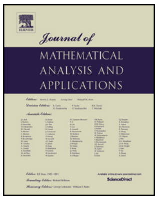
J. Math. Anal. Appl. 426 (2015) 330–339

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



Existence, computability and stability for solutions of the diffusion equation with general piecewise constant argument

Tomas Veloz [a,b,*], Manuel Pinto [c]

[a] *University of British Columbia – Okanagan, Mathematics Department, University Way 3333, Kelowna, V1V 1V7, Canada*
[b] *Insituto de Filosofía y Ciencias de la Complejidad – IFICC, Los Alerces 3024, Ñuñoa, Santiago, Chile*
[c] *Departamento de Matematicas, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile*

ARTICLE INFO

Article history:
Received 4 June 2013
Available online 20 November 2014
Submitted by S.A. Fulling

Keywords:
Piecewise constant argument of generalized type
Partial differential equations
Boundary value problems
Stability

ABSTRACT

We study the solution of a class of PDE with piecewise constant argument of generalized type. Separation of variables leads to a solution formed by a series of products. In previous works, the convergence and bounds of the solution could be obtained from the study of the solution on the first constancy interval only. In the general case however, each term of the series may be unbounded at every interval, implying that the solution is not computable. We establish conditions where the convergence of the solution can be verified computing a finite number of terms of the series in each constancy interval, without requiring any regularity on the initial condition. Moreover, we combine asymptotic properties for each variable of the equation to obtain an exponential bound for the solution.

1. Introduction

Functional differential equations with deviated argument provide a mathematical model for systems where the changes of state depend upon its past history or its future. In [16], Myshkis proposed to study delayed differential equations with piecewise constant argument: DEPCA. These equations appear as an attempt to extend the theory of functional differential equations to systems with discontinuous argument deviations. DEPCA also arises in the process of replacing some terms of a differential equation by their piecewise constant approximations. This point of view has applications in impulsive or loaded differential equations of control theory, and stabilization of systems with discrete (sample) control [11,14,20,23,19]. A typical DEPCA is of the form

$$x'(t) = f\big(t, x(t), x\big(\gamma(t)\big)\big), \tag{1.1}$$

---

* Corresponding author at: University of British Columbia – Okanagan, Mathematics Department, University Way 3333, Kelowna, V1V 1V7, Canada.

E-mail addresses: tomas.veloz@ubc.ca (T. Veloz), pintoj.uchile@gmail.com (M. Pinto).

http://dx.doi.org/10.1016/j.jmaa.2014.10.045

where the argument $\gamma(t)$ has intervals of constancy. For example, delayed equations with $\gamma(t) = [t], [t-n]$, $t - n[t]$, advanced equations with $\gamma(t) = [t+n]$, and alternately retarded and advanced equations with $\gamma(t) = m[\frac{t+k}{m}]$ were investigated in [12,21,25] respectively, where $n, k, m \in \mathbb{N}$, $[\cdot]$ denotes the lowest-integer (floor) function, and $0 < k < m$.

Akhmet introduced in [1] the piecewise constant argument of generalized type, and the study of DEPCA with general deviation, so-called DEPCAG. Let $\{\theta_i\}$, $\{\zeta_i\}$ be real sequences, $I_i = [\theta_i, \theta_{i+1})$ and $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(t) = \zeta_i$, where $t \in I_i$, and $i \in \mathbb{N}_0$. The piecewise constant argument of generalized type assumes that

$$\begin{aligned} &\theta_0 = 0 < \theta_i \le \zeta_i \le \theta_{i+1}, \qquad \lim_{i\to\infty} \theta_i \to \infty, \quad \text{and} \\ &\text{there exist} \quad 0 < l^* \le L^* < \infty \quad \text{such that} \quad l^* \le \theta_{i+1} - \theta_i \le L^*, \quad i \in \mathbb{N}. \end{aligned} \tag{1.2}$$

For each $i \in \mathbb{N}_0$, $l_i = \theta_{i+1} - \theta_i$ represents the length of the interval of constancy, and ζ_i represents the deviation in I_i. The cases $\zeta_i = \theta_i, \theta_{i+1}$ correspond to the delayed and advanced cases respectively, and the case $\theta_i < \zeta_i < \theta_{i+1}$ corresponds to a general alternated advanced and retarded case.

Analytical results in DEPCA and DEPCAG focus on the existence, oscillation, and bounds of solutions [3,17,8,18,7,15,28,29,6]. These results have been applied in models of physical and artificial systems [5,14,31,9]. Moreover, the stability of numerical solutions of ordinary differential equations approximated by DEPCAG has received attention recently [10,30,22]. A detailed introduction to DEPCAG is in [4,5,25].

Partial differential equations with piecewise constant argument (PDEPCA) were introduced in [26] to study the existence, oscillation, and asymptotic bounds of the solutions of initial value problems with piecewise constant delays. For example, the wave equation with piecewise constant delays, boundary value problems, and initial value problems assuming piecewise constant arguments were investigated in [27,13,28] respectively. Complete descriptions for DEPCA and PDEPCA are in [23,24].

In this article we introduce the PDEPCA of generalized type, or PDEPCAG. Our aim is to extend the study of DEPCAG to partial differential systems. We deal with the following equation:

$$\begin{aligned} &u_t(x,t) = a^2(t)u_{xx}(x,t) - b(t)u\big(x,\gamma(t)\big), \\ &u(0,0) = u(1,0) = 0, \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.3}$$

where u_0 is a continuous function on $[0,1]$, $a(t)$ and $b(t)$ are locally integrable functions on $[0,\infty)$ and $\gamma(t)$ is a step function as in (1.2). Eq. (1.3) describes the heat flow in a rod with a diffusion term $a^2(t)u_{xx}(x,t)$ along the rod, with gain or loss across lateral sides of the rod, measured at discrete times $-b(t)u(x,\gamma(t))$. An important example is given by the term $-b(t)u(x,t-2)$, which is simplified considering a step function $\gamma(t)$ near of $t-2$ [10]. In [13], Eq. (1.3) is studied considering two different piecewise constant arguments $\gamma(t) = [t]$ and $\gamma(t) = [t+\frac{1}{2}]$, with both $a(t) = a$ and $b(t) = b$ constant functions.

Definition 1. $u(x,t)$ is a *solution* of PDEPCAG (1.3) if $u(x,t)$ is continuous in $[0,1] \times [0,\infty)$, the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ exist and are continuous in $[0,1] \times [0,\infty)$, with the possible exception of points (x,θ_n) where one sided partial derivatives exist, and $u(x,t)$ satisfies Eq. (1.3) with the possible exception of points (x,θ_n) $(n = 1,2,...)$.

Applying separation of variables on a PDEPCA leads to a solution defined by a series of products. In PDEPCA, the existence and bounds of the solution can be obtained from the study of the solution on the first constancy interval only [13,23,26,28]. This is because the length and deviation of the non-general piecewise constant argument assume the same values on each constancy interval. In particular, if a term of the series is bounded on the first constancy interval, then it will be bounded on every constancy interval.

In the generalized case however, even if a and b are constant, the boundedness of a term of the solution series in a constancy interval I_i, $i \in \mathbb{N} \cup \{0\}$ does not ensure its boundedness on I_{i+1} because the terms of the solution series can be extremely sensitive to the values that l_i and ζ_i assume at each constancy interval I_i. Hence, the convergence of the series defining the solution of (1.3) has to be verified in each constancy interval. However, this corresponds to compute infinite terms, implying that the solution of (1.3) cannot be approximated nor computed in general. Our aim is to establish a computability criterion, and bounds for the solution of the PDEPCAG (1.3). To do so, we show that under general assumptions for γ, a and b, the terms of the solution series behave as $(\frac{1}{j})^n$ for $t \in I_n$, when j is large enough.

The paper is organized as follows: We obtain the solution $u(x,t)$ of Eq. (1.3) in Section 2. In Section 3 we study the convergence of $u(x,t)$, and establish a criterion, that requires the computation of finite terms of the series, to ensure that

$$\left|u(x,t)\right| \le Ce^{-\alpha t}\|u_0\|, \quad \text{where } C, \alpha > 0, \text{ and } \|\cdot\| \text{ is a suitable norm.} \tag{1.4}$$

2. Solution of Eq. (1.3)

Separation of variables $u(x,t) = X(x)T(t)$ in (1.3) leads to the boundary value problem

$$X'' + \lambda^2 X = 0, \qquad X(0) = X(1) = 0,$$

which has solutions given by the orthonormal set

$$X_j(x) = \sqrt{2}\sin(\pi j x), \quad j \in \mathbb{N},\ x \in [0,1]. \tag{2.1}$$

Writing the solution $u(x,t)$ as superposition of $T_j(t)X_j(x)$, we obtain

$$u(x,t) = \sum_{j=1}^{\infty} T_j(t)X_j(x). \tag{2.2}$$

Now, for t in $[0,\infty)$ and $j \in \mathbb{N}$, $T_j(t)$ satisfies the DEPCAG

$$\begin{aligned} T_j'(t) &= -a^2(t)\pi^2 j^2 T_j(t) - b(t)T_j\big(\gamma(t)\big) \\ T_j(0) &= T_{j0}. \end{aligned} \tag{2.3}$$

If $t = 0$, (2.2) becomes

$$u_0(x) = u(x,0) = \sqrt{2}\sum_{j=1}^{\infty} T_j(0)\sin(\pi j x). \tag{2.4}$$

Then, $T_j(0)$ is the j-th Fourier coefficient of $u_0(x)$, i.e.:

$$T_j(0) = \sqrt{2}\int_0^1 \sin(\pi j x)u_0(x)dx = T_{j0}. \tag{2.5}$$

The properties of the solutions of (2.3) in combination with (2.1) form the basis of study for (1.3). Eq. (2.3) with $j = 1$ is studied in [2]. Indeed, let

$$M_j(\tau,t) = \exp\left(-\pi^2 j^2 \int_\tau^t a^2(r)dr\right)\left(1 - \int_\tau^t b(s)\exp\left(\pi^2 j^2 \int_\tau^s a^2(r)dr\right)ds\right). \tag{2.6}$$

The function $M_l(t)$ defined in [2] corresponds to $M_1(\zeta_l, t)$.

We recall Lemma 2.2 from [2] which summarizes the solution of Eq. (2.3).

Lemma 2.1. *Let $j \in \mathbb{N}$ and*

$$M_j(\zeta_i, \theta_i) \neq 0 \quad \text{for all } i \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

The solution of Eq. (2.3) is well defined for all $t \geq 0$ and is given by

$$T_j(t) = T_{j0} \prod_{i=0}^{n-1} \left(\frac{M_j(\zeta_i, \theta_{i+1})}{M_j(\zeta_i, \theta_i)}\right) \frac{M_j(\zeta_n, t)}{M_j(\zeta_n, \theta_n)}, \quad t \in I_n. \tag{2.8}$$

Lemma 2.1 establishes the conditions for the existence of a solution for Eq. (2.3). If we assume a non-general piecewise constant argument γ, i.e. $l^- = \theta_{i+1} - \zeta_i$ and $l^+ = \zeta_i - \theta_i$ are constant sequences, and $a(t) = a$ and $b(t) = b$ are constant functions, the analysis of $T_j(t)$ is considerably simpler. From now on we refer to these conditions as the *constancy case*, or **CC**.

Corollary 2.1. *Let $j \in \mathbb{N}$ and assume* **CC**. *If*

$$l^+ \neq \frac{1}{a^2\pi^2 j^2} \ln\left(\frac{b}{a^2\pi^2 j^2 - b}\right), \tag{2.9}$$

and

$$\left|\frac{(a^2\pi^2 j^2 + b)e^{-a^2\pi^2 j^2 l^-} - b}{(a^2\pi^2 j^2 + b)e^{a^2\pi^2 j^2 l^+} - b}\right| < 1, \tag{2.10}$$

then, there exist $C, \alpha > 0$ such that $|T_j(t)| \leq Ce^{-\alpha t}$ for all $x \in [0,1]$, $t > 0$.

Proof. Under **CC**, we have that $M_j(\tau,t) = (1 + \frac{b}{a^2\pi^2 j^2})e^{-a^2\pi^2 j^2(t-\tau)} - \frac{b}{a^2\pi^2 j^2}$, and hence (2.9) is equivalent to (2.7). Moreover, Eq. (2.10) implies that $|\frac{M_j(\zeta_n,\theta_{n+1})}{M_j(\zeta_n,\theta_n)}| = \rho_j < 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$\left|T_j(t)\right| \leq |T_{j0}|\rho_j^n \max_{t \in I_0}\left(\left|\frac{(a^2\pi^2 j^2 + b)e^{-a^2\pi^2 j^2 t} - b}{(a^2\pi^2 j^2 + b)e^{a^2\pi^2 j^2 l^+} - b}\right|\right) \leq Ce^{-\alpha t},$$

where $C = |T_{j0}| \max_{t \in I_0}(|\frac{(a^2\pi^2 j^2 + b)e^{-a^2\pi^2 j^2 t} - b}{(a^2\pi^2 j^2 + b)e^{a^2\pi^2 j^2 l^+} - b}|)$, and $\alpha = \frac{1}{l^+ + l_-} \ln(\frac{1}{\rho_j})$. □

In previous work in PDEPCA [13,26,23], exponential bounds of (2.8) have been obtained under **CC** assuming particular deviations such as $\gamma(t) = [t]$ and $\gamma(t) = [t + \frac{1}{2}]$. In [28], although a more complex case assuming $b(t) = b\cos(\pi t)$ is considered, the analysis of $T_j(t)$ is essentially the same because of the regularity of the piecewise constant argument $\gamma(t) = [t + \frac{1}{2}]$ they assume.

We claim that the analysis of $T_j(t)$ in PDEPCA and in PDEPCAG are fundamentally different due to (2.7), even for the simplest case when $a(t)$ and $b(t)$ are constant. Indeed, note that (2.7) is equivalent

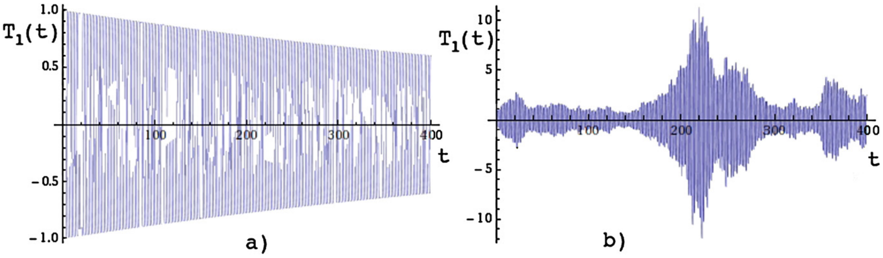


Fig. 1. PDEPCA vs PDEPCAG.

to condition (2.9) under **CC**, but to an infinite number of inequalities in PDEPCAG. This implies that the criteria for the existence of the solution $T_j(t)$ in $[0, \infty)$ is not computable in general for PDEPCAG. Although the solution can be recursively computed verifying condition (2.7) for each interval I_i, $i \in \mathbb{N} \cup \{0\}$, for real applications this is an important algorithmic obstacle because if condition (2.7) is not fulfilled for some interval I_i, then the computation of $T_j(t)$ must stop. Hence, the domain where $T_j(t)$ is well defined cannot be known in advance. To illustrate this problem, we plot $T_1(t)$ for a non-general and for a general piecewise constant argument in Fig. 1, with $a(t)$ and $b(t)$ being constant functions. Namely, plot **a)** assumes a piecewise constant argument γ of constant length and deviation, i.e. **CC** is hold, and plot **b)** considers a small perturbation of γ, and thus a general piecewise constant argument. Note that this slightly perturbed piecewise argument, which can be for example assumed when natural imperfections of a control system are considered, radically modifies the behaviour of $T_1(t)$, and hence the behaviour of the solution of Eq. (1.3).

Theorem 2.1. *Assuming condition (2.7) for all $j \in \mathbb{N}$, the solution $u(x, t)$ of the PDEPCAG (1.3) can be written as*

$$u(x,t) = \sum_{j=1}^{\infty} T_{j0} \left(\prod_{i=0}^{n-1} \frac{M_j(\zeta_i, \theta_{i+1})}{M_j(\zeta_i, \theta_i)} \right) \frac{M_j(\zeta_n, t)}{M_j(\zeta_n, \theta_n)} \sqrt{2} \sin(\pi j x), \quad t \in I_n, \tag{2.11}$$

where M_j is given by (2.6) and T_{j0} is given by (2.5).

Theorem 2.1 constitutes the explicit solution of equation for the PDEPCAG (1.3). The simplified case under **CC** is stated in the following corollary.

Corollary 2.2. *Assuming* **CC**, *then Eq. (2.11) can be written as*

$$u(x,t) = \sqrt{2} \sum_{j=0}^{\infty} T_{j0} \left(\frac{\hat{M}_j(l^-)}{\hat{M}_j(-l^+)} \right)^n \frac{\hat{M}_j((t-\theta_n) - l^+)}{\hat{M}_j(-l^+)} \sin(\pi j x), \tag{2.12}$$

where $\hat{M}_j(t) = (1 + \frac{b}{a^2\pi^2 j^2}) e^{-a^2\pi^2 j^2 t} - \frac{b}{a^2\pi^2 j^2}$.

In previous studies of PDEPCA, it is trivial and implicit the fact that $|T_j(t)|$ is monotonically decreasing with respect to j because **CC** is assumed. This allows to estimate $|u(x, t)|$ from an estimation of $|T_1(t)|$. See for example the proofs of Theorems 2.1 and 2.2 in [13]. In PDEPCAG however, this is not the case. First, the well-definiteness of $T_j(t)$ has to be verified for each $j \in \mathbb{N}$. This entails a non-computable task as it requires infinite calculations in general. Secondly, even when $T_j(t)$ is well defined for all $j \in \mathbb{N}$, the series (2.11) may not converge. In the following section we will elucidate general conditions under which the convergence of the series $u(x, t)$ can be verified from a finite number of calculations, and hence providing a computationally feasible existence criteria. Moreover, we study the cases where $|u(x, t)|$ can be bounded by an exponentially decreasing function.

3. Convergence and bounds of $u(x,t)$

Definition 2. Let $t \in I_n$. We define $i(t) = n$ and $\rho_j(t) = |\frac{M_j(\zeta_n,t)}{M_j(\zeta_n,\theta_n)}|$.

The following lemma is a general condition for the exponential convergence to zero of $T_j(t)$ when $t \to \infty$.

Lemma 3.1. *Let j be a fixed integer, $\bar{t}_j > 0$, $0 < \varrho_j < 1$ and $\varOmega_j < \infty$ such that for $t > \bar{t}_j$*

$$\rho_j(\theta_{i(t)}) \leq \varrho_j \quad \textit{and} \quad \rho_j(t) \leq \varOmega_j. \tag{3.1}$$

Then, $|T_j(t)| \leq |T_{j0}|c_j e^{-\alpha_j t}$ for $t > \theta_{i(\bar{t}_j)}$, where $c_j > 0$ and $\alpha_j > 0$.

Proof. Let $t \in I_n = [\theta_n, \theta_{n+1})$, $n > N_j = i(\bar{t}_j)$. From (3.1) we obtain

$$\left|T_j(t)\right| \leq |T_{j0}|\left|\prod_{i=0}^{N_j} \rho(\theta_{i+1})\right| \varOmega_j \varrho_j^{n-N_j-1} \leq |T_{j0}|c_j e^{-\alpha_j t}, \tag{3.2}$$

where

$$c_j = \left|\prod_{i=0}^{N_j} \rho(\theta_{i+1})\right| \varOmega_j \varrho_j^{-N_j-1}, \qquad \alpha_j = \left(\frac{1}{L^*}\right) \ln\left(\frac{1}{\varrho_j}\right) > 0, \tag{3.3}$$

with L^* defined in (1.2) ☐

Proposition 3.1. *If $\sup_{j\in\mathbb{N}}(\rho_j(t)) < \infty$ and $\|u_0\|_1 = \sum_{j=1}^{\infty} |T_{j0}| < \infty$, then the series (2.2) defining the solution $u(x,t)$ of Eq. (1.3) converges uniformly for $x \in [0,1]$ for all $t > 0$. Moreover, let $k \in \mathbb{N}$, $\bar{\varOmega} > 0$ such that for all $j \in \mathbb{N}$ there exist $\bar{t}_j, \varrho_j > 0$ such that for $t > \theta_{i(\bar{t}_j)}$*

$$\rho_j(t) \leq \bar{\varOmega} t^k, \quad \textit{and} \quad \rho_j(\theta_n) \leq \varrho_j. \tag{3.4}$$

If $\varrho = \sup_{j\in N}(\varrho_j) < 1$ and $\bar{t} = \sup_{j\in\mathbb{N}}(\bar{t}_j) < \infty$, then $u(x,t) \leq Ce^{-\alpha t}\|u_0\|_1$ for $t > \max(\bar{t}, \frac{k}{\alpha})$, where $\alpha = \frac{1}{L^*}\ln(\frac{1}{\varrho})$, with L^* defined in (1.2), and C is a constant.

Proof. Note that

$$\left|u(x,t)\right| \leq \left|\sum_{j=1}^{\infty} \sqrt{2} T_{j0} \prod_{i=0}^{i(t)} \rho_j(\theta_i)\rho_j(t)\sin(\pi j x)\right|. \tag{3.5}$$

We have that

$$\left|u(x,t)\right| \leq \sqrt{2}\Big(\max_{0\leq s\leq t}\Big(\sup_{j\in\mathbb{N}} \rho_j(s)\Big)\Big)^{i(t)+1} \|u_0\|_1.$$

To prove the second assertion, we apply condition (3.4) in Eq. (3.5) and obtain that for $t > \bar{t}$ we have

$$\left|u(x,t)\right| \leq \sqrt{2}\Big(\max_{0\leq s\leq \bar{t}}\Big(\sup_{j\in\mathbb{N}} \rho_j(s)\Big)\Big)^{i(\bar{t})} \varrho^{i(t)-i(\bar{t})} \varOmega t^k \sum_{j=1}^{\infty} |T_{j0}| = \bar{C} t^k e^{-\alpha t}\|u_0\|_1 \tag{3.6}$$

where $\bar{C} = \sqrt{2}(\max_{o\leq s\leq \bar{t}}(\sup_{j\in\mathbb{N}} \rho_j(s)))^{i(\bar{t})} \varrho^{-i(\bar{t})} \varOmega$. We have that $\bar{C} t^k e^{-\alpha t}$ is a decreasing function for $t \geq \max(\bar{t}, \frac{k}{\alpha})$. Hence, for $t \geq \tau = \max(\bar{t}, \frac{k}{\alpha})$ we have

$$\left|u(x,t)\right| \leq \bar{C}\tau^k e^{-\alpha t}\|u_0\|_1 = Ce^{-\alpha t}\|u_0\|_1, \quad \text{where } C = \bar{C}\tau^k. \qquad \square$$

Proposition 3.1 establishes conditions under which the solution of the PDEPCAG (1.3) converges, and also supply extra conditions under which the series of the solution is exponentially bound. Is important to note that Proposition 3.1 requires $\|u_0\|_1 < \infty$. This condition has been implicitly assumed in estimations of $|u(x,t)|$ done in previous works in PDEPCA (e.g. see Theorems 2.1 and 2.2 of [13]). Therefore, Proposition 3.1 generalizes any **CC** and previous work where the convergence of $u(x,t)$ is ensured [13,26,23].

However, note that Proposition 3.1 requires the estimation of $\sup_{j\in\mathbb{N}}(\rho_j(t))$ for each t. This is necessary because each term of the series (2.11) may be unbounded. This occurs for example, when for some $i,j \in \mathbb{N}$ (2.7) is violated. Hence, the criterion established in Proposition 3.1 is not computable in general. The following proposition establishes conditions under which the convergence of the series (2.11) can be verified computing a finite number of terms of the series only, without requiring $\|u_0\|_1 < \infty$, or any other regularity condition on u_0.

Proposition 3.2. *Let $J \in \mathbb{N}$, $\bar{t} > 0$. If there exist $\Omega > 0$ and a function $\bar{\Omega}(\cdot) \geq 0$ such that for $j \geq J$*

$$\rho_j(t) \leq \bar{\Omega}(t) \quad \textit{for } t \leq \bar{t}, \quad \textit{and} \quad \rho_j(t) \leq \frac{\Omega}{j} < 1 \quad \textit{for } t > \bar{t}, \tag{3.7}$$

and condition (3.1) is satisfied for $j < J$. Then

$$\big|u(x,t)\big| \leq Ce^{-\alpha t}\|u_0\|_2, \quad \textit{for } t > \theta_{i(\bar{t})} \tag{3.8}$$

where $C, \alpha > 0$, and $\|u_0\|_2 = (\sum_{j=1}^{\infty} |T_{j0}|^2)^{\frac{1}{2}}$.

Proof. Note that $\sum_{j=1}^{\infty} |\frac{T_{j0}}{j}|$ converges since $\sum_{j=1}^{\infty} |T_{j0}|^2$ converges. For $t > \theta_{i(\bar{t})}$ we have that

$$\begin{aligned}
\big|u(x,t)\big| &\leq \left|\sum_{j=1}^{J-1} \sqrt{2}T_j(t)\sin(\pi jx)\right| + \left|\sum_{j=J}^{\infty} \sqrt{2}\prod_{i=0}^{i(t)} \rho_j(\theta_i)\rho_j(t)T_{j0}\sin(\pi jx)\right| \\
&\leq \left|\sum_{j=1}^{J-1} \sqrt{2}T_j(t)\sin(\pi jx)\right| + \sqrt{2}\max_{s\leq\bar{t}}\big(\sup_j\big(\bar{\Omega}(s)\big)\big)^{i(\bar{t})}\Omega\left(\frac{\Omega}{j}\right)^{(i(t)-i(\bar{t})-1)} \sum_{j=J}^{\infty}\left|\frac{T_{j0}}{j}\right|.
\end{aligned} \tag{3.9}$$

Applying Lemma 3.1 on (3.9) we obtain

$$\big|u(x,t)\big| \leq \sum_{j=1}^{J-1} jc_j e^{-\alpha_j t}\left|\frac{T_{j0}}{j}\right| + \bar{C}e^{-\bar{\alpha}t}\|u_0\|_2 \leq Ce^{-\alpha t}\|u_0\|_2, \tag{3.10}$$

where $C = \max_{j<J}(jc_j; \bar{C})$, and $\alpha = \min_{j<J}(\alpha_j; \bar{\alpha})$, with c_j and α_j given by (3.3),

$$\bar{C} = \sqrt{2}\max_{s\leq\bar{t}}\big(\bar{\Omega}(s)\big)^{i(\bar{t})}\Omega\left(\frac{\Omega}{J}\right)^{-(i(\bar{t})+1)}, \quad \text{and} \quad \bar{\alpha} = \frac{1}{L^*}\ln\left(\frac{J}{\Omega}\right),$$

with L^* defined in (1.2). □

Proposition 3.2 entails a novel methodology to study the solution of Eq. (1.3). In particular, it does not require any special regularity to the initial condition u_0. To prove the uniform convergence of (2.11), Proposition 3.2 combines the convergence of $|T_j(t)|$ in the variable t by Lemma 3.1, to control the first $J-1$ terms of (2.11), with a condition that bounds the value of $\rho_j(t)$ for $j \geq J$ and $t > \bar{t}$, and hence insures the convergence of $u(x,t)$.

Note that if condition (3.7) is hold, then the convergence of $u(x,t)$ is verified computing the first $J-1$ terms of only. We now establish a general case under which Proposition 3.2 can be applied.

Lemma 3.2. *Let*

$$0 < a_- = \inf_{t\in[0,\infty)} a(t)^2 \quad \textit{and} \quad \sup_{t\in[0,\infty)} \bigl|b(t)\bigr| = B < \infty. \tag{3.11}$$

Then, for $j > \frac{1}{\pi}\sqrt{\frac{B}{a_-}}$, $\rho_j(t)$ *of Definition 2, satisfies*

$$\rho_j(t) \le e^{-a_j(t-\theta_n)} + \frac{3B}{a_j - B}, \quad \textit{with } a_j = \pi^2 j^2 a_-,\ t \in I_n. \tag{3.12}$$

Proof. Note that

$$\begin{aligned}\left|\int\limits_{\theta_n}^{\zeta_n} b(s)\exp\left(-\pi^2 j^2 \int\limits_{s}^{\zeta_n} a^2(r)dr\right)ds\right| &\le \int\limits_{\theta_n}^{\zeta_n} Be^{-a_j(\zeta_n - s)}ds \\ &= \frac{B}{a_j}\bigl(1 - e^{-a_j(\zeta_n - \theta_n)}\bigr).\end{aligned}$$

Then for $j > \frac{1}{\pi}\sqrt{\frac{B}{a_-}}$, we have

$$1 + \int\limits_{\theta_n}^{\zeta_n} b(s)\exp\left(-\pi^2 j^2 \int\limits_{s}^{\zeta_n} a^2(r)dr\right)ds \ge 1 - \frac{B}{a_j} > 0.$$

Now, for $j > \frac{1}{\pi}\sqrt{\frac{B}{a_-}}$ and $\theta_n < t < \zeta_n$, we have

$$\begin{aligned}&\exp\left(-\pi^2 j^2 \int\limits_{\theta_n}^{t} a^2(r)dr\right)\left(1 + \left|\int\limits_{t}^{\zeta_n} b(s)\exp\left(-\pi^2 j^2 \int\limits_{s}^{\zeta_n} a^2(r)dr\right)ds\right|\right) \\ &\quad \le e^{-a_j(t-\theta_n)}\left(1 + B\int\limits_{t}^{\zeta_n} e^{-a_j(\zeta_n - s)}ds\right) \le e^{-a_j(t-\theta_n)}\left(1 + \frac{B}{a_j}\right) + \frac{B}{a_j}.\end{aligned}$$

For $\theta_n \le \zeta_n < t$, we have

$$\begin{aligned}&\left|\exp\left(-\pi^2 j^2 \int\limits_{\theta_n}^{t} a^2(r)dr\right)\left(1 - \int\limits_{\zeta_n}^{t} b(s)\exp\left(\pi^2 j^2 \int\limits_{\zeta_n}^{s} a^2(r)dr\right)ds\right|\right) \\ &\quad \le e^{-a_j(t-\theta_n)} + e^{-a_j(\zeta_n - \theta_n)}B\int\limits_{\zeta_n}^{t} e^{-\pi^2 j^2 \int_s^t a^2(r)dr}ds \le e^{-a_j(t-\theta_n)}\left(1 + \frac{B}{a_j}\right) + \frac{B}{a_j}.\end{aligned}$$

Thus, for $j > \frac{1}{\pi}\sqrt{\frac{B}{a_-}}$ we have

$$\rho_j(t) \le \frac{e^{-a_j(t-\theta_n)}(a_j + B) + B}{a_j - B} = e^{-a_j(t-\theta_n)} + \frac{3B}{a_j - B}. \qquad \square$$

Corollary 3.1. *Assume (3.11) holds. Let*

$$\Omega_j = \frac{1}{\pi\sqrt{2ea_- l^*}} + \frac{3Bj}{j^2\pi^2 a_- - B}. \tag{3.13}$$

Then, there exists J^* such that for $j > J^*$ we have

$$\rho_j(\theta_{i(t)}) \le \frac{\Omega_j}{j} < 1, \quad \textit{for } t > 0. \tag{3.14}$$

Proof. Solving $\frac{\partial}{\partial j} j e^{-\pi^2 j^2 (t-\theta_n)a_-} = 0$, we find that $je^{-\pi^2 j^2 (t-\theta_n)a_-}$ is a decreasing function with respect to j, for $j > j^* = \frac{1}{\pi}\sqrt{\frac{1}{2a_- l^*}}$, and that $je^{-\pi^2 j^2 (t-\theta_n)a_-} \le \frac{1}{\pi\sqrt{2ea_- l^*}}$. Then from (3.12) we have that for $j > j^*$

$$\rho_j(l^*) \le \frac{1}{j}\left(\frac{1}{j^*\sqrt{e}} + \frac{3Bj}{\pi^2 j^2 a_- - B}\right) \le \frac{\Omega_j}{j}. \qquad \square \tag{3.15}$$

The following result summarizes the convergence and exponential bound for the solution of Eq. (1.3) under condition (3.11).

Theorem 3.1. *If (3.11) holds, then for all $t \in [0,\infty)$ the series (2.11) of $u(x,t)$ converges absolutely and uniformly for $x \in [0,1]$. Moreover, let Ω_j be defined by (3.13) and $j = J^* > 0$ be a solution of Eq. (3.15). If condition (3.1) holds for $t > t_j$, $j \le J^*$, then $u(x,t)$ satisfies (3.8) for $t > \max_{j\le J^*}(t_j)$.*

Proof. The first assertion follows from Proposition 3.1 and (3.12). To prove the second assertion we apply the bounds $\rho_j(t) \le 1 + \frac{3B}{a_{J^*} - B}$ from (3.12), and $\rho_j(\theta_{i(t)}) < \frac{\Omega_{J^*}}{j} < 1$ by (3.14), on Proposition 3.2 for $j > J^*$. $\square$

4. Conclusion

We study Eq. (1.3) with the general piecewise constant argument $\gamma(t)$ detailed in (1.2). Separation of variables lead to a solution formed by a series of products. Contrasting previous work in PDEPCA [12,13,21,25,26] and for all **CC** case, the convergence for $t > 0$ of the solution $u(x,t)$ for the PDEPCAG (1.3) cannot be deduced from the first constancy interval of γ. Moreover, each term of the solution series (2.11) may be unbounded. Therefore, a computable convergence criterion for the solution is necessary in PDEPCAG. To illustrate the strong differences between PDEPCA and PDEPCAG, we plot the solutions of Eq. (2.3) for a non-general piecewise constant argument γ, and for a general piecewise constant argument which corresponds to a small perturbation of γ, for $j = 1$, with $a(t)$ and $b(t)$ being constant functions.

Concerning the solution $u(x,t)$ of Eq. (1.3), in Proposition 3.1 we present a criterion for the convergence of $|u(x,t)|$ in t, uniformly for $x \in [0,1]$, that requires $\|u_0\|_1$ to be finite, and extend this criterion obtaining an exponential bound for the series of $|u(x,t)|$. The results in Proposition 3.1 are a direct generalization of what has been developed in PDEPCA. However, the conditions of Proposition 3.1 are non-computable in general because require the estimation of $\sup_{j\in\mathbb{N}}(\rho_j(t))$. Hence, Proposition 3.1 cannot be considered as a useful criterion for PDEPCAG.

As an alternative, in Proposition 3.2 we establish a criterion which allows to decide the convergence, and to provide an exponential bound to the solution of (1.3), from finite terms of the series only, and without requiring any regularity on the initial condition $u_0(x)$. Moreover, in Corollary 3.1 we show that under the ample conditions specified in (3.11), $|T_j(t)|$ behaves as $(\frac{1}{j})^{i(t)}$, when j is large enough, and hence Proposition 3.2 can be applied (see Theorem 3.1).

The rationale of Proposition 3.2 and Theorem 3.1 allows to develop computationally tractable numerical estimations for the solution of (1.3) as well as for the solution of other PDEPCAG. Thus, it has the potential to become general piecewise approximation method for PDE.

Acknowledgments

One of the authors has been supported by: FONDECYT 1080034 and 1120709.

References

[1] M.U. Akhmet, On the integral manifolds of the differential equations with piecewise constant argument of generalized type, Nonlinear Anal. 66 (2) (2007) 367–383.
[2] M.U. Akhmet, Stability of differential equations with piecewise constant arguments of generalized type, Nonlinear Anal. 68 (4) (2008) 794–803.
[3] M.U. Akhmet, Almost periodic solutions of differential equations with piecewise constant argument of generalized type, Nonlinear Anal. Hybrid Syst. 2 (2) (2008) 456–467.
[4] M.U. Akhmet, Principles of Discontinuous Dynamical Systems, Springer, New York, 2010.
[5] M.U. Akhmet, Nonlinear Hybrid Continuous/Discrete-Time Models, Atlantic Press, 2011.
[6] M.U. Akhmet, C. Büyükadall, Differential equations with state-dependent piecewise constant argument, Nonlinear Anal. 72 (11) (2010) 4200–4210.
[7] A. Chavez, S. Castillo, M. Pinto, Discontinuous almost automorphic functions and almost automorphic solutions of differential equations with piecewise constant argument, Electron. J. Differential Equations 56 (2014) 1–13.
[8] K.-S. Chiu, M. Pinto, Periodic solutions of differential equations with a general piecewise constant argument and applications, Electron. J. Qual. Theory Differ. Equ. 46 (2010) 1–19.
[9] K.-S. Chiu, M. Pinto, J.-C. Jeng, Existence and global convergence of periodic solutions in recurrent neural network models with a general piecewise alternately advanced and retarded argument, Acta Appl. Math. (2013) 1–20.
[10] K.L. Cooke, I. Györi, Numerical approximation of the solutions of delay differential equations on an infinite interval using piecewise constant arguments, Comput. Math. Appl. 28 (1994) 81–92.
[11] K.L. Cooke, J. Turi, G. Turner, Stabilization of Hybrid Systems in the Presence of Feedback Delays, IMA Preprint Series 906, Institute of Mathematics and Its Applications, University of Minnesota, 1991.
[12] K.L. Cooke, J. Wiener, Retarded differential equations with piecewise constant delays, J. Math. Anal. Appl. 99 (1984) 265–297.
[13] L. Debnath, J. Wiener, Boundary value problems for the diffusion equation with piecewise continuous time delay, Int. J. Math. Math. Sci. 20 (1) (1997) 187–195.
[14] Z. Huang, X. Wang, Feng Gao, The existence and global attractivity of almost periodic sequence solution of discrete-time neural networks, Phys. Lett. A 350 (3) (2006) 182–191.
[15] Z. Huang, Y. Xia, X. Wang, The existence and exponential attractivity of k-almost periodic sequence solution of discrete time neural networks, Nonlinear Dynam. 50 (1–2) (2007) 13–26.
[16] A.D. Myshkis, On certain problems in the theory of differential equations with deviating argument, Uspekhi Mat. Nauk 32 (1977) 173–202.
[17] M. Pinto, Asymptotic equivalence of non-linear and quasilinear differential equations with piecewise constant argument, Math. Comput. Modelling 49 (2009) 1750–1758.
[18] M. Pinto, Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems, J. Difference Equ. Appl. 17 (2) (2011) 235–254.
[19] M. Pinto, G. Robledo, Controllability and observability for a linear time varying system with piecewise constant delay, Acta Appl. Math. (2014) 1–24.
[20] F. Qiuxiang, Y. Rong, On the Lasota–Wazewska model with piecewise constant argument, Acta Math. Sci. Ser. B Engl. Ed. 26 (2) (2006) 371–378.
[21] S.M. Shah, J. Wiener, Advanced differential equations with piecewise constant argument deviations, Int. J. Math. Math. Sci. 6 (1983) 671–703.
[22] M.H. Song, Z.W. Yang, M.Z. Liu, Stability of θ-methods for advanced differential equations with piecewise continuous arguments, Comput. Math. Appl. 49 (9–10) (2005) 1295–1301.
[23] J. Wiener, Generalized Solutions of Functional Differential Equations, World Scientific Publishing Co. Pte. Ltd., Singapore, 1993.
[24] J. Wiener, A survey of partial differential equations with piecewise continuous arguments, Int. J. Math. Math. Sci. 18 (2) (1995) 209–228.
[25] J. Wiener, A.R. Aftabizadeh, Differential equations alternately of retarded and advanced type, J. Math. Anal. Appl. 129 (1988) 243–255.
[26] J. Wiener, L. Debnath, Partial differential equations with piecewise constant delay, Int. J. Math. Math. Sci. 14 (1991) 485–496.
[27] J. Wiener, L. Debnath, A wave equation with discontinuous time delay, Int. J. Math. Math. Sci. 15 (4) (1992) 781–788.
[28] J. Wiener, W. Heller, Oscillatory and periodic solutions to a diffusion equation of neutral type, Int. J. Math. Math. Sci. 22 (2) (1999) 313–348.
[29] Y. Xia, Z. Huang, M. Han, Existence of almost periodic solutions for forced perturbed systems with piecewise constant argument, J. Math. Anal. Appl. 333 (2) (2007) 798–816.
[30] Z.W. Yang, M.Z. Liu, J.J. Nieto, Runge–Kutta methods for first-order periodic boundary value differential equations with piecewise constant arguments, J. Comput. Appl. Math. 233 (4) (2009) 990–1004.
[31] R. Yuan, On Favard's theorems, J. Difference Equ. Appl. 249 (2010) 1884–1916.