The Ordered Weighted Average in the Variance and the Covariance

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The ordered weighted average (OWA) is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. This paper analyzes the use of the OWA in the variance and the covariance. It presents several extensions by using a unified framework between the weighted average and the OWA. Furthermore, it also develops other generalizations with induced aggregation operators and by using quasi-arithmetic means. Several measures of correlation by using the OWA are introduced including a new type of Pearson coefficient. The paper ends with some numerical examples focused on the construction of interval and fuzzy numbers with the variance and the covariance.

1. INTRODUCTION

The variance and the covariance are basic concepts in statistics for measuring the dispersion of data. The variance is used when dealing with one set of variables, whereas the covariance is used with two sets. They are fundamentally related to the notion of distance and, as such, it has been at the center of virtually all data analysis applications.[1] Usually, the variance and the covariance are defined as an averaging process of the individual dispersions. The most common tools for doing so are the simple average and the weighted average (WA).

However, it is possible to introduce some additional information on the definition of weights by using more sophisticated tools. For example, the ordered weighted average (OWA)[2,3] can be used for that purpose. The OWA provides a parameterized family of aggregation operators between the minimum and the maximum. Since its

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INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 30, 985–1005 (2015)

View this article online at wileyonlinelibrary.com. • DOI 10.1002/int.21716

introduction, it has been studied and generalized by many authors. For example, Yager and Filev[4] introduced the induced OWA (IOWA) operator by using complex ordering processes assessed with order-inducing variables. Fodor et al.[5] presented the quasi-arithmetic OWA (Quasi-OWA) operator that generalized a wide range of aggregation operators including the OWA and the quadratic OWA operator. Merigó and Gil-Lafuente[6] suggested a unified framework of the previous approaches named induced generalized OWA operator. Some other authors[7–9] have studied several extensions under imprecise information that can be assessed with interval and fuzzy numbers.

The aim of this paper is to analyze the use of the OWA operator in the variance and in the covariance. The main advantage of doing so is that the variance can be studied considering a wide range of scenarios from the minimum to the maximum, that is, from the most optimistic to the most pessimistic scenario. Currently, the main studies in this direction have been developed by Yager[10,11] with the use of the OWA and the IOWA operator in the variance and by Merigó[12] who focused on the use of the ordered weighted averaging–weighted average (OWAWA) and the induced ordered weighted averaging–weighted average (IOWAWA) operator. This work reviews these approaches and suggests some additional ones by using the weighted OWA (WOWA),[13] the hybrid average,[14] and the immediate weights.[15–17] Furthermore, the use of generalized aggregation operators is also introduced by using quasi-arithmetic means. Therefore, it is possible to consider a wide range of particular cases including quadratic and cubic aggregations.

Some additional results are also presented by using the Pearson coefficient (PC). Moreover, several analyses for the construction of interval and fuzzy numbers with the variance and covariance are also developed under the framework of the OWA operator. A numerical example is also presented to understand the main advantages of the new approach.

This paper is organized as follows. Section 2 reviews the OWA operator, some of its extensions, and its implementation in the variance and the covariance. Section 3 studies the use of the OWAWA operator and some related extensions in the variance. Section 4 develops a similar analysis with the covariance and Section 5 studies some measures of correlation with the OWAWA operator. Section 6 presents some numerical examples focused on the construction of interval numbers with the variance and the covariance. Section 7 summarizes the main results and conclusions of the paper.

2. PRELIMINARIES

This section presents a brief description of some basic aggregation operators, the variance, the covariance, and the use of the OWA operator in the variance and the covariance.

2.1. Aggregation Operators

2.1.1. The Weighted Average

The WA is one of the most well-known aggregation operators. It has been used in a wide range of applications including statistics, economics, and engineering. It can be defined as follows.

DEFINITION 1. *A WA operator of dimension n is a mapping WA:* $R^n \rightarrow R$ *that has an associated weighting vector V, with* $v_i \in [0, 1]$ *and* $\sum_{i=1}^{n} v_i = 1$ *, such that*

$$\mathrm{WA}\,(a_1,\ \ldots, a_n) = \sum_{i=1}^{n} v_i = 1, \tag{1}$$

where a_i *represents the argument variable.*

The WA operator satisfies the common properties of aggregation operators. For further reading on different extensions and generalizations (see, e.g.,Refs. 18–21).

An important issue when integrating the WA with the OWA operator is that one of them has to adapt his initial ordering to the other one. Therefore, it is useful to see how the WA would be formulated if it has to adapt his ordering to the OWA operator.[12] In this case, it can be defined as follows:

$$\mathrm{WA}\,(a_1,\ \ldots, a_n) = \sum_{j=1}^{n} v_j b_j, \tag{2}$$

where b_j is the jth smallest argument a_i and v_j is the weight v_i reordered according to the reordering of the arguments a_i in the form of b_j such that $v_j \in [0, 1]$ and $\sum_{j=1}^{n} v_j = 1$.

Obviously, Equations 1 and 2 provides the same aggregated result although the reordering process of each method is different. This is a key feature for unifying the OWA with the WA and will be explained in Section 2.1.3.

2.1.2. The OWA Operator

The OWA operator[2] is an aggregation operator that provides a parameterized family of aggregation operators between the minimum and the maximum. It can be defined as follows.

DEFINITION 2. *An OWA operator of dimension n is a mapping OWA:* $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n with* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, such that*

$$\mathrm{OWA}\,(a_1,\ \ldots, a_n) = \sum_{j=1}^{n} w_j b_j, \tag{3}$$

where b_j *is the jth smallest of the* a_i*.*

Note that different properties could be studied including the distinction between descending and ascending orders, different measures for characterizing the weighting vector and different families of OWA operators. For further reading, refer to Refs. 3,7,22, and [23].

In most of the OWA literature, the arguments are reordered according to an established weighting vector. However, it is also possible to reorder the weighting vector according to the initial positions of the arguments a_i,[24] that is

$$\text{OWA}\,(a_1,\ \ldots, a_n) = \sum_{i=1}^{n} a_i w_i, \tag{4}$$

where w_i is the ith weight w_j reordered according to the positions of the a_i.

2.1.3. The OWAWA Operator

The OWAWA operator is an aggregation operator that integrates the OWA operator and the WA in the same formulation and considering the degree of importance that each concept has in the analysis. It can be defined as follows.

DEFINITION 3. *An OWAWA operator of dimension n is a mapping OWAWA:* $R^n \rightarrow R$ *that has an associated weighting vector W of dimension n such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$, *according to the following formula:*

$$\text{OWAWA}\,(a_1,\ \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{5}$$

where b_j *is the jth smallest of the* a_i, *each argument* a_i *has an associated weight* v_i *with* $\sum_{i=1}^{n} v_i = 1$ *and* $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ *with* $\beta \in [0, 1]$ *and* v_j *is the weight* v_i *ordered according to* b_j, *that is, according to the jth smallest of the* a_i.

As we can see, if $\beta = 1$, *we get the OWA operator and if* $\beta = 0$, *the WA. The OWAWA operator accomplishes similar properties than the usual aggregation operators. Note that we can distinguish between descending and ascending orders, extend it by using mixture operators, and so on.*

Observe that Equation (5) *has been presented adapting the ordering of the WA to the OWA operator. However, it is also possible to formulate the OWAWA operator integrating the ordering of the OWA operator to the WA as*

$$\text{OWAWA}\,(a_1,\ \ldots, a_n) = \sum_{i=1}^{n} \hat{v}_i a_i, \tag{6}$$

where each argument a_i *has an associated weight* v_i *with* $\sum_{i=1}^{n} v_i = 1$ *and* $v_i \in [0, 1]$, $\hat{v}_i = \beta w_i + (1-\beta) v_i$ *with* $\beta \in [0, 1]$ *and* w_i *is the weight* w_j *ordered according to the ordering of the arguments* a_i.

2.1.4. The IOWAWA Operator

The IOWAWA operator is a model that unifies the IOWA operator and the WA in the same formulation and considering a complex reordering process based on order-inducing variables. Therefore, both concepts can be seen as a particular case of a more general one. It can also be seen as a unification between decision-making problems under uncertainty (with IOWA operators) and under risk (with probabilities). Note that the motivation for using this approach instead of the OWAWA operator is especially useful when dealing with complex interpretations of the information. It can be defined as follows.

DEFINITION 4. *An IOWAWA operator of dimension n is a mapping IOWAWA: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, according to the following formula:*

$$\text{IOWAWA}\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle\right) = \sum_{j=1}^{n} \hat{v}_j b_j, \tag{7}$$

where b_j is the a_i value of the IOWAWA pair $\langle u_i, a_i\rangle$ having the jth smallest u_i, u_i is the order-inducing variable, and a_i is the argument variable, each argument a_i has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to b_j, that is, according to the jth smallest u_i.

Note that it is also possible to formulate the IOWAWA operator adapting the reordering of the IOWA operator to the WA as it has been shown in Equation 6 for the OWAWA operator. If $\beta = 1$, it becomes the IOWA operator[4] and if $\beta = 0$, the WA (for further reading, see Merigó[12]).

2.2. Variance and Covariance

2.2.1. The Variance and the Covariance

The variance and the covariance are two statistical measures of variability. The variance measures how far the numbers lie from the mean. It has been applied in a wide range of problems and it can be defined as:

$$\text{Var}\left(a_1, \ldots, a_n\right) = \sum_{i=1}^{n} v_i (a_i - \mu)^2, \tag{8}$$

where a_i is the argument variable, μ is the average, and each argument $(a_i - \mu)^2$ has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$.

Note that in this formulation it is implicitly assumed that the variance uses a WA. However, it is also possible to consider it with arithmetic means, when $v_i = 1/n$, for all i, in the following way:

$$\text{Var}\left(a_1, \ldots, a_n\right) = \frac{1}{n} \sum_{i=1}^{n} (a_i - \mu)^2. \tag{9}$$

Furthermore, the variance is often transformed into the standard deviation (SD) to obtain a more representative measure of dispersion as

$$\mathrm{SD} = \sqrt{\sum_{i=1}^{n} v_i (a_i - \mu)^2}. \tag{10}$$

The covariance is a measure of how much two variables change together. It can be defined as follows:

$$\mathrm{Cov}\left(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle\right) = \sum_{i=1}^{n} v_i (x_i - \mu)(y_i - \nu), \tag{11}$$

where x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, and μ and ν are the average of the sets X and Y, respectively, each argument$(x_i - \mu)$ $(y_i - \nu)$ has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$.

In the covariance, it is also possible to consider the situation where all the weights are equal, that is, $v_i = 1/n$, for all i. Note that the variance is a particular case of the covariance when the two set of variables are equal $\mathrm{Cov}(X, X) = \mathrm{Var}(X)$.

2.2.2. *Variance and Covariance with the OWA Operator*

The OWA operator can also be used in the variance and covariance providing two measures that provide a parameterized family of measures that range from the minimum dispersion to the maximum one. The use of the OWA operator in the variance was suggested by Yager[10] and it can be defined as follows:

$$\mathrm{Var} - \mathrm{OWA}\left(a_1, \ldots, a_n\right) = \sum_{j=1}^{n} w_j D_j, \tag{12}$$

where D_j is the jth smallest of the $(a_i - \mu)^2$, a_i is the argument variable, μ is the average (in this case, the OWA operator), $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

The Var-OWA accomplishes similar properties than other OWA operators including commutativity, idempotency, monotonicity, and the boundary condition. Since it is a measure of variability, it also accomplishes similar properties than the OWA distance[25] such as reflexivity and commutativity of a distance measure. Moreover, it includes the classical variance as a particular case when $w_j = 1/n$ for all i.

The Var-OWA can be extended by using induced aggregation operators[11] to deal with complex reordering processes as follows:

$$\mathrm{Var} - \mathrm{IOWA}\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle\right) = \sum_{j=1}^{n} w_j D_j, \tag{13}$$

where D_j is the $(a_i - \mu)^2$ value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth smallest u_i, u_i is the order-inducing variable, a_i is the argument variable, μ is the average (in this case, the OWA operator), $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

The Var-IOWA operator also accomplishes similar properties than other IOWA operators[4,26] such as commutativity, the boundary condition, idempotency, monotonicity, and reflexivity. Note that in the case of ties between order-inducing variables, the aggregation needs an adjustment. Yager and Filev[4] suggested replacing the arguments of the tied order-inducing variables by their average. Thus, in the case of ties in the Var-IOWA, the arguments $(a_i - \mu)^2$ with tied inducing variables can be replaced by the average, that is, $[(a_i - \mu)^2 + (a_k - \mu)^2]/2$. Note that the Var-IOWA includes the classical variance with the WA when the reordering of the order-inducing variables is equal to the ordering of the arguments a_i.

The covariance can also be formulated using a similar methodology as shown for the variance. Note that the use of the OWA operator in the covariance was suggested by Merigó[12] as a particular case of the OWAWA operator. It is formulated as follows:

$$\text{Cov} - \text{OWA}\,(X, Y) = \sum_{j=1}^{n} w_j K_j, \tag{14}$$

where K_j is the jth smallest of the $(x_i - \mu)(y_i - \nu)$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, y_i is the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, and μ and ν are the average (or the OWA operator) of the sets X and Y, respectively, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Furthermore, it is also possible to extend this approach by using induced aggregation operators[12] as follows:

$$\text{Cov} - \text{IOWA}\,(U,\ X, Y) = \sum_{j=1}^{n} w_j K_j, \tag{15}$$

where K_j is the $(x_i - \mu)(y_i - \nu)$ value of the IOWA triplet $\langle u_i, x_i, y_i \rangle$, having the jth smallest u_i, and u_i is the order-inducing variable of the set of elements $U = \{u_1, \ldots, u_i, \ldots, u_n\}$.

Note that both the Cov-OWA and Cov-IOWA accomplish similar properties than the usual OWA and IOWA operator.

3. OWAWA OPERATORS IN THE VARIANCE

3.1. The Var-OWAWA Operator

Recently, Merigó[12,27] has suggested a new aggregation approach that integrates the OWA operator with the WA, considering the degree of importance that each concept has in the specific problem considered. He called it the OWAWA operator.

He also briefly showed that it is possible to use the OWAWA in the variance as follows:

$$\mathrm{Var} - \mathrm{OWAWA}\,(a_1,\ \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j D_j, \tag{16}$$

where D_j is the jth smallest of the $(a_i - \mu)^2$, a_i is the argument variable, μ is the average (in this case, the OWAWA operator), $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, each argument$(a_i - \mu)^2$ has an associated weight (WA) v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight (WA) v_i ordered according to D_j, that is, according to the jth smallest of the $(a_i - \mu)^2$.

Note that the Var-OWAWA operator can also be formulated separating the part that affects the OWA operator and the part concerning the WA as:

$$\mathrm{Var} - \mathrm{OWAWA}\,(a_1,\ \ldots, a_n) = \beta \sum_{j=1}^{n} w_j D_j + (1 - \beta) \sum_{i=1}^{n} v_i (a_i - \mu)^2. \tag{17}$$

Observe that this formulation can be summarized with the following expression:

$$\mathrm{Var} - \mathrm{OWAWA} = \beta \times \mathrm{Var} - \mathrm{OWA} + (1 - \beta) \times \mathrm{Var} - \mathrm{WA}. \tag{18}$$

Moreover, it is also possible to adapt the reordering of the OWA operator to the WA as:

$$\mathrm{Var} - \mathrm{OWAWA}\,(a_1,\ \ldots, a_n) = \sum_{i=1}^{n} \hat{v}_i (a_i - \mu)^2, \tag{19}$$

where $\hat{v}_i = \beta w_i + (1 - \beta) v_i$ with $\beta \in [0, 1]$ and w_i is the weight w_j ordered according to the ordering of the arguments a_i and j is the ordering of the jth smallest $(a_i - \mu)^2$.

Obviously, once we have the variance, it is straightforward to obtain the SD with the OWAWA operator by using

$$\mathrm{SD} = \sqrt{\sum_{j=1}^{n} \hat{v}_j D_j}. \tag{20}$$

The Var-OWAWA operator accomplishes similar properties as the OWAWA operator including the boundary and semiboundary condition, monotonicity, and idempotency. It is worth noting that if $\beta = 1$, it becomes the Var-OWA operator and if $\beta = 0$, the classical variance with the WA. The more of β located to the top, the more importance we give to the OWA aggregation and vice versa.

An important issue when analyzing the Var-OWAWA operator is the use of measures for characterizing the weighting vector. In the OWA literature,[2,27–29] there are several measures for doing so including the or-ness measure, the entropy of dispersion, the balance operator, and the divergence. The or-ness measure for an OWAWA operator can be formulated as follows:

$$\alpha(\hat{V}) = \beta \sum_{j=1}^{n} w_j \left(\frac{j-1}{n-1}\right) + (1-\beta) \sum_{j=1}^{n} v_j \left(\frac{j-1}{n-1}\right). \tag{21}$$

It is straightforward to calculate the and-ness measure by using the dual: andness$(\hat{V}) = 1 - \alpha(\hat{V})$. Note that $\alpha \in [0, 1]$.

There are different methods for defining the entropy of dispersion.[2,27,30,31] In this study, the combined entropy is defined as follows:

$$R(\hat{V}) = -\left(\beta \sum_{j=1}^{n} w_j \ln(w_j) + (1-\beta) \sum_{i=1}^{n} v_j \ln(v_j)\right). \tag{22}$$

Note that if $\beta = 1$, this measure becomes the Yager[2] entropy of dispersion and if $\beta = 0$, the Shannon[31] entropy.

The balance operator can be formulated as follows:

$$\text{Bal}(\hat{V}) = \beta \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1}\right) w_j + (1-\beta) \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1}\right) v_j. \tag{23}$$

And the divergence of $\hat{V}$ is given as

$$\text{Div}(\hat{V}) = \beta \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2\right) + (1-\beta) \left(\sum_{j=1}^{n} v_j \left(\frac{n-j}{n-1} - \alpha(V)\right)^2\right). \tag{24}$$

Another interesting feature of the Var-OWAWA is that it includes a wide range of particular cases by selecting a different manifestation in the weighting vector. Among others, the following cases are included as particular cases.

Moreover, note that in the literature there are other methods for integrating the OWA operator and the WA in the same formulation. The main approaches are the WOWA, the hybrid average, and the immediate weights[16] that are based on the immediate probabilities.[15–17] Thus, these approaches could also be considered when constructing new types of variance measures. By using the WOWA operator, the variance can be expressed as follows.

Let P and W be two weighting vectors of dimension n $[P = (p_1, p_2, \ldots, p_n)]$, $[W = (w_1, w_2, \ldots, w_n)]$, such that $p_i \in [0, 1]$ and $\sum_{i=1}^{n} p_i = 1$, and $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$. In this case, a mapping Var-WOWA: $R^n \rightarrow R$ is a Var-WOWA operator of dimension n if:

$$\text{Var} - \text{WOWA}\,(a_1, \ldots, a_n) = \sum_{i=1}^{n} \omega_i D_{\sigma(i)}, \tag{25}$$

where $\{\sigma(1), \ldots, \sigma(n)\}$ is a permutation of $\{1, \ldots, n\}$ such that $D_{\sigma(i-1)} \geq D_{\sigma(i)}$ for all $i = 2, \ldots, n$ (i.e., $D_{\sigma(i)}$ is the ith largest in the collection $D_1, \ldots, D_n$ and $D_i = (a_i - \mu)^2$), and the weight ω_i is defined as:

$$\omega_i = w^* \left(\sum_{j \leq i} p_{\sigma(j)} \right) - w^* \left(\sum_{j < i} p_{\sigma(j)} \right), \tag{26}$$

with w^* a monotonically increasing function that interpolates the points $(i/n, \sum_{j \leq i} w_j)$ together with the point (0, 0). Note that it is necessary for w^* to be a straight line when the points are interpolated in this way.

By using the hybrid average, it becomes the hybrid averaging variance (Var-HA) and it can be formulated as follows. A Var-HA operator of dimension n is a mapping Var-HA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$\text{Var} - \text{HA}\,(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j D_j, \tag{27}$$

where D_j is the jth largest of the $\hat{a}_i (\hat{a}_i = n\omega_i (a_i - \mu)^2$, i = 1,2, . . . ,$n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighting vector of the a_i, with$\omega_i \in [0, 1]$ and the sum of the weights is 1.

Finally, the immediate WA may form the immediate variance (Var-IWA) in the following way. A Var-IWA operator of dimension n is a mapping Var-IWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$\text{Var} - \text{IWA}\,(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j D_j, \tag{28}$$

where D_j is the jth largest of the $(a_i - \mu)^2$, each a_i has associated a weight v_i, v_j is the weight v_i ordered according to b_j, and $\hat{v}_j = (w_j v_j / \sum_{j=1}^{n} w_j v_j)$.

It is worth noting that there are other methods for dealing with OWA operators and WAs in the same formulation that could be considered in this analysis.[2,3,32]

3.2. Induced and Generalized Aggregation Operators in the Variance

The IOWAWA operator can also be implemented in the variance. Thus, a more general formulation is developed that may consider complex reordering processes. This is important because many times the highest or lowest result do not need to go first or last in the aggregation.

The variance of a population (discrete case) using the IOWAWA operator can be expressed with the following formulation:

$$\mathrm{Var} - \mathrm{IOWAWA}\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ \ldots, \langle u_n, a_n\rangle\right) = \sum_{j=1}^{n} \hat{v}_j D_j, \tag{29}$$

where D_j is the $(a_i - \mu)^2$ value of the IOWAWA pair $\langle u_i, a_i\rangle$ having the jth smallest u_i, u_i is the order-inducing variable, a_i is the ith argument variable of the set X, and μ is the IOWAWA operator.

Obviously, it is straightforward to obtain the SD with the IOWAWA operator by using the following expression:

$$\mathrm{SD} = \sqrt{\sum_{j=1}^{n} \hat{v}_j D_j}. \tag{30}$$

In the case of ties between order-inducing variables in the reordering process of the arguments $(a_i - \mu)^2$, it is recommended the use of the average.[4] Note that we can also develop similar families of Var-IOWAWA operators to those explained for the Var-OWAWA operator.

More generally, it is possible to extend them by using quasi-arithmetic means. The main advantage of this approach is that it can represent a wide range of aggregation operators based on the variance.

$$\begin{aligned}\mathrm{Var} - \mathrm{Quasi} - \mathrm{OWAWA}\left(a_1, \ \ldots, a_n\right) = {} & \beta g^{-1}\left(\sum_{j=1}^{n} w_j g(D_j)\right) \\ & +(1-\beta) h^{-1}\left(\sum_{i=1}^{n} v_i h((a_i - \mu)^2)\right),\end{aligned} \tag{31}$$

where g and h are strictly continuous monotonic functions. In Table I, some of the main particular cases of the Var-Quasi-OWAWA operators are presented.

Conventions for geometric-Var-OWAWA do not consider those arguments with $(a_i - \mu)^2 = 0$. Furthermore, observe that many other extensions could be developed by using induced aggregation operators[4,6,33] and many other approaches.[3,34–36]

Table I. Families of Var-OWA operators

Weights	Particular case
$w_j = 1/n$ and $v_i = 1/n$, for all i, j	Simple variance
$w_1 = 1$ and $w_j = 0$ for all $j \neq 1$	Minimum weighted variance
$w_n = 1$ and $w_j = 0$ for all $j \neq n$	Maximum weighted variance
$w_j = 1/n$, for all j	Arithmetic weighted variance
$v_i = 1/n$, for all i	Arithmetic Var-OWA
$w_k = 1$ and $w_j = 0$ for all $j \neq k$	Step-Var-OWA weighted variance
$w_1 = 1-\alpha$, $w_n = \alpha$ and $w_j = 0$ for all $j \neq 1, n$	Hurwicz weighted variance
$w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m, jk$	Window-Var-OWA weighted variance
If n is odd, we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others	Median-Var-OWA weighted variance
If n is even we assign, for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_j = 0$ for all others	Median-Var-OWA weighted variance
$w_1 = w_n = 0$, and for all others $w_j = 1/(n-2)$	Olympic-Var-OWA weighted variance

By using the IOWAWA operator, it is formed the Var-Quasi-IOWAWA operator as

$$\text{Var} - \text{Quasi} - \text{IOWAWA} = \beta \times \text{Var} - \text{Quasi} - \text{IOWA} + (1-\beta) \times \text{Var} - \text{Quasi} - \text{WA}. \quad (32)$$

4. OWAWA OPERATORS IN THE COVARIANCE

The OWAWA operator can also be implemented in the covariance.[12] Thus, it is possible to consider a covariance that takes into account the importance of the variables and the attitudinal character of the decision maker. It can be represented by using the OWAWA operator as follows:

$$\text{Cov} - \text{OWAWA}\,(X, Y) = \sum_{j=1}^{n} \hat{v}_j K_j, \quad (33)$$

where K_j is the jth smallest of the $(x_i - \mu)(y_i - \nu)$, x_i is the argument variable of the first set of elements $X = \{x_1, \ldots, x_n\}$, and y_i the argument variable of the second set of elements $Y = \{y_1, \ldots, y_n\}$, μ and ν are the OWAWA operator of the sets X and Y, respectively, $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, each argument$(x_i - \mu)(y_i - \nu)$ has an associated weight v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0, 1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0, 1]$ and v_j is the weight v_i ordered according to K_j, that is, according to the jth smallest of the $(x_i - \mu)(y_i - \nu)$.

Table II. Families of Var-Quasi-OWAWA operators

Function	Particular case
$g = H^{\lambda}$ and $h = ((a_i - \mu)^2)^{\delta}$	Generalized Var-OWAWA operator
$g = H$ and $h = (a_i - \mu)^2$	Var-OWAWA operator
$g = H^2$ and $h = ((a_i - \mu)^2)^2$	Quadratic Var-OWAWA
$g \to H^0$ and $h \to ((a_i - \mu)^2)^0$	Geometric Var-OWAWA
$g = H^{-1}$ and $h = ((a_i - \mu)^2)^{-1}$	Harmonic Var-OWAWA
$g = H^3$ and $h = ((a_i - \mu)^2)^3$	Cubic Var-OWAWA
$g = H$ and $h = ((a_i - \mu)^2)^2$	Var-OWA weighted quadratic variance
$g = H^2$ and $h = (a_i - \mu)^2$	Ordered weighted quadratic averaging variance weighted variance
$g = H$ and $h = ((a_i - \mu)^2)^3$	Var-OWA weighted cubic variance
$g = H^2$ and $h = ((a_i - \mu)^2)^3$	Ordered weighted quadratic averaging variance weighted cubic variance

The Cov-OWAWA operator can also be extended by using induced aggregation operators. In this case, it is formed the Cov-IOWAWA operator, which is defined as follows:

$$\text{Cov} - \text{IOWAWA}\,(U,\ X,\ Y) = \sum_{j=1}^{n} \hat{v}_j K_j, \tag{34}$$

where K_j is the $(x_i - \mu)(y_i - \nu)$ value of the IOWAWA triplet $\langle u_i, x_i, y_i \rangle$ having the jth smallest u_i and u_i is the order-inducing variable of the set of elements $U = \{u_1, \ldots, u_i, \ldots, u_n\}$.

More generally, it is possible to extend this approach by using quasi-arithmetic means. The main advantage of this approach is that we can represent a wide range of aggregation operators based on the covariance.

$$\text{Cov} - \text{Quasi} - \text{OWAWA}\,(X, Y) = \beta \times \text{Cov} - \text{Quasi} - \text{OWA} \\ +(1 - \beta) \times \text{Cov} - \text{Quasi} - \text{WA}, \tag{35}$$

where g and h are strictly continuous monotonic functions. Observe that a similar analysis as it has been done for the Var-Quasi-OWAWA operator in Table II could also be developed for the Cov-Quasi-OWAWA operator obtaining a wide range of particular cases including geometric, quadratic, and harmonic aggregations.

5. MEASURES OF CORRELATION WITH OWAWA OPERATORS

A wide range of measures of correlation that use the variance and the covariance could be extended with the new approach developed in this paper. Among others, it is worth noting the PC. The PC with the OWAWA (PC – OWAWA) is formulated as follows:

$$\text{PC} - \text{OWAWA} = \frac{\text{Cov} - \text{OWAWA}(X, Y)}{\sqrt{\text{Var} - \text{OWAWA}(X) \times \text{Var} - \text{OWAWA}(Y)}}. \tag{36}$$

The PC – OWAWA is 1 if it presents an increasing linear relationship, −1 in a decreasing linear relationship, and 0 if the variables *X* and *Y* are independent.

Next, we could study the IOWAWA covariance matrix and a lot of other aspects. For example, the PC with the IOWAWA (PC – IOWAWA) can be formulated in the following way:

$$\text{PC} - \text{IOWAWA} = \frac{\text{Cov} - \text{IOWAWA}(X, Y)}{\sqrt{\text{Var} - \text{IOWAWA}(X) \times \text{Var} - \text{IOWAWA}(Y)}}. \tag{37}$$

The PC – IOWAWA is 1 in the case of an increasing linear relationship and −1 in a decreasing linear relationship. If the variables *X* and *Y* are independent, the PC – IOWAWA is 0.

Note that if $\beta = 1$, they become the PC-OWA and the PC-IOWA and if $\beta = 0$, the classical PC with the WA. The higher β, the more importance is given to the OWA and IOWA aggregation and vice versa. More specifically, for the OWA operator the PC can be expressed as follows:

$$\text{PC} - \text{OWA} = \frac{\text{Cov} - \text{OWA}(X, Y)}{\sqrt{\text{Var} - \text{OWA}(X) \times \text{Var} - \text{OWA}(Y)}}. \tag{38}$$

And for the IOWA operator:

$$\text{PC} - \text{IOWA} = \frac{\text{Cov} - \text{IOWA}(X, Y)}{\sqrt{\text{Var} - \text{IOWA}(X) \times \text{Var} - \text{IOWA}(Y)}}. \tag{39}$$

Finally, let us present a similar extension by using the Var-Quasi-OWAWA and the Cov-Quasi-OWAWA operator. In this case, we get the PC with the Quasi-OWAWA operator (PC-Quasi-OWAWA):

$$\text{PC} - \text{Quasi} - \text{OWAWA} = \frac{\text{Cov} - \text{Quasi} - \text{OWAWA}(X, Y)}{\sqrt{\text{Var} - \text{Quasi} - \text{OWAWA}(X) \times \text{Var} - \text{Quasi} - \text{OWAWA}(Y)}}. \tag{40}$$

This formulation provides a general representation that includes a wide range of particular cases including arithmetic, quadratic, and harmonic aggregations.

6. CONSTRUCTION OF INTERVAL AND FUZZY NUMBERS IN THE VARIANCE AND THE COVARIANCE

By using the OWA and the OWAWA operator, the aggregation process can provide a parameterized family of aggregation operators between the minimum and the maximum variance. Thus, it is possible to consider all the possible individual dispersions and select the one that it is in closest accordance with our interests and beliefs. As it was shown by Merigó,[27] with the OWA operator we can create many intervals including triplets, quadruplets, and quintuplets. Following this methodology, with the Var-OWA we can also create many interval numbers as shown in Table III.

Table III. Interval numbers formed with the Var-OWA operator

Interval number	Formulation
Var-OWA	
Triplet	[Min, Var-OWA, Max]
Quadruplet	[Min, Var-OWA$_*$, Var-OWA*, Max]
Quintuplet	[Min, Var-OWA$_*$, Var-OWA, Var-OWA*, Max]
Var-IOWA	
Triplet	[Min, Var-IOWA, Max]
Quadruplet	[Min, Var-IOWA$_*$, Var-IOWA*, Max]
Quintuplet	[Min, Var-IOWA$_*$, Var-IOWA, Var-IOWA*, Max]

Table IV. Fuzzy numbers built with the Var-OWA operator

Fuzzy number	Formulation
Var-OWA	
Triangular FN	[Min + (Var-OWA – Min) $\times$ α, Max – (Max – Var-OWA) $\times$ α]
Trapezoidal FN	[Min + (Var-OWA$_*$ – Min) $\times$ α, Max – (Max – Var-OWA*) $\times$ α]
Var-IOWA	
Triangular FN	[Min + (Var-IOWA – Min) $\times$ α, Max – (Max – Var-IOWA) $\times$ α]
Trapezoidal FN	[Min + (Var-IOWA$_*$ – Min) $\times$ α, Max – (Max – Var-IOWA*) $\times$ α]

Note that with the Var-IOWA operator the assumption is that the information moves between the minimum and the maximum but the attitudinal character is more complex and cannot be measured with a simple numerical order.

The interval numbers can be studied in a deeper way using the knowledge of a fuzzy number where it is also considered the possibility that the internal values between the minimum and the maximum will occur.[37] For example, by using the α-cut representation in [0, 1], we get the expressions shown in Table IV.

This framework can also be extended to the Var-OWAWA operator to consider the importance of the variables in the analysis together with the attitudinal character. Note that in this case we also find semibounds when the OWA operator is minimum and maximum but the WA is considered. It is worth noting that in this case subjective interval numbers (SIN) are formed. They can be constructed as shown in Table V. Note that Table V also includes some representative fuzzy numbers that in this formulation becomes subjective fuzzy numbers. Moreover, observe that a distinction is made between quintuplets formed with the OWA (quintuplets) and quintuplets formed with the WA (quintuplets*).

In a similar way, it is also possible to construct a wide range of interval and fuzzy numbers by using the covariance with the OWA and the OWAWA operator as shown in Table VI.

Next, let us look into some numerical examples when dealing with OWA and OWAWA operators in the variance and in the covariance. Assume the following data shown in Table VII regarding an investment problem with five alternatives $A = \{A_1, A_2, A_3, A_4, A_5\}$ and five states of nature $S = \{S_1, S_2, S_3, S_4, S_5\}$. Note that for the OWA operator the weighting vector is $W = (0.1, 0.2, 0.2, 0.2, 0.3)$ and for the WA

Table V. Interval and fuzzy numbers formed with the Var-OWAWA operator

Construction	Formulation
Var-OWAWA	
Triplet	[Min, Var-OWAWA, Max]
Quadruplet	[Min, Var-OWAWA$_*$, Var-OWAWA*, Max]
Quintuplet	[Min, Var-OWAWA$_*$, Var-OWAWA, Var-OWAWA*, Max]
Quintuplet*	[Min, Min-Var-WA, Var-OWAWA, Max-Var-WA, Max]
Triangular FN	[Min + (Var-OWAWA – Min) $\times\ \alpha$, Max – (Max – Var-OWAWA) $\times\ \alpha$]
Trapezoidal FN	[Min + (Var-OWAWA$_*$ – Min) $\times\ \alpha$, Max – (Max – Var-OWAWA*) $\times\ \alpha$]
Var-IOWAWA	
Triplet	[Min, Var-IOWAWA, Max]
Quadruplet	[Min, Var-IOWAWA$_*$, Var-IOWAWA*, Max]
Quintuplet	[Min, Var-IOWAWA$_*$, Var-IOWAWA, Var-IOWAWA*, Max]
Quintuplet*	[Min, Min-Var-WA, Var-IOWAWA, Max-Var-WA, Max]
Triangular FN	[Min + (Var-IOWAWA – Min) $\times\ \alpha$, Max – (Max – Var-IOWAWA) $\times\ \alpha$]
Trapezoidal FN	[Min + (Var-IOWAWA$_*$ – Min) $\times\ \alpha$, Max – (Max – Var-IOWAWA*) $\times\ \alpha$]

we assume a simple average where all the states of nature have the same importance 1/5. In this example, the OWA operator has a degree of importance of 40%.

With this information, we can calculate the variance with the OWA and the OWAWA operator and build some triplets and triangular fuzzy numbers (TFN). The results are shown in Table VIII. Observe that we use the average μ to calculate the individual dispersions in the variance and the covariance.

Next, we can also analyze these data by comparing the different investments through the covariance. The results are shown in Table IX.

As we can see, this analysis permits us to consider the relation between the different investments to see if they are similar or not. The example has been adapted so all the alternatives provide the same average. However, each investment has a different dispersion concerning the variance. In general terms, A_3 seems to be the alternative with the highest dispersion, whereas A_5 seems to be the investment with the lowest one. The covariance permits to show the dispersion between the alternatives.

Finally, to provide a more complete picture of the information found in the calculation of the Var-OWA and Var-OWAWA, let us analyze their triplets shown in Table VIII by using a box plot system.[38,39] Thus, the information can be classified from the minimum to the maximum and also considering the position of the central values. Note that in the literature there are many different methodologies when building a box plot such as the use of a violin plot.[40] The results are shown in Figure 1.

As we can see, the minimum and the maximum appear both in the triplets and in the box plot. The differences appear in the central values where the box plot shows the median and the first and third quarter, whereas the triplet uses the Var-OWA and the Var-OWAWA. The main advantage of both methodologies is that they do not loose information in the analysis so it is possible to consider all the potential scenarios that may occur in the uncertain future and select the one in closest accordance with our interests.

Table VI. Interval and fuzzy numbers formed with the covariance

Construction	Formulation
Cov-OWA	
Triplet	[Min, Cov-OWA, Max]
Quadruplet	[Min, Cov-OWA$_*$, Cov-OWA*, Max]
Quintuplet	[Min, Cov-OWA$_*$, Cov-OWA, Cov-OWA*, Max]
Triangular FN	[Min + (Cov-OWA – Min) × α, Max – (Max – Cov-OWA) × α]
Trapezoidal FN	[Min + (Cov-OWA$_*$ – Min) × α, Max – (Max – Cov-OWA*) × α]
Cov-IOWA	
Triplet	[Min, Cov-IOWA, Max]
Quadruplet	[Min, Cov-IOWA$_*$, Cov-IOWA*, Max]
Quintuplet	[Min, Cov-IOWA$_*$, Cov-IOWA, Cov-IOWA*, Max]
Triangular FN	[Min + (Cov-IOWA – Min) × α, Max – (Max – Cov-IOWA) × α]
Trapezoidal FN	[Min + (Cov-IOWA$_*$ – Min) × α, Max – (Max – Cov-IOWA*) × α]
Cov-OWAWA	
Triplet	[Min, Cov-OWAWA, Max]
Quadruplet	[Min, Cov-OWAWA$_*$, Cov-OWAWA*, Max]
Quintuplet	[Min, Cov-OWAWA$_*$, Cov-OWAWA, Cov-OWAWA*, Max]
Quintuplet*	[Min, Min-Cov-WA, Cov-OWAWA, Max-Cov-WA, Max]
Triangular FN	[Min + (Cov-OWAWA – Min) × α, Max – (Max – Cov-OWAWA) × α]
Trapezoidal FN	[Min + (Cov-OWAWA$_*$ – Min) × α, Max – (Max – Cov-OWAWA*) × α]
Cov-IOWAWA	
Triplet	[Min, Cov-IOWAWA, Max]
Quadruplet	[Min, Cov-IOWAWA$_*$, Cov-IOWAWA*, Max]
Quintuplet	[Min, Cov-IOWAWA$_*$, Cov-IOWAWA, Cov-IOWAWA*, Max]
Quintuplet*	[Min, Min-Cov-WA, Cov-IOWAWA, Max-Cov-WA, Max]
Triangular FN	[Min + (Cov-IOWAWA – Min) × α, Max – (Max – Cov-IOWAWA) × α]
Trapezoidal FN	[Min + (Cov-IOWAWA$_*$ – Min) × α, Max – (Max – Cov-IOWAWA*) × α]

Table VII. Initial information

	S_1	S_2	S_3	S_4	S_5	μ	OWA	OWAWA
A_1	30	60	80	10	20	40	33	37.2
A_2	50	80	20	40	10	40	33	37.2
A_3	20	0	20	110	70	40	33	37.2
A_4	80	30	20	60	10	40	33	37.2
A_5	50	30	30	20	50	40	33	37.2

Table VIII. Different aggregated results with the variance

	Var	Triplet-Var-OWA	TFN-Var-OWA	Triplet-Var-OWAWA	TFN-Var-OWAWA
A_1	680	(100, 530, 1600)	(100 + 430α, 1600 – 1070α)	(100, 620, 1600)	(100 + 520α, 1600 – 980α)
A_2	600	(0, 400, 1600)	(400α, 1600 –1200α)	(0, 520, 1600)	(520α, 1600 –1080α)
A_3	1640	(400, 1190, 4900)	(400 + 790α, 4900 – 3710α)	(400, 1460, 4900)	(400 + 1060α, 4900 – 3440α)
A_4	680	(100, 530, 1600)	(100 + 430α, 1600 – 1070α)	(100, 620, 1600)	(100 + 520α, 1600 – 980α)
A_5	160	(100, 130, 400)	(100 + 30α, 400 –270α)	(100, 148, 400)	(100 + 48α, 400 –252α)

Table IX. Aggregated results with the covariance

	Cov	Triplet-Cov-OWA	TFN-Cov-OWA	Triplet-Cov-OWAWA	TFN-Cov-OWAWA
Cov(A_1,A_2)	460	(0, 380, 800)	$(380\alpha, 800 - 420\alpha)$	(0, 428, 800)	$(428\alpha, 800 - 372\alpha)$
Cov(A_1,A_3)	900	(200, 710, 2100)	$(200 + 510\alpha, 2100 - 1390\alpha)$	(200, 824, 2100)	$(200 + 624\alpha, 2100 - 1276\alpha)$
Cov(A_1,A_4)	520	(200, 460, 800)	$(200 + 260\alpha, 800 - 340\alpha)$	(200, 496, 800)	$(200 + 296\alpha, 800 - 304\alpha)$
Cov(A_1,A_5)	300	(100, 250, 600)	$(100 + 150\alpha, 600 - 350\alpha)$	(100, 280, 600)	$(100 + 180\alpha, 600 - 320\alpha)$
Cov(A_2,A_3)	620	(0, 460, 1600)	$(460\alpha, 1600 - 1140\alpha)$	(0, 556, 1600)	$(556\alpha, 1600 - 1044\alpha)$
Cov(A_2,A_4)	420	(0, 330, 900)	$(330\alpha, 900 - 570\alpha)$	(0, 384, 900)	$(384\alpha, 900 - 516\alpha)$
Cov(A_2,A_5)	200	(0, 160, 400)	$(160\alpha, 400 - 240\alpha)$	(0, 184, 400)	$(184\alpha, 400 - 216\alpha)$
Cov(A_3,A_4)	780	(400, 680, 1400)	$(400 + 280\alpha, 1400 - 720\alpha)$	(400, 740, 1400)	$(400 + 340\alpha, 1400 - 660\alpha)$
Cov(A_3,A_5)	500	(200, 380, 1400)	$(200 + 180\alpha, 1400 - 1020\alpha)$	(200, 452, 1400)	$(200 + 252\alpha, 1400 - 948\alpha)$
Cov(A_4,A_5)	280	(100, 250, 400)	$(100 + 150\alpha, 400 - 150\alpha)$	(100, 268, 400)	$(100 + 168\alpha, 400 - 132\alpha)$

7. CONCLUSIONS

This paper has presented an overview regarding the use of the OWA operator in the variance and the covariance and some fundamental extensions. The main advantage of this approach is that it provides a parameterized family of aggregation operators between the minimum and the maximum. Thus, in uncertain environments it is possible to reconsider the traditional variance under a wide range of scenarios that may occur from the most pessimistic to the most optimistic one. Moreover, it has been demonstrated that the classical variance is included as a particular case. Several extensions and generalizations have been introduced by using the OWAWA, the WOWA, the hybrid average, and immediate weights. Furthermore, some extensions with induced aggregation operators and quasi-arithmetic means have been presented.

Additional extensions have also been presented by using the OWAWA and the IOWAWA operators in the PC. The main advantage of this approach is the possibility of analyzing the data considering the importance of the variables and the attitudinal character of the decision maker. Some numerical examples have been developed to understand numerically this approach. Special attention has been given to the construction of interval and fuzzy numbers with the Var-OWAWA and the Cov-OWAWA operator. Moreover, a related approach by using a box plot analysis has also been considered.

In future research, further extensions and generalizations will be considered in the analysis by using additional statistical tools and aggregation operators.[41,42]

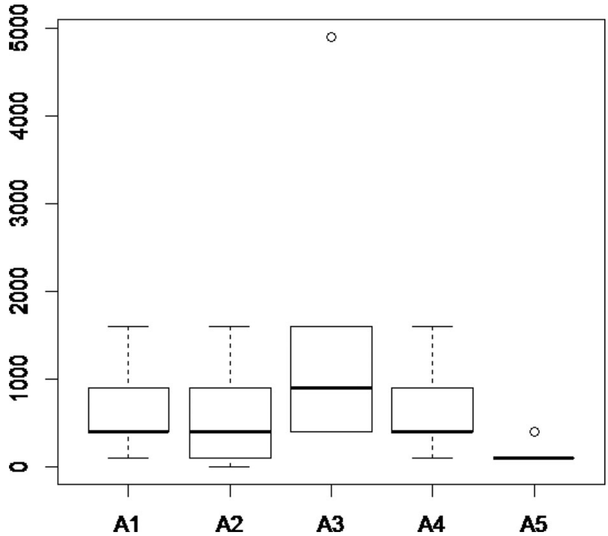

Figure 1. Box plots of the individual square deviation of the alternatives.

Moreover, several applications will be studied in a wide range of fields including economics and engineering.

Acknowledgments

Support from the European Commission through the project PIEF-GA-2011-300062, ECO2010-21787-C03-01, ECO2013-48326-C2-1-P and ECO 2013-48326-C2-2-P from the Spanish Government and ICREA Academia is gratefully acknowledged.

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