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A formulation of the wide partition conjecture using the atom problem in discrete tomography

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ABSTRACT

The Wide Partition Conjecture (WPC) was introduced by Chow and Taylor as an attempt to prove inductively Rota's Basis Conjecture, and in the simplest case tries to characterize partitions whose Young diagram admits a "Latin" filling. Chow et al. (2003) showed how the WPC is related to problems such as edge-list coloring and multi-commodity flow. As far as we know, the conjecture remains widely open.

We show that the WPC can be formulated using the k-atom problem in Discrete Tomography, introduced in Gardner et al. (2000). In this approach, the WPC states that the sequences arising from partitions admit disjoint realizations if and only if any combination of them can be realized independently. This realizability condition can be checked in polynomial time, although is not sufficient in general Chen and Shastri (1989), Guíñez et al. (2011). In fact, the problem is NP-hard for any fixed $k \geqslant 2$ Dürr et al. (2012). A stronger condition, called the saturation condition, was introduced in Guíñez et al. (2011) to solve instances where the realizability condition fails. We prove that in our case, the saturation condition is implied by the realizability condition. Moreover, we show that the saturation condition can be obtained as the Lagrangian dual of the linear programming relaxation of a natural integer programming formulation of the k-atom problem.

1. Introduction

A partition is a sequence of integers $\lambda = (\lambda_1, \dots, \lambda_\ell)$ ordered as $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell > 0$. It can be described through the Young tableau Y_λ, which is a collection of cells arranged in left justified rows, with λ_i cells in row i. We say that a partition λ is *Latin* if the cells of Y_λ can be labeled such that row i contains the integers $1, \dots, \lambda_i$ and such that no two squares in the same row or column have the same value. See Fig. 1 for an example.

The *conjugate* of λ is the partition λ^* with $\lambda^*_j = |\{i : \lambda_i \geqslant j\}|$. Here Y_{λ^*} is obtained from Y_λ by interchanging rows and columns. A subpartition μ of λ, denoted $\mu \subseteq \lambda$, is a partition obtained by deleting some parts of λ. Equivalently, Y_μ is obtained from Y_λ by deleting some rows and making the remaining rows adjacent.

We say that $\lambda = (\lambda_1, \dots, \lambda_\ell)$ *dominates* $\mu = (\mu_1, \dots, \mu_k)$, written $\lambda \succcurlyeq \mu$, if $\sum_{i=1}^{j} \lambda_i \geqslant \sum_{i=1}^{j} \mu_i$ for each $j = 1, \dots, \ell$, and $\sum_{i=1}^{\ell} \lambda_i = \sum_{i=1}^{k} \mu_i$. A partition λ is *wide* if $\mu \succcurlyeq \mu^*$ for each $\mu \subseteq \lambda$.

As noticed in [3], every Latin partition is wide. The Wide Partition Conjecture states that this necessary condition is also sufficient.

Conjecture 1 (*WPC, See [3]*). *λ is Latin if and only if it is wide.*

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5	4	3	1	2
3	1	2		
2	3	1		
1	2			

Fig. 1. The tableau $Y_{(5,3,3,2)}$ is Latin.

The WPC was introduced by Chow and Taylor and originally motivated as an attempt to prove Rota's basis conjecture [14,3]. Chow et al. [3] showed how the WPC is related to some problems such as edge-list coloring, multi-commodity network flows and the Greene–Kleitman theorem. They also mentioned some connections with the invariant theory, although they did not make it explicit.

Let us denote $[n] = \{1, \dots, n\}$ for convenience. The row projection of a subset M of the grid $[m] \times [n]$ is the vector $r \in \mathbb{Z}_+^m$ such that $r_i = |\{j : (i, j) \in M \text{ for some } j\}|$. The column projection $s \in \mathbb{Z}_+^n$ is defined analogously. Given a pair $b = (r, s) \in \mathbb{Z}_+^m \oplus \mathbb{Z}_+^n$, any subset M having r and s as row and column projection is a *realization* of b. If b admits a realization then we say that b is *realizable*. Notice that using the analogy between the $m \times n$ grid and the complete bipartite graph $K_{m,n}$, realizations of b correspond to b-factors of $K_{m,n}$ (see Section 3).

The k-atom problem consists in, given sequences $b^i = (r^i, s^i), i = 1, \dots, k$, finding pairwise disjoint realizations $M^1, \dots, M^k$ of $b^1, \dots, b^k$, respectively. If such realizations exist then we say that $(M^1, \dots, M^k)$ is a $(b^1, \dots, b^k)$-*packing*. This problem is motivated by the reconstruction problem of a polyatomic structure organized on a grid, where b^i are the projections of the atoms of type i. Gardner et al. [10] introduced it as a generalization of the binary matrix reconstruction problem under given row and columns sums, first studied by Ryser [16]. Many problems in discrete tomography can be modeled using the k-atom problem [4,5,11,1,8], which makes it a central problem in the area; see [12,13] for the foundations and recent advances in discrete tomography. Moreover, the k-atom problem can be seen as a k-commodity flow problem over a bipartite directed network and also as a problem of finding a 3-way consistency table of size $m \times n \times (k + 1)$ with specified line sums (or 2-margins as they are known in the statistical context); see [6,7] for definitions and complexity results and [17] for a summary of necessary conditions for the existence of such tables.

A necessary condition for the existence of a $(b^1, \dots, b^k)$-packing is that $b^J = (r^J, s^J)$ is individually realizable for each $J \subseteq [k]$, where $r^J = \sum_{j \in J} r^j$ and $s^J = \sum_{j \in J} s^j$. We refer to this as the *realizability condition*. This condition can be checked in polynomial time using Ryser's algorithm [16,11]. Although this condition ensures the existence of k-packings for some special instances, it does not suffice in general, even for $k = 2$ [2]. In fact, the k-atom problem is NP-hard for any fixed $k \geqslant 2$ [8].

In [11], a stronger necessary condition was introduced and proved to be sufficient for a family of instances of the 2-atom problem. For $A \subseteq [m] \times [n]$ and a given sequence $b = (r, s)$, let $\min_b(A) = \min\{|M \cap A| : M \text{ is a realization of } b\}$. Guíñez et al. [11] showed how to calculate $\min_b(A)$ in polynomial time for any set A using the minimum-weight max-flow algorithm. Observe that if $(M^1, \dots, M^k)$ is a $(b^1, \dots, b^k)$-packing then $\sum_{i=1}^k \min_{b^i}(A) \leqslant \sum_{i=1}^k |M^i \cap A| \leqslant |A|$. Then we say that $(b^1, \dots, b^k)$ satisfy the *saturation condition* if b^i is realizable for each $i \in [k]$, and for every set $A \subseteq [m] \times [n]$

$$\min_{b^1}(A) + \dots + \min_{b^k}(A) \leqslant |A|. \tag{1}$$

Notice that the saturation condition requires to check a number of inequalities which is exponential in the size of the grid. It is still unknown if we can check it for a fixed number k of sequences in polynomial time.

As first contribution, we present an equivalent formulation of the WPC using the k-atom problem, which might import new tools and ideas to solve it. This formulation is presented in Section 2, where we define an instance $b^\lambda = (b^1, \dots, b^\ell)$ for each partition λ such that there exists a b^λ-packing of a given grid if and only if λ is Latin. We also show that the wideness of λ is equivalent to the realizability condition of b^λ. In Section 3 we show that the saturation condition is strictly stronger than the realizability condition for the general k-atom problem. The main result of this section is however that for instances arising from partitions, these two conditions are equivalent. Finally, in Section 4 we discuss how the saturation condition can be obtained as a combinatorial analog of the dual of the relaxation of a Linear Program formulation (P) of the k-atom problem. The following scheme outlines our main contributions.

$$\begin{array}{ccccccc}
\boxed{\lambda \text{ is Latin}} & & & & & & \boxed{\lambda \text{ is wide}} \\
\Updownarrow & & & & & & \Updownarrow \\
\boxed{\exists b^\lambda\text{-packing}} & \Rightarrow & \boxed{(\text{P}) \text{ is feasible}} & \Rightarrow & \boxed{b^\lambda \text{ satisfies saturation}} & \Leftrightarrow & \boxed{b^\lambda \text{ satisfies realizability}}
\end{array}$$

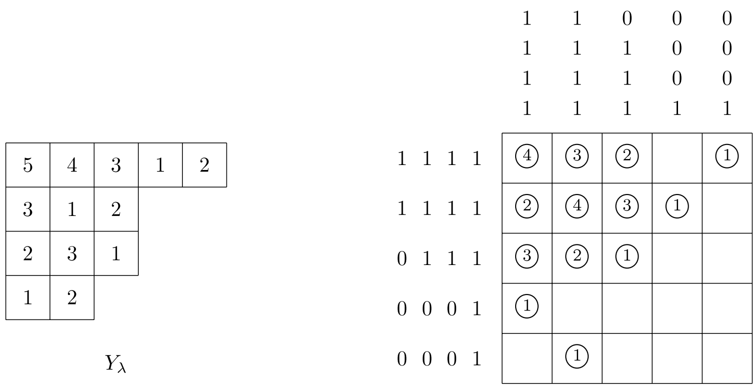

Fig. 2. The figure on the right shows the $(b^1, \ldots, b^4)$-packing of the 5×5 grid obtained from the Latin assignment of the Young tableau Y_λ in Fig. 1 (on the left). For each $k \in \{1, \ldots, 4\}$, the row and column projections r^k and s^k are listed on the left and above the grid, respectively. Also, the cells labeled with ⓚ represent the realization M^k of $b^k = (r^k, s^k)$ as defined in the proof of Theorem 1. Observe that $(i, j) \in M^k$ (i.e. the cell (i, j) in the packing is labeled with ⓚ) if and only if the cell (k, i) in Y_λ has label j.

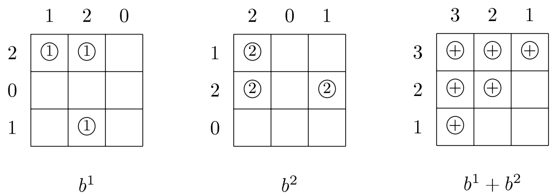

Fig. 3. An example of a 3×3 instance (b^1, b^2) that satisfies the realizability condition but does not admit disjoint realizations. For $k = 1, 2$, there is only one realization of the sequence b^k, which is represented in the grid by the cells labeled with ⓚ. Also note that $b^1 + b^2$ is realized by the set of cells labeled with $\oplus$. Thus, (b^1, b^2) satisfies the realizability condition. However, observe that the top left cell is used in the unique realizations of both b^1 and b^2. That is, there is no (b^1, b^2)-packing.

2. WPC and the k-atom problem

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition and let $n = \lambda_1$. For $t \in [\ell]$, let $b^t = (r^t, s^t)$, where $r^t = s^t = (\overbrace{1, \ldots, 1}^{\lambda_t}, 0, \ldots, 0)$. We denote $b^\lambda = (b^1, \ldots, b^\ell)$. The following result shows that a partition λ is Latin if and only if there exists pairwise disjoint realizations of the sequences $b^1, \ldots, b^\ell$. The idea of the proof is to map the k-th row in a Latin assignment of Y_λ to a realization M^k of b^k in the $n \times n$ grid.

Theorem 1. *λ is Latin if and only if there is a b^λ-packing in $[n] \times [n]$.*

Proof. Let f be a Latin assignment in Y_λ. For each $k = 1, \ldots, \ell$, let $M^k = \{(i, j) \in [n] \times [n] : f(k, i) = j\}$. In other words, for each $i = 1, \ldots, \lambda_k$, $(i, f(k, i)) \in M^k$. Since the assignments in row k of Y_λ are precisely the integers in $[\lambda_k]$, there is no column i such that $f(k, i) > \lambda_k$. This shows that M^k is a realization of b^k.

Let us prove that these realizations are pairwise disjoint. Assume that $(i, j) \in M^k \cap M^{k'}$. Then $f(k, i) = j = f(k', i)$. But the assignments in column i of Y_λ are all distinct, and then $k = k'$. It follows that $M^1, \ldots, M^\ell$ are pairwise disjoint and then $(M^1, \ldots, M^\ell)$ is a b^λ-packing.

The converse can be proved using a similar idea, since all the arguments above are in fact equivalences (see Fig. 2 for an example of the construction). □

For any $\mu \subseteq \lambda$, let J_μ be the indices of the parts of μ. Observe that $r_i^{J_\mu} = s_i^{J_\mu} = |\{j : \mu_j \geqslant i\}| = \mu_i^*$ and then $r^{J_\mu} = s^{J_\mu} = \mu^*$. Then by the Gale–Ryser condition, b^{J_μ} is realizable if and only if $\mu = (\mu^*)^* \succcurlyeq \mu^*$ [9,16]. That is,

Theorem 2. *λ is wide if and only if b^λ satisfies the realizability condition.*

Written in these terms, the WPC states that the sequences arising from partitions admit a packing provided they satisfy the realizability condition. As we mention in the introduction, the realizability condition does not suffice in general. An example by Chen and Shastri [2] is illustrated in Fig. 3.

However, we remark that the instance $b^\lambda = (b^1, \ldots, b^\ell)$ arising from partition λ holds two special properties:

(i) it is *binary*: for each $c \in [\ell]$, the row and column projections r^c, s^c are 0, 1-vectors. That is, the realizations of $b^c = (r^c, s^c)$, if there exist, are sub-matrices of a $n \times n$ permutation matrix;

(ii) it is *non-increasing* in rows and columns: both r^c and s^c are non-increasing, simultaneously for each $c \in [\ell]$.

Observe that there exists a realization of the sequence b^c in the $n \times n$ grid if and only if the number of ones in the row projection is the same than in the column projection. If this is the case for each $b^1, \ldots, b^k$, then properties (i) and (ii) characterize the instances of the k-atom problem arising from partitions. Moreover, we observe that the instance b^λ is *symmetric*, that is, $r_i^c = s_i^c$ for each $i \in [n]$ and $c \in [\ell]$. Theorem 3.1 in [8] can be easily modified to show that the k-atom problem restricted to symmetric instances is NP-hard for each $k \geqslant 2$. Considering this, and assuming that the WPC is true, an interesting question is to determine if any of these properties suffices for the k-atom problem to become tractable.

3. The saturation condition

We first prove that the saturation condition is stronger than the realizability condition. The proof is just an extension of Theorem 7 in [11] that uses a well-known characterization of bipartite b-factors by Ore [15]. Any bipartite graph G with color classes of size m and n can be embedded in an $m \times n$ grid by assigning to each edge a grid cell. Thus, we can think of G as a set $E_G \subseteq [m] \times [n]$. For sets $I \subseteq [m], J \subseteq [n]$ and $F \subseteq E_G$, we denote $F(I, J) = |F \cap (I \times J)|$. Observe that, given a sequence $b = (r, s)$ with $r \in \mathbb{Z}_+^m$ and $s \in \mathbb{Z}_+^n$, $F \subseteq E_G$ is a realization of b on the $m \times n$ grid if $|F(\{i\}, [n])| = r_i$ for each $i \in [m]$, and $|F([m], \{j\})| = s_j$ for each $j \in [n]$. If we think b as a function on the vertices of G, then $F \subseteq E_G$ corresponds to a b-factor of G. Using the grid notation, Ore's characterization says there exists a realization of b included in E_G (i.e. G admits a b-factor) if and only if $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j$ and $\sum_{i \in I} r_i \leqslant \sum_{j \in [n] \setminus J} s_j + |E_G(I, J)|$, for each $I \subseteq [m]$ and $J \subseteq [n]$.

Lemma 1. *If $b^1, \ldots, b^k$ satisfy the saturation condition on the $m \times n$ grid, then b^L is realizable for each $L \subseteq [k]$.*

Proof. First, observe that $\sum_{i=1}^m r_i^L = \sum_{j=1}^n s_j^L$ is straightforward since for each $l \in L$, $\sum_{i=1}^m r_i^l = \sum_{j=1}^n s_j^l$ by the realizability of b^l. Also notice that for every $I \subseteq [m]$ and $J \subseteq [n]$, each realization F_l of b^l is a b^l-factor of the bipartite graph H such that $E_H = F_l$. Then by Ore's characterization, it satisfies the inequality $\sum_{i \in I} r_i^l \leqslant \sum_{j \in [n] \setminus J} s_j^l + |F_l(I, J)|$. In particular, if we take the realization F_l that minimizes the intersection with $I \times J$, we conclude that $\sum_{i \in I} r_i^l \leqslant \sum_{j \in [n] \setminus J} s_j^l + \min_{b^l}(I \times J)$. Then

$$
\begin{aligned}
\sum_{i \in I} r_i^L &= \sum_{l \in L} \sum_{i \in I} r_i^l \\
&\leqslant \sum_{l \in L} \sum_{j \in [n] \setminus J} s_j^l + \sum_{l \in L} \min_{b^l}(I \times J) \\
&\leqslant \sum_{j \in [n] \setminus J} s_j^L + |I \times J|,
\end{aligned}
$$

where the last inequality follows from the saturation inequality (1) applied to set $A = I \times J$. If we denote by G the bipartite graph such that $E_G = [m] \times [n]$, then $E_G(I, J) = I \times J$. Since this holds for each $I \subseteq [m]$ and $J \subseteq [n]$, Ore's characterization shows that b^L is realizable in the $m \times n$ grid. $\square$

Observe that in the example in Fig. 3, the saturation inequality is not satisfied for the set $A = \{(1, 1)\}$ (the top left cell). Since (b^1, b^2) satisfies the realizability condition, we conclude that the saturation is a strictly stronger condition, even for $k = 2$.

The saturation condition was introduced in a weaker version in [11], where it is proved to be sufficient for a class of instances of the 2-atom problem. Unfortunately, from the proof of NP-hardness in [8], one can construct examples showing that it is not sufficient. We prove that for the special instances arising from partitions they are equivalent. The proof goes in several steps. First, we prove that the saturation inequality is satisfied for some sets of the cells of the grid we refer to as rectangles. Then we use this to prove it for the union of rectangles, which we call as tableau sets. Finally, we show how to derive the saturation inequality for each set of cells from the result for tableau sets.

3.1. The saturation inequality for rectangles

We first prove that the saturation inequality (1) is satisfied for every set $A = [p] \times [q]$, where $1 \leqslant p, q \leqslant n$. We refer to any of these sets as a *rectangle*. Without any loss of generality, we can assume that $p \leqslant q$.

For each $x = 1, \ldots, n$, let $b(x) = (r, s)$ with $r = s = (\overbrace{1, \ldots, 1}^{x}, 0, \ldots, 0)$. Observe that the realizations of $b(x)$ are permutation matrices of $[x] \times [x]$.

Lemma 2. *The value $\min_{b(x)}(A)$ is given by the function*

$$
f(x) = \begin{cases} x, & \text{if } x \leqslant p \\ p, & \text{if } p < x \leqslant q \\ p + q - x, & \text{if } q < x \leqslant p + q \\ 0, & \text{otherwise,} \end{cases}
$$

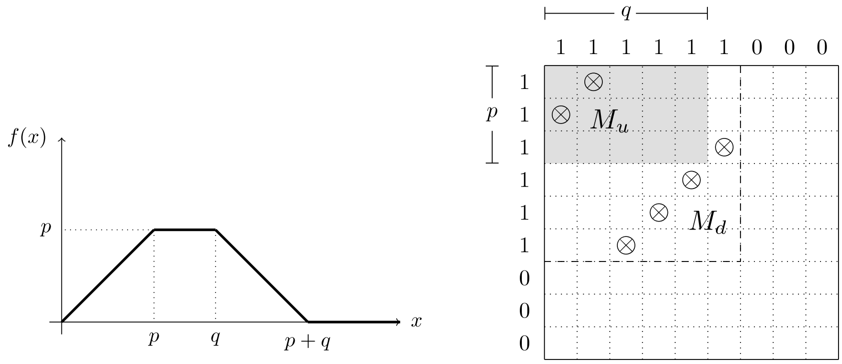

Fig. 4. On the left we have depicted the graphic of the function f introduced in Lemma 2. We recall that $f(x) = \min_{b(x)}(A)$ for the rectangle $A = [p] \times [q]$. The figure on the right shows the realization $M = M_u \cup M_d$ of $b(x)$ that, as proved in Lemma 2, minimizes the intersection with rectangle A, where the parameters are $n = 9$, $p = 2$, $q = 5$ and $x = 6$.

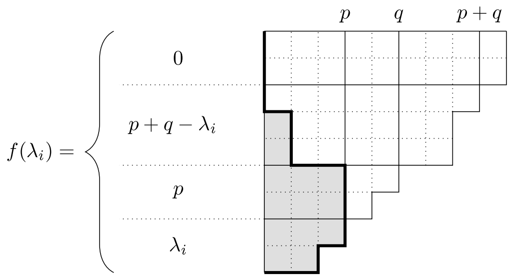

Fig. 5. Graphical interpretation of Lemma 3 for $\lambda = (9, 9, 8, 7, 7, 5, 4, 3, 2)$, $p = 3$ and $q = 5$. The rows of the Young tableau Y_λ are subdivided with respect to their length and according to the intervals used in the definition of function f in Lemma 2. The value of each $f(\lambda_i)$ is indicated on the left of the tableau and corresponds to the number of cells in row i on the left of the dark line. Then $\sum_{i=1}^{\ell} f(\lambda_i)$ is equal to the number of cells in the shaded region. Observe that this quantity corresponds to the number of cells in the first p columns minus the number of cells between column $q + 1$ and $p + q$.

with the minimum value attained by the realization $M = M_u \cup M_d$ of $b(x)$, where $M_u = \{(i, f(x) + 1 - i) : 1 \leqslant i \leqslant f(x)\}$ and $M_d = \{(i, f(x) + x + 1 - i) : f(x) < i \leqslant x\}$.

Proof. By the definition of f, $f(x) \leqslant x$. Then M is well defined and for each $1 \leqslant i \leqslant x$, there is exactly one column j such that $(i, j) \in M$. It is immediate that $j \leqslant x$ for each i, and that all the j's take different values. This shows that M is a realization of $b(x)$.

Let $(i, j) \in M_d$. Since $M_d \neq \emptyset$, we have $f(x) < x$. Therefore we can assume that $x > p$. If $x \leqslant q$, $j = p + x + 1 - i \geqslant p + 1$ and then $(i, j) \notin A$. Otherwise, $i + j = f(x) + x + 1 \geqslant p + q + 1$ and then $(i, j) \notin A$. We conclude that $M_d \cap A = \emptyset$. Also, and since $f(x) \leqslant x$, $M_u \subseteq A$. Then $\min_{b(x)}(A) \leqslant |M \cap A| = |M_u| = f(x)$.

Let M' be any realization of $b(x)$. Observe that $|M' \cap A| \geqslant f(x)$ trivially for $x > p + q$. First, assume that $x \leqslant q$ and let $(i, j) \in M'$. Then $(i, j) \in A$ if and only if $i \leqslant \min\{x, p\}$. That is, $|M' \cap A| = \min\{x, p\} = f(x)$. Finally, consider $q < x \leqslant p + q$ and let $\tilde{M}$ be the set of cells in $M' \setminus A$ in the first p rows. Clearly $|\tilde{M}| \leqslant x - q$. Then $|M' \cap A| = p - |\tilde{M}| \geqslant p - (x - q) = f(x)$. This proves that $|M' \cap A| \geqslant f(x)$ and we conclude that M attains $\min_{b(x)}(A)$. □

A graphic of the function f is depicted on the left of Fig. 4. On the right, we have illustrated the realization $M = M_u \cup M_d$ of $b(x)$ that minimizes the intersection with rectangle A. Observe that the intersection is exactly the set of cells in M_u.

In Fig. 5 we have represented $f(\lambda_i) = \min_{b(\lambda_i)}(A) = \min_{b^i}(A)$ with respect to the length λ_i of row i in the Young tableau Y_λ. We observe that $\sum_{i=1}^{\ell} f(\lambda_i)$ corresponds to the shaded region in the tableau. From the figure it is not difficult to deduce the following identity.

Lemma 3. *Let f be the function defined in Lemma 2. For any partition λ,*

$$\sum_{i=1}^{\ell} f(\lambda_i) = \sum_{j=1}^{p} \lambda_j^* - \sum_{j=q+1}^{p+q} \lambda_j^*.$$

Proof. The following is a well-known equality satisfied by the conjugate

$$\sum_{j=1}^{a} \lambda_j^* = \sum_{i=1}^{\ell} \min\{a, \lambda_i\}$$

for any integer $a \geqslant 1$. By applying this equality for values $a = p, q$ and $p + q$, we obtain that

$$\begin{aligned}\sum_{j=1}^{p} \lambda_j^* - \sum_{j=q+1}^{p+q} \lambda_j^* &= \sum_{j=1}^{p} \lambda_j^* - \sum_{j=1}^{p+q} \lambda_j^* + \sum_{j=1}^{q} \lambda_j^* \\ &= \sum_{i=1}^{\ell} \min\{p, \lambda_i\} - \sum_{i=1}^{\ell} \min\{p+q, \lambda_i\} + \sum_{i=1}^{\ell} \min\{q, \lambda_i\} \\ &= \sum_{i=1}^{\ell} (\min\{p, \lambda_i\} - \min\{p+q, \lambda_i\} + \min\{q, \lambda_i\}).\end{aligned}$$

Then the result follows easily by observing that $f(x) = \min\{p, x\} + \min\{q, x\} - \min\{p + q, x\}$ for each $x \geqslant 0$. □

Notice that

$$\sum_{j=1}^{p} \lambda_j^* + \sum_{j=p+1}^{p+q} \lambda_j^* = \sum_{j=1}^{p+q} \lambda_j^* = \sum_{j=1}^{q} \lambda_j^* + \sum_{j=q+1}^{p+q} \lambda_j^*,$$

and then

$$\sum_{i=1}^{\ell} f(\lambda_i) = \sum_{j=1}^{q} \lambda_j^* - \sum_{j=p+1}^{p+q} \lambda_j^*.$$

That is, the roles of p and q are exchangeable in Lemma 3, as expected. In particular, $\min_{b(x)}([p] \times [q]) = \min_{b(x)}([q] \times [p])$, for each value x.

Lemma 4. *If λ is a wide partition, then $\sum_{j=1}^{p} \lambda_j^* - \sum_{j=q+1}^{p+q} \lambda_j^* \leqslant pq$ for each p and q.*

Proof. Let μ be the partition obtained from λ by deleting its first $t = \lambda_{p+q}^*$ parts. Let $\ell' = \ell - t$ be the length of μ and $n' = \mu_1 = \lambda_{t+1}$. It is clear from the definitions of t and μ that $\mu_j^* = \lambda_j^* - t$ for each $1 \leqslant j \leqslant n'$. In particular,

$$\sum_{j=1}^{p} \lambda_j^* = pt + \sum_{j=1}^{p} \mu_j^* \quad \text{and} \quad \sum_{j=q+1}^{p+q} \lambda_j^* = pt + \sum_{j=q+1}^{n'} \mu_j^*. \tag{2}$$

Observe that

$$\sum_{j=1}^{p} \mu_j^* = \sum_{i=1}^{\ell'} \min\{p, \mu_i\} \leqslant pq + \sum_{i=q+1}^{\ell'} \mu_i. \tag{3}$$

From (2) and (3), it remains to show that $\sum_{i=q+1}^{\ell'} \mu_i \leqslant \sum_{j=q+1}^{n'} \mu_j^*$. But since $\sum_{i=1}^{\ell'} \mu_i = \sum_{j=1}^{n'} \mu_j^*$ and $\mu \succcurlyeq \mu^*$,

$$\sum_{i=q+1}^{\ell'} \mu_i - \sum_{j=q+1}^{n'} \mu_j^* = \sum_{j=1}^{q} \mu_j^* - \sum_{i=1}^{q} \mu_i \leqslant 0. \quad \square$$

Lemma 5. *Let $b^\lambda = (b^1, \ldots, b^\ell)$ be as defined in Section 2. If λ is wide then $\sum_{i=1}^{\ell} \min_{b^i}(A) \leqslant |A|$, for each rectangle A.*

Proof. Let $A = [p] \times [q]$ be a rectangle. By Lemmas 2–4, we have that for any wide partition λ

$$\sum_{i=1}^{\ell} \min_{b^i}(A) = \sum_{i=1}^{\ell} f(\lambda_i) = \sum_{j=1}^{p} \lambda_j^* - \sum_{j=q+1}^{p+q} \lambda_j^* \leqslant pq = |A|. \quad \square$$

3.2. *The saturation inequality for tableau sets*

We say that $A \subseteq [n] \times [n]$ is a *tableau set* if it can be represented as the union of rectangles $A_j = [p_j] \times [q_j], j = 1, \ldots, t$, for some t. Let us assume that the rectangles are numbered such that $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_t$. Note that we can actually assume that $p_1 < p_2 < \cdots < p_t$ and $q_1 > q_2 > \cdots > q_t$; otherwise, there exist a pair of rectangles such that one is contained

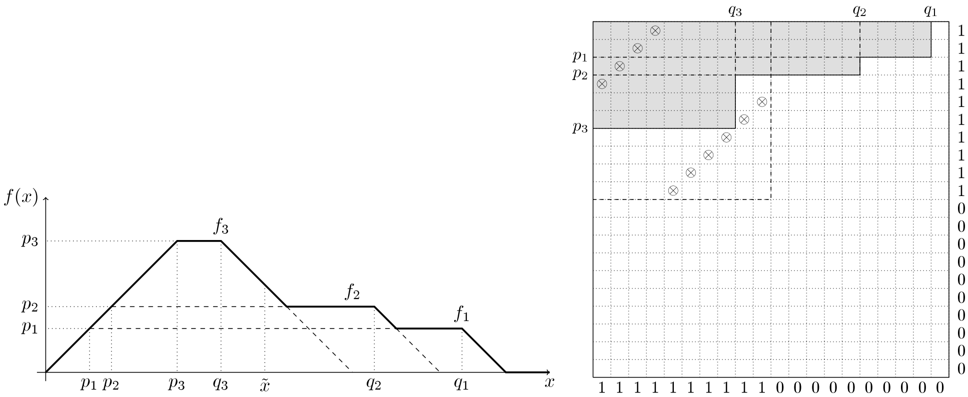

Fig. 6. The value of $\min_{b(\cdot)}(A)$ for tableau set $A = A_1 \cup A_2 \cup A_3$, where A_1, A_2 and A_3 are rectangles of dimensions $p_1 \times q_1 = 2 \times 19$, $p_2 \times q_2 = 3 \times 15$ and $p_3 \times q_3 = 6 \times 8$, respectively. On the left, the graphic of $f(x) = \min_{b(x)}(A)$. By Lemma 6, $f(x) = \max\{f_1(x), f_2(x), f_3(x)\}$, where $f_i(x) = \min_{b(x)}(A_i)$. On the right, A is represented by a shaded region in the $[n] \times [n]$ grid, where $n = 20$. We have also drawn a realization of $b(\tilde{x})$ that attains the minimum of f at $\tilde{x} = 10$. Observe that this realization is exactly the one that minimizes f_3 at $\tilde{x}$, since $f(\tilde{x}) = f_3(\tilde{x})$.

in the other, so we can omit the smaller one in the definition. Let $b(x)$ be defined as in Section 3.1, $f(x) = \min_{b(x)}(A)$ and $f_j(x) = \min_{b(x)}(A_j)$, for each $j = 1, \dots, t$. We recall that the value $f_j(x)$ is given by the function in Lemma 2 with parameters p_j and q_j. We denote by $M^j = M_u^j \cup M_d^j$ the realization of $b(x)$ defined in that lemma that attains the value $f_j(x)$.

Lemma 6. *For each x, $f(x) = \max\{f_1(x), \dots, f_t(x)\}$. Moreover, if $j \in \arg\max f_j(x)$ for a fixed integer x, then M^j is such that $f(x) = |M^j \cap A|$.*

Proof. Let $j \in \arg\max f_j(x)$. We clearly have that $f_j(x) \leqslant f(x)$, since $A_j \subseteq A$. We claim that M^j is such that $|M^j \cap A| \leqslant f_j(x)$. Observe that this suffices because $f(x) \leqslant |M^j \cap A|$. We claim that for each $j' = 1, \dots, t$, $M_d^j \cap A_{j'} = \emptyset$. By contradiction, assume that $(i, f_j(x) + x + 1 - i) \in A_{j'}$ for some $i > f_j(x)$. Since $A_{j'}$ is a rectangle and $f_{j'}(x) \leqslant f_j(x)$, we have that $(i, f_{j'}(x) + x + 1 - i) \in A_{j'}$. This contradicts the fact that $M_d^{j'} \cap A_{j'} = \emptyset$. Because $A = A_1 \cup \dots \cup A_t$, we deduce that $M_d^j \cap A = \emptyset$. Then $|M^j \cap A| = |M_u^j \cap A| \leqslant |M_u^j| = f_j(x)$. $\square$

Fig. 6 shows a tableau set A obtained as the union of rectangles A_1, A_2 and A_3. By the previous lemma, if we denote $f(x) = \min_{b(x)}(A)$ and $f_i(x) = \min_{b(x)}(A_i)$, for $i = 1, 2, 3$, then $f(x) = \max\{f_1(x), f_2(x), f_3(x)\}$.

From Lemma 6, and since the function and realization for rectangle $[p_j] \times [q_j]$ coincide with those for rectangle $[q_j] \times [p_j]$, we can assume that $p_j \leqslant q_j$ for each $j \in [t]$. Observe that by assuming this, the size of A cannot increase. So if the saturation inequality holds under this assumption, it certainly does in all the cases.

In order to prove the saturation inequality for tableau set $A = A_1 \cup \dots \cup A_t$ we proceed by induction on the number t of rectangles that form A. By Lemma 5, it holds for $t = 1$. So let us assume the saturation inequality holds for any tableau set formed by at most $t-1$ rectangles and let us prove for tableau set A. We recall that $f(x) = \min_{b(x)}(A)$ and $f_i(x) = \min_{b(x)}(A_i)$, for each $i = 1, \dots, t$. By Lemma 6, $f(x) = \max\{f_1(x), \dots, f_t(x)\}$, for each x.

We now consider two cases. First, assume that for every x, $f_{t-1}(x) \leqslant f_t(x)$. Let $A' = A_1 \cup \dots \cup A_{t-2} \cup A_t$ and $f'(x) = \min_{b(x)}(A')$. By Lemma 6, we have that $f'(x) = \max\{f_1(x), \dots, f_{t-2}(x), f_t(x)\}$, for every x. Then $f' = f$ and we have

$$\sum_{i=1}^{\ell} \min_{b^i}(A) = \sum_{i=1}^{\ell} f(\lambda_i) = \sum_{i=1}^{\ell} f'(\lambda_i) \leqslant |A'| \leqslant |A|,$$

where the first inequality follows by the induction hypothesis.

By Lemma 2 and because $p_{t-1} \leqslant p_t$, the previous case occurs if and only if $p_{t-1} + q_{t-1} \leqslant p_t + q_t$. Thus, we can assume that $p_{t-1} + q_{t-1} > p_t + q_t$. Let $A' = A_1 \cup \dots \cup A_{t-1}$ and $\tilde{A} = [p_t - p_{t-1}] \times [q_t]$. We denote $f' = \min_{b(\cdot)}(A')$ and $\tilde{f} = \min_{b(\cdot)}(\tilde{A})$.

Claim 1. *For every x, $f(x) \leqslant f'(x) + \tilde{f}(x)$.*

Proof. Let x be such that $f(x) > f'(x)$. Observe that $f(x) = f_t(x)$; this is immediate since $f'(x) = \max\{f_1(x), \dots, f_{t-1}(x)\}$ by Lemma 6. By Lemma 2 applied to rectangles A_{t-1} and A_t, $f_t(x) = f_{t-1}(x)$ for $x \leqslant p_{t-1}$ and $f_t(x) \leqslant f_{t-1}(x)$ for $x \geqslant p_t + q_t - p_{t-1}$. Thus, we can assume that $p_{t-1} < x < p_t + q_t - p_{t-1}$. If $x \leqslant p_t$ then $f(x) = x \leqslant p_{t-1} + \min\{x, p_t - p_{t-1}\} = f_{t-1}(x) + \tilde{f}(x) \leqslant f'(x) + \tilde{f}(x)$. Otherwise, $f(x) = \min\{p_t, p_t + q_t - x\} = p_{t-1} + \min\{p_t - p_{t-1}, p_t - p_{t-1} + q_t - x\} = p_{t-1} + \tilde{f}(x)$. By a previous assumption, $p_t + q_t - p_{t-1} < q_{t-1}$. Then $x < q_{t-1}$ and so $f_{t-1}(x) = p_{t-1}$. We conclude that $f(x) \leqslant f_{t-1}(x) + \tilde{f}(x) \leqslant f'(x) + \tilde{f}(x)$, which ends the proof of the claim. $\square$

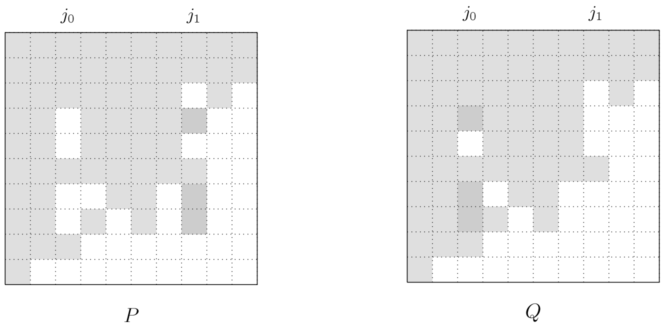

Fig. 7. Q is obtained from P by a left-compression from column j_1 to column j_0. Observe that only three cells are shifted (in dark gray), the ones in row i such that $(i, j_1) \in P$ and $(i, j_0) \notin P$.

Lemma 7. *Let $b^\lambda = (b^1, \dots, b^\ell)$ be as defined in Section 2. If λ is wide then $\sum_{i=1}^{\ell} \min_{b^i}(A) \leqslant |A|$, for each tableau set A.*

Proof. By a previous observation, we only need to prove the inductive step when $p_t + q_t < p_{t-1} + q_{t-1}$. But in this case Claim 1 applies. In particular, we have that $f(\lambda_i) \leqslant f'(\lambda_i) + \tilde{f}(\lambda_i)$ for every $i = 1, \dots, \ell$. Then

$$\sum_{i=1}^{\ell} \min_{b^i}(A) \leqslant \sum_{i=1}^{\ell} \min_{b^i}(A') + \sum_{i=1}^{\ell} \min_{b^i}(\tilde{A}).$$

By the induction hypothesis, we have that $\sum_{i=1}^{\ell} \min_{b^i}(A') \leqslant |A'|$, and by Lemma 5, $\sum_{i=1}^{\ell} \min_{b^i}(\tilde{A}) \leqslant |\tilde{A}|$ since $\tilde{A}$ is a rectangle. Then the result follows after observing that $|\tilde{A}| = (p_t - p_{t-1})q_t = |A \setminus A'|$. $\square$

3.3. Reduction to tableau sets

So far, we have proved that the saturation inequality holds for any tableau set. In this section we show that these sets are the worst case, in the sense that the left-hand side of (1) is maximal for tableau sets in a certain partial order on the grid.

Let $P, Q \subseteq [m] \times [n]$. We say that Q is obtained from P by a *left-compression* if there exist columns $j_0 < j_1$ such that:

(i) for each row i such that $(i, j_0) \notin P$ and $(i, j_1) \in P$, $(i, j_0) \in Q$ and $(i, j_1) \notin Q$, and
(ii) for any other row i, $(i, j) \in P$ if and only if $(i, j) \in Q$.

Fig. 7 shows an example of a set Q obtained from a set P by a left-compression. We say that Q is obtained from P by an *up-compression* if P^T is obtained from Q^T by a left-compression. Let $b = (r, s)$ be realizable in $[m] \times [n]$.

Lemma 8. *Assume that s is non-increasing. If Q is obtained from P by a left-compression, then $\min_b(P) \leqslant \min_b(Q)$.*

Proof. Let $j_0 < j_1$ be the columns used in the left-compression from P to Q. Let us consider the set $I = \{i : (i, j_1) \in P \setminus Q\}$, that is, the indexes of the rows that are modified. We remark that $(i, j_0) \in Q \setminus P$ for each $i \in I$. Let F be a realization of b. For $k = 0, 1$, let $R_k = \{i : (i, j_k) \in F, (i, j_{1-k}) \notin F\}$ and $I_k = I \cap R_k$. Observe that $F \cap Q$ and $F \cap P$ only differ in the rows in $I_0 \cup I_1$. In fact,

$$|F \cap Q| - |F \cap P| = |I_0| - |I_1|. \tag{4}$$

We claim that there exists a realization F' of b such that

$$|F' \cap P| \leqslant |F \cap P| - |I_1| + |I_0|. \tag{5}$$

The proof is constructive. Since $s_{j_0} \geqslant s_{j_1}$, $|R_0| \geqslant |R_1|$. In particular, $|R_0| \geqslant |I_1|$ and then we can choose $S \subseteq R_0$ such that $|S| = |I_1|$. We define

$$F' = F \cup \{(i, j_0), (i', j_1) : i \in I_1, i' \in S\} \setminus \{(i', j_0), (i, j_1) : i \in I_1, i' \in S\}.$$

It is not difficult to check that F' is a realization of b. Moreover, observe that $|F' \cap P| \leqslant |F \cap P| - |I_1| + |I \cap S|$. But $I \cap S \subseteq I \cap R_0 = I_0$, which proves the inequality (5).

From (4) and (5), $\min_b(P) \leqslant |F' \cap P| \leqslant |F \cap Q|$. Then the lemma follows by taking a realization F such that $\min_b(Q) = |F \cap Q|$. $\square$

Clearly the same result holds when r is non-increasing and Q is obtained from P by an up-compression.

Lemma 9. *Each $A \subseteq [m] \times [n]$ can be transformed into a tableau set $\tilde{A} \subseteq [m] \times [n]$ by a sequence of left- and up-compressions.*

Proof. Let us first show that A can be transformed into a set A' that is *left-justified*, that is, such that there are no cells $(i, j_0) \notin A'$ and $(i, j_1) \in A'$ with $j_0 < j_1$, through a sequence of left-compressions.

For each $A' \subseteq [m] \times [n]$, we denote $\sigma_i(A')$ the number of cells $(i, j_0) \notin A'$ such that there exists $(i, j_1) \in A$ with $j_0 < j_1$. We also denote $\sigma(A') = \sum_{i=1}^{n} \sigma_i(A')$. Observe that if A is left-justified then $\sigma(A) = 0$. Otherwise consider the smallest i such that $\sigma_i(A) > 0$ and let j_0 be the smallest integer for which $(i, j_0) \notin A$ and there exists $(i, j_1) \in A$ with $j_0 < j_1$. We choose j_1 as the largest integer greater than j_0 having this property. We now perform a left-compression using columns j_0 and j_1. Let us call A' the set we obtain. We claim that $\sigma(A') < \sigma(A)$. Let I be the set of rows that were modified by the left-compression. Clearly $\sigma_{i'}(A') = \sigma_{i'}(A)$ for each $i' \notin I$. Also, $\sigma_{i'}(A') \leqslant \sigma_{i'}(A)$ for each $i' \in I$ and by the choice of j_1, we have that $\sigma_i(A') < \sigma_i(A)$. We conclude that $\sigma(A') < \sigma(A)$. By repeating this process we will end up with a left-justified set A' obtained from A by a sequence of left-compressions.

Using an analogous method, we can use up-compressions to push cells up to obtain the desired tableau set $\tilde{A}$. □

Theorem 3. *If λ is wide, then b^λ satisfies the saturation condition.*

Proof. Let $A \subseteq [m] \times [n]$ and let $\tilde{A}$ be a tableau set obtained from A by a sequence of left- and up-compressions. Since for each $b^t = (r^t, s^t)$, $t = 1, \ldots, \ell$, both r^t and s^t are non-increasing, by Lemma 8 we have that $\min_{b^i}(A) \leqslant \min_{b^i}(\tilde{A})$. Then using Lemma 7, and the fact that left- and up-compressions preserve the cardinality of sets, we conclude that

$$\sum_{i=1}^{\ell} \min_{b^i}(A) \leqslant \sum_{i=1}^{\ell} \min_{b^i}(\tilde{A}) \leqslant |\tilde{A}| = |A|. \quad \square$$

4. Final comments and open problems

In this work we have related the Wide Partition Conjecture to the k-atom problem in discrete tomography. In these terms, the WPC is equivalent to the existence of solutions for the instances of the k-atom problem arising from partitions provided they satisfy the necessary condition of realizability. This condition is sufficient only in very restricted cases, and it has been applied mainly for $k = 2$ [11]. Then, if the conjecture is true, we obtain a whole family of instances for which the realizability condition suffices.

Our main result shows the equivalence of the realizability and saturation conditions for the instances arising from partitions. This implies that the saturation condition can be checked in polynomial time for these instances, which is still open in general even for $k = 2$. Also, it is known that each instance of the 2-atom problem where at least one of the sequences is binary is realizable provided it satisfies the realizability condition [2]. On the other hand, the 2-atom problem remains NP-hard for symmetric instances as we mentioned in Section 2. Thus, an interesting open question is to determine the computational complexity of the problem restricted to non-increasing instances.

We finish this section showing an approach to the k-atom problem using a linear programming (LP) formulation. As we will see below, the saturation condition appears naturally when we interpret combinatorially the Lagrangian dual of this LP. For each $t = 1, \ldots, k$, let us consider sequences $b^t = (r^t, s^t)$, where $r^t \in \mathbb{Z}_+^m$ and $s^t \in \mathbb{Z}_+^n$, and let $\mathcal{P}_t$ be the convex-hull of characteristic vectors of all realizations of b^t in the $[m] \times [n]$ grid. We also consider $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_k$ and $f : \mathcal{P} \to \mathbb{R}$ a linear function. Let us denote $q = mn$ and consider the following optimization problem

$$\begin{aligned} &\max f(x) \\ &\text{s.t. } x_i^1 + \cdots + x_i^k \leqslant 1, \; i = 1, 2, \ldots, q \\ &x \in \mathcal{P}. \end{aligned} \tag{P}$$

Observe that $(b^1, \ldots, b^k)$-packings are the integral feasible solutions of (P). We define the Lagrangian function for each $\pi \in \mathbb{R}^q$ as

$$L(x, \pi) = f(x) + \sum_{i=1}^{q} \pi_i (1 - x_i^1 - \cdots - x_i^k),$$

and the Lagrangian dual function as $h(\pi) = \max_{x \in \mathcal{P}} L(x, \pi)$.

By weak duality, we know that if x^* is an optimal solution for (P), then $h(\pi) \geqslant f(x^*)$ for each $\pi \geqslant 0$. That is, for each non negative Lagrangian multipliers, the value of the dual function provides an upper bound on the optimal objective value of (P). To obtain the best upper bound we consider the dual problem

$$\min_{\pi \geqslant 0} h(\pi). \tag{D}$$

Since we are only interested in the feasibility of (P), we can take $f \equiv 0$. Because $h(0) = 0$, the optimum of (D) is non-positive. In fact, it is zero if and only if (P) is feasible. Observe that

$$h(\pi) = \max_{x \in \mathcal{P}} \sum_{i=1}^{q} \pi_i (1 - x_i^1 - \cdots - x_i^k) = \sum_{i=1}^{q} \pi_i - \sum_{j=1}^{k} \min_{x^j \in \mathcal{P}^j} \sum_{i=1}^{q} \pi_i x_i^j.$$

Because $\mathcal{P}_j$ is integral for each j, if we take $A \subseteq [m] \times [n]$ and $\pi \in \{0, 1\}^q$ such that $\mathcal{X}_A = \pi$, then $\min_{b^j}(A) = \min_{x \in \mathcal{P}^j} \sum_{i=1}^{q} \pi_i x_i$. Then $(b^1, \ldots, b^k)$ satisfies the saturation condition if and only if $h(\pi) \geqslant 0$ for every $\pi \in \{0, 1\}^q$. That is, (D) can be thought as a *weighted version* of the saturation condition.

By a continuity argument, in order to check the infeasibility of (P), that is, to find $\pi \geqslant 0$ such that $h(\pi) < 0$, we can assume that $\pi \in \mathbb{Z}_+^q$. However, it is an open question to determine if we can restrict our attention only to vectors $\pi \in \{0, 1\}^q$. Observe that if this is true, it would imply that the saturation condition could be checked in polynomial time through the previous linear feasibility problem. Otherwise, it would be interesting to find larger families of instances for which the restriction to binary vectors is sufficient.

We remark that from the NP-hardness proof for the 2-atom problem in [8], we can construct instances that are feasible for (P), but that are not integral. Because of this, any proof of integrality for the instances arising from partitions must rely on some of their special properties that they satisfy.

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