-perfect graphs is defined by requiring equality in a min–max type inequality related to the Kőnig property of the family of maximal cliques. Consider a family $\mathcal{F}$ of nonempty subsets of a finite ground set X, then the *transversal number* $\tau(\mathcal{F})$ is the minimum number of elements of X needed to intersect every member of $\mathcal{F}$ and the *matching number* $\nu(\mathcal{F})$ of $\mathcal{F}$ is the maximum size of a collection of pairwise disjoint members of $\mathcal{F}$. If these two numbers coincide, the family $\mathcal{F}$ is said to have the *Kőnig property* [4].

Let $\mathcal{Q}$ be the family of all maximal cliques of G. A collection of pairwise disjoint maximal cliques of a graph is a *clique-independent set* and a vertex set intersecting every maximal clique of a graph is a *clique-transversal*. Accordingly, we call $\nu(\mathcal{Q})$ the *clique-independence number* $\alpha_c(G)$ and $\tau(\mathcal{Q})$ the *clique-transversal number* $\tau_c(G)$. Clearly, the min–max type inequality $\alpha_c(G) \leq \tau_c(G)$ holds for every graph G. A graph G is *clique-perfect* [30] if $\alpha_c(H) = \tau_c(H)$ holds for each induced subgraph H of G. In other words, a graph G is clique-perfect if and only if, for each induced subgraph of G, the family of all maximal cliques has the Kőnig property.

The Kőnig property has its origins in the study of matchings and transversals in bipartite graphs. The *matching number* $\nu(G)$ of a graph G is the maximum size of a *matching* (a set of vertex-disjoint edges) and the *transversal number* $\tau(G)$ is the minimum size of a *vertex cover* (a set of vertices intersecting every edge). Clearly, the min–max type inequality $\nu(G) \leq \tau(G)$ holds for every graph G. In 1931, Kőnig [36] and Egerváry [27] proved that every bipartite graph B satisfies $\nu(B) = \tau(B)$. This result is now known as the *Kőnig–Egerváry Theorem*. Notice that if B is bipartite, then $\alpha_c(B) = \nu(B)+i(B)$ and $\tau_c(B) = \tau(B)+i(B)$ where $i(B)$ denotes the number of isolated vertices of B; consequently, $\alpha_c(B) = \tau_c(B)$ if and only if $\nu(B) = \tau(B)$. Therefore, since each induced subgraph of a bipartite graph is also bipartite, the Kőnig–Egerváry Theorem can be restated by saying that every bipartite graph is clique-perfect. Apart from bipartite graphs, some other graph classes are known to be clique-perfect: comparability graphs [1], balanced graphs [5], complements of forests [7], and distance-hereditary graphs [37].

It is important to mention that not all clique-perfect graphs are perfect and that not all perfect graphs are clique-perfect. For instance, the even antihole $\overline{C_{6k+2}}$ is perfect but not clique-perfect, whereas the odd antihole $\overline{C_{6k+3}}$ is clique-perfect but not perfect, for each $k \geq 1$. In fact, we have:

Theorem 1.2 (*[26,30]*). *A hole C_n is clique-perfect if and only if n is even. An antihole $\overline{C_n}$ is clique-perfect if and only if n is a multiple of* 3.

Notice also that if the equality $\alpha_c(G) = \tau_c(G)$ holds for a graph G, then the same equality may not hold for all its induced subgraphs. For instance, every graph G in the class of *dually chordal graphs* [14] satisfies the equality $\alpha_c(G) = \tau_c(G)$, but dually chordal graphs are not clique-perfect in general; e.g., the 5-*wheel* (the graph that arises from C_5 by adding a vertex adjacent to every other vertex) is dually chordal but it is not clique-perfect because it contains an induced C_5, for which $\alpha_c(C_5) = 2$ but $\tau_c(C_5) = 3$.

Unlike perfect graphs, the class of clique-perfect graphs is not closed under graph complementation; e.g., the net and the 3-sun (see Fig. 1) are complement graphs of each other such that the former is clique-perfect but the latter is not clique-perfect. Moreover, a complete characterization of clique-perfect graphs by forbidden induced subgraphs is not known either. Another open question regarding clique-perfect graphs is the computational complexity of their recognition problem. Nevertheless, some partial results in this direction appeared in [8,9,11,38], where necessary and sufficient conditions for a graph G to be clique-perfect in terms of forbidden induced subgraphs as well as polynomial-time algorithms for deciding whether a given graph G is clique-perfect were found when restricting G to belong to one of several different graph classes. Interestingly, the problems of determining $\alpha_c(G)$ and $\tau_c(G)$ are both NP-hard even if G is a split graph [17] and determining $\tau_c(G)$ is NP-hard even if G is a triangle-free graph [28]. More NP-hardness results of this type for α_c and τ_c were proved in [30]. Some polynomial-time algorithms for determining $\alpha_c(G)$ and $\tau_c(G)$ when G belongs to one of several different graph classes were devised in [1,13,17,22,24–26,30,37].

The *line graph* $L(H)$ of a graph H is the graph whose vertices are the edges of H and such that, for every two different edges e and f of H, ef is an edge of $L(H)$ if and only if e and f share an endpoint. A graph G is a *line graph* [51] if it is the line graph of some graph H; if so, H is called a *root graph* of G. Perfectness of line graphs (or, equivalently, of their complements) was studied in [42,43]. In [8], clique-perfectness of line graphs was characterized by forbidden induced subgraphs, as follows (see Fig. 1 for a 3-sun).

Theorem 1.3 (*[8]*). *If G is a line graph, then G is clique-perfect if and only if G contains no induced* 3*-sun and has no odd hole.*

Since the class of clique-perfect graphs is not closed under graph complementation, the above result does not determine which complements of line graphs are clique-perfect. The main result of this work is the theorem below which gives necessary and sufficient conditions for the complement of a line graph to be clique-perfect, in terms of forbidden induced subgraphs.

Theorem 1.4. *If G is the complement of a line graph, then G is clique-perfect if and only if G contains no induced 3-sun and has no antihole $\overline{C_k}$ for any $k \geq 5$ such that k is not a multiple of 3.*

Let G be the complement of the line graph of a graph H. In order to prove Theorem 1.4, we profit from the fact that the maximal cliques of G are precisely the maximal matchings of H. (In this work, *maximal* means inclusion-wise maximal, whereas *maximum* means maximum-sized.) We call any set of edges intersecting all the nonempty maximal matchings of H a *matching-transversal* of H, and any collection of edge-disjoint nonempty maximal matchings of H a *matching-independent set* of H. We define the *matching-transversal number* $\tau_{\mathrm{m}}(H)$ of H as the minimum size of a matching-transversal of H and the *matching-independence number* $\alpha_{\mathrm{m}}(H)$ of H as the maximum size of a matching-independent set of H.[1] Clearly, $\alpha_{\mathrm{c}}(G) = \alpha_{\mathrm{m}}(H)$ and $\tau_{\mathrm{c}}(G) = \tau_{\mathrm{m}}(H)$. We say that H is *matching-perfect* if $\alpha_{\mathrm{m}}(H') = \tau_{\mathrm{m}}(H')$ for every subgraph H' (induced or not) of H. Equivalently, H is matching-perfect if and only if the nonempty maximal matchings of H' have the Kőnig property for each subgraph H' (induced or not) of H. Hence, G is clique-perfect if and only if H is matching-perfect, and Theorem 1.4 can be reformulated as follows (see Fig. 1 for a bipartite claw).

Theorem 1.5. *A graph H is matching-perfect if and only if H contains no bipartite claw and the length of each cycle of H is at most 4 or a multiple of 3.*

In this work, 'H contains no J' means that H contains no subgraph (induced or not) isomorphic to J.

The structure of the paper is as follows. In the next subsection, we give basic definitions and preliminaries. In Section 2, we collect all structural theorems needed to establish our main results. In Section 2.1, we give a precise description of the linear and circular structure of those graphs containing no bipartite claw, which is used all along this work. In Section 2.2, we give a structure theorem for those graphs containing no bipartite claw that are Class 2 with respect to edge-coloring. This structure theorem is key for finding the matching-independent sets needed for the proofs given in Section 2.3 of the main results of this work (Theorems 1.4 and 1.5). This leads to a linear-time recognition algorithm for matching-perfect graphs and an $O(n^2)$-time algorithm for deciding whether or not any given complement of a line graph is clique-perfect, that follow from our main results. In Section 3, we present the proofs for all results. The main results of this paper appeared in the extended abstract [10].

Basic definitions and preliminaries

All graphs in this work are finite, undirected, without loops, and without multiple edges. For all basic graph-theoretic definitions and notations not defined in this section, we refer to West [50]. The only exceptions to this rule are the notions of minors and tree-width, which we will mention only incidentally; for a gentle introduction to these notions, see [23, Chapter 12].

Let G be a graph. The vertex set of G is denoted by $V(G)$, the edge set by $E(G)$, and the complement by $\overline{G}$. The neighborhood of a vertex v in G is denoted by $N_G(v)$, whereas $N_G[v]$ denotes $N_G(v) \cup \{v\}$. We denote by $E_G(v)$ the set of edges of G incident to a vertex v. Two nonadjacent vertices v and w of G are *false twins* if $N_G(v) = N_G(w)$, whereas two adjacent vertices v and w are *true twins* if $N_G[v] = N_G[w]$. For any set S, $|S|$ denotes its cardinality. The *degree* $d_G(v)$ of a vertex v of G is $|N_G(v)|$. The *maximum degree* among the vertices of G is denoted by $\Delta(G)$ and the *minimum degree* by $\delta(G)$. A vertex is *pendant* if its degree is 1. An edge is *pendant* if at least one of its endpoints is a pendant vertex. The *center* of the bipartite claw is its vertex of degree 3. A vertex of G is *universal* if it is adjacent to every other vertex of G. A graph is *complete* if its vertices are pairwise adjacent and K_n denotes the complete graph on n vertices. A *clique* of a graph is a set of pairwise adjacent vertices. A *stable set* of a graph is a set of pairwise nonadjacent vertices.

Let Z be a path or a cycle. We denote by $E(Z)$ the set of edges joining two consecutive vertices of Z and the *length* of Z is $|E(Z)|$. A *chord* of Z is an edge joining two nonconsecutive vertices of Z and Z is *chordless* if Z has no chords. A *chord* ab of Z is *short* if there is some vertex c of Z that is consecutive to each of a and b in Z; if so, c is called a *midpoint* of the short chord ab. Three short chords are *consecutive* if they admit three consecutive vertices of Z as their midpoints. A chord of Z which is not short is called *long*. Two chords ab and cd of a cycle C such that their endpoints are four different vertices of C that appear in the order a, c, b, d when traversing C are called *crossing*. An *n-path* (or *n-cycle*) is a path (or cycle, respectively) on n vertices. The chordless path (or cycle) on n vertices is denoted by P_n (or C_n, respectively). The *endpoints* of a path are the initial and final vertices of the path. If $P = v_1v_2 \ldots v_n$ is a path and v is a vertex adjacent to v_1, then vP denotes the path $vv_1v_2 \ldots v_n$. If $P = v_1v_2 \ldots v_n$ and $P' = w_1w_2 \ldots w_m$ are two paths whose only common vertex is $v_n = w_1$, then PP' denotes the path $v_1v_2 \ldots v_nw_2w_3 \ldots w_m$.

Let G and H be two graphs. We say that G *contains* H if H is a subgraph (induced or not) of G and that G *contains an induced* H if H is an induced subgraph of G. A graph G is *spanned* by H if H is a subgraph of G and $V(H) = V(G)$. We say that G is

[1] Notice that only nonempty maximal matchings are considered when defining matching-transversals and matching-independent sets to guarantee that for every edgeless graph H the equality $\alpha_m(H) = \tau_m(H)$ holds because both parameters are equal to 0; otherwise, $\alpha_{\mathrm{m}}(H)$ would be 1 because of the empty set being a maximal matching of H.

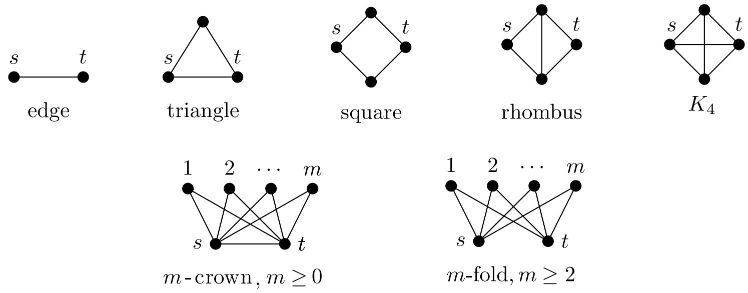

Fig. 2. Basic two-terminal graphs with terminals s and t.

H-*free* if G contains no induced H. A graph is *triangle-free* if it contains no K_3. The subgraph of G induced by a subset W of vertices of G is denoted by $G[W]$ and $G-W$ denotes $G[V(G)-W]$. A *cut-vertex* of a graph is a vertex whose removal increases the number of components. A component is *trivial* if it has precisely one vertex. A *block* of a graph is a maximal connected subgraph that has no cut-vertex. We say that a subset W of the vertices of a graph H is *edge-dominating* in H if each edge of H has at least one endpoint in W. A subgraph J of a graph H is *edge-dominating* in H if $V(J)$ is edge-dominating in H.

If F is a subset of the edge set of a graph G, $G-F$ denotes the graph that arises from G by removing the members of F from the edge set of G. If G is a graph and e is an edge of G, we denote $G-\{e\}$ simply by $G-e$. For each $n \geq 2$, $K_n - e$ denotes the graph that arises from K_n by removing exactly one edge from its edge set.

By *contracting* a subgraph H of G we mean replacing $V(H)$ with a new vertex h and making each vertex $v \in V(G)-V(H)$ adjacent to h if and only if v was adjacent in G to some vertex of H. By *identifying* two vertices u and v of a graph G we mean contracting the subgraph H of G induced by $\{u, v\}$. Let G and H be two graphs and assume, without loss of generality, that $V(G) \cap V(H) = \emptyset$. The *disjoint union* $G+H$ of G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If t is a positive integer, we denote by tG the disjoint union of t graphs, each of which isomorphic to G.

A vertex v is *saturated* by a matching M if v is the endpoint of some edge of M. A graph H is *bipartite* if its vertex set is the union of two disjoint (possibly empty) stable sets X and Y; if so, $\{X, Y\}$ is called a *bipartition* of H. The following is a well-known result about matchings in bipartite graphs.

Theorem 1.6 (*Hall's Theorem [31]*). *If H is a bipartite graph with bipartition $\{X, Y\}$, then there is a matching M of H that saturates each vertex of X if and only if*

$$\left|\bigcup_{a \in A} N_H(a)\right| \geq |A| \quad \text{for each } A \subseteq X.$$

2. Structural characterizations

2.1. Linear and circular structure of graphs containing no bipartite claw

In this subsection, we present a structure theorem for graphs containing no bipartite claw that will turn out to be very useful along this work.

The linear and circular structure of net-free $\cap$ claw-free graphs is studied in [15]. As the line graphs of those graphs containing no bipartite claw are the net-free $\cap$ line graphs, the main result of this subsection (Theorem 2.4) may be regarded as describing a more explicit linear and circular structure for the more restricted class of net-free $\cap$ line graphs.

Our structure theorem will be stated in terms of linear and circular concatenations of two-terminal graphs that we now introduce. A *two-terminal graph* is a triple $\Gamma = (H, s, t)$, where H is a graph and s and t are two different vertices of H, called the *terminals* of Γ.

We now introduce in some detail the two-terminal graphs depicted in Fig. 2. For each $m \geq 0$, the m-*crown* is the two-terminal graph (H, s, t) where $V(H) = \{s, t, a_1, a_2, \ldots, a_m\}$ and $E(H) = \{st\} \cup \{sa_i\colon 1 \leq i \leq m\} \cup \{ta_i\colon 1 \leq i \leq m\}$. The 0-crown and the 1-crown are called *edge* and *triangle*, respectively. For each $m \geq 2$, the m-*fold* is the two-terminal graph (H, s, t) where $V(H) = \{s, t, a_1, a_2, \ldots, a_m\}$ and $E(H) = \{sa_i\colon 1 \leq i \leq m\} \cup \{ta_i\colon 1 \leq i \leq m\}$. The 2-fold is also called *square*. By a *crown* we mean an m-crown for some $m \geq 0$ and by a *fold* we mean an m-fold for some $m \geq 2$. Finally, K_4 will also denote the two-terminal graph (K_4, s, t) for any two vertices s and t of the K_4. We will refer to the crowns, the folds, the rhombus, and the K_4 as the *basic two-terminal graphs*.

If $\Gamma = (H, s, t)$ is a two-terminal graph, then H is the *underlying* graph of Γ, s is the *source* of Γ, and t is the *sink* of Γ. If $\Gamma_1 = (H_1, s_1, t_1)$ and $\Gamma_2 = (H_2, s_2, t_2)$ are two-terminal graphs, the p-*concatenation* $\Gamma_1 \,\&_p\, \Gamma_2$ is the two-terminal graph (H, s_1, t_2) where H arises from the disjoint union $H_1 + H_2$ by identifying t_1 and s_2 into one vertex u and then attaching p pendant vertices adjacent to u. The 0-concatenation $\Gamma_1 \,\&_0\, \Gamma_2$ is denoted simply by $\Gamma_1 \,\&\, \Gamma_2$. If a two-terminal graph $\Gamma = (H, s, t)$ is such that $N_H[s] \cap N_H[t] = \emptyset$, we define its p-*closure*, denoted $\Gamma \,\&_p \circlearrowleft$, as the graph that arises by identifying s and t into one vertex u and then attaching p pendant vertices adjacent to u. The 0-closure $\Gamma \,\&_0 \circlearrowleft$ is simply denoted by $\Gamma \,\& \circlearrowleft$.

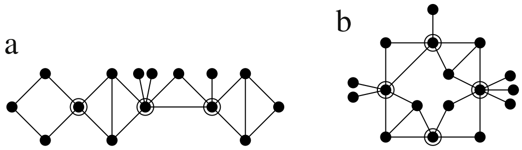

Fig. 3. A linear and a circular concatenation of the sequence $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ of two-terminal graphs, where Γ_1 is a square, Γ_2 and Γ_4 are rhombi, and Γ_3 is a triangle: (a) Underlying graph of $\Gamma_1 \,\&\, \Gamma_2 \,\&_2\, \Gamma_3 \,\&_1\, \Gamma_4$ and (b) $\Gamma_1 \,\&\, \Gamma_2 \,\&_2\, \Gamma_3 \,\&_1\, \Gamma_4 \,\&_3 \circlearrowleft$. Concatenation vertices are circled.

Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be a sequence of two-terminal graphs. A *linear concatenation* of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is the underlying graph of the two-terminal graph $\Gamma_1 \,\&_{p_1}\, \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{n-1}}\, \Gamma_n$ for some nonnegative integers $p_1, p_2, \ldots, p_{n-1}$. The two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are called the *links* of the linear concatenation. The *concatenation vertices* of such a linear concatenation are the $n-1$ vertices that arise by identifying the sink of Γ_i with the source of Γ_{i+1} for each $i \in \{1, 2, \ldots, n-1\}$. The two links Γ_i and Γ_{i+1} are called *adjacent* in the linear concatenation, for each $i \in \{1, 2, \ldots, n-1\}$. The graph K_1 will be regarded as the linear concatenation of an empty sequence of two-terminal graphs. See Fig. 3(a) for an example of a linear concatenation. A *circular concatenation* of $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ is any graph $\Gamma_1 \,\&_{p_1}\, \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{n-1}}\, \Gamma_n \,\&_{p_n} \circlearrowleft$ for some nonnegative integers $p_1, p_2, \ldots, p_n$. The two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ are called the *links* of the circular concatenation. The *concatenation vertices* of such a circular concatenation are the $n-1$ vertices that arise by identifying the sink of Γ_i with the source of Γ_{i+1} for each $i \in \{1, 2, \ldots, n-1\}$, as well as the vertex that arises by identifying the sink of Γ_n with the source of Γ_1. The two links Γ_i and Γ_{i+1} are called *adjacent* in the circular concatenation, for each $i \in \{1, 2, \ldots, n-1\}$, as well as the links Γ_n and Γ_1. See Fig. 3(b) for an example of a circular concatenation.

A *caterpillar* [32] is a connected graph containing no bipartite claw and having no cycle. The fact that caterpillars have edge-dominating chordless paths, gives them a very simple linear structure that can be expressed using our notion of linear concatenation, as follows.

Theorem 2.1 (*[33]*)**.** *A graph is a caterpillar if and only if it is the linear concatenation of edge links.*

We say *fat caterpillars* to those connected graphs containing no bipartite claw and having no cycle of length greater than 4. Our first result characterizes fat caterpillars depending on whether or not they contain an A or a net:

Theorem 2.2. *If H is a graph, then each of the following holds:*

(i) *H is a fat caterpillar containing no A and no net if and only if H is a linear concatenation of crowns, folds, rhombi, and K_4's where the K_4 links may occur only as the first and/or last links of the concatenation.*
(ii) *H is a fat catepillar containing A if and only if H has an edge-dominating 4-cycle $C = v_1v_2v_3v_4v_1$ and two different vertices $x_1, x_2 \in V(H) - V(C)$ such that x_i is adjacent to v_i for each $i \in \{1, 2\}$, each non-pendant vertex in $V(H) - V(C)$ is a false twin of v_4 of degree 2, and one of the following holds: C is chordless; v_1v_3 is the only chord of C and $d_H(v_4) = 2$; C has two chords and $d_H(v_3) = d_H(v_4) = 3$.*
(iii) *H is a fat caterpillar containing a net but no A if and only if H has some edge-dominating triangle C such that for each vertex $v \in V(C)$ there is a pendant vertex x adjacent to v and every vertex in $V(H) - V(C)$ is pendant.*

In summary, we proved the following structure of fat caterpillars that will be useful in the proof of the main result of this subsection.

Corollary 2.3. *A graph H is a fat caterpillar if and only if exactly one of the following conditions holds:*

(i) *H is a linear concatenation of crowns, folds, rhombi, and K_4's where the K_4 links may occur only as the first and/or last links of the concatenation.*
(ii) *H is the circular concatenation* edge $\&_{p_1}$ edge $\&_{p_2}$ edge $\&_{p_3}$ edge $\&_{p_4} \circlearrowleft$ *for some nonnegative integers p_1, p_2, p_3, p_4 such that $p_1, p_2 \geq 1$.*
(iii) *H is the circular concatenation* edge $\&_{p_1}$ edge $\&_{p_2}$ m-fold $\&_{p_3} \circlearrowleft$ *for some $m \geq 2$ and some nonnegative integers p_1, p_2, p_3 such that $p_1, p_2 \geq 1$.*
(iv) *H is the circular concatenation* edge $\&_{p_1}$ edge $\&_{p_2}$ m-crown $\&_{p_3} \circlearrowleft$ *for some $m \geq 1$ and some nonnegative integers p_1, p_2, p_3 such that $p_1, p_2 \geq 1$.*
(v) *H is the underlying graph of* edge $\&_{p_1}$ K_4 $\&_{p_2}$ edge *for some nonnegative integers p_1, p_2.*
(vi) *H is the circular concatenation* edge $\&_{p_1}$ edge $\&_{p_2}$ edge $\&_{p_3} \circlearrowleft$ *for some positive integers p_1, p_2, p_3.*

This enables us to prove that, except for a few sporadic cases (assertions (i), (ii), and (iii)), connected graphs containing no bipartite claw are linear and circular concatenations of basic two-terminal graphs (assertion (iv)).

Theorem 2.4. *If H is a connected graph, then H contains no bipartite claw if and only if at least one of the following assertions holds:*

(i) *H is spanned by a 6-cycle having a long chord or three consecutive short chords.*
(ii) *H has a 5-cycle C and a vertex $u \in V(C)$ such that: (1) each $v \in V(H) - V(C)$ is a pendant vertex adjacent to u and (2) C has three consecutive short chords or u is the midpoint of a chord of C.*

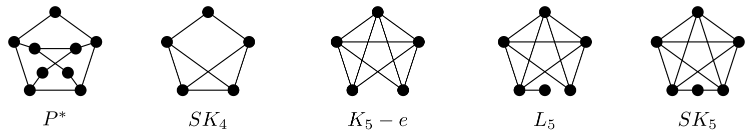

Fig. 4. Graphs P^*, SK_4, $K_5 - e$, L_5, and SK_5.

(iii) *H has a clique Q of size* 4 *and* $q_1, q_2 \in Q$ *such that:* (1) *each* $v \in V(H) - V(Q)$ *is a pendant vertex adjacent to* q_1 *or* q_2 *and* (2) *there is at least one pendant vertex adjacent to* q_i *for each* $i \in \{1, 2\}$.
(iv) *H is a linear or circular concatenation of crowns, folds, rhombi, and* K_4*'s, where the* K_4 *links may occur only in the case of linear concatenation and only as the first and/or last links of the concatenation.*

Notice that, although those graphs satisfying (iii) are also linear concatenations of basic two-terminal graphs (namely, the underlying graphs of edge $\&_{p_1} K_4 \&_{p_2}$ edge for some positive integers p_1 and p_2), we prefer to consider (iii) a sporadic case.

2.2. Edge-coloring graphs containing no bipartite claw

The *chromatic index* $\chi'(H)$ of a graph H is the minimum number of colors needed to color all the edges of H so that no two incident edges receive the same color. Clearly, $\chi'(H) \geq \Delta(H)$. In fact, Vizing [48] proved that for every graph H either $\chi'(H) = \Delta(H)$ or $\chi'(H) = \Delta(H) + 1$. The problem of deciding whether or not any given graph H satisfies $\chi'(H) = \Delta(H)$ is NP-complete even for graphs having only vertices of degree 3 [35]. Interestingly, if H contains no bipartite claw, then $\chi'(H)$ can be computed in linear-time via the algorithm devised in [52] (in fact, H has bounded tree-width because the bipartite claw is not a minor of H [44,45]). Here, we give a structure theorem for those graphs containing no bipartite claw and satisfying $\chi' \neq \Delta$.

We need to introduce some terminology related to edge-coloring. A *major vertex* of a graph is a vertex of maximum degree. If H is a graph, the *core* H_Δ of H is the subgraph of H induced by the major vertices of H. Graphs H for which $\chi'(H) = \Delta(H)$ are *Class* 1, and otherwise they are *Class* 2. A graph H is *critical* if H is Class 2, connected, and $\chi'(H - e) < \chi'(H)$ for each $e \in E(H)$. Some graphs needed in what follows are introduced in Fig. 4.

We rely on the following results.

Theorem 2.5 ([34]). *If H is a connected Class* 2 *graph with* $\Delta(H_\Delta) \leq 2$*, then the following conditions hold:*
(i) *H is critical.*
(ii) $\delta(H_\Delta) = 2$.
(iii) $\delta(H) = \Delta(H) - 1$*, unless H is an odd chordless cycle.*
(iv) *Every vertex of H is adjacent to some major vertex of H.*

Theorem 2.6 ([16]). *If H is a connected graph such that* $\Delta(H_\Delta) \leq 2$ *and* $\Delta(H) = 3$*, then H is Class 1, unless H is* P^*.

Theorem 2.7 ([49]). *If H is a graph of Class* 2*, then H contains a critical subgraph of maximum degree k for each k such that* $2 \leq k \leq \Delta(H)$.

Theorem 2.8 ([2]). *There are no critical graphs having* 4 *or* 6 *vertices. The only critical graphs having 5 vertices are* C_5, SK_4*, and* $K_5 - e$.

By exploiting our structure theorem for graphs containing no bipartite claw (Theorem 2.4) and the results above, we give a structure theorem for all connected graphs containing no bipartite claw that are Class 2.

Theorem 2.9. *If H is a connected graph containing no bipartite claw, then* $\chi'(H) = \Delta(H)$ *if and only if none of the following statements holds:*
(i) $\Delta(H) = 2$ *and H is an odd chordless cycle.*
(ii) $\Delta(H) = 3$ *and H is the circular concatenation of a sequence of edges, triangles, and rhombi, where the number of edge links equals one plus the number of rhombus links.*
(iii) $\Delta(H) = 4$ *and H is* $K_5 - e$, K_5, L_5*, or* SK_5.

As a corollary, we obtain the complete list of critical graphs containing no bipartite claw.

Corollary 2.10. *The critical graphs containing no bipartite claw are the odd chordless cycles,* $K_5 - e$*, and those graphs H satisfying* $\Delta(H) = 3$ *that are circular concatenations of edges, triangles, and rhombi having exactly one more edge link than rhombus links and without pendant edges.*

2.3. Matching-perfect graphs

Notice that the bipartite claw is not matching-perfect (because it satisfies $\alpha_m = 1$ but $\tau_m = 2$) and that the cycles of length $k \geq 5$ such that k is not a multiple of 3 are not matching-perfect (because they satisfy $\alpha_m = 2$ but $\tau_m = 3$). Hence, since the class of matching-perfect graphs is *monotone* (i.e., all the subgraphs of matching-perfect graphs are matching-perfect) by definition, in order to prove Theorem 1.5 (and hence Theorem 1.4), it will be enough to show that if H is a graph containing no bipartite claw and the length of each cycle of H is at most 4 or is a multiple of 3, then $\alpha_m(H) = \tau_m(H)$. Moreover, we can assume that H is connected because $\alpha_m(H)$ (resp. $\tau_m(H)$) is the minimum of $\alpha_m(H')$ (resp. $\tau_m(H')$) among the nontrivial components H' of H (except when H has only trivial components, in which case $\alpha_m(H) = \tau_m(H) = 0$). Therefore, it suffices to prove the theorem below:

Theorem 2.11. *If H is a connected graph containing no bipartite claw and such that the length of each cycle of H is at most 4 or is a multiple of 3, then $\alpha_m(H) = \tau_m(H)$.*

For the proof, we will consider several different cases and in all of them we will prove the existence of a matching-transversal and a matching-independent set of the same size, which means that $\alpha_m(H) = \tau_m(H)$. To produce these matching-independent sets, we strongly rely on edge-coloring H or some graphs derived from it, via Theorem 2.9. In fact, the proof of Theorem 2.11 splits into the following two parts:

Theorem 2.12. *Let H be a connected graph containing no bipartite claw and such that the length of each cycle of H is at most 4 or is a multiple of 3. If H has some cycle of length $3k$ for some $k \geq 2$, then $\alpha_m(H) = \tau_m(H)$.*

Theorem 2.13. *If H is a fat caterpillar, then $\alpha_m(H) = \tau_m(H)$.*

Theorem 2.13 together with Theorem 2.12, implies Theorem 2.11, from which the main results of this work (Theorems 1.4 and 1.5) follow.

Finally, the reader acquainted with the theory of tree-width and second-order logic may notice the following. Since forbidding the bipartite claw as a subgraph or as a minor is equivalent, graphs containing no bipartite claw have bounded tree-width [44] and have a linear-time recognition algorithm [6]. Moreover, as our characterization of matching-perfect graphs given in Theorem 1.5 can be expressed in counting monadic second-order logic with edge set quantifications (see [21]), its validity can be verified in linear time within any graph class of bounded tree-width [12,20]. In particular, matching-perfect graphs can be recognized in linear time. Nevertheless, the resulting algorithm is not elementary. Instead, we propose an elementary linear-time recognition algorithm for matching-perfect graphs which relies on depth-first search only.

We first show that there is a simple linear-time algorithm to recognize fat caterpillars. Let H be a graph. We denote by H_1 the graph that arises from H by removing all vertices that are pendant in H. We denote by H_2 some maximal induced subgraph of H having no vertices that are pendant in H and no two vertices that are false twins of degree 2 in H. Finally, we denote by H_3 some maximal induced subgraph of H having no two vertices that are false twins of degree 2 in H. We claim that there is an elementary linear-time algorithm that either computes H_3 or determines that H contains a bipartite claw. Let us consider an algorithm that keeps a list $L(v)$ for each vertex v of H and that stores at each vertex v of H a boolean variable indicating whether or not the vertex is marked for deletion. Initially, all the lists are empty and no vertex is marked for deletion. The algorithm proceeds by visiting every vertex v of H and, for each neighbor $u \in N_H(v)$ that was not marked for deletion and such that $N_H(u) = \{v, w\}$ for some $w \in V(H)$, we do the following: if w is already in the list of $L(v)$, then we mark u for deletion, otherwise we add w to $L(v)$. To make the algorithm linear-time, we stop whenever we attempt to add a third vertex to any of the lists $L(v)$, as this means that v is the center of a bipartite claw. If all vertices of H are visited and no bipartite claw is detected, then we output as H_3 the subgraph of H induced by those vertices not marked for deletion. The algorithm is clearly correct and linear-time. It follows that there is an elementary algorithm that either computes H_1, H_2, and H_3 in linear time or detects that H contains a bipartite claw. By relying on this algorithm and analyzing the structure of the graphs H_1, H_2, and H_3, we further prove the following.

Theorem 2.14. *There is a simple linear-time algorithm that decides whether a given graph H is matching-perfect and, if affirmative, computes a matching-transversal of H of minimum size within the same time bound.*

In particular, if H is matching-perfect, we can determine the common value of $\alpha_m(H)$ and $\tau_m(H)$ in linear time. We do not know if it is possible to also compute a matching-independent set of maximum size of any given matching-perfect graph within the same time bound. Notice however that the only non-constructive argument used in the proofs of Section 3.3 is the existence of optimal edge-colorings for some Class 1 graphs containing no bipartite claw. This means that using an algorithm such as the one given in [52] to produce the necessary edge-colorings, our proofs in Section 3.3 can actually be turned into an algorithm to compute a matching-independent set of maximum size of any given matching-perfect graph.

Let G be graph on n vertices which is the complement of a line graph. We can compute a *root graph* H of $\overline{G}$ in $O(n^2)$ time by relying on [39,46] and then decide whether G is clique-perfect by determining whether H is matching-perfect as above. Thus, we conclude the following.

Theorem 2.15. *There is an $O(n^2)$-time algorithm that given any graph G, which is the complement of a line graph, decides whether or not G is clique-perfect and, if affirmative, computes a clique-transversal of G of minimum size within the same time bound.*

Notice that the bottleneck of the algorithm is computing a root graph H of $\overline{G}$.

3. Proofs of the structure theorems

In this section, we present all proofs of the previously stated structure theorems.

3.1. Proofs for the structure of graphs containing no bipartite claw

Our first result below shows that fat caterpillars containing no A and no net are linear concatenations of basic two-terminal graphs, as is the case of the graph depicted in Fig. 3(a).

Lemma 3.1. *A graph H is a fat caterpillar containing no A and no net if and only if H is a linear concatenation of crowns, folds, rhombi, and K_4's where the K_4 links may occur only as the first and/or last links of the concatenation.*

The proof of Lemma 3.1 will follow from Lemmas 3.2 and 3.3.

Lemma 3.2. *If H is a fat caterpillar containing no A and no net, then H has an edge-dominating path $P = u_0u_1 \dots u_\ell$ having no long chords and no three consecutive short chords, and such that each vertex $v \in V(H) - V(P)$ satisfies one the following assertions:*

(i) *v is a pendant vertex and the only neighbor of v is neither an endpoint of P nor the midpoint of any short chord of P.*
(ii) *v has degree 2 and is a false twin of u_j for some $j \in \{1, 2, \dots, \ell - 1\}$.*
(iii) *v has degree 3 and is a true twin of u_j for some $j \in \{1, \ell - 1\}$ such that u_{j-1} is adjacent to u_{j+1}.*

Proof. If H is the underlying graph of an m-crown for some $m \geq 3$, then the lemma holds trivially by letting P be any path of H of length 2 whose endpoints are the two vertices of H of degree $m+1$. Therefore, without loss of generality, we will assume that H is not the underlying graph of an m-crown for any $m \geq 3$. Among the longest paths of H without long chords, let us choose some path $P = u_0u_1u_2 \dots u_\ell$ that maximizes $d_H(u_0) + d_H(u_\ell)$ and, among those with maximum $d_H(u_0) + d_H(u_\ell)$, we choose one that minimizes $\min\{d_H(u_0), d_H(u_\ell)\}$. We will show that P satisfies the thesis of the lemma. Notice that P has no long chords by construction and that P has no three consecutive short chords simply because H has no 5-cycle. We make the following claims.

Claim 1. *P is edge-dominating.*

Proof. Suppose, by the way of contradiction, that P is not edge-dominating. Since H is connected, there is some edge vw of H such that none of v and w is a vertex of P and v is adjacent to u_j for some $j \in \{0, 1, 2, \dots, \ell\}$. Since H contains no bipartite claw, $j \in \{0, 1, \ell - 1, \ell\}$. Let us consider first the case $j = 0$. Hence, the path vP must have some long chord because it is longer than P. Since P has no long chords and H has no cycle of length greater than 4, necessarily v is adjacent to u_2. Thus, as H contains no A, $\ell = 2$. Hence, as $P' = u_1u_0vw$ is a path longer than P, P' must have some long chord; i.e., w is adjacent to u_1. In addition, $\{u_0, u_2, w\}$ is a stable set because H has no 5-cycles. Moreover, $N_H(u_0) = N_H(u_2) = N_H(w) = \{u_1, v\}$ because H contains no A. Now, $P'' = u_1u_0v$ is a path of the same length than P but the sum of the degrees of the endpoints of P'' is $d_H(u_1) + d_H(v) > 4 = d_H(u_0) + d_H(u_2)$, which contradicts the choice of P. The contradiction arose from assuming that $j = 0$. Hence, $j \neq 0$ and, symmetrically, $j \neq \ell$. Therefore, also by symmetry, we assume, without loss of generality, that $j = 1$. As $P''' = wvu_1u_2 \dots u_\ell$ is longer than P, P''' must have some long chord. Hence, as H is a fat caterpillar containing no A and no net, this means that w is adjacent to u_2 and $\ell = 2$. But then, we find ourselves in the case $j = \ell$ by letting w play the role of v and vice versa, which leads again to a contradiction. As this contradiction arose from assuming that P was not edge-dominating, Claim 1 follows. ◇

Claim 2. *If $v \in V(H) - V(P)$ is pendant, then (i) holds.*

Proof. Suppose that $v \in V(H) - V(P)$ is pendant. As P is edge-dominating, $N_H(v) = \{u_j\}$ for some $j \in \{0, 1, 2, \dots, \ell\}$. If $j = 0$, then vP would be a path longer than P and without long chords, contradicting the choice of P. This contradiction proves that $j \neq 0$ and, by symmetry, $j \neq \ell$. Suppose, by the way of contradiction, that u_j is the midpoint of some short chord of P; i.e., u_{j-1} is adjacent to u_{j+1}. Since H contains no net and by symmetry, we assume, without loss of generality, that $j = 1$. As $vu_1u_0u_2u_3 \dots u_\ell$ is longer than P, it must have some long chord and, necessarily, u_1 is adjacent to u_3. Hence, as H contains no A and P has no long chords, $\ell = 3$ and $d_H(u_0) = d_H(u_3) = 2$. Thus, $P' = vu_1u_0u_2$ is a path of the same length than P without long chords and such that $d_H(v) + d_H(u_2) \geq 4 = d_H(u_0) + d_H(u_3)$ but $\min\{d_H(v), d_H(u_2)\} = 1 < \min\{d_H(u_0), d_H(u_3)\}$, which contradicts the choice of P. This contradiction arose from assuming that v was adjacent to the midpoint of some short chord of P. Now, Claim 2 follows. ◇

Claim 3. *If $v \in V(H) - V(P)$ has degree 2, then (ii) holds.*

Proof. Let $v \in V(H) - V(P)$ of degree 2 and suppose, by the way of contradiction, that v is adjacent to two consecutive vertices of P; i.e., $N_H(v) = \{u_j, u_{j+1}\}$ for some $j \in \{0, 1, 2, \dots, \ell - 1\}$. If $j = 0$, then vP would be a path without long chords and longer than P, contradicting the choice of P. Therefore, $j \geq 1$ and, by symmetry, $j \leq \ell - 1$. The path $u_0u_1 \dots u_jvu_{j+1}u_{j+2} \dots u_\ell$ must have some long chord because it is longer than P and, as P has no long chords, this means that u_ju_{j+2} or $u_{j+1}u_{j-1}$ is a chord of P. By symmetry, suppose, without loss of generality, that u_ju_{j+2} is a chord of P. Thus, $j = \ell - 2$ since otherwise H would contain A. Moreover, $N_H(u_\ell) = \{u_{\ell-2}, u_{\ell-1}\}$ because P has no long chords and H contains no A. Hence, $d_H(u_\ell) = 2 < d_H(u_{\ell-1})$. Now, $P' = u_0u_1 \dots u_{\ell-2}vu_{\ell-1}$ is a path of the same length than P but $d_H(u_0) + d_H(u_{\ell-1}) > d_H(u_0) + d_H(u_\ell)$. Because of the choice of P, P' must have some long chord and, necessarily, $u_{\ell-1}$ is adjacent to $u_{\ell-3}$; i.e., u_{j+1} is adjacent to u_{j-1}. As we derived from the adjacency of u_j and u_{j+2} that $j = \ell - 2$ and $d_H(u_\ell) = 2$, symmetrically we can prove the fact that u_{j+1} and u_{j-1} are adjacent implies that $j = 1$ and $d_H(u_0) = 2$. Therefore, $\ell = 3$, $d_H(u_0) = d_H(u_\ell) = 2$, and $N_H(v) = \{u_1, u_2\}$. Hence, as H is connected and P is edge-dominating, every vertex $v \in V(H) - V(P)$ is adjacent to u_1

and/or to u_2 only. If some vertex $w \in V(H) - V(P)$ were adjacent to u_1 but not to u_2, then $P'' = wu_1u_0u_2$ would be a path without long chords of the same length than P and such that $d_H(w) + d_H(u_2) > 4 = d_H(u_0) + d_H(u_3)$, contradicting the choice of P. By symmetry, this proves that each vertex $w \in V(H) - V(P)$ satisfies $N_H(w) = \{u_1, u_2\}$. We conclude that H is the underlying graph of an m-crown for some $m \geq 3$, which contradicts our initial hypothesis. This contradiction arose from assuming that v was adjacent to two consecutive vertices of P. Hence, as P is edge-dominating and H has no cycle of length greater than 4, necessarily $N_H(v) = \{v_{j-1}, v_{j+1}\}$ for some $j \in \{1, 2, \ldots, \ell - 1\}$. Suppose, by the way of contradiction, that $d_H(u_j) > 2$ and let w be a neighbor of u_j different from u_{j-1} and u_{j+1}. Thus, since H contains no A and has no 5-cycle, $\ell = 2$ and $j = 1$. But then, wu_1u_2v is a path longer than P and without long chords, contradicting the choice of P. This contradiction arose from assuming that $d_H(u_j) > 2$. Consequently, u_j is a false twin of v and (ii) holds. Hence, Claim 3 follows. ◇

Claim 4. *If $v \in V(H) - V(P)$ has degree at least 3, then (iii) holds.*

Proof. Let $v \in V(H) - V(P)$ of degree at least 3. As P is edge-dominating and H has no cycles of length greater than 4, $N_H(v) = \{u_{j-1}, u_j, u_{j+1}\}$ for some $j \in \{1, 2, \ldots, \ell - 1\}$. Since the paths $u_0u_1 \ldots u_{j-1}vu_ju_{j+1} \ldots u_\ell$ and $u_0u_1 \ldots u_{j-1}u_jvu_{j+1} \ldots u_\ell$ are longer than P, they have at least one long chord each. Thus, if u_{j-1} were nonadjacent to u_{j+1}, then u_j would be adjacent to u_{j-2} and to u_{j+2} and $vu_{j+1}u_{j+2}u_ju_{j-2}u_{j-1}v$ would be a 6-cycle of H, a contradiction. Therefore, u_{j-1} is adjacent to u_{j+1}. As H contains no A, $j = 1$ or $j = \ell - 1$. By symmetry, assume that $N_H(v) = \{u_0, u_1, u_2\}$. Suppose, by the way of contradiction, that u_1 is not a true twin of v. Hence, there is some $w \in N_H(u_1) - \{v, u_0, u_2\}$ and, since P is edge-dominating and H has no cycle of length greater than 4, w is pendant. But then, $wu_1u_0u_2u_3 \ldots u_\ell$ is a path longer than P and without long chords, a contradiction with the choice of P. This contradiction proves that v is a true twin of u_1 and (iii) holds. This completes the proof of Claim 4. ◇

Now, the lemma follows from the four above claims. □

Lemma 3.3. *If H is a fat caterpillar containing no A and no net, $P = u_0u_1 \ldots u_\ell$ is as in the statement of Lemma 3.2, and $\ell \geq 1$, then H is the underlying graph of $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{n-1}} \Gamma_n$ for some basic two-terminal graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ and some nonnegative integers $p_1, p_2, \ldots, p_{n-1}$ such that the source of Γ_1 is u_0 and the sink of Γ_n is u_ℓ.*

Proof. The proof will be by induction on ℓ. If $\ell = 1$, then H is the underlying graph of an edge link with source u_0 and sink u_1. Let $\ell \geq 2$ and assume that the lemma holds whenever the edge-dominating path has length less than ℓ. We will define a two-terminal graph Γ_1 by considering several cases. In each case, we assume, without loss of generality, that none of the preceding cases holds.

Case 1: *u_0 is adjacent to some vertex $v \in V(H) - V(P)$ of degree 3.* By assertions (i)–(iii) of Lemma 3.2, we have that v is a true twin of u_1 and $N_H(u_0) = \{v, u_1, u_2\}$. We define Γ_1 to be the two-terminal graph with source u_0 and sink u_2 and whose underlying graph is the subgraph of H induced by $N_H[v]$. Hence, Γ_1 is a K_4.

Case 2: *u_0 is adjacent to some vertex in $v \in V(H) - V(P)$ of degree 2.* By assertions (i)–(iii) of Lemma 3.2, we have that v is a false twin of u_1 and each neighbor of u_0 in $V(H) - V(P)$ is also a false twin of u_1. We define Γ_1 as the two-terminal graph with source u_0 and sink u_2, and whose underlying graph is the subgraph of H induced by $N_H[u_0] \cup \{u_2\}$. Notice that Γ_1 is a crown or a fold, depending on whether or not u_0 is adjacent to u_2.

As Lemma 3.2 implies that each neighbor of u_0 in $V(H) - V(P)$ has degree 2 or 3, in the cases below we are assuming, without loss of generality, that u_0 has no neighbors in $V(H) - V(P)$. Hence, since P has no long chords, either $N_H(u_0) = \{u_1, u_2\}$ or $N_H(u_0) = \{u_1\}$, depending on whether u_0 is adjacent to u_2 or not.

Case 3: *u_0 is adjacent to u_2 and u_1 is adjacent to u_3.* By assertions (i)–(iii) of Lemma 3.2, $N_H(u_1) = \{u_0, u_2, u_3\}$ and $N_H(u_2) = \{u_0, u_1, u_3\}$. Let Γ_1 be the two-terminal graph with source u_0 and sink u_3, and whose underlying graph is the subgraph of H induced by $\{u_0, u_1, u_2, u_3\}$. Thus, Γ_1 is a rhombus.

Case 4: *u_0 is adjacent to u_2 and u_1 is nonadjacent to u_3.* As u_1 is the midpoint of the short chord u_0u_2 and we are assuming that u_0 has no neighbors in $V(H) - V(P)$, assertions (i)–(iii) of Lemma 3.2 imply that u_1 has no neighbors in $V(H) - V(P)$. Therefore, as u_1 is nonadjacent to u_3 and P has no long chords, $N_H(u_1) = \{u_0, u_2\}$. Let Γ_1 be the two-terminal graph whose source is u_0 and sink u_2, and whose underlying graph is the subgraph of H induced by $\{u_0, u_1, u_2\}$. Thus, Γ_1 is a triangle.

Case 5: *u_0 is nonadjacent to u_2.* In this case, $N_H(u_0) = \{u_1\}$ and we define Γ_1 as the two-terminal graph with source u_0, sink u_1, and whose underlying graph is the induced subgraph of H induced by $\{u_0, u_1\}$. Hence, Γ_1 is an edge.

Once defined Γ_1 as prescribed in Cases 1 to 5 above, we let j be such that u_j is the sink of Γ_1, $v_1, v_2, \ldots, v_{p_1}$ be the pendant vertices adjacent to u_j, $P' = u_ju_{j+1} \ldots u_\ell$, and $H' = H - ((V(\Gamma_1) - \{u_j\}) \cup \{v_1, \ldots, v_{p_1}\})$. Notice that, unless $V(\Gamma_1) = V(H)$, v_j is a cut-vertex of H because we have proved that each vertex of Γ_1 different from v_j has only neighbors in Γ_1. By construction, H' and P' satisfy the statement of Lemma 3.2 by letting H' and P' play the roles of H and P, respectively. If $j = \ell$, then H is the underlying graph of Γ_1 with source u_0 and sink u_ℓ and the lemma holds for H. If $j < \ell$, by induction hypothesis, H' is the underlying graph of some $\Gamma_2 \,\&_{p_2} \Gamma_3 \,\&_{p_3} \cdots \,\&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph, each $p_i \geq 0$, the source of Γ_2 is u_j, and the sink of Γ_n is u_ℓ. Thus, H is the underlying graph of $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \Gamma_3 \,\&_{p_3} \cdots \,\&_{p_{n-1}} \Gamma_n$ where u_0 is the source of Γ_1 and u_ℓ is the sink of Γ_n. Now, Lemma 3.3 follows by induction. □

As a consequence of the two above results, we now prove Lemma 3.1.

Proof (*of* Lemma 3.1)**.** Suppose that H is a linear concatenation of a sequence $\Gamma_1, \ldots, \Gamma_n$ of basic two-terminal graphs such that if Γ_j is a K_4, then $j \in \{1, n\}$. Let v_0 be the source of Γ_1, v_n be the sink of Γ_n and, for each $i \in \{1, \ldots, n-1\}$, let v_i be the concatenation vertex of H that arose by identifying the sink of Γ_i and the source of Γ_{i+1}. Notice that H contains no A

and no net because each 4-cycle of H has two nonconsecutive vertices adjacent to vertices of the 4-cycle only, each triangle contained in a K_4 link of H has at least two vertices u and v of degree 3 in H each such that $N_H[u] = N_H[v]$, and each triangle contained in any other link of H has at least one vertex of degree 2. Moreover, H has no cycle of length greater than 4 because each cycle of H is contained in one of the links and we are assuming that the links are basic. Suppose, by the way of contradiction, that H contains a bipartite claw B. Let b_0 be the center of B and let b_1, b_2, and b_3 be the neighbors of b_0 in B. As b_0 has degree at least 3 in H, the vertex b_0 is either v_j for some $j \in \{0, \ldots, n\}$ or a non-terminal vertex of a rhombus link. If b_0 were the non-terminal vertex of a rhombus link, then the remaining non-terminal vertex of the rhombus link is b_k for some $k \in \{1, 2, 3\}$ and $N_H(b_k) = \{b_0, b_2, b_3\}$, which contradicts the choice of b_0, b_1, b_2, and b_3. Therefore, $b_0 = v_j$ for some $j \in \{0, \ldots, n\}$ and, by symmetry, we assume without loss of generality that $j \neq n$. As b_1, b_2, and b_3 are non-pendant vertices, at least two of them belong to the same link of H. By symmetry, we assume, without loss of generality, that b_1 and b_2 are two vertices of Γ_j. By construction, $b_1, b_2 \in N_H(b_0)$, $N_H(b_1) - \{b_0, b_2\} \neq \emptyset$, $N_H(b_2) - \{b_0, b_1\} \neq \emptyset$, and $|(N_H(b_1) \cup N_H(b_2)) - \{b_0, b_1, b_2\}| \geq 2$. Thus, since Γ_j is basic, necessarily Γ_j is a K_4 and either b_1 or b_2 is v_{j+1}. Since the K_4 links may only occur at the beginning or end of the concatenation, necessarily $j = 1$, b_1 is the sink of Γ_1, and b_2 and b_3 are the non-terminal vertices of Γ_1. Hence, $N_H[b_2] = N_H[b_3] = \{b_0, b_1, b_2, b_3\}$, contradicting the choice of b_0, b_1, b_2, and b_3. This contradiction shows that H contains no bipartite claw and we conclude that H is a fat caterpillar.

Conversely, let H be a fat caterpillar containing no A and no net. If H is K_1, then, by definition, H is the linear concatenation of an empty sequence of two-terminal graphs. Otherwise, there is some path $P = u_0u_1 \ldots u_\ell$ as in the statement of Lemma 3.2 for some $\ell \geq 1$. Thus, Lemma 3.3 implies that H is the linear concatenation of basic two-terminal graphs. Moreover, as H contains no A, the K_4 links, if any, occur as first and/or last links of the concatenation, which completes the proof of Lemma 3.1. □

The next lemmas describe the structure of the remaining fat caterpillars.

Lemma 3.4. *A graph H is a fat caterpillar containing A if and only if H has an edge-dominating 4-cycle $C = v_1v_2v_3v_4v_1$ and two different vertices $x_1, x_2 \in V(H) - V(C)$ such that x_i is adjacent to v_i for each $i \in \{1, 2\}$, each non-pendant vertex in $V(H) - V(C)$ is a false twin of v_4 of degree 2, and one of the following holds:*

(i) *C is chordless.*
(ii) *v_1v_3 is the only chord of C and $d_H(v_4) = 2$.*
(iii) *C has two chords and $d_H(v_3) = d_H(v_4) = 3$.*

Proof. The 'if' part is clear. In order to prove the 'only if', suppose that H is a fat caterpillar containing A. Thus, there is some 4-cycle $C = v_1v_2v_3v_4v_1$ and two different vertices $x_1, x_2 \in V(H) - V(C)$ such that x_i is adjacent to v_i for each $i \in \{1, 2\}$. As H contains no bipartite claw and H is connected, C is edge-dominating in H. Therefore, as H has no 5-cycle, each vertex in $V(H) - V(C)$ is pendant or has exactly two neighbors which are two nonconsecutive vertices of C. If there are two non-pendant vertices $w_1, w_2 \in V(H) - V(C)$, then w_1 and w_2 are false twins because H contains no bipartite claw. Hence, we assume, without loss of generality, that each non-pendant vertex in $V(H) - V(C)$ is adjacent in H precisely to v_1 and v_3. Thus, if there is some non-pendant vertex $w \in V(H) - V(C)$, then v_4 has degree 2 and is a false twin of w because H contains no bipartite claw and has no 5-cycle. If C is chordless, then (i) holds. If C has two chords, then, as H contains no bipartite claw, $d_H(v_3) = d_H(v_4) = 3$ and (iii) holds. Suppose that C has exactly one chord and assume, without loss of generality, that v_1v_3 is the only chord of C. As H has no 5-cycle and contains no bipartite claw, $d_H(v_4) = 2$ and (ii) holds. □

Lemma 3.5. *A graph H is a fat caterpillar containing net but containing no A if and only if H has some edge-dominating triangle C such that for each vertex $v \in V(C)$ there is a pendant vertex x adjacent to v and every vertex in $V(H) - V(C)$ is pendant.*

Proof. The 'if' part is clear. For the converse, suppose that H contains no bipartite claw. Since H contains net, there are six different vertices $v_1, v_2, v_3, x_1, x_2, x_3$ such that v_1, v_2, v_3 are pairwise adjacent and v_i is adjacent to x_i for each $i \in \{1, 2, 3\}$. As H contains no bipartite claw and H is connected, $C = v_1v_2v_3v_1$ is edge-dominating in H. In addition, as H contains no A, each vertex in $V(H) - V(C)$ is pendant. □

Combining the assertions of Lemmas 3.1, 3.4 and 3.5 yields the statement of Theorem 2.2, which can be rephrased to the structure of fat caterpillars given in Corollary 2.3 that will be useful in the proof of the main result of this subsection, Theorem 2.4.

This theorem proves that, except for a few sporadic cases (assertions (i), (ii), and (iii)), connected graphs containing no bipartite claw are linear and circular concatenations of basic two-terminal graphs (assertion (iv)). For the proof of these assertions, we need the following lemma.

Lemma 3.6. *Let H be a connected graph containing no bipartite claw and having some cycle of length at least 5. Assume further that the 5-cycles of H are chordless and the 6-cycles of H have no long chords and no three consecutive short chords. If $C = u_1u_2 \ldots u_\ell u_1$ is a longest cycle of H, then C has no long chords and no three consecutive short chords and, for each vertex $v \in V(H) - V(C)$, one of the following assertions holds:*

(i) *v is pendant and its only neighbor is not the midpoint of any short chord of C.*
(ii) *v has degree 2 and is a false twin of u_j for some $j \in \{1, 2, \ldots, \ell\}$.*

As a result, H is a circular concatenation of crowns, folds, and rhombi.

Proof. By hypothesis, C has length at least 5. Notice also that C is edge-dominating in H because H contains no bipartite claw. Moreover, C has no long chords and no three consecutive short chords, since otherwise C would have length at least 7 (because we are assuming that the 6-cycles have no long chords and no consecutive short chords) and, as a consequence, H would contain a bipartite claw, a contradiction.

Let $v \in V(H) - V(C)$. As C is edge-dominating and H is connected, $d_H(v) \geq 1$. Suppose first that v is pendant. If the only neighbor of v were the midpoint of some short chord of C, then C should have length at least 6 (because we are assuming that 5-cycles are chordless) and, consequently, H would contain a bipartite claw, a contradiction. Hence, if v is pendant, then (i) holds. Suppose now that v is non-pendant. As C is a longest cycle of H, no two consecutive vertices of C are adjacent to v. Moreover, as H contains no bipartite claw, v has no two neighbors at distance larger than 2 within C. Thus, the neighbors of v are at distance 2 in C from each other. This means that if v had at least three neighbors, then C would be a 6-cycle and v would be adjacent to every second vertex of C, but then H would contain a bipartite claw. We conclude that v has exactly two neighbors and that these two neighbors are at distance 2 within C; i.e., $N_H(v) = \{u_{j-1}, u_{j+1}\}$ for some $j \in \{1, \ldots, \ell\}$ (from this point on, subindices should be understood modulo ℓ) and, due to the fact that H contains no bipartite claw and its 5-cycles are chordless, u_j is a false twin of v. This proves that if v is not pendant, then (ii) holds.

It only remains to prove that H is a circular concatenation of crowns, folds, and rhombi.

We claim that there is some $k \in \{1, 2, \ldots, \ell\}$ such that u_k is neither the midpoint of any short chord of C nor a false twin of any vertex outside $V(C)$. Indeed, if no vertex of C is a false twin of a vertex outside $V(C)$, the existence of k is guaranteed by the fact that C has no three consecutive short chords. Suppose, on the contrary, that there is some $j \in \{1, \ldots, \ell\}$ such that u_j is a false twin of a vertex outside $V(C)$. Thus, as C is a longest cycle of H, u_{j-1} is not the midpoint of a short chord of C and u_{j-1} is not the false twin of any vertex outside $V(C)$ because $d_H(u_{j-1}) > 2$. Therefore, the claim holds by letting $k = j - 1$. This concludes the proof of the claim.

Assume, without loss of generality, that u_ℓ is neither the midpoint of any short chord nor a false twin of any vertex outside $V(C)$. Let $v_1, v_2, \ldots, v_q$ be the pendant vertices of H incident to u_ℓ. We create a new vertex u_0 and we add the edge u_0u_1 and the edges joining u_0 to every false twin of u_1 outside $V(C)$ (if any). If u_ℓ is adjacent to u_2, then we also add an edge joining u_0 to u_2. Finally, we remove every edge joining u_ℓ to a neighbor of u_0. Let H' be the graph that arises this way and let $P' = u_0u_1u_2 \ldots u_\ell$. Clearly, H' and P' satisfy Lemma 3.2 by letting H' and P' play the roles of H and P, respectively. Hence, by Lemma 3.3 and its proof, H' is the underlying graph of some $\Gamma_1 \mathbin{\&}_{p_1} \Gamma_2 \mathbin{\&}_{p_2} \mathbin{\&} \cdots \mathbin{\&}_{p_{n-1}} \Gamma_n$ where each Γ_i is a crown, a fold, or a rhombus, and each $p_i \geq 0$. (Indeed, no Γ_i is a K_4 because no vertex $v \in V(H') - V(P')$ has degree 3.) Finally, H is the circular concatenation $\Gamma_1 \mathbin{\&}_{p_1} \Gamma_2 \mathbin{\&}_{p_2} \mathbin{\&} \cdots \mathbin{\&}_{p_{n-1}} \Gamma_n \mathbin{\&}_q \circlearrowright$, where each link is a crown, a fold, or a rhombus. □

Now we are ready to give the proof of Theorem 2.4.

Proof (*of* Theorem 2.4)**.** Suppose that H contains no bipartite claw and we will prove that at least one of the assertions (i)–(iv) holds. Since H contains no bipartite claw and H is connected, every cycle of H of length at least 5 is edge-dominating in H.

If H contains a 6-cycle C having a long or three consecutive short chords, then, as H contains no bipartite claw, H is spanned by C and assertion (i) holds. Hence, from now on, we assume, without loss of generality, that H contains no 6-cycle having a long or three consecutive short chords.

Suppose now that H contains antenna. Thus, H has some 5-cycle $C = v_1v_2v_3v_4v_5v_1$ and some vertex $v \in V(H) - V(C)$ such that v is adjacent to v_2 and v_1 is adjacent to v_3. If v were adjacent to any vertex of C different from v_2, then H would have a 6-cycle having a long chord, contradicting our assumption. If any vertex of C different from v_2 were adjacent to some vertex outside $V(C)$ different from v, then H would contain a bipartite claw. Thus, as H is connected and C is edge-dominating, each vertex $v \in V(H) - V(C)$ is a pendant vertex adjacent to v_2. Hence, (ii) holds. Therefore, from now on, we assume, without loss of generality, that H contains no antenna.

Suppose now H has a 5-cycle C with three consecutive short chords. If there were any vertex $v \in V(H) - V(C)$ adjacent to the two vertices v_1 and v_2 of C that are no midpoints of any of these three short chords, then H would have a 6-cycle with three consecutive short chords, contradicting our assumption. Since H contains no antenna, the midpoints of the chords of C have neighbors in $V(C)$ only. Therefore, as C is edge-dominating, each $v \in V(H) - V(C)$ is a pendant vertex adjacent to v_1 or v_2. If there were two different vertices $u_1, u_2 \in V(H) - V(C)$ such that u_i is adjacent to v_i for each $i \in \{1, 2\}$, then H would contain a bipartite claw. Hence, without loss of generality, each $v \in V(H) - V(C)$ is a pendant vertex adjacent to v_1 and (ii) holds. From now on, we assume without loss of generality that H has no 5-cycle with three consecutive short chords.

Suppose now that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with at least three chords. By hypothesis, C has exactly three chords and, without loss of generality, the chords of C are v_1v_3, v_1v_4, and v_3v_5. As C is edge-dominating and H contains no antenna, each vertex $v \in V(H) - V(C)$ is adjacent to v_1 and/or to v_3 only. Thus, $H =$ rhombus $\mathbin{\&}_{p_1} m$-crown $\mathbin{\&}_{p_2} \circlearrowright$ for some $p_1, p_2 \geq 0$ and some $m \geq 1$ and, in particular, (iv) holds. Hence, from now on, we assume, without loss of generality, that each 5-cycle of H has at most two chords.

Suppose that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with two crossing chords. Without loss of generality, let v_2v_4 and v_3v_5 be the chords of C. As H contains no antenna, v_3 and v_4 have neighbors in $V(C)$ only. Suppose that there is some vertex $v \in V(H) - V(C)$ such that v is adjacent simultaneously to v_1, v_2, and v_5. Since H contains no bipartite claw, it follows that the only neighbors of v_1 are v, v_2, and v_5, and the only vertex outside $V(C)$ adjacent simultaneously to v_2 and v_5 is v. Thus, since C is edge-dominating, we conclude that $H =$ rhombus $\mathbin{\&}_{p_1}$ rhombus $\mathbin{\&}_{p_2} \circlearrowright$ for some $p_1, p_2 \geq 0$ and, in particular, (iv) holds. Therefore, without loss of generality, suppose that there is no vertex outside $V(C)$ adjacent simultaneously to v_1, v_2, and v_5. Suppose now that there is some vertex $v \in V(H) - V(C)$ which is adjacent to v_2 and v_5 and nonadjacent to v_1. Since H contains no bipartite claw, v_1 has no neighbors apart from v_2 and v_5. Thus, since C is edge-dominating, we conclude that $H =$ rhombus $\&_{p_1} m$-fold $\&_{p_2} \circlearrowleft$ for some $p_1, p_2 \geq 0$ and $m \geq 2$ and, in particular, (iv) holds. Finally, assume, without loss of generality, that there is no vertex $v \in V(H) - V(C)$ adjacent to v_2 and v_5 simultaneously. Hence, since C is edge-dominating, $H =$ rhombus $\&_{p_1} m_1$-crown $\&_{p_2} m_2$-crown $\&_{p_3} \circlearrowleft$ for some $p_1, p_2, p_3, m_1, m_2 \geq 0$ and (iv) holds.

Suppose that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with two noncrossing chords. Without loss of generality, assume that v_1v_3 and v_1v_4 are the chords of C. Since H contains no antenna, vertices v_2 and v_5 have neighbors in $V(C)$ only. If there were a vertex outside $V(C)$ which were adjacent to v_1, v_3, and v_4, then H would have a 6-cycle with a long chord, contradicting our assumption. Hence, as C is edge-dominating, $H = m_1$-crown $\&_{p_1} m_2$-crown $\&_{p_2} m_3$-crown $\&_{p_3} \circlearrowleft$ for some $p_1, p_2, p_3, m_1 \geq 0$ and some $m_2, m_3 \geq 1$ and (iv) holds. Therefore, from now on, we assume, without loss of generality, that each 5-cycle of H has at most one chord.

Suppose now that H has a 5-cycle $C = v_1v_2v_3v_4v_5v_1$ with exactly one chord. Without loss of generality, assume that the only chord is v_1v_3. Since H has no antenna, no vertex outside $V(C)$ is adjacent to v_2. Moreover, each vertex outside $V(C)$ is adjacent to at most two vertices of C, since otherwise H would have a 5-cycle with at least two chords, contradicting our hypothesis. Suppose that there is some vertex $v \in V(H) - V(C)$ which is adjacent to two nonconsecutive vertices of C but $N_H(v) \neq \{v_1, v_3\}$. By symmetry, assume that the two neighbors of v are v_1 and v_4. Since H contains no bipartite claw, v_5 has no neighbors outside $V(C)$. As C is edge-dominating, we conclude that $H = m_1$-fold $\&_{p_1} m_2$-crown $\&_{p_2} m_3$-crown $\&_{p_3} \circlearrowleft$ for some $m_1 \geq 2$, $m_2 \geq 1$, and some $m_3, p_1, p_2, p_3 \geq 0$. If, on the contrary, every vertex $v \in V(H) - V(C)$ adjacent to two nonconsecutive vertices of C satisfies $N_H(v) = \{v_1, v_3\}$, then $H = m_1$-crown $\&_{p_1} m_2$-crown $\&_{p_2} m_3$-crown $\&_{p_3} m_4$-crown $\&_{p_4} \circlearrowleft$ for some $m_1 \geq 1$ and some $m_2, m_3, m_4, p_1, p_2, p_3, p_4 \geq 0$. In either case, (iv) holds. Hence, from now on, we assume that every 5-cycle of H is chordless.

Since we are assuming that H has no 6-cycle having a long chord or three consecutive short chords and that each 5-cycle of H is chordless, if H has a cycle of length at least 5, then Lemma 3.6 implies that H is a circular concatenation of crowns, folds, and rhombi, which means that (iv) holds. Therefore, we assume, without loss of generality, that each cycle of H has length at most 4. But then, H is a fat caterpillar and assertion (iii) or (iv) holds by virtue of Theorem 2.2.

Conversely, if H satisfies one of the assertions (i)–(iii), then clearly H contains no bipartite claw. Finally, if H satisfies assertion (iv), then also H contains no bipartite claw by reasoning as in the first part of the proof of Lemma 3.1. This completes the proof of Theorem 2.4. □

3.2. *Proofs for edge-coloring graphs containing no bipartite claw*

By exploiting our structure theorem for graphs containing no bipartite claw (Theorem 2.4) and Theorems 2.5–2.8, we arrive at the structure of all connected graphs containing no bipartite claw that are Class 2 (Theorem 2.9):

Proof (*of* Theorem 2.9**)**. Let H be a connected graph containing no bipartite claw and such that $\chi'(H) \neq \Delta(H)$. We need to prove that H satisfies (i), (ii), or (iii). Since the result holds trivially if $\Delta(H) \leq 2$, we assume, without loss of generality, that $\Delta(H) \geq 3$. The proof splits into three cases.

Case 1: $\Delta(H_\Delta) \leq 2$. We claim that H is $K_5 - e$. Since P^* contains a bipartite claw, Theorem 2.6 implies that if $\Delta(H) = 3$, then H would be Class 1, contradicting the hypothesis. Hence, $\Delta(H) \geq 4$. Thus, by Theorem 2.5, H is critical, $\delta(H_\Delta) = 2$, and $\delta(H) = \Delta(H) - 1 \geq 3$. Suppose, by the way of contradiction, that assertion (iv) of Theorem 2.4 holds for H. Since the vertices of H that are not concatenation vertices have degree at most 3, all major vertices of H are concatenation vertices. Since $\delta(H_\Delta) = 2$, H is necessarily a circular concatenation of crowns. Finally, since $\delta(H) \geq 3$, each of the crowns of the concatenation is an edge and H has no pendant vertices; i.e., H is a chordless cycle, contradicting $\Delta(H) \geq 4$. This contradiction proves that assertion (iv) of Theorem 2.4 does not hold. Thus, assertion (i), (ii), or (iii) of Theorem 2.4 holds. As $\delta(H) \geq 3$, H has no pendant vertices and necessarily $|V(H)|$ is 5 or 6. Thus, since H is critical and $\Delta(H) \geq 4$, it follows from Theorem 2.8 that H is $K_5 - e$, as claimed.

Case 2: $\Delta(H_\Delta) \geq 3$ *and* $\Delta(H) \geq 4$. We claim that H is K_5, L_5, or SK_5. Suppose first that H has a 6-cycle C having a long chord. This implies that C is spanning in H because H is connected and contains no bipartite claw. In particular, $|V(H)| \leq 6$. Hence, since we are assuming that $\Delta(H) \geq 4$, Theorems 2.7 and 2.8 imply that H contains $K_5 - e$ and $\Delta(H) = 4$. Therefore, as H has a spanning 6-cycle, H arises from $K_5 - e$ by adding one vertex adjacent precisely to the two vertices of degree 3 of the $K_5 - e$; i.e., H is SK_5. Thus, for the remaining of this case, we assume that H has no 6-cycle having a long chord.

As $\Delta(H_\Delta) \geq 3$, there is some major vertex w_0 of H that is adjacent in H to three other major vertices w_1, w_2, w_3 of H and let $W = \{w_0, w_1, w_2, w_3\}$. Let B be the bipartite graph with bipartition $\{X, Y\}$ and edge set F, where $X = \{w_1, w_2, w_3\}$, $Y = (N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) - W$, and $F = E(H) \cap (X \times Y)$. Notice that, by construction, $\delta(B) \geq 1$.

We claim that $d_B(w_j) = 1$ for some $j \in \{1, 2, 3\}$ and, in particular, $\Delta(H) = 4$. Suppose, on the contrary, that $d_B(w_i) \geq 2$ for each $i \in \{1, 2, 3\}$. If $|Y| \geq 3$, then Theorem 1.6 would imply that B has a matching that saturates every vertex of X and, consequently, w_0 would be the center of a bipartite claw contained in H, a contradiction. This contradiction implies $|Y| < 3$. Thus, $N_B(w_1) = N_B(w_2) = N_B(w_3) = Y = \{y_1, y_2\}$ where y_1 and y_2 are two different vertices. Hence, $C = w_0w_1y_1w_2y_2w_3w_0$ is a 6-cycle in H having three long chords, a contradiction. This contradiction proves the claim; i.e., $d_B(w_j) = 1$ for some $j \in \{1, 2, 3\}$ and, in particular, $\Delta(H) = 4$. By symmetry, from now on, we assume that $d_B(w_3) = 1$.

Suppose that $|Y| = 1$ and we will prove that H is K_5 or L_5. The fact that the vertices of W are major vertices and $\Delta(H) = 4$ implies that $N_H[w_1] = N_H[w_2] = N_H[w_3] = W \cup \{y\}$ where $Y = \{y\}$. If w_0 is adjacent to y, then H is K_5. If, on the contrary, the neighbor of w_0 outside W is a vertex z different from y, then, as H contains no bipartite claw and has no 6-cycle having a long chord, H is L_5. Hence, from now on, we assume without loss of generality that $|Y| \geq 2$.

Suppose that $|N_B(w_1) \cup N_B(w_2)| = 1$ and we will prove that H is L_5. Let y_1 be the only neighbor in B of w_1 and w_2. By construction, $Y = \{y_0, y_1\}$ where y_0 is the only neighbor of w_3 in B and $y_0 \neq y_1$ (because we are assuming that $d_B(w_3) = 1$ and $|Y| \geq 2$). Since w_1, w_2, and w_3 are major vertices and $\Delta(H) = 4$, W is a clique. Since H is connected and contains no bipartite claw, $V(H) = W \cup Y$. Moreover, as H has no 6-cycle with a long chord, w_0 is nonadjacent to y_0, respectively. Hence, since w_0 is a major vertex and $\Delta(H) = 4$, $N_H(w_0) = \{w_1, w_2, w_3, y_1\}$. Since H has no 6-cycle with a long chord, y_0 is nonadjacent to y_1. Therefore, H is L_5, as desired. Thus, from now on, we assume without loss of generality that $|N_B(w_1) \cup N_B(w_2)| \geq 2$.

Since $|N_B(w_1) \cup N_B(w_2)| \geq 2$, there are two different vertices $y_1, y_2 \in Y$ such that w_i is adjacent to y_i for each $i \in \{1, 2\}$. As w_3 is a major vertex and we are assuming that $d_B(w_3) = 1$, w_3 is necessarily adjacent to w_1 and w_2. As $\Delta(H) = 4$ and H contains no bipartite claw, for each of w_3 and w_0 its only neighbor outside W is either y_1 or y_2. By symmetry, we assume, without loss of generality, that $N_H[w_3] = W \cup \{y_1\}$. Thus, as H contains no bipartite claw and has no 6-cycle having a long chord, $N_H[w_0] = W \cup \{y_1\}$, $N_H[w_1] = W \cup \{y_1\}$, $N_H[w_2] = W \cup \{y_2\}$, $N_H(y_1) = \{w_0, w_1, w_3\}$, and $N_H(y_2) = \{w_2\}$. Hence, H is L_5.

We have verified that if $\Delta(H_\Delta) \geq 3$ and $\Delta(H) \geq 4$, then H is K_5, L_5, or SK_5, as claimed.

Case 3: *$\Delta(H_\Delta) \geq 3$ and $\Delta(H) = 3$.* As $\Delta(H) = 3$, assertion (iii) of Theorem 2.4 does not hold. Suppose, by the way of contradiction, that assertion (i) or (ii) of Theorem 2.4 holds for H. Thus, $|V(H)|$ is 5 or 6 and, by Theorems 2.7 and 2.8, H contains SK_4. Hence, since H contains no bipartite claw, H is connected, and $\Delta(H) = 3$, it follows that either H is SK_4 or H arises from SK_4 by adding a pendant vertex adjacent to the vertex of degree 2 of the SK_4, contradicting the assumption that assertion (i) or (ii) of Theorem 2.4 holds. We conclude that, necessarily, H is a linear or circular concatenation as described in assertion (iv) of Theorem 2.4. As $\Delta(H) = 3$, no link of the linear or circular concatenation is an m-crown for any $m \geq 3$ or an m-fold for any $m \geq 4$. Moreover, if any of the links in the linear or circular concatenation were a 2-crown, 3-fold, or K_4, then H would be precisely the underlying graph of a 2-crown, 3-fold, or K_4, and H would be Class 1, a contradiction. Therefore, H is a linear or circular concatenation of edges, triangles, squares, and rhombi. As $\Delta(H) = 3$, if any link of the concatenation is a triangle, square, or rhombus, then its adjacent links in the concatenation are edges. Hence, it is clear that there is a 3-edge-coloring of H if and only if there is a coloring of only the edge links of H such that:

(1) Each two edge links that are adjacent to the same triangle link are colored with different colors.
(2) Each two edge links that are adjacent to the same rhombus link are colored with the same color.
(3) Each two adjacent edge links are colored with different colors.

Thus, if H is a linear concatenation, a greedy coloring of only the edge links following the order of their occurrence in the linear concatenation and following rules (1)–(3) above, ends up successfully, implying that H has a 3-edge-coloring, a contradiction with the fact that H is Class 2. Since the links adjacent to the same square may receive the same or different colors, if H is a circular concatenation where some link is a square, then also a greedy coloring of only the edge links, following rules (1)–(3) around the concatenation starting at one of the edge links adjacent to the square and ending at the other one, ends up successfully, contradicting the fact that H is Class 2. These contradictions prove that H is a circular concatenation of edges, triangles, and rhombi only.

We will now prove that if H is a circular concatenation of edges, triangles, and rhombi such that $\Delta(H_\Delta) \geq 3$ and $\Delta(H) = 3$, then H is Class 2 if and only if H has exactly one more edge link than rhombus links. As $\Delta(H_\Delta) \geq 3$, H has at last one rhombus link. Thus, without loss of generality, $H = \text{edge} \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \text{edge} \,\&\, \text{rhombus} \,\&\, \circlearrowright$. Notice that H is Class 2 if and only if there is no 3-edge-coloring of the edge links of $H' = \text{edge} \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \text{edge}$ satisfying rules (1)–(3) above and such that the first and the last link of H' are colored with the same color. Moreover, H' is not 3-edge-colorable satisfying rules (1)–(3) above if and only if the graph H'', that arises from H' by contracting each triangle link to a vertex and contracting each pair formed by a rhombus link followed by an edge link also to a vertex, consists of precisely two edges; i.e., H' has two more edge links than rhombus links. We conclude that H has exactly one more edge link than rhombus links; i.e., (ii) holds. This completes Case 3 and the proof of the 'only if' part of the theorem.

Notice also that we have just proved that if assertion (ii) holds, then H is Class 2. As a result, the 'if' part of the theorem is also proved, because if assertion (i) or (iii) holds, then H is clearly Class 2. □

3.3. Proofs for matching-perfect graphs

We start with the proof of Theorem 2.11. For that, we will consider several cases and in all of them we will ensure the existence of a matching-transversal and a matching-independent set of the same size, which means that $\alpha_m(H) = \tau_m(H)$. To produce these matching-independent sets, we strongly rely on edge-coloring H or some graphs derived from it, via Theorem 2.9.

The next lemma states a simple yet useful upper bound on $\tau_m(H)$.

Lemma 3.7. *If H is a graph and v_1 and v_2 are two adjacent vertices of H, then the set of edges of H that are incident to v_1 and/or to v_2 is a matching-transversal of H and, in particular, $\tau_m(H) \leq d_H(v_1) + d_H(v_2) - 1$.*

Proof. No matching M of H disjoint from $E_H(v_1) \cup E_H(v_2)$ is maximum because $M \cup \{v_1v_2\}$ is a larger matching of H. □

Let k be a nonnegative integer. A *partial k-edge-coloring* of a graph H is a map $\phi : E(H) \to \{0, 1, 2, \dots, k\}$ such that, for each pair of incident edges e_1, e_2 of H, $\phi(e_1) = \phi(e_2)$ implies $\phi(e_1) = \phi(e_2) = 0$. If $\phi(e) \neq 0$, then e is said to be *colored with color* $\phi(e)$; otherwise, e is said to be *uncolored*. A *k-edge-coloring* of H is a partial k-edge-coloring that colors all edges of H. The *color classes* of a partial k-edge-coloring are the sets $\xi_1, \xi_2, \dots, \xi_k$ where ξ_j is the set of edges of H with color j, for each $j \in \{1, 2, \dots, k\}$.

We complement the upper bounds on τ_{m} with lower bounds on α_{m} obtained with the help of a special kind of partial edge-colorings that we call profuse-colorings. A *k-profuse-coloring* of a graph H is a partial k-edge-coloring $\phi : E(H) \to \{0, 1, 2, \dots, k\}$ satisfying the following conditions:

(1) If $k = 1$, then there is at least one edge e of H colored with color 1.
(2) If $k \geq 2$, then each edge e (either colored or not) of H is incident to edges colored with at least $k - 1$ different colors.

We say that a k-profuse-coloring ϕ is *maximal* if, for each uncolored edge, there are edges incident to it that are colored with the k different colors (i.e., no uncolored edge can be colored while keeping ϕ a k-profuse-coloring). We first show that every k-profuse-coloring uses all the colors $1, \dots, k$.

Lemma 3.8. *Each k-profuse-coloring of a graph H colors some edge of H with color i for each* $i \in \{1, \dots, k\}$.

Proof. If $k = 0$, there is nothing to prove. If $k = 1$, then the lemma holds by condition (1) of the definition. Thus, assume that $k \geq 2$ and let e be any edge of H. Since $k - 1 \geq 1$, condition (2) implies that e is incident to some edge e_j colored with some color $j \in \{1, \dots, k\}$. Since ϕ is a partial edge-coloring and by virtue of condition (2), e_j is incident to some edge e_i colored with color i for each $i \in \{1, \dots, k\} - \{j\}$. By construction, edge e_i is colored with color i for each $i \in \{1, \dots, k\}$. □

We now show that the maximum value of k for which a graph H has a k-profuse-coloring is $k = \alpha_{\mathrm{m}}(H)$. Hence, in order to prove that $\alpha_{\mathrm{m}}(H) \geq k$ it will suffice to exhibit a k-profuse-coloring of H.

Lemma 3.9. *For each graph H and each nonnegative integer k, the following assertions are equivalent:*

(i) $\alpha_{\mathrm{m}}(H) \geq k$.
(ii) *H has a k-profuse-coloring.*
(iii) *H has a maximal k-profuse-coloring.*

Moreover, the collection of color classes of a maximal k-profuse-coloring of H is a matching-independent set of size k.

Proof. If $k = 0$, the three assertions (i)–(iii) are true; in fact, for every graph H, the constant 0 function is the only 0-profuse-coloring of H and it is also maximal. Hence, we assume that $k \geq 1$.

Let us prove first that (i) ⇒ (iii). Suppose that $\alpha_{\mathrm{m}}(H) \geq k$ and let $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ be a collection of k pairwise disjoint nonempty maximal matchings of H. Let $\phi_{\mathcal{M}} : E(H) \to \{0, 1, 2, \dots, k\}$ be defined for each $e \in E(H)$ and each $i \in \{1, \dots, k\}$ by

$$\phi_{\mathcal{M}}(e) = i \quad \text{if and only if } e \in M_i.$$

Notice that $\phi_{\mathcal{M}}(e) = 0$ if and only if $e \notin M_1 \cup M_2 \cup \cdots \cup M_k$. We claim that $\phi_{\mathcal{M}}$ is a maximal k-profuse-coloring of H. Since each M_i is a matching, $\phi_{\mathcal{M}}$ is a k-partial edge-coloring of H. If $k = 1$, then $\phi_{\mathcal{M}}$ is a maximal 1-profuse-coloring because the fact that M_1 is nonempty and maximal implies that there is at least one edge of H colored by $\phi_{\mathcal{M}}$ with color 1 and that each uncolored edge is incident to an edge colored with color 1. Thus, we are left to consider the case $k \geq 2$. Let e be any edge of H. Assume first that $e \in M_j$ for some $j \in \{1, 2, \dots, k\}$. For each $i \in \{1, 2, \dots, k\}$ such that $i \neq j$, the maximality of M_i implies that there is some edge e_i of H incident to e such that $\phi_{\mathcal{M}}(e_i) = i$. Hence, the set $\{e_i \colon i \neq j\}$ consists of $k - 1$ edges incident to e that are colored with $k-1$ different colors. Suppose now that $e \notin M_1 \cup M_2 \cup \cdots \cup M_k$. For each $i \in \{1, 2, \dots, k\}$, the maximality of M_i implies that there is some edge e_i of H incident to e such that $\phi_{\mathcal{M}}(e_i) = i$. We conclude that $\phi_{\mathcal{M}}$ is a maximal k-profuse-coloring of H and (iii) holds.

We now prove that (ii) ⇒ (i). Suppose (ii) holds and let $\phi : E(H) \to \{0, 1, 2, \dots, k\}$ be a k-profuse-coloring of H. Thus, for each $i \in \{1, 2, \dots, k\}$, the color class $\xi_i = \{e \in E(H) : \phi(e) = i\}$ is a matching of H and $\xi_i \neq \emptyset$ by Lemma 3.8. For each $i \in \{1, 2, \dots, k\}$, let M_i be any maximal matching of H containing ξ_i. If $k = 1$, then $\alpha_{\mathrm{m}}(G) \geq 1$ because the fact that $M_1 \neq \emptyset$ implies that $\{M_1\}$ is a clique-independent set of H. Hence, assume that $k \geq 2$. Let e be any edge of H. As ϕ is a k-profuse-coloring, there are $k - 1$ edges $e_1, e_2, \dots, e_{k-1}$ of H incident to e such that $\phi(e_1), \phi(e_2), \dots, \phi(e_{k-1})$ are positive and pairwise different. Hence, as $e_i \in \xi_{\phi(e_i)}$ and $M_{\phi(e_i)}$ is a matching containing $\xi_{\phi(e_i)}$, $e \notin M_{\phi(e_i)}$ for each $i \in \{1, 2, \dots, k-1\}$. This proves that each edge e of H belongs to at most one of $M_1, M_2, \dots, M_k$. Thus, by construction, $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ is a collection of k disjoint nonempty maximal matchings of H and $\alpha_{\mathrm{m}}(H) \geq k$; i.e., (i) holds, as desired.

Since (iii) trivially implies (ii), this completes the proof of the equivalence among (i)–(iii). Finally, notice that, in the preceding paragraph, if ϕ is maximal, then $M_i = \xi_i$ because each $e \in E(H) - \xi_i$ is incident to some edge in ξ_i. Therefore, if ϕ is maximal, then $\{\xi_1, \dots, \xi_k\}$ is a collection of k disjoint nonempty maximal matchings, proving the last statement of Lemma 3.9. □

As immediate consequence of Lemma 3.9 we obtain:

Corollary 3.10. *If ϕ is a maximal k-profuse-coloring of a graph H, then every matching-transversal of H has at least one edge colored with color i for each $i \in \{1, 2, \dots, k\}$.*

More upper bounds on τ_m and lower bounds on α_m will be proved later in this subsection. Some of them depend on the degrees of what we call hubs. The *hubs* of a graph are the vertices of degree at least 3. The *minimum hub degree* $\delta_h(H)$ of a graph H is the infimum of the degrees of the hubs of H. Notice that $\delta_h(H) \geq 3$ for any graph H and that $\delta_h(H) = +\infty$ if and only if H has no hubs. A hub is *minimum* if its degree is the minimum hub degree. An edge of a graph is *hub-covered* if at least one of its endpoints is a hub. A graph H is *hub-covered* if each of its edges is hub-covered. Equivalently, H is hub-covered if and only if its hub set is edge-dominating. A graph is *hub-regular* if all its hubs have the same degree. Equivalently, a graph H is hub-regular if and only if either $\delta_h(H) = \Delta(H)$ or $\delta_h(H) = +\infty$.

The proof of Theorem 2.11 splits into two parts: Theorem 2.12, the case when H has some cycle of length greater than 4 (which is necessarily a cycle of length $3k$ for some $k \geq 2$), and Theorem 2.13, the case when H has no cycle of length greater than 4.

Theorem 2.12 will follow by considering separately the cases when the graph is hub-covered (Lemma 3.16) or not (Lemma 3.17). The lemma below implies that if a graph H containing no bipartite claw has a cycle of a certain length, then H is triangle-free.

Lemma 3.11. *Let H be a connected graph containing no bipartite claw such that the length of each cycle is at most 4 or a multiple of 3. If H contains a cycle C of length 3k for some $k \geq 2$, then one of the following assertions holds:*

(i) *H arises from C_6 by adding 1, 2, or 3 long chords.*
(ii) *C is chordless and each vertex $v \in V(H) - V(C)$ is either: (1) a false twin of a vertex of C of degree 2 in H or (2) a pendant vertex adjacent to a vertex of C.*

In particular, H is triangle-free.

Proof. Let C' be any cycle of H of length ℓ for any $\ell \geq 5$. By hypothesis, ℓ is a multiple of 3. Moreover, C' has no short chords since otherwise H would have a cycle of length $\ell - 1$, where $\ell - 1$ is at least 5 and not a multiple of 3. Thus, if C' has a chord, then this chord must be long and, as H contains no bipartite claw and is connected, C' is a spanning 6-cycle of H and (i) holds. Hence, we assume, without loss of generality, that every cycle of H of length at least 5 is chordless. It now follows from Lemma 3.6 that (ii) holds. □

We start considering the case of hub-covered graphs with the following upper bound on τ_m.

Lemma 3.12. *Let H be a triangle-free graph containing no bipartite claw. If v is a hub of H, then $E_H(v)$ is a matching-transversal of H. In particular, if H has at least one hub, then $\tau_m(H) \leq \delta_h(H)$.*

Proof. Let v be any hub of H and let w_1, w_2 and w_3 be three of its neighbors in H. Suppose, by the way of contradiction, that $E_H(v)$ is not a matching-transversal of H and let M be a maximal matching M of H disjoint from $E_H(v)$. In particular, for each $i \in \{1, 2, 3\}$, there is some $e_i \in M$ incident to w_i and non-incident to v. As H is triangle-free, w_i is the only endpoint of e_i in $\{w_1, w_2, w_3\}$, for each $i \in \{1, 2, 3\}$. Thus, $\{vw_1, vw_2, vw_3, e_1, e_2, e_3\}$ is the edge set of a bipartite claw contained in H, a contradiction. This contradiction proves that $E_H(v)$ is a matching-transversal of H and that $\tau_m(H) \leq \delta_h(H)$. □

The counterpart of the above upper bound on $\tau_m(H)$ is the following lemma from which we deduce sufficient conditions for $\delta_h(H)$ to be also a lower bound on $\alpha_m(H)$.

Lemma 3.13. *In a triangle-free graph H containing no bipartite claw, there exists a set F of hub-covered edges such that the graph $H' = H - F$ is hub-regular and has the same hub set and the same minimum hub degree as H.*

Proof. Let H be a counterexample to the lemma with minimum number of edges. If H were hub-regular, the lemma would hold by letting $F = \emptyset$. Hence, H is not hub-regular; i.e., $\Delta(H) > \delta_h(H)$. Let v be any hub of H that is not minimum.

We claim that v has some neighbor w in H which is not a minimum hub. Suppose, by the way of contradiction, that all the neighbors of v are minimum hubs. By construction, v has at least four neighbors w_1, w_2, w_3, w_4 and let $W = \{v, w_1, w_2, w_3, w_4\}$. As H is triangle-free and w_i is a hub, $|N_H(w_i) - W| = \delta_h(H) - 1 \geq 2$ for each $i \in \{1, 2, 3\}$. Hence, $|(N_H(w_1) \cup N_H(w_2) \cup N_H(w_3)) - W| \leq 2$, since otherwise $|\bigcup_{a \in A} N_H(a) - W| \geq |A|$ for every nonempty $A \subseteq \{w_1, w_2, w_3\}$ and Theorem 1.6 (applied to the bipartite graph with bipartition $\{X, Y\}$ and edge set $E(H) \cap (X \times Y)$, where $X = \{w_1, w_2, w_3\}$ and $Y = V(H) - W$) would imply that v is the center of a bipartite claw contained in H. Therefore, $\delta_h(H) = 3$ and there are two different vertices y_1, y_2 outside W such that $N_H(w_1) = N_H(w_2) = N_H(w_3) = \{v, y_1, y_2\}$ and, by symmetry, also $N_H(w_4) = \{v, y_1, y_2\}$. But this means that w_4 is the center of a bipartite claw contained in H, a contradiction. This contradiction proves that v has some neighbor w which is not a minimum hub, as claimed.

Let w be a neighbor of v which is not a minimum hub of H. Thus, vw is a hub-covered edge of H and $H_1 = H - \{vw\}$ has the same hub set and the same minimum hub degree as H. By minimality of the counterexample H, the lemma holds for H_1. Hence, there exists a set F_1 of hub-covered edges of H_1 such that $H' = H_1 - F_1$ is hub-regular and has the same hub set and the same minimum hub degree as H_1. By construction, $F = F_1 \cup \{vw\}$ is a set of hub-covered edges of H such that $H' = H - F$ is hub-regular and H' has the same hub set and the same minimum hub degree as H. Therefore, the lemma holds for H, contradicting the choice of H. This contradiction proves the lemma. □

Lemma 3.14. *Let H be a triangle-free graph containing no bipartite claw. If H is hub-covered and has at least one edge, then $\alpha_{\mathrm{m}}(H) \geq \delta_{\mathrm{h}}(H)$.*

Proof. By Lemma 3.13, there exists a set F of hub-covered edges of H such that $H' = H - F$ is hub-regular and has the same hub set and the same minimum hub degree as H. Since H has at least one edge and H is hub-covered, H has at least one hub; i.e., $3 \leq \delta_{\mathrm{h}}(H) < +\infty$. By construction, H' is also hub-covered and $\Delta(H') = \delta_{\mathrm{h}}(H') = \delta_{\mathrm{h}}(H) \geq 3$. Since H' is a subgraph of H, H' is also triangle-free and contains no bipartite claw. By Theorem 2.9, $\chi'(H') = \Delta(H')$; i.e., there is an edge-coloring ϕ' of H' using $\Delta(H') = \delta_{\mathrm{h}}(H)$ colors. Let $\phi : E(H) \to \{0, 1, 2, \ldots, \delta_{\mathrm{h}}(H)\}$ be defined by $\phi(e) = \phi'(e)$ for each $e \in E(H')$ and $\phi(e) = 0$ for each $e \in E(H) - E(H')$. Since H is hub-covered, ϕ is a $\delta_{\mathrm{h}}(H)$-profuse-coloring of H by construction and Lemma 3.9 implies that $\alpha_{\mathrm{m}}(H) \geq \delta_{\mathrm{h}}(H)$. □

From Lemmas 3.12 and 3.14, we can determine α_{m} and τ_{m} for all connected hub-covered triangle-free graphs containing no bipartite claw and having at least one edge.

Lemma 3.15. *If H is a hub-covered triangle-free graph containing no bipartite claw and having at least one edge, then $\alpha_{\mathrm{m}}(H) = \tau_{\mathrm{m}}(H) = \delta_{\mathrm{h}}(H)$.*

Using the above lemma and Lemma 3.11, we prove Theorem 2.12 for hub-covered graphs, as follows.

Lemma 3.16. *Let H be a connected graph containing no bipartite claw and such that the length of each cycle of H is at most 4 or is a multiple of 3. If H has a cycle of length $3k$ for some $k \geq 2$ and H is hub-covered, then $\alpha_{\mathrm{m}}(H) = \tau_{\mathrm{m}}(H) = \delta_{\mathrm{h}}(H)$.*

Finally, we also settle Theorem 2.12 for graphs that are not hub-covered.

Lemma 3.17. *Let H be a connected graph containing no bipartite claw and such that the length of each cycle of H is at most 4 or is a multiple of 3. If H has a cycle of length $3k$ for some $k \geq 2$ and H is not hub-covered, then $\alpha_{\mathrm{m}}(H) = \tau_{\mathrm{m}}(H) = 3$.*

Proof. Since H is not hub-covered and H has at least one edge, Lemma 3.7 implies $\tau_{\mathrm{m}}(H) \leq 3$. Thus, we just need to prove that $\alpha_{\mathrm{m}}(H) \geq 3$. Since the length of C is a multiple of 3, there is a 3-edge-coloring of C, $\phi' : E(C) \to \{1, 2, 3\}$ such that each three consecutive edges of C are colored with three different colors by ϕ'. Let $\phi : E(H) \to \{0, 1, 2, 3\}$ be defined by $\phi(e) = \phi'(e)$ for each $e \in E(C)$ and $\phi(e) = 0$ for each $e \in E(H) - E(C)$. Since H is connected and contains no bipartite claw, C is edge-dominating in H and, consequently, ϕ is a 3-profuse-coloring of H. By virtue of Lemma 3.9, $\alpha_{\mathrm{m}}(H) \geq 3$, as needed. □

Clearly, Lemmas 3.16 and 3.17 together imply Theorem 2.12.

As Theorem 2.12 is now proved, to complete the proof of Theorem 2.11, it only remains to prove Theorem 2.13.

To begin with, the next lemma provides several upper bounds on τ_{m}.

Lemma 3.18. *If H is a graph containing no bipartite claw and having no 5-cycle and v is a hub of H, then each of the following holds:*

(i) *If v has degree at least 5 in H, then $E_H(v)$ is a matching-transversal of H and, in particular, $\tau_{\mathrm{m}}(H) \leq d_H(v)$.*
(ii) *If v has degree 4 in H, then $\tau_{\mathrm{m}}(H) \leq 5$. Moreover, if v has degree 4 and $N_H(v)$ does not induce $2K_2$ in H, then $E_H(v)$ is a matching-transversal of H and, in particular, $\tau_{\mathrm{m}}(H) \leq 4$.*
(iii) *If v has degree 3 in H, then $\tau_{\mathrm{m}}(H) \leq 5$. Moreover, if $N_H(v)$ induces $3K_1$ in H, then $E_H(v)$ is a matching-transversal of H and, in particular, $\tau_{\mathrm{m}}(H) \leq 3$. If, instead, $N_H(v)$ induces $K_2 + K_1$ in H, then $\tau_{\mathrm{m}}(H) \leq 4$.*

Proof. If $E_H(v)$ is a matching-transversal of H, then $\tau_{\mathrm{m}}(H) \leq d_H(v)$ and there is nothing left to prove. Hence, we assume, without loss of generality, that $E_H(v)$ is not a matching-transversal of H. Therefore, there exists a maximal matching M of H such that $M \cap E_H(v) = \emptyset$. Because of the maximality of M, for each neighbor w of v there is exactly one edge $e_w \in M$ that is incident to w. Notice that there could be two different neighbors w_1 and w_2 of v such that $e_{w_1} = e_{w_2}$.

We claim that $|\{e_w : w \in N_H(v)\}| \leq 2$. In fact, if $e_{w_1}, e_{w_2}, e_{w_3}$ were three different edges for some $w_1, w_2, w_3 \in N_H(v)$, then v would be the center of a bipartite claw contained in H with edge set $\{vw_1, e_{w_1}, vw_2, e_{w_2}, vw_3, e_{w_3}\}$, a contradiction. This contradiction proves the claim. Therefore, as each edge e_w is incident to at most two vertices of $N_H(v)$, in particular, $d_H(v) \leq 4$. So far, we have proved (i).

Suppose that $d_H(v) = 3$ and let $N_H(v) = \{w_1, w_2, w_3\}$. We denote by $F_H(v)$ the set of edges of H joining two neighbors of v. Suppose, by the way of contradiction, that $E_H(v) \cup F_H(v)$ is not a matching-transversal of H. Thus, there is some maximal matching M' such that $M' \cap (E_H(v) \cup F_H(v)) = \emptyset$. Because of the maximality of M', for each $i \in \{1, 2, 3\}$, there is an edge $e'_{w_i} \in M'$ and v is the center of a bipartite claw whose edge set is $\{vw_1, e'_{w_1}, vw_2, e'_{w_2}, vw_3, e'_{w_3}\}$, a contradiction. This contradiction proves that $E_H(v) \cup F_H(v)$ is a matching-transversal of H. In particular, $\tau_{\mathrm{m}}(H) \leq 3 + |F_H(v)|$. This proves (iii) when $N_H(v)$ is not a clique. Thus, assume that $N_H(v)$ is a clique. Since H has no 5-cycle, every vertex $x \in V(H) - N_H[v]$ having at least one neighbor in $N_H(v)$, has exactly one neighbor in $N_H(v)$. Hence, since H contains no bipartite claw, there is at least one vertex w in $N_H(v)$ that has degree 3 in H and, by Lemma 3.7, $\tau_{\mathrm{m}}(H) \leq d_H(v) + d_H(w) - 1 = 5$. This completes the proof of (iii).

Finally, we consider the case $d_H(v) = 4$. Since $|\{e_w : w \in N_H(v)\}| \leq 2$ and each edge e_w is incident to at most two neighbors of v, we assume, without loss of generality, that $e_{w_1} = e_{w_2} = w_1w_2$ and $e_{w_3} = e_{w_4} = w_3w_4$. In particular, the graph induced by $N_H(v)$ contains $2K_2$. Moreover, since H has no 5-cycle, $N_H(v)$ induces $2K_2$. To complete the proof of (ii) it only remains to prove that $\tau_m(H) \leq 5$. Suppose, by the way of contradiction, that $E_H(v) \cup \{w_1w_2\}$ is not a matching-transversal; i.e., there is a maximal matching M' of H such that $M' \cap (E_H(v) \cup \{w_1w_2\}) = \emptyset$. Because of the maximality of M', for each $w \in N_H(v)$, there is some edge $e'_w \in M'$ incident to w. Since $w_1w_2 \notin M'$, $e'_{w_1} \neq e'_{w_2}$. Since w_3 is nonadjacent to w_1 and w_2, e'_{w_3} is different from e'_{w_1} and e'_{w_2}. We conclude that v is the center of a bipartite claw contained in H whose edge set is $\{vw_1, e'_{w_1}, vw_2, e'_{w_2}, vw_3, e'_{w_3}\}$. This contradiction proves that $E_H(v) \cup \{w_1w_2\}$ is a matching-transversal, which means that $\tau_m(H) \leq 5$. This completes the proof of (ii) and of the lemma. □

We now prove a lower bound on α_m (Lemma 3.21), which will be the last of the three lemmas below.

Lemma 3.19. *Let H be a graph. If v is a vertex of H that is neither the center of a bipartite claw nor a vertex of a 5-cycle, then at most two of the neighbors of v have degree at least 4 each.*

Proof. Suppose, by the way of contradiction, that there is some vertex v of H that is neither the center of a bipartite claw nor a vertex of a 5-cycle and such that v has three different neighbors w_1, w_2, w_3 in H such that $d_H(w_i) \geq 4$ for each $i \in \{1, 2, 3\}$. In particular, for each $i \in \{1, 2, 3\}$, w_i is adjacent to at least one vertex x_i different from v, w_1, w_2, w_3.

We claim that $\{w_1, w_2, w_3\}$ is a stable set of H. Suppose, by the way of contradiction, that $\{w_1, w_2, w_3\}$ is not a stable set of H. By symmetry, we assume, without loss of generality, that w_1 is adjacent to w_2. Since there is no 5-cycle passing through v, x_3 is different from x_1 and x_2. Thus, $x_1 = x_2$ and $N_H(w_1) \subseteq \{v, w_2, w_3, x_1\}$ because v is not the center of a bipartite claw. Hence, since $d_H(w_1) \geq 4$, necessarily w_1 is adjacent to w_3 and $w_1x_1w_2vw_3w_1$ is a 5-cycle of H passing through v, which is a contradiction. This contradiction proves that $\{w_1, w_2, w_3\}$ is a stable set of H.

Since $\{w_1, w_2, w_3\}$ is a stable set and $d_H(w_i) \geq 4$, there are three pairwise different vertices $x_{i1}, x_{i2}, x_{i3} \in N_H(w_i) - \{v, w_1, w_2, w_3\}$, for each $i \in \{1, 2, 3\}$. By Theorem 1.6, there are some $j_1, j_2, j_3 \in \{1, 2, 3\}$ such that $M = \{w_1x_{1j_1}, w_2x_{2j_2}, w_3x_{3j_3}\}$ is a matching of H of size 3. Therefore, $\{vw_1, vw_2, vw_3\} \cup M$ is the edge set of a bipartite claw with center v, a contradiction. This contradiction completes the proof of the lemma. □

Lemma 3.20. *Let H be a graph containing no bipartite claw and having no 5-cycle. If $\delta_h(H) \geq 4$, then there exists a set F of hub-covered edges of H such that the graph $H' = H - F$ is hub-regular and has the same hub set and the same minimum hub degree as H.*

Proof. Suppose, by the way of contradiction, that the lemma is false and let H be a counterexample to the lemma with minimum number of edges. If H were hub-regular, then the lemma would hold for H by letting $F = \emptyset$, a contradiction. Hence, H is not hub-regular; i.e., $\Delta(H) > \delta_h(H)$. Let v be a hub of H that is not minimum. As $\delta_h(H) \geq 4$, the vertex v has at least 5 neighbors. Thus, since H contains no bipartite claw and has no 5-cycle, Lemma 3.19 implies that v has some neighbor w that is not a hub (recall that $\delta_h(H) \geq 4$). Hence, since vw is not incident to any minimum hub of H, $H_1 = H - \{vw\}$ has the same hub set and the same minimum hub degree as H. The proof ends exactly in the same way as the proof of Lemma 3.13. □

Lemma 3.21. *Let H be a graph containing no bipartite claw and having no 5-cycle. If H is hub-covered, has at least one edge, and $\delta_h(H) \geq 4$, then $\alpha_m(H) \geq \delta_h(H)$.*

Proof. By Lemma 3.20, there exists a set F of hub-covered edges of H such that $H' = H - F$ is hub-regular and has the same hub set and the same minimum hub degree as H. Since H is hub-covered and has at least one edge, $\delta_h(H) < +\infty$. Hence, H' is also hub-covered and $\Delta(H') = \delta_h(H') = \delta_h(H) \geq 4$. Since H' is a subgraph of H, H' contains no bipartite claw and has no 5-cycle. Therefore, by Theorem 2.9, $\chi'(H') = \Delta(H')$; i.e., there is an edge-coloring ϕ' of H' using $\Delta(H') = \delta_h(H)$ colors. Let $\phi : E(H) \to \{0, 1, 2, \ldots, \delta_h(H)\}$ be such that $\phi(e) = \phi'(e)$ for each $e \in E(H')$ and $\phi(e) = 0$ for each $e \in E(H) - E(H')$. Since H is hub-covered, ϕ is a $\delta_h(H)$-profuse-coloring of H by construction. Thus, by Lemma 3.9, $\alpha_m(H) \geq \delta_h(H)$. □

We now use Lemmas 3.18 and 3.21 to prove the two lemmas below which settle Theorem 2.13 for fat caterpillars containing A or net.

Lemma 3.22. *Let H be a fat caterpillar containing A. Hence, $\alpha_m(H) = \tau_m(H)$. More precisely, there are some $C = v_1v_2v_3v_4v_1$ and $x_1, x_2 \in V(H) - V(C)$ as in the statement of Lemma 3.4 and one of the following assertions holds:*

(i) *C is chordless and*

$$\alpha_m(H) = \tau_m(H) = \begin{cases} 3 & \text{if } d_H(v_3) = d_H(v_4) = 2 \\ \delta_h(H) & \text{otherwise.} \end{cases}$$

(ii) *v_1v_3 is the only chord of C, $d_H(v_4) = 2$, and*

$$\alpha_m(H) = \tau_m(H) = \begin{cases} 4 & \text{if } d_H(v_2) \geq 4 \text{ and } \delta_h(H) = 3 \\ \delta_h(H) & \text{otherwise.} \end{cases}$$

(iii) *C has two chords, $d_H(v_3) = d_H(v_4) = 3$, and*

$$\alpha_m(H) = \tau_m(H) = \begin{cases} 5 & \text{if each of } v_1 \text{ and } v_2 \text{ has degree at least 5} \\ 4 & \text{otherwise.} \end{cases}$$

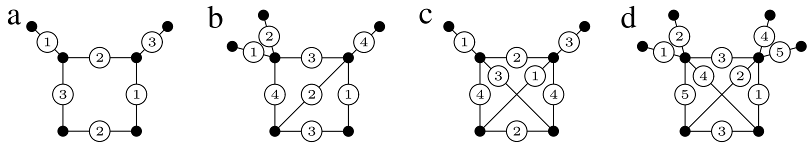

Fig. 5. Some profuse-colorings for the proof of Lemma 3.22.

Proof. Let $C = v_1v_2v_3v_4v_1$ and $x_1, x_2 \in V(H) - V(C)$ as in the statement of Lemma 3.4. In particular, each non-pendant vertex in $V(H) - V(C)$ is a false twin of v_4 of degree 2. Notice that $\alpha_m(H) \geq 3$ because a 3-profuse-coloring of H arises by coloring the edges in $E(C) \cup \{v_1x_1, v_2x_2\}$ as in Fig. 5(a) and leaving the remaining edges of H uncolored.

We now claim that if $\delta_h(H) \geq 4$, then $\tau_m(H) \leq \delta_h(H)$. On the one hand, if some minimum hub of H is adjacent to a pendant vertex, then $\tau_m(H) \leq \delta_h(H)$ due to Lemma 3.7. On the other hand, if $\delta_h(H) \geq 4$ and no minimum hub of H is adjacent to a pendant vertex, then v_3 is the only minimum hub of H and Lemma 3.18 implies that $\tau_m(H) \leq \delta_h(H)$ because $d_H(v_3) = \delta_h(H) \geq 4$ and $N_H(v_3)$ does not induce $2K_2$ in H. Thus, the claim follows.

The proof splits into three cases corresponding to assertions (i)–(iii) of Lemma 3.4.

Case 1: *C is chordless.* Suppose first that neither $d_H(v_3) = d_H(v_4) = 2$ nor $\delta_h(H) = 3$ holds. Thus, H is hub-covered and $\delta_h(H) \geq 4$, which implies that $\alpha_m(H) = \tau_m(H) = \delta_h(H)$ because $\alpha_m(H) \geq \delta_h(H)$ (by Lemma 3.21) and $\tau_m(H) \leq \delta_h(H)$ (by the above claim). Hence, (i) holds.

Suppose now that $d_H(v_3) = d_H(v_4) = 2$ or $\delta_h(H) = 3$. If $d_H(v_3) = d_H(v_4) = 2$ or some vertex of degree 3 is adjacent to a pendant vertex, then $\alpha_m(H) = \tau_m(H) = 3$ because $\tau_m(H) \leq 3$ by Lemma 3.7 and we have already seen that $\alpha_m(H) \geq 3$. Otherwise, the only minimum hub is v_3 and $N_H(v_3)$ induces $3K_1$ which also leads to $\alpha_m(H) = \tau_m(H) = 3$ because $\tau_m(H) \leq 3$ by Lemma 3.18 and we have seen that $\alpha_m(H) \geq 3$. We conclude again that (i) holds.

Case 2: *v_1v_3 is the only chord of C and $d_H(v_4) = 2$.* Assume first that $d_H(v_2) \geq 4$ and $\delta_h(H) = 3$. Necessarily, $d_H(v_3) = 3$. Thus, as $d_H(v_4) = 2$, Lemma 3.7 implies that $\tau_m(H) \leq 4$. Let y_2 be a neighbor of v_2 outside $V(C)$ different from x_2. Hence, $\alpha_m(H) \geq 4$ because a 4-profuse-coloring of H arises by coloring the edges of the subgraph of H induced by $V(C) \cup \{x_1, x_2, y_2\}$ as in Fig. 5(b) and leaving the remaining edges of H uncolored. We have proved that if $d_H(v_2) \geq 4$ and $\delta_h(H) = 3$, then (ii) holds (because $\alpha_m(H) = \tau_m(H) = 4$).

Assume now that, on the contrary, $d_H(v_2) = 3$ or $\delta_h(H) \geq 4$. If the former holds, then $\alpha_m(H) = \tau_m(H) = 3 = \delta_h(H)$ because we know that $\alpha_m(H) \geq 3$ and Lemma 3.7 would imply that $\tau_m(H) \leq 3$. If the latter holds, then $\alpha_m(H) = \tau_m(H) = \delta_h(H)$ because, since H is hub-covered, Lemma 3.21 would imply that $\alpha_m(H) \geq \delta_h(H)$ and because we have proved that $\tau_m(H) \leq \delta_h(H)$ whenever $\delta_h(H) \geq 4$. We conclude that if $d_H(v_1) = 3$ or $\delta_h(H) \geq 4$, then $\alpha_m(H) = \tau_m(H) = \delta_h(H)$ and (ii) holds.

Case 3: *C has two chords and $d_H(v_3) = d_H(v_4) = 3$.* If v_1 or v_2 has degree 4, then $\tau_m(H) \leq 4$ (by Lemma 3.18) and a 4-profuse-coloring of H arises by coloring the edges of the subgraph of H induced by $V(C) \cup \{x_1, x_2\}$ as in Fig. 5(c) and leaving all the remaining edges of H uncolored. Therefore, if v_1 or v_2 has degree 4, then $\alpha_m(H) = \tau_m(H) = 4$ and (iii) holds.

Assume now that each of v_1 and v_2 has degree at least 5 and, for each $i \in \{1, 2\}$, let y_i be a neighbor of v_i outside $V(C)$ different from x_i. As $d_H(v_3) = d_H(v_4) = 3$, Lemma 3.7 implies that $\tau_m(H) \leq 5$. In addition, $\alpha_m(H) \geq 5$ because a 5-profuse-coloring of H arises by coloring the edges of the subgraph of H induced by $V(C) \cup \{x_1, x_2, y_1, y_2\}$ as in Fig. 5(d) and leaving the remaining edges of H uncolored. Hence, $\alpha_m(H) = \tau_m(H) = 5$ and we conclude again that (iii) holds. □

Now we deal with the case of fat caterpillars containing net but no A.

Lemma 3.23. *If H is a fat caterpillar containing net but containing no A, then $\alpha_m(H) = \tau_m(H) = \delta_h(H)$.*

Proof. That H has an edge-dominating triangle C such that each vertex $v \in V(C)$ is adjacent to some pendant vertex and each vertex in $V(H) - V(C)$ is pendant follows from Lemma 3.5. As the hubs of H are the vertices of C and each of them is adjacent to some pendant vertex, Lemma 3.7 implies that $\tau_m(H) \leq \delta_h(H)$. For the proof of the lemma to be complete, it suffices to show that $\alpha_m(H) \geq \delta_h(H)$. On the one hand, if $\delta_h(H) \geq 4$, then as H is hub-covered, $\alpha_m(H) \geq \delta_h(H)$ by Lemma 3.21. On the other hand, if $\delta_h(H) = 3$, then $\alpha_m(H) \geq 3$ because a 3-profuse-coloring of H arises by 3-edge-coloring the net induced in H by $\{v_1, v_2, v_3, u_1, u_2, u_3\}$ and leaving the remaining edges of H uncolored, where u_i is some pendant neighbor of v_i for each $i \in \{1, 2, 3\}$. □

Given the two lemmas above, in order to settle Theorem 2.13, it only remains to prove the following result.

Theorem 3.24. *If H is a fat caterpillar containing no A and no net and $k \geq 1$, then $\alpha_m(H) \geq k$ if and only if $\tau_m(H) \geq k$.*

By Lemma 3.1, fat caterpillars containing no A and not net are certain linear concatenations of basic two-terminal graphs. To begin with, the following lemma, whose proof is straightforward, enumerates the values of α_m and τ_m for the underlying graphs of each of the basic two-terminal graphs.

Lemma 3.25. *The underlying graph of each of the basic two-terminal graphs satisfies $\alpha_m = \tau_m$. Moreover, the following assertions hold:*

(i) *For the underlying graph of the edge, $\alpha_m = \tau_m = 1$.*
(ii) *For the underlying graphs of the triangle, the rhombus, and the K_4, $\alpha_m = \tau_m = 3$.*

(iii) *For each* $m \geq 2$, *the underlying graph of the* m-*crown has* $\alpha_{\mathrm{m}} = \tau_{\mathrm{m}} = m + 1$.
(iv) *For each* $m \geq 2$, *the underlying graph of the* m-*fold has* $\alpha_{\mathrm{m}} = \tau_{\mathrm{m}} = m$.

Our proof of Theorem 3.24 is indirect. The theorem clearly holds for $k = 1$. In the remaining of this subsection, we deal separately with the cases $k = 2$, $k = 3$, $k = 4$, $k = 5$, and $k \geq 6$.

Case $k = 2$ of Theorem 3.24 can be derived from [41]. For each $n \geq 1$, let Q_{2n+1} be the graph having $4n + 2$ vertices $u_1, u_2, \ldots, u_{2n+1}, v_1, v_2, \ldots, v_{2n+1}$ such that $Q_{2n+1}[\{v_1, v_2, \ldots, v_{2n+1}\}] = \overline{C_{2n+1}}$ and $N_{Q_{2n+1}}(u_i) = V(Q_{2n+1}) - \{v_i\}$, for each $i \in \{1, 2, \ldots, 2n + 1\}$. These graphs Q_{2n+1} were introduced in [41] in connection with the following result.

Theorem 3.26 (*[41]*). *For each* $n \geq 1$, $\alpha_c(Q_{2n+1}) = 1$ *and* $\tau_c(Q_{2n+1}) = 2$. *Moreover, if* G *is a graph such that* $\alpha_c(G) = 1$ *but* $\tau_c(G) > 1$, *then* G *contains an induced* Q_{2n+1} *for some* $n \geq 1$.

Now we are ready to prove the case $k = 2$ of Theorem 3.24.

Lemma 3.27. *Let* H *be a fat caterpillar. Hence,* $\alpha_{\mathrm{m}}(H) \geq 2$ *if and only if* $\tau_{\mathrm{m}}(H) \geq 2$.

Proof. The 'only if' part is trivial. For the converse, suppose, by the way of contradiction, that $\tau_{\mathrm{m}}(H) \geq 2$ but $\alpha_{\mathrm{m}}(H) \leq 1$. Hence, if $G = \overline{L(H)}$, then $\tau_c(G) \geq 2$ and $\alpha_c(G) \leq 1$. Thus, by Theorem 3.26, G contains an induced Q_{2n+1} for some $n \geq 1$. As G is the complement of a line graph but $Q_{2n+1}[\{v_1, v_2, v_3, u_2\}]$ is the complement of the claw, necessarily G contains an induced Q_3 (i.e., 3-sun) and, as a result, H contains a bipartite claw, a contradiction. This contradiction proves the 'if' part and completes the proof of the lemma. □

Case $k = 3$ can be dealt as follows.

Lemma 3.28. *Let* H *be a fat caterpillar containing no* A *and no net and having at least one edge. Hence,* $\alpha_{\mathrm{m}}(H) \geq 3$ *if and only if* $\tau_{\mathrm{m}}(H) \geq 3$. *In fact, each of the inequalities holds if and only if* H *satisfies all of the following assertions:*

(i) *For each pair of adjacent vertices* v_1 *and* v_2, $d_H(v_1) + d_H(v_2) - 1 \geq 3$.
(ii) *Each* 4-*cycle of* H *has at most two vertices of degree* 2 *in* H.
(iii) H *is not the underlying graph of* triangle $\&_p$triangle *for any* $p \geq 0$.

Proof. Since $\alpha_{\mathrm{m}}(H) \leq \tau_{\mathrm{m}}(H)$, clearly $\alpha_{\mathrm{m}}(H) \geq 3$ implies $\tau_{\mathrm{m}}(H) \geq 3$. Suppose that $\tau_{\mathrm{m}}(H) \geq 3$. Hence, (i) holds because of Lemma 3.7. If there were some 4-cycle $C = v_1v_2v_3v_4v_1$ such that $d_H(v_1) = d_H(v_2) = d_H(v_3) = 2$, then $\{v_1v_2, v_2v_3\}$ would be a matching-transversal of H, contradicting $\tau_{\mathrm{m}}(H) \geq 3$. Similarly, if H were the underlying graph of triangle $\&_p$triangle for some $p \geq 0$, then the set consisting of the two edges of H non-incident to the concatenation vertex would be a matching-transversal of H, another contradiction. These contradictions prove that (ii) and (iii) also hold.

To complete the proof of the lemma, let us assume that (i)–(iii) hold and we will prove that $\alpha_{\mathrm{m}}(H) \geq 3$, or, equivalently, that H has a 3-profuse-coloring. As H is a fat caterpillar containing no A and no net, Lemma 3.1 implies that H is the underlying graph of $\Gamma_1 \&_{p_1} \Gamma_2 \&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geq 0$. If $n = 1$, then H is the underlying graph of some two-terminal graph different from an edge and a square and H admits a 3-profuse-coloring by Lemma 3.25. Hence, from now on we assume that $n \geq 2$.

Case 1: *H is the underlying graph of $\Gamma_1 \&_p \Gamma_2$ where each of Γ_1 and Γ_2 is an edge or a triangle and $p \geq 0$.* By (iii), assume, without loss of generality, that Γ_1 is an edge. If Γ_2 is also an edge, then (i) implies that $p \geq 1$ and clearly $\alpha_{\mathrm{m}}(H) \geq 3$ because a 3-profuse-coloring of H arises by coloring with three different colors any three edges of H and leaving the remaining edges of H uncolored. If, on the contrary, Γ_2 is a triangle, then also $\alpha_{\mathrm{m}}(H) \geq 3$ because a 3-profuse-coloring of H arises by coloring the edge of Γ_1 and the two edges of Γ_2 incident to the concatenation vertex with three different colors and leaving the remaining edges of H uncolored.

Case 2: *H does not fulfill Case 1.* For each $i \in \{1, \ldots, n\}$, let P_i be some shortest path in Γ_i joining its two terminal vertices. Thus, $P = P_1P_2 \ldots P_n$ is a chordless path in H and let $P = u_0u_1 \ldots u_\ell$ where u_0 is the source of Γ_1 and u_ℓ is the sink of Γ_n. Consider a coloring of the edges of P with the colors 1, 2, and 3, such that any three consecutive edges of P receive three different colors. As P is edge-dominating, every edge of H is incident to at least two differently colored edges, except for the edges incident to u_0 and u_ℓ. Assume without loss of generality that u_0u_1 is colored with color 1 and u_1u_2 with color 2. We make the edges incident to u_0 adjacent to at least two differently colored edges as follows:

(1) If there are at least two edges joining u_0 to vertices outside P, we color two of these edges using colors 2 and 3.
(2) If there is exactly one vertex u' outside P adjacent to u_0, then Γ_1 is a triangle or a rhombus (because (ii) ensures that Γ_1 is not a square). In particular, u_1 is also adjacent to u'. We color u_1u' with color 3.
(3) If there is no vertex outside P adjacent to u_0, then Γ_1 is an edge and, by (i), u_1 is adjacent to some vertex u' outside P. We color u_1u' with color 3.

Symmetrically, let x be the color of $u_{\ell-1}u_\ell$, y be the color of $u_{\ell-2}u_{\ell-1}$, and $z \in \{1, 2, 3\} - \{x, y\}$. We make the edges incident to u_ℓ adjacent to at least two differently colored edges as follows:

(1′) If there are at least two edges joining u_ℓ to vertices outside P, we color two of these edges using colors y and z.
(2′) If there is exactly one vertex u'' outside P adjacent to u_ℓ, then u'' is adjacent to $u_{\ell-1}$ (as in (2)). If there were an edge incident to $u_{\ell-1}$ colored with color z, then $n = 2$, Γ_2 is a triangle, and either Γ_1 is a triangle or an edge, contradicting the hypothesis. Thus, we color the edge $u_{\ell-1}u''$ with color z.

(3′) If there is no vertex outside P adjacent to u_ℓ, then Γ_n would be an edge and $u_{\ell-1}$ is adjacent to some vertex u'' outside P (as in (3)). If there were some edge incident to $u_{\ell-1}$ colored with color z, then $n = 2$ and Γ_1 is an edge or a triangle, which would contradict our hypothesis because Γ_2 is a triangle. We color $u_{\ell-1}u''$ with color z.

The resulting partial 3-edge-coloring is a 3-profuse-coloring of H because each edge of H is incident to at least two differently colored edges. Hence, $\alpha_{\mathrm{m}}(H) \geq 3$, as needed. □

For case $k = 4$, we prove the following.

Lemma 3.29. *Let H be a fat caterpillar containing no net and no A and having at least one edge. Hence, $\alpha_{\mathrm{m}}(H) \geq 4$ if and only if $\tau_{\mathrm{m}}(H) \geq 4$. In fact, each of the inequalities holds if and only if H satisfies all of the following conditions:*

(i) *For each pair of adjacent vertices v_1 and v_2, $d_H(v_1) + d_H(v_2) - 1 \geq 4$.*
(ii) *No block of H is a clique on four vertices.*
(iii) *Each vertex of degree 3 that is not a cut-vertex has only neighbors of degree at least 3.*
(iv) *The neighborhood of each cut-vertex of degree 3 induces $K_2 + K_1$ in H.*

Proof. By Lemma 3.1, H is the underlying graph of some $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geq 0$. For each $i \in \{1, 2, \ldots, n-1\}$, let v_i be the concatenation vertex of H that arose by identifying the sink of Γ_i with the source of Γ_{i+1} and let v_0 be the source of Γ_1 and v_n be the sink of Γ_n. Clearly, the cut-vertices of H are the concatenation vertices $v_1, v_2, \ldots, v_{n-1}$ and the underlying graph of each Γ_i is a block of H.

Since $\alpha_{\mathrm{m}}(H) \leq \tau_{\mathrm{m}}(H)$, $\alpha_{\mathrm{m}}(H) \geq 4$ implies that $\tau_{\mathrm{m}}(H) \geq 4$. Suppose now that H satisfies $\tau_{\mathrm{m}}(H) \geq 4$. Thus, H satisfies (i) because of Lemma 3.7. If some block of H were a clique of size four, this block would have at least three vertices of degree 3 in H (because H contains no A and has no 5-cycle) and the edges of the K_3 induced by these three vertices would be a matching-transversal of H. Hence, since $\tau_{\mathrm{m}}(H) \geq 4$, H satisfies (ii). If there were a vertex v of H of degree 3 that were not a cut-vertex and had a neighbor of degree less than 3, then, up to symmetry, either: (1) v is a non-terminal vertex of Γ_1 and Γ_1 is a rhombus, or (2) v is the source of Γ_1 and Γ_1 is a 2-crown or a 3-fold. If (1) holds, then the edges of the triangle induced by $N_H[v_0]$ form a matching-transversal of H of size 3. If (2) holds, then $E_H(v_0)$ is a matching-transversal of H of size 3. In either case, we reach a contradiction with $\tau_{\mathrm{m}}(H) \geq 4$. This contradiction proves that H satisfies (iii). Finally, if v is a cut-vertex of H of degree 3, then $N_H(v)$ induces a disconnected graph on three vertices; i.e., $N_H(v)$ induces $3K_1$ or $K_2 + K_1$. But, if $N_H(v)$ induces $3K_1$, then Lemma 3.18 implies that $\tau_{\mathrm{m}}(H) \leq 3$, a contradiction. This proves that H satisfies (iv). Altogether, we have proved that if $\tau_{\mathrm{m}}(H) \geq 4$, then H satisfies conditions (i)–(iv).

To complete the proof of the lemma, we assume that H satisfies conditions (i)–(iv) and show that $\alpha_{\mathrm{m}}(H) \geq 4$ or, equivalently, by Lemma 3.9, that H has a 4-profuse-coloring. To start, we prove the following claims about H.

Claim 1. *Each of Γ_1 and Γ_n is either an edge, m-crown for some $m \geq 3$, or m-fold for some $m \geq 4$.*

Proof. Each of Γ_1 and Γ_n is different from triangle and square because of (i), different from 2-crown, 3-fold, and rhombus because of (iii), and different from K_4 because of (ii). As Γ_1 and Γ_n are basic, the claim follows. ◇

Claim 2. *If there is a maximal 4-profuse-coloring ϕ of H and there are at least three edges of Γ_j incident to the same terminal vertex of Γ_j, then each terminal vertex of Γ_j is incident to four edges of H colored by ϕ.*

Proof. Without loss of generality, assume that there are at least three edges of Γ_j incident to v_j. As Γ_j is basic, there are also at least three edges of Γ_j incident to v_{j-1} and Γ_j is either an m-crown for some $m \geq 2$ or an m-fold for some $m \geq 3$. Hence, if $d_H(v_j) = 3$, then $j = n$ and either Γ_n would be a 2-crown or a 3-fold, contradicting Claim 1. Therefore, $d_H(v_j) \geq 4$ and, symmetrically, $d_H(v_{j-1}) \geq 4$. In addition, neither $N_H(v_j)$ nor $N_H(v_{j-1})$ induces $2K_2$ in H and, by Lemma 3.18, $E_H(v_j)$ and $E_H(v_{j-1})$ are matching-transversals of H. Hence, by Corollary 3.10, the maximality of ϕ implies that each of v_j and v_{j-1} is incident to four edges of H colored by ϕ. ◇

Claim 3. *If $n \geq 2$, Γ_{n-1} and Γ_n are both edges, $p_{n-1} = 2$, and there is some 4-profuse-coloring of H, then either $n = 2$ or there is some 4-profuse-coloring of H that colors at least two of the edges incident to v_{n-2}.*

Proof. Suppose that $n \geq 3$ and we have to prove that there is a 4-profuse-coloring of H that colors at least two edges incident to v_{n-2}. Let ϕ be a 4-profuse-coloring of H that maximizes the number of colored edges incident to v_{n-2} and, without loss of generality, assume that ϕ is maximal. Suppose, by the way of contradiction, that ϕ colors at most one edge incident to v_{n-2}. As ϕ is maximal, the four edges incident to v_{n-1} are colored by ϕ and, in particular, $v_{n-2}v_{n-1}$ is colored. Hence, by hypothesis, all edges incident to v_{n-2} different from $v_{n-2}v_{n-1}$ are uncolored. If there were an edge joining v_{n-2} to some non-cut-vertex of H, then this edge would be uncolored and, at the same time, incident to at most three colored edges, contradicting the maximality of ϕ. Therefore, $p_{n-2} = 0$ and Γ_{n-2} is an edge. As $v_{n-3}v_{n-2}$ is uncolored and $v_{n-2}v_{n-1}$ is the only colored edge incident to v_{n-2}, there are at least three colored edges incident to v_{n-3} such that each of them is colored differently from $v_{n-2}v_{n-1}$. If there were some pendant edge q incident to v_{n-3} and colored differently from $v_{n-2}v_{n-1}$, then, by coloring $v_{n-3}v_{n-2}$ with the color of q and uncoloring q, a new 4-profuse-coloring of H arises that colors at least two edges incident to v_{n-2}, a contradiction with the choice of ϕ. This contradiction shows that there are at least three colored edges of Γ_{n-2} incident to v_{n-3}. Hence, by Claim 2, v_{n-4} is incident to four colored edges. Let e be any of the colored edges incident to v_{n-3} but not to v_{n-4} such that e is colored differently from $v_{n-2}v_{n-1}$. Thus, coloring $v_{n-3}v_{n-2}$ with the color of e and uncoloring

e, a new 4-profuse-coloring of H arises that colors two of the edges incident to v_{n-2}, contradicting the choice of ϕ. This contradiction arose from assuming that ϕ does not color at least two edges incident to v_{n-2}. Hence, the claim follows. ◇

Claim 4. *If H has a 4-profuse-coloring, Γ_1 is an edge, $n \geq 2$, $p_1 = 1$, and $N_H(v_1)$ induces $K_2 + 2K_1$ in H, then there is a 4-profuse-coloring ϕ of H that colors the only edge of H joining two neighbors of v_1.*

Proof. Let ϕ' be a maximal 4-profuse-coloring of H and let e be the only edge of H joining two vertices in $N_H(v_1)$. As $d_H(v_1) = 4$ and $N_H(v_1)$ does not induce $2K_2$, Lemma 3.18 implies that $E_H(v_1)$ is a matching-transversal of H and the four edges incident to v_1 are colored by ϕ' because of the maximality of ϕ' and because of Corollary 3.10. If ϕ' colors e, then the claim holds by letting $\phi = \phi'$. Hence, suppose that e is not colored by ϕ'. Thus, the maximality of ϕ' implies that e is incident to at least four other edges of H.

Suppose first that e is incident to exactly four edges of H; i.e., either Γ_2 is triangle and $d_H(v_2) = 4$, or Γ_2 is rhombus. Let w be an endpoint of e different from v_2 and let $e' = v_1w$. Let e'' be a pendant edge incident to v_1 and colored differently from each of the colored edges incident to w. Notice that the maximality of ϕ, Lemma 3.7, and Corollary 3.10 imply that the four edges of H incident to e are colored by ϕ' using four different colors. Hence, if we define $\phi : E(H) \to \{0, 1, 2, 3, 4\}$ to coincide with ϕ' except that ϕ colors e and e'' with color $\phi'(e')$ and e' with color $\phi'(e'')$, then ϕ is a 4-profuse-coloring of H that colors e, as claimed.

It only remains to consider the case where e is incident to more than four edges of H. Necessarily, Γ_2 is a triangle and $d_H(v_2) \geq 5$. In particular, Lemma 3.18 and Corollary 3.10 imply that there are four edges incident to v_2 colored by ϕ'. Let w be the non-terminal vertex of Γ_2. Suppose that there is some pendant edge q incident to v_2 that is colored by ϕ'. By permuting, if necessary, the colors of the edges of H incident to v_1 that are different from v_1v_2, we assume, without loss of generality, that v_1w is colored differently from q and, then, by coloring e with the color of q and uncoloring q, a new 4-profuse-coloring of H arises that colors e, as claimed. Hence, from now on, we assume, without loss of generality, that there is no pendant edge incident to v_2 colored by ϕ'. Since there are four edges incident to v_2 colored by ϕ', necessarily three of them are edges of Γ_3. By Claim 2, there are four colored edges incident to v_3. Therefore, if we let e' be any edge of Γ_3 incident to v_2 but not to v_3 and colored by ϕ' differently from v_1w, then by coloring e with the color of e' and uncoloring e', a new 4-profuse-coloring of H arises that colors e, as claimed. ◇

Claim 5. *If H has a 4-profuse-coloring, Γ_1 is an edge, $n \geq 2$, and $p_1 \geq 1$, then there is a 4-profuse-coloring of H that colors at least two pendant edges incident to v_1.*

Proof. Suppose, by the way of contradiction, that ϕ is a 4-profuse-coloring of H that maximizes the number of colored pendant edges incident to v_1 and that, nevertheless, ϕ colors at most one pendant edge incident to v_1. Since $p_1 \geq 1$, there is at least one uncolored pendant edge incident to v_1. Thus, the maximality of ϕ means that there are four colored edges incident to v_1. Hence, there are at least three colored edges of Γ_2 incident to v_1 and, by Claim 2, there are four colored edges incident to v_2. Let e be any colored edge of Γ_2 incident to v_1 but not to v_2 and let q be any of the uncolored pendant edges incident to v_1. If we color q with the color of e and uncolor e, a new 4-profuse-coloring of H arises that colors one more pendant edge incident to v_1 than ϕ, contradicting the choice of ϕ. This contradiction proves that the claim holds. ◇

We turn back to the proof of the lemma. The proof proceeds by induction on the number of cut-vertices of H. Clearly, the cut-vertices of H are the $n - 1$ vertices $v_1, \ldots, v_{n-1}$. Consider first the case where H has no cut-vertices; i.e., $n = 1$ and H is the underlying graph of Γ_1 which, by Claim 1, is an edge, m-crown for some $n \geq 3$, or m-fold for some $m \geq 2$. If Γ_1 were an edge, then $d_H(v_0) + d_H(v_1) - 1 = 1$, contradicting (i). Therefore, if $n = 1$, then H is m-crown for some $m \geq 3$ or m-fold for some $m \geq 4$ and, by Lemma 3.25, $\alpha_{\mathrm{m}}(H) \geq 4$.

Assume that $n \geq 2$ and that the lemma holds for graphs with less than $n - 1$ cut-vertices. Suppose that H has some cut-vertex of degree 3; i.e., there is some $j \in \{1, 2, \ldots, n-1\}$ such that v_j has degree 3 in H. By (iv), $N_H(v_j)$ induces $K_2 + K_1$ in H. Therefore, $p_j = 0$ and, by symmetry, assume, without loss of generality, that Γ_j is an edge and Γ_{j+1} is either a triangle or a rhombus. Let H_1 be the graph that arises from H by first removing all vertices and edges from $\Gamma_{j+1}, \Gamma_{j+2}, \ldots, \Gamma_n$, except for the vertices of $N_H[v_j]$ and the edges incident to v_j, and then adding one pendant edge q incident to v_j. Notice that H_1 can be regarded as the underlying graph of $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_1} \ldots \,\&_{p_{j-1}} \Gamma_j \,\&_2$edge. Clearly, H_1 satisfies (i)–(iv) and, by induction hypothesis, there is a maximal 4-profuse-coloring of H_1. By Claim 3, there is a 4-profuse-coloring ϕ_1 of H_1 that colors at least two of the edges of H_1 incident to v_{j-1}. Thus, by permuting, if necessary, the colors of the pendant edges incident to v_j in H_1, we assume, without loss of generality, that ϕ_1 colors some edge incident to v_{j-1} with color $\phi_1(q)$. Let H_2 be the graph that arises from H by first removing all vertices and edges of $\Gamma_1, \Gamma_2, \ldots, \Gamma_j$, except for the vertices of $N_H[v_j]$ and the edges incident to v_j, and then adding one pendant edge incident to v_j. The graph H_2 can also be regarded as the underlying graph of edge $\&_1 \Gamma_{j+1} \,\&_{p_{j+1}} \Gamma_{j+2} \,\&_{p_{j+2}} \cdots \,\&_{p_{n-1}} \Gamma_n$. By Claim 4, there is a maximal 4-profuse-coloring ϕ_2 of H_2 that colors the only edge e joining two neighbors of v_j. By permuting, if necessary, the pendant edges incident to v_j, we assume, without loss of generality, that ϕ_2 colors e differently from the edge of Γ_j. Moreover, by permuting, if necessary, the colors of ϕ_2, we assume without loss of generality, that ϕ_1 and ϕ_2 color exactly in the same way the edge of Γ_j and each of the edges of Γ_{j+1} incident to v_j. Thus, there is no edge of H where ϕ_1 and ϕ_2 differ and the partial edge-coloring ϕ that results by merging ϕ_1 and ϕ_2 is easily seen to be a 4-profuse-coloring of H, as desired. Therefore, from now on, we assume, without loss of generality, that H has no cut-vertex of degree 3.

Suppose now that there is some $j \in \{1, 2, \dots, n\}$ such that Γ_j is a rhombus. Let H_1 be the graph that arises from H by removing all the vertices and edges from $\Gamma_j, \Gamma_{j+1}, \dots, \Gamma_n$ except for the vertices of $N_H[v_{j-1}]$ and the edges incident to v_{j-1}, and let H_2 be the graph that arises from H by removing all vertices and edges from $\Gamma_1, \Gamma_2, \dots, \Gamma_j$ except for the vertices of $N_H[v_j]$ and the edges incident to v_j. Moreover, as H has no cut-vertex of degree 3, $d_{H_1}(v_{j-1}) \geq 4$, from which it follows that H_1 satisfies (i)–(iv) and, by induction hypothesis, H_1 admits a 4-profuse-coloring ϕ_1. Similarly, $d_{H_2}(v_{j+1}) \geq 4$ and H_2 admits a 4-profuse-coloring ϕ_2. By Claim 5, we assume, without loss of generality, that ϕ_i colors both edges of Γ_j that belong to H_i, for each $i \in \{1, 2\}$. By permuting, if necessary, the colors of ϕ_2, we assume, without loss of generality, that ϕ_1 and ϕ_2 color the four edges of Γ_j that belong to H_1 or H_2 using 4 different colors. Let $\phi : E(H) \to \{0, 1, 2, 3, 4\}$ be defined as ϕ_1 in $E(H_1)$, as ϕ_2 in $E(H_2)$, and that leaves the only edge of Γ_j that belongs neither to H_1 nor to H_2 uncolored. Clearly, ϕ is a 4-profuse-coloring of H, as desired.

It only remains to consider the case where H has no cut-vertices of degree 3 and no Γ_j is a rhombus; i.e., the case where $\delta_h(H) \geq 4$. Since (i) ensures that H is hub-covered and since H has at least one edge, Lemma 3.21 implies that $\alpha_m(H) \geq \delta_h(H) \geq 4$, which completes the proof of the lemma. □

The following lemma settles case $k = 5$.

Lemma 3.30. *Let H be a fat caterpillar containing no A and no net and having at least one edge. Hence, $\alpha_m(H) \geq 5$ if and only if $\tau_m(H) \geq 5$. In fact, each of the inequalities holds if and only if H satisfies all of the following assertions:*

(i) *For each pair of adjacent vertices v_1 and v_2, $d_H(v_1) + d_H(v_2) - 1 \geq 5$.*
(ii) *No block of H is a clique on four vertices.*
(iii) *No cut-vertex of H has degree 3 in H.*
(iv) *The neighborhood of each vertex of degree 4 induces $2K_2$ in H.*

Proof. Since $\alpha_m(H) \leq \tau_m(H)$, $\alpha_m(H) \geq 5$ implies $\tau_m(H) \geq 5$. Suppose now that H satisfies $\tau_m(H) \geq 5$. Thus, H satisfies (i) because of Lemma 3.7. If there were some block of H of size four, it would have at least three vertices of degree 3 in H (because H contains no A and has no 5-cycle) and the edges of the K_3 induced by these three vertices would be a matching-transversal of H, contradicting $\tau_m(H) \geq 5$. Thus, H satisfies (ii). Since the neighborhood of a cut-vertex induces a disconnected graph, if H had some cut-vertex of degree 3, then by Lemma 3.18, $\tau_m(H) \leq 4$. Hence, H satisfies (iii). Finally, Lemma 3.18 implies that H satisfies (iv). Hence, we have proved that if $\tau_m(H) \geq 5$, then H satisfies (i)–(iv). To complete the proof of the lemma, we assume that H satisfies assertions (i)–(iv) and we will show that $\alpha_m(H) \geq 5$, or, equivalently, by Lemma 3.9, that H has a 5-profuse-coloring.

By virtue of Lemma 3.1, H is the underlying graph of some $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geq 0$. Clearly, the underlying graph of each Γ_i is a block of H. Therefore, because of (ii), none of $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ is a K_4. For each $i \in \{1, 2, \dots, n-1\}$, let v_i be the concatenation vertex of H that arises by identifying the sink of Γ_i with the source of Γ_{i+1}. Let v_0 be the source of Γ_1 and let v_n be the sink of Γ_n. We make the following claims.

Claim 6. *Each of Γ_1 and Γ_n is either an edge, m-crown for some $m \geq 4$, or m-fold for some $m \geq 5$.*

Proof. Indeed, each of Γ_1 and Γ_n is different from triangle, square, 2-crown, 3-fold, and rhombus because of (i), different from 3-crown and 4-fold because of (iv), and different from K_4 because of (ii). The claim follows. ◇

Claim 7. *If there is a maximal 5-profuse-coloring ϕ of H and there are at least three edges of Γ_j incident to the same terminal vertex of Γ_j, then each terminal vertex of Γ_j is incident to five edges of H colored by ϕ.*

Proof. Without loss of generality, suppose that there are at least three edges of Γ_j incident to v_j. As Γ_j is basic, there are also at least three edges of Γ_j incident to v_{j-1} and Γ_j is either and m-crown for some $m \geq 2$ or an m-fold for some $m \geq 3$. If $d_H(v_j) = 3$, then $j = n$ and Γ_n is either a 3-crown or a 4-fold, contradicting Claim 1. Thus, $d_H(v_j) \geq 4$ and, symmetrically, $d_H(v_{j-1}) \geq 4$. In addition, neither $N_H(v_j)$ nor $N_H(v_{j-1})$ induces $2K_2$ and, by (iv), $d_H(v_j) \geq 5$ and $d_H(v_{j-1}) \geq 5$. Hence, Lemma 3.18, Corollary 3.10, and the maximality of ϕ imply that each of v_j and v_{j-1} is incident to five edges colored of H by ϕ, as claimed. ◇

Claim 8. *If H has a 5-profuse-coloring and Γ_j is a triangle of H, then there is a 5-profuse-coloring of H that colors the three edges of Γ_j.*

Proof. By the way of contradiction, suppose that the claim is false. Hence, there is some link Γ_j that is a triangle and some 5-profuse-coloring ϕ of H that maximizes the number of colored edges of Γ_j such that, nevertheless, ϕ does not color the three edges of Γ_j. Without loss of generality, assume that ϕ is maximal. Let w be the non-terminal vertex of Γ_j. By Claim 1 and (iii), $d_H(v_{j-1}) \geq 4$ and $d_H(v_j) \geq 4$. Suppose, by the way of contradiction, that $d_H(v_j) = 4$. Thus, Lemma 3.7 implies that the set of five edges $E_H(v_j) \cup E_H(w)$ is a matching-transversal of H and, by the maximality of ϕ and Corollary 3.10, these five edges are colored by ϕ, contradicting the fact that not all the edges of Γ_j are colored. Thus, necessarily $d_H(v_j) \geq 5$ and, symmetrically, $d_H(v_{j-1}) \geq 5$. Let e be any uncolored edge of Γ_j and assume, without loss of generality, that e is incident to v_j. As $d_H(v_j) \geq 5$, there are five colored edges incident to v_j because of Lemma 3.18, Corollary 3.10, and the maximality of ϕ. If there were some pendant edge q incident to v_j and colored differently from $v_{j-1}w$ (if colored), then, by coloring e with

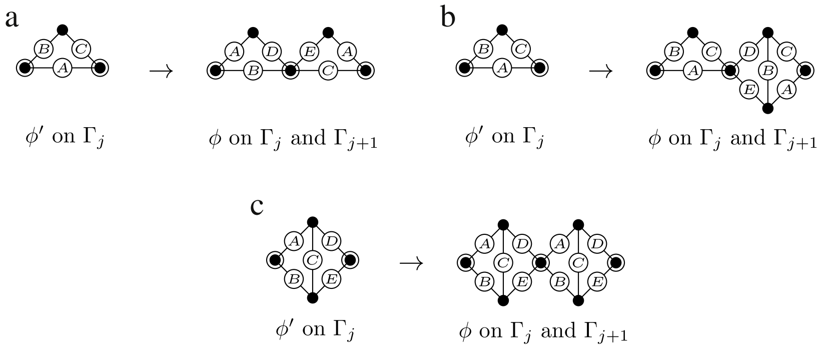

Fig. 6. Rules for transforming ϕ' into ϕ in the proof of Lemma 3.30. Here A, B, C, D, E represents any permutation of the colors 1, 2, 3, 4, 5 and rule (a), (b), or (c) apply depending on whether each of Γ_j and Γ_{j+1} is a triangle or a rhombus.

the color of q and uncoloring q, a new 5-profuse-coloring of H that colors one more edge of Γ_j would arise, contradicting the choice of ϕ. This contradiction proves that among the colored edges incident to v_j, there are at least three of them that are edges of Γ_{j+1}. Therefore, by Claim 2, there are five colored edges incident to v_{j+1}. Symmetrically, if e were incident to v_{j-1}, then there would be five colored edges incident to v_{j-2}. Finally, let $c \in \{1, 2, 3, 4, 5\}$ be different from the colors of the colored edges of Γ_j and different from the colors of v_jv_{j+1} (if present and colored) and $v_{j-2}v_{j-1}$ (if present and colored). Let ϕ' be the partial edge-coloring of H defined as ϕ except that ϕ' colors e with color c and uncolors the edge of H incident to e colored by ϕ with color c. By construction, ϕ' is a 5-profuse-coloring of H and ϕ' colors one more edge of Γ_j than ϕ, a contradiction with the choice of ϕ. This contradiction proves that ϕ colors all the edges of Γ_j and the claim holds. $\diamond$

Claim 9. *If H has a 5-profuse-coloring, Γ_1 is an edge, $n \geq 2$, and $p_1 \geq 1$, then there is a 5-profuse-coloring ϕ of H that colors at least two pendant edges incident to v_1.*

Proof. By the way of contradiction, suppose that there is a 5-profuse-coloring ϕ of H that maximizes the number of colored pendant edges incident to v_1 and that, nevertheless, ϕ colors at most one pendant edge incident to v_1. Without loss of generality, assume that ϕ is maximal. Since $p_1 \geq 1$, there is still at least one uncolored pendant edge incident to v_1. Thus, the maximality of ϕ implies that there are five colored edges incident to v_1 and, as there is at most one colored pendant edge incident to v_1, there are at least four colored edges of Γ_2 incident to v_1. By Claim 2, there are five colored edges incident to v_2. Let e be any of the colored edges of Γ_2 incident to v_1 but not to v_2 and let q be any of the uncolored pendant edges incident to v_1. If we color q with the color of e and uncolor e, a new 5-profuse-coloring of H arises that colors one more pendant edge incident to v_1 than ϕ, contradicting the choice of ϕ. This contradiction proves the claim. $\diamond$

We turn back to the proof of the lemma. The proof proceeds by induction on the number of cut-vertices of H. Consider the case H has no cut-vertices; i.e., $n = 1$ and H is the underlying graph of Γ_1 which, by Claim 1, is an edge, m-crown for some $m \geq 4$, or m-fold for some $m \geq 5$. If H were an edge, v_0 and v_1 would be two adjacent pendant vertices of H and $d_H(v_0) + d_H(v_1) - 1 = 1$, which would contradict (i). Hence, H is m-crown for some $m \geq 4$ or m-fold for some $m \geq 5$ and, by Lemma 3.25, $\alpha_{\mathrm{m}}(H) \geq 5$.

Assume now that $n \geq 2$ and that the lemma holds for graphs with less than $n - 1$ cut-vertices. Suppose first that H has a cut-vertex of degree 4 and let $j \in \{1, 2, 3, \dots, n-1\}$ such that $d_H(v_j) = 4$. Because of (iv), $N_H(v_j)$ induces $2K_2$ in H. Therefore, $p_j = 0$ and each of Γ_j and Γ_{j+1} is a triangle or a rhombus. If one of Γ_j and Γ_{j+1} is a triangle and the other is a rhombus, we assume, without loss of generality, that Γ_j is the one that is a triangle. Let H' be the graph that arises from H by contracting Γ_{j+1} to a vertex. Thus, H' is the underlying graph of $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{j-1}} \Gamma_j \,\&_{p_{j+1}} \Gamma_{j+2} \,\&_{p_{j+2}} \cdots \,\&_{p_{n-1}} \Gamma_n$ and H' satisfies (i)–(iv). By induction hypothesis, H' has a 5-profuse-coloring ϕ'. Without loss of generality, assume that ϕ' is maximal. Moreover, we can further assume that ϕ' colors all the edges of Γ_j. (In fact, if Γ_j is a rhombus then it is true by the maximality of ϕ', whereas if Γ_j is a triangle then it can be assumed by Claim 3.) We define a new partial 5-edge-coloring $\phi : E(H) \to \{0, 1, 2, 3, 4, 5\}$ as follows. Let ϕ coincide with ϕ' in those edges of H that are neither of Γ_j nor of Γ_{j+1} and we define ϕ on the edges of Γ_j and Γ_{j+1} depending on how ϕ' colors the edges of Γ_j as described in Fig. 6, where A, B, C, D, E is any permutation of the colors 1, 2, 3, 4, 5. Clearly, ϕ is a 5-profuse-coloring of H and $\alpha_{\mathrm{m}}(H) \geq 5$, as desired. Therefore, from now on, we assume that $d_H(v_i) \geq 5$ for each $i \in \{1, 2, \dots, n-1\}$.

Next, we suppose that Γ_j is a rhombus for some j. As Claim 1 implies that neither Γ_1 nor Γ_n is rhombus, $2 \leq j \leq n-1$. Let H_1 be the graph that arises from H by removing all the vertices and edges of $\Gamma_j, \Gamma_{j+1}, \dots, \Gamma_n$ except for the vertices of $N_H[v_{j-1}]$ and the edges incident to v_{j-1}. Let H_2 be the graph that arises from H by removing all the vertices and edges of $\Gamma_1, \Gamma_2, \dots, \Gamma_j$ except the vertices of $N_H[v_j]$ and the edges incident to v_j. Thus, we can regard H_1 as the underlying graph of $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{j-2}} \Gamma_{j-1} \,\&_{p_{j-1}+1}$edge and H_2 as the underlying graph of edge $\&_{p_j+1} \Gamma_{j+1} \,\&_{p_{j+1}} \Gamma_{j+2} \,\&_{p_{j+2}} \cdots \,\&_{p_{n-1}} \Gamma_n$. Since we are assuming that $d_H(v_{j-1}) \geq 5$ and $d_H(v_j) \geq 5$, H_1 and H_2 satisfy conditions (i)–(iv). By induction hypothesis, there are 5-profuse-colorings of H_1 and H_2. By Claim 4, we can assume that the 5-profuse-colorings of H_1 and H_2 are such that the two edges of Γ_j incident to v_{j-1} are colored by the 5-profuse-coloring of H_1 and the two edges of Γ_j incident to v_j

are colored by the 5-profuse-coloring of H_2. By permuting, if necessary, the colors in the 5-profuse-coloring of H_2, we can assume that the four edges of Γ_j that are incident to some terminal vertex of Γ_j are colored by these profuse-colorings using four different colors. Thus, a 5-profuse-coloring of H arises by merging the profuse-colorings of H_1 and H_2 and letting the edge joining the two non-terminal vertices of Γ_j uncolored. Hence, Lemma 3.9 implies $\alpha_{\mathrm{m}}(H) \geq 5$. Therefore, from this point on, we assume that no Γ_i is a rhombus.

Because of (iii) and because we are assuming that no cut-vertex of H has degree 4, each of the vertices $v_1, v_2, \ldots, v_{n-1}$ has either degree 2 or degree at least 5. In addition, since each of Γ_1 and Γ_n is either an edge, m-crown for some $m \geq 4$, or m-fold for some $m \geq 5$, each of v_0 and v_n has degree 1 or at least 5. Finally, since no Γ_i is rhombus or K_4, each vertex of H different from $v_0, v_1, \ldots, v_n$ has degree at most 2. Hence, $\delta_{\mathrm{h}}(H) \geq 5$. Since H has at least one edge and H is hub-covered (because of (i)), Lemma 3.21 implies that $\alpha_{\mathrm{m}}(H) \geq \delta_{\mathrm{h}}(H) \geq 5$, which completes the proof of Lemma 3.30. □

Finally, for the case $k \geq 6$ we prove the following.

Lemma 3.31. *Let H be a fat caterpillar containing no A and no net and having at least one edge. If $k \geq 6$, then the following assertions are equivalent:*

(i) $\alpha_{\mathrm{m}}(H) \geq k$.
(ii) $\tau_{\mathrm{m}}(H) \geq k$.
(iii) *H is hub-covered and $\delta_{\mathrm{h}}(H) \geq k$.*

Proof. Clearly, (i) implies (ii) because $\alpha_{\mathrm{m}}(H) \leq \tau_{\mathrm{m}}(H)$. As $k \geq 6$ and H has at least one edge, Lemma 3.21 shows that (iii) implies (i). For the proof to be complete, it suffices to show that (ii) implies (iii). Suppose that $\tau_{\mathrm{m}}(H) \geq k$. Since $k \geq 6$, H is hub-covered because of Lemma 3.7. By virtue of Lemma 3.1, H is the underlying graph of some $\Gamma_1 \,\&_{p_1} \Gamma_2 \,\&_{p_2} \cdots \,\&_{p_{n-1}} \Gamma_n$ where each Γ_i is a basic two-terminal graph and each $p_i \geq 0$. If there were some $i \in \{1, 2, \ldots, n\}$ such that Γ_i is a rhombus or K_4, then the two non-terminal vertices of Γ_i would be two adjacent vertices of degree 3 and Lemma 3.7 would imply that $\tau_{\mathrm{m}}(H) \leq 5$, a contradiction. Therefore, each Γ_i is an m-crown for some $m \geq 0$ or an m-fold for some $m \geq 2$. Let v_i be the vertex of H that arises by identifying the sink of Γ_i and the source of Γ_{i+1} and let v_0 be the source of Γ_1 and v_n be the sink of Γ_n. Thus, for each $i \in \{1, \ldots, n-1\}$, v_i has degree 2 in H or has a neighbor in H of degree 2 in H and, consequently, Lemma 3.7 implies that either $d_H(v_i) = 2$ or $d_H(v_i) \geq k-1$. Notice that either $d_H(v_0) = 1$ or $d_H(v_0) \geq k$ because if v_0 is not pendant then H has a matching-transversal of size at most $\max\{5, d_H(v_0)\}$ (by Lemmas 3.7 and 3.18) but we are assuming $\tau_{\mathrm{m}}(H) \geq k \geq 6$. Symmetrically, either $d_H(v_n) = 1$ or $d_H(v_n) \geq k$. Finally, all vertices of H different from $v_0, v_1, \ldots, v_n$ are vertices of degree 2 because no block of H is a rhombus or K_4. We conclude that $\delta_{\mathrm{h}}(H) \geq k-1$. Since $k-1 \geq 5$, Lemma 3.18 implies that $\tau_{\mathrm{m}}(H) \leq \delta_{\mathrm{h}}(H)$. Since we are assuming $\tau_{\mathrm{m}}(H) \geq k$, $\delta_{\mathrm{h}}(H) \geq k$. Thus, (ii) implies (iii) and the proof is complete. □

As we have proved Lemmas 3.22 and 3.23 and all the cases of Theorem 3.24, now Theorem 2.13 follows.

This, together with Theorem 2.12, implies Theorem 2.11, from which the main results of this work (Theorems 1.4 and 1.5) follow. It only remains to prove Theorem 2.14, i.e., to present the elementary linear-time recognition algorithm for matching-perfect graphs:

Proof (*of* Theorem 2.14)**.** We claim that there is an elementary linear-time algorithm that decides whether a given graph is a fat caterpillar and, if affirmative, computes a matching-transversal of minimum size. To begin with, we proceed as in the paragraph preceding the statement of Theorem 2.14 in order to either compute H_1, H_2, and H_3, or detect that H contains a bipartite claw. If the latter occurs, we can be certain that H is not a fat caterpillar and stop. Hence, without loss of generality, assume that H_1, H_2, and H_3 were successfully computed in linear time. If H_1 is a triangle and each vertex of H_1 has some neighbor in H outside H_1, then Theorem 2.2(iii), 3.23 imply that H is a fat caterpillar and the set of edges incident to any minimum hub of H is a matching-transversal of minimum size. Suppose now that H_2 is spanned by a 4-cycle C having at least two consecutive vertices that are adjacent in H to some vertex outside H_2. In this case, it is straightforward to determine whether or not H is a fat caterpillar thanks to Theorem 2.2(ii) and, if affirmative, compute a matching-transversal of minimum size in linear time by means of Lemma 3.22. Assume now that neither H_1 is a triangle such that each vertex of H_1 is adjacent in H to some vertex outside H_1, nor H_2 is spanned by a 4-cycle having at least two consecutive vertices adjacent in H to vertices outside H_2. Thus, by Theorem 2.2, H is a fat caterpillar if and only if H is a linear concatenation of basic two-terminal graphs where the K_4 links may occur only as the first and/or last links of the concatenation. Therefore, H is a fat caterpillar if and only if H_3 is a linear concatenation of edge, triangle, rhombus, and K_4 links where the K_4 links may occur only as the first/and or last link of the concatenation and no vertex of a rhombus link has a false twin of degree 2 in H. Equivalently, H is a fat caterpillar if and only if H_3 satisfies each of the following conditions:

(1) H_3 is connected.
(2) Each of the blocks of H_3 is K_2, K_3, $K_4 - e$, or K_4.
(3) Each block of H_3 has at most two cut-vertices.
(4) The cut-vertices of each $K_4 - e$ block are vertices of degree 2 in the block.
(5) Each K_4 block has at most one cut-vertex.
(6) Each cut-vertex of H_3 belongs to at most two blocks of H_3 that are not pendant edges.
(7) No vertex of a $K_4 - e$ block of H_3 of degree 2 in H has a false twin in H.

All these conditions can be easily verified in linear time once the blocks and the cut-vertices of H_3 are determined, which in turn can be done in linear time by performing a depth-first search [47]. Finally, if all the above conditions are met, H is a fat caterpillar containing no A and no net and a matching-transversal of H of minimum size can be determined in linear time as follows from the characterizations given in Lemmas 3.28–3.31.

Suppose now that we need to determine whether a given graph H is matching-perfect and assume, without loss of generality, that H is connected and has more than 6 vertices. We begin by deciding whether H is a fat caterpillar as in the preceding discussion. If H is found to be a fat caterpillar, we are done because we know that H is matching-perfect and stop. Therefore, assume without loss of generality that H is not a fat caterpillar. Hence, H is matching-perfect if and only if H is a matching-perfect graph containing a cycle of length $3k$ for some $k \geq 2$. Thus, by Lemma 3.11, if H is matching-perfect, then H_3 is a chordless cycle of length $3k$ for some $k \geq 2$. Conversely, if H_3 is a chordless cycle of length $3k$ for some $k \geq 2$, then H contains no bipartite claw (because H_3 contains no claw) and, moreover, H is matching-perfect by Theorem 1.5. This shows that we can decide in linear time whether H is matching-perfect. Finally, if there is any edge $e = uv$ of H_3 that is not hub-covered in H, then $E_H(u) \cup E_H(v)$ is a matching-transversal of H of minimum size by Lemma 3.17; otherwise, if v is any minimum hub v of H, then $E_H(v)$ is a matching-transversal of H of minimum size by Lemma 3.16. □

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