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Moreau envelopes of s-lower regular functions

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ABSTRACT

The paper investigates various properties of Moreau envelopes and proximal mappings of s-lower regular functions. In particular, for such a function on uniformly convex space, the Moreau envelope is shown to be differentiable with its derivative locally Hölder continuous.

1. Introduction

Let H be a Hilbert space and $f : H \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. The Moreau envelope function denoted by $e_\lambda f$ (see Section 3 for definition) for some positive real λ, plays important roles in optimization, dynamic systems and differential inclusions. The strong connection in convex analysis between lower semicontinuous convex functions and their Moreau envelopes as well as the nice properties of Moreau envelopes of such functions motivated several authors to investigate properties of Moreau envelopes of functions of some classes of nonconvex functions.

L. Thibault and D. Zagrodny [32] showed how Moreau envelopes can allow us to obtain the subdifferential determination (or integration) for many classes of nonconvex functions. R.A. Poliquin and R.T. Rockafellar [28] deeply studied the Moreau envelope $e_\lambda f$ of a *prox-regular function* f in finite-dimensional spaces.

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Later, F. Bernard and L. Thibault [4] adapted the concept of prox-regularity to infinite dimensional Hilbert spaces and proved in this setting several fundamental results of [28] concerning the Moreau envelope and its proximal mapping (see also [3]); we refer to [29,6,7] for similar studies concerning the class of *prox-regular sets*. More recently and again in the Hilbert context, S. Marcellin and L. Thibault [23] and M. Mazade and L. Thibault [24] studied the class of primal lower nice functions (pln for short) which has been introduced by R.A. Poliquin [27] in $\mathbb{R}^n$. They showed that suitable versions of the main results of [27] still hold true in the setting of Hilbert space in proving that $e_\lambda f$ is $\mathcal{C}^{1,1}$ on a neighborhood of each point u_0 where f is pln and in giving several properties concerning the proximal mapping $P_\lambda f$ (see Section 3 for definition). A. Jourani, L. Thibault and D. Zagrodny [18] established various subregularity properties of $e_\lambda f$ as well as the description of the subdifferential of $e_\lambda f$ when the norm $\|\cdot\|$ of X is Gâteaux differentiable.

Our aim in this paper consists in extending in some sense the study of the Moreau envelope and its proximal mapping to another important new class of functions, namely the *s-lower regular functions*, continuing in this way our work [19]. We generalize the results obtained in [28,4,23,24] in more general spaces as uniformly convex and uniformly smooth Banach spaces, in view of further applications to evolution problems in L^p-spaces. We prove that in this setting, the Moreau envelope $e_{\lambda,s}f$ is $\mathcal{C}^{1,\frac{\alpha}{2}}$ on some neighborhood U of a point $u_0 \in \operatorname{dom} f$ whenever f is s-lower regular near u_0 and $\alpha+1$ is the power type of the modulus of smoothness of the norm of X (see Section 2.2 for the definition). The $\mathcal{C}^{1,\frac{\alpha}{2}}$ property of the function $e_{\lambda,s}f$ means that it is differentiable on U and its derivative is Hölder continuous therein with $\frac{\alpha}{2}$ as Hölder power.

In Section 2, we give some definitions and properties which will be used throughout the paper. In Section 3, we state the definitions of Moreau envelope and proximal mapping $e_{\lambda,s}f$ and $P_{\lambda,s}f$, and prove many properties of these functions as the Lipschitz property when the function f is bounded from below by a suitable function. The expression of the Fréchet subdifferential of $e_{\lambda,s}f$ is also obtained in terms of $P_{\lambda,s}f$ when the norm $\|\cdot\|$ is Fréchet differentiable off zero.

In the last section, we introduce the notion of (s,r)-hypomonotone operator. In the context of uniformly convex and uniformly smooth Banach spaces we prove via this operator that the Moreau envelope $e_{\lambda,s}f$ of a lower s-regular function f is of class $\mathcal{C}^{1,\frac{\alpha}{2}}$, where $\alpha+1$ is the power of the modulus of smoothness of the norm of X.

2. Preliminaries

Throughout the paper, unless otherwise stated, X is a Banach space endowed with the norm $\|\cdot\|$ and X^* its topological dual equipped with the dual norm $\|\cdot\|_*$. The closed unit ball of X centered at zero will be denoted by $\mathbb{B}_X$ or $\mathbb{B}$, and $B[x,r]$ (resp. $B(x,r)$) is the closed (resp. open) ball of radius $r>0$ centered at the point $x \in X$.

2.1. Subdifferential properties of s-lower regular functions

First, we need to recall some definitions and properties concerning the concepts of subdifferentials and s-lower regular functions. In order to define the Clarke subdifferential, let us start with the Clarke tangent cone of a set. Let S be a set of X and $\overline{x} \in S$. The *Clarke tangent cone* of S at $\overline{x}$ is defined as the *Painlevé–Kuratowski limit inferior* of the set-differential quotient

$$T^C(S,\overline{x}) := \operatorname*{Lim\,inf}_{t\downarrow 0,\, x \xrightarrow[S]{} \overline{x}} \frac{1}{t}(S-x),$$

or equivalently a vector $h \in T^C(S,\overline{x})$ if and only if, for any sequence $(t_n)_{n\in\mathbb{N}}$ tending to 0 with $t_n>0$ and $(x_n)_{n\in\mathbb{N}}$ in S converging to $\overline{x}$, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ converging to h such that

$$x_n + t_n h_n \in S \quad \text{for all } n \in \mathbb{N}.$$

The *Clarke normal cone* $N^C(S,\overline{x})$ of S at $\overline{x}$ is the negative polar $(T^C(S,\overline{x}))^\circ$ of the Clarke tangent cone, that is,

$$N^C(S,\overline{x}) := \{x^* \in X^* : \langle x^*, h\rangle \le 0, \forall h \in T^C(S,\overline{x})\}.$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and let $\overline{x} \in \operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. The *Clarke subdifferential* $\partial_C f(\overline{x})$ of f at $\overline{x}$ is defined by

$$\partial_C f(\overline{x}) := \{x^* \in X : (x^*, -1) \in N^C(\operatorname{epi} f, (\overline{x}, f(\overline{x})))\},$$

where $\operatorname{epi} f := \{(x, r) \in H \times \mathbb{R} : f(x) \le r\}$ is the epigraph of f. When $\overline{x} \notin \operatorname{dom} f$, one puts $\partial_C f(\overline{x}) = \emptyset$. Denoting by $\operatorname{Dom} T$ the (effective) domain of a set-valued operator $T : X \rightrightarrows X^*$, that is,

$$\operatorname{Dom} T := \{x \in X : T(x) \ne \emptyset\},$$

whenever f is lower semicontinuous on the Banach space X, then $\operatorname{Dom} \partial_C f$ is *graphically dense* in $\operatorname{dom} f$, in the sense that for each $x \in \operatorname{dom} f$ there exists a sequence $(x_n)_n$ with $x_n \in \operatorname{Dom} \partial_C f$ and such that $x_n \to_f x$ as $n \to \infty$, that is, $x_n \to x$ and $f(x_n) \to f(x)$.

The Fermat rule $0 \in \partial_C f(\overline{x})$ is known to hold true whenever $\overline{x}$ is a local minimizer of f.

When f is finite on some neighborhood of $\overline{x}$ and Lipschitz continuous therein, one has (see [9,26]) that

$$\partial_C f(\overline{x}) = \{x^* \in X^* : \langle x^*, h\rangle \le f^o(\overline{x}; h),\ \forall h \in X\}, \tag{2.1}$$

where $f^o(\overline{x}; h) := \limsup_{t\downarrow 0, x\to\overline{x}} \frac{f(x+th)-f(x)}{t}$. So, under the Lipschitz continuity of f near $\overline{x}$, the set $\partial_C f(\overline{x})$ is nonempty, weak* compact and convex in X^* since $f^o(\overline{x}; \cdot)$ is continuous, convex and positively homogeneous. The general sum formula

$$\partial_C(f + g)(\overline{x}) \subset \partial_C f(\overline{x}) + \partial_C g(\overline{x}) \tag{2.2}$$

is also fulfilled whenever the function f is finite and Lipschitz continuous near $\overline{x}$.

As usual, we will denote by $\delta_S(\cdot)$ the *indicator function* of the closed set S, i.e., $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = +\infty$ otherwise. It is easily checked that $\partial_C \delta_S(x) = N^C(S, x)$.

An element $x^* \in X^*$ is said to be a *Fréchet subgradient* of f at $\overline{x} \in \operatorname{dom} f$ if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\langle x^*, x - \overline{x}\rangle \le f(x) - f(\overline{x}) + \varepsilon\|x - \overline{x}\| \quad \text{for all } x \in B(\overline{x}, \delta).$$

The set of all Fréchet subgradients of f at $\overline{x}$ is called the *Fréchet subdifferential* of f at $\overline{x}$ and is denoted by $\partial_F f(\overline{x})$; as above one also sets $\partial_F f(\overline{x}) = \emptyset$ if $\overline{x} \notin \operatorname{dom} f$.

The *Mordukhovich limiting subdifferential* $\partial_L f(\overline{x})$ is defined by

$$\partial_L f(\overline{x}) := \{{}^{w^*}\lim x_n^* : x_n^* \in \partial_F f(x_n),\ x_n \to_f \overline{x}\},$$

that is, a continuous linear functional $x^* \in \partial_L f(\overline{x})$ provided there are a sequence $(x_n)_n$ in X converging to $\overline{x}$ with $f(x_n) \to f(\overline{x})$ and a sequence $(x_n^*)_n$ in X^* converging weakly* to x^* with $x_n^* \in \partial_F f(x_n)$ for all $n \in \mathbb{N}$. One has $\partial_L f(\overline{x}) \subset \partial_C f(\overline{x})$ (see [25]), so

$$\partial_F f(\overline{x}) \subset \partial_L f(\overline{x}) \subset \partial_C f(\overline{x}).$$

Those three subdifferentials coincide with the usual subdifferential in the sense of convex analysis

$$\partial f(\overline{x}) = \{x^* \in X^* : \langle x^*, u - \overline{x}\rangle \le f(u) - f(\overline{x}),\ \forall u \in U\}$$

whenever f is convex on an open convex set U containing $\overline{x}$. When X is an Asplund space and f is finite and Lipschitz continuous near $\overline{x}$, it is known (see [25]) that

$$\partial_L f(\overline{x}) \ne \emptyset \quad \text{and} \quad \partial_C f(\overline{x}) = \overline{\text{co}}^*\big(\partial_L f(\overline{x})\big), \tag{2.3}$$

where $\overline{\mathrm{co}}^*$ denotes the weak* closed convex hull. Recall that a Banach space is an Asplund space whenever the topological dual of every separable subspace is separable.

If X is an Asplund space and f is lower semicontinuous on X, then $\mathrm{Dom}\, \partial_F f$ is graphically dense in $\mathrm{dom}\, f$.

When f is Lipschitz continuous near $\overline{x}$, from (2.1) it is not difficult to see that $\partial_C f(\overline{x})$ is a singleton if and only if f is strictly Gâteaux differentiable at $\overline{x}$; in such a case the equality $\partial_C f(\overline{x}) = \{D_G f(\overline{x})\}$ holds true. In particular, if f is a continuous convex function which is Gâteaux differentiable at $\overline{x}$, then

$$\partial_C f(\overline{x}) = \partial f(\overline{x}) = \{D_G f(\overline{x})\}, \tag{2.4}$$

where $\partial f(\overline{x})$ denotes as above the subdifferential in the sense of convex analysis.

For $\alpha > 0$, the function f is said to be $\mathcal{C}^{1,\alpha}$ on an open set $\mathcal{O}$ where it is finite, when it is Fréchet differentiable on $\mathcal{O}$ and the Fréchet derivative Df is α-Hölderian on $\mathcal{O}$, that is, there is some real constant $L \geq 0$ such that

$$\|Df(x) - Df(y)\| \leq L\|x-y\|^\alpha \quad \text{for all } x, y \in \mathcal{O}.$$

We recall now the concept of s-lower regular functions that we developed in [19]. The class of such functions is an adaptation of the class of primal lower nice functions introduced in [27] and of similar classes in [11]; see also [2,5,6,15–17,21,24,31].

Definition 2.1. For a real $s > 0$, we say that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is *s-lower regular* on an open convex set $\mathcal{O}$ of the Banach space X with $\mathcal{O} \cap \mathrm{dom} f \neq \emptyset$, when it is lower semicontinuous (lsc, for short) on $\mathcal{O}$ and there exists some real coefficient $c \geq 0$ such that for all $x \in \mathcal{O} \cap \mathrm{Dom}\, \partial_C f$ and for all $x^* \in \partial_C f(x)$ we have

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - c\,(1 + \|x^*\|)\, \|x-y\|^{s+1}, \quad \forall y \in \mathcal{O}. \tag{2.5}$$

When $s = 1$, one just says that the function f is *primal lower regular or primal lower nice* (pln, for short) on $\mathcal{O}$.

Obviously, convex functions and $\mathcal{C}^{1,s}$ functions are s-lower regular. One of the fundamental other examples of s-lower regular functions is provided (see [19]) by convexly $\mathcal{C}^{1,s}$-composite functions which are qualified. Let $G : X \to Y$ be a mapping which is of class $\mathcal{C}^{1,s}$ on an open convex set $\mathcal{O}$ of X for some real $s > 0$ and let $g : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The *convexly composite function* $g \circ G$ is said to be *qualified* at $\overline{x} \in \mathcal{O} \cap G^{-1}(\mathrm{dom}\, g)$ whenever the following Robinson qualification condition holds:

$$\mathbb{R}_+\big(\mathrm{dom}\, g - G(\overline{x})\big) - DG(\overline{x})(X) = Y, \tag{2.6}$$

where as usual $\mathbb{R}_+ := [0, +\infty[$.

The following theorem from [19] shows that the definition of s-lower regularity can be given with any of the above subdifferentials. Further, this property can be characterized by some local hypomonotonicity of $\partial_C f$ (or $\partial_F f$, $\partial_L f$).

Theorem 2.1 (*See [19]*)**.** *Let X be an Asplund space, $s > 0$, $\mathcal{O}$ be an open convex set of X, and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function which is lsc on $\mathcal{O}$ with $\mathcal{O} \cap$ dom $f \neq \emptyset$. Then the following hold:*

(a) *If f is s-lower regular on $\mathcal{O}$, then for all $x \in \mathcal{O}$, one has*

$$\partial_F f(x) = \partial_L f(x) = \partial_C f(x).$$

(b) *The function f is s-lower regular on $\mathcal{O}$ with coefficient $c \geq 0$ if and only if for all $x \in \mathcal{O} \cap$ Dom $\partial_F f$ (resp. $x \in \mathcal{O} \cap$ Dom $\partial_L f$) and for all $x^* \in \partial_F f(x)$ (resp. $x^* \in \partial_L f(x)$) one has*

$$f(y) \geq f(x) + \langle x^*, y - x\rangle - c(1 + \|x^*\|)\|x-y\|^{s+1}, \quad \forall y \in \mathcal{O}.$$

(c) *f is s-lower regular near $\overline{x}$ if and only if there exist reals $\varepsilon > 0$ and $c \geq 0$ such that for all $x_i^* \in \partial_C f(x_i)$ with $\|x_i - \overline{x}\| < \varepsilon,\ i = 1, 2,$ one has*

$$\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq -c(1 + \|x_1^*\| + \|x_2^*\|)\|x_1 - x_2\|^{s+1}.$$

The latter equivalence is still true with $\partial_F f$ (resp. $\partial_L f$) in place of $\partial_C f$.

We need also in our study the following lemma from [19] adapted from a result of [32].

Lemma 2.1. *Let X be a normed vector space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function with $f(\overline{x}) < +\infty$. Let r be a positive number such that f is bounded from below over $B[\overline{x}, r]$ by some real α. Let $s > 0, \beta \in \mathbb{R}$ and θ be a non-negative number. For each real $c \geq 0$, let*

$$F_{\beta,c}(x^*, x, y) := f(y) + \beta\langle x^*, x - y\rangle + c(1 + \|x^*\|)\|x - y\|^{s+1},$$

for all $x, y \in X$ and $x^* \in X^*$. Let any real

$$c_0 \geq \frac{|\beta|4^{s+1}}{(2^{s+1} - 1)r^s} \quad \text{such that } c_0 > \frac{4^{s+1}}{(2^{s+1} - 1)r^{s+1}}(f(\overline{x}) + \theta - \alpha).$$

Then, for any $c \geq c_0$, for any $x^* \in X^*$ and for any $x \in B[\overline{x}, \frac{r}{4}]$, any point $u \in B[\overline{x}, r]$ such that

$$F_{\beta,c}(x^*, x, u) \leq \inf_{y \in B[\overline{x}, r]} F_{\beta,c}(x^*, x, y) + \theta$$

must belong to $B(\overline{x}, \frac{3r}{4})$.

2.2. Uniformly convex and uniformly smooth spaces

Let $(X, \|\cdot\|)$ be a normed space. We recall that the norm $\|\cdot\|$ is called *strictly convex* (or *rotund*) if for any $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ one has $\frac{\|x+y\|}{2} < 1$. This is equivalent to saying that, for any $\lambda \in]0, 1[$ and for any $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ one has $\|\lambda x + (1 - \lambda)y\| < 1$.

The *modulus of convexity* of $\|\cdot\|$ is the function $\delta_{\|\cdot\|}(\cdot) : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta_{\|\cdot\|}(\varepsilon) := \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\right\}.$$

The norm $\|\cdot\|$ is said to be *uniformly convex* (or *uniformly rotund*) whenever

$$\delta_{\|\cdot\|}(\varepsilon) > 0, \quad \forall \varepsilon \in]0, 2].$$

One also says that the space $(X, \|\cdot\|)$ is a *uniformly convex* space.

The modulus of convexity is said to be of *power type q* if there exists some real $k > 0$ such that

$$\delta_{\|\cdot\|}(\varepsilon) \geq k\varepsilon^q, \quad \forall \varepsilon \in]0, 2];$$

for such a power type q one always has $q \geq 2$ (see, e.g., [14]).

One defines the *modulus of smoothness* $\rho_{\|\cdot\|}(\cdot)$ of $\|\cdot\|$ by

$$\rho_{\|\cdot\|}(\tau) := \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\} \quad \text{for all } \tau \geq 0.$$

The norm $\|\cdot\|$ is said to be *uniformly smooth* if the following property holds

$$\lim_{\tau \downarrow 0} \frac{\rho_{\|\cdot\|}(\tau)}{\tau} = 0.$$

The modulus of smoothness is of *power type* p if there exists some real $c > 0$ such that

$$\rho_{\|\cdot\|}(\tau) \leq c\tau^p, \quad \forall \tau \geq 0;$$

for such a power type p one always has $1 < p \leq 2$ (see, e.g., [22]).

It is known (see [13]) that all Hilbert spaces and the Banach spaces ℓ^p, L^p, W^p_m, $1 < p < \infty$ (with their usual norms) are both uniformly convex and uniformly smooth with modulus of convexity and smoothness of power type.

Now we recall some properties of the duality mapping. For a real $p > 1$, the set-valued mapping $J_p : X \rightrightarrows X^*$ defined by $J_p(x) := \partial(\frac{1}{p}\|\cdot\|^p)(x)$, for all $x \in X$, is called the *duality mapping* with gauge function $\varphi(t) := t^{p-1}$. In particular when $p = 2$, J_2 is generally named the *normalized duality mapping* and is denoted by J instead of J_2. Moreover, it is clear that the set $J_p(x)$ is nonempty for all $x \in X$.

The next well-known proposition provides a useful description of J_p (see, e.g., [20]).

Proposition 2.1. *For any real $p > 1$ the following hold:*

(a) *For each $x \in X$ the set $J_p(x)$ is nonempty, convex and* weakly* *compact.*
(b)

$$J_p(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|\,\|x\|, \|x^*\| = \|x\|^{p-1}\} \quad \textit{for all } x \in X.$$

(c) *For each $x \in X$ one has $J_p(tx) = t^{p-1}J_p(x)$ for all $t > 0$, as well as the equality $J_p(-x) = -J_p(x)$.*

We also need to recall some characteristic properties of uniformly convex and uniformly smooth Banach spaces related to the mapping J_p. From [30] we have the following properties:

(i) If the norm $\|\cdot\|$ is uniformly convex, then there exists a constant $K_1 \geq 0$ such that for all $x, y \in X$

$$\langle J_px - J_py, x - y\rangle \geq K_1(\max(\|x\|, \|y\|))^p \delta_{\|\cdot\|}\left(\frac{\|x - y\|}{2\max(\|y\|, \|x\|)}\right).$$

In particular, when the modulus of convexity is of power type p, there exists some real $L_1 \geq 0$ such that

$$\langle J_px - J_py, x - y\rangle \geq L_1\|x - y\|^p \quad \text{for all } x, y \in X. \tag{2.7}$$

(ii) If the norm $\|\cdot\|$ is uniformly smooth, then there exists a constant $K_2 \geq 0$ such that for all $x, y \in X$

$$\|J_qx - J_qy\| \leq K_2\frac{(\max(\|x\|, \|y\|))^q}{\|x - y\|}\rho_{\|\cdot\|}\left(\frac{\|x - y\|}{\max(\|x\|, \|y\|)}\right).$$

In particular, when the modulus of smoothness is of power type $\alpha + 1$ with $0 < \alpha \leq 1$, there exists some real $L_2 \geq 0$ such that for all $x, y \in X$

$$\|J_qx - J_qy\| \leq L_2(\max(\|x\|, \|y\|))^{q-\alpha-1}\|x - y\|^\alpha. \tag{2.8}$$

Recall that one says that the norm $\|\cdot\|$ has the sequential *Kadec–Klee property* if for every sequence $(x_n)_n$ in X with $\|x_n\| \to \|x\|$ and $x_n \to x$ weakly, one has $x_n \to x$ strongly, that is, $\|x_n - x\| \to 0$ as $n \to \infty$.

The following proposition recalls several classical properties which can be found, e.g., in [12,13].

Proposition 2.2. (a) *Every uniformly convex normed space is strictly convex.*
(b) *Every uniformly convex (resp. uniformly smooth) Banach space is reflexive.*

(c) *If* $(X, \|\cdot\|)$ *is a uniformly convex Banach space, then* $\|\cdot\|$ *has the Kadec–Klee property.*
(d) *If* $(X, \|\cdot\|)$ *is a uniformly smooth space, then* J_p *is a single-valued mapping and this mapping is uniformly continuous on any bounded set of* X*.*
(e) *If the norm* $\|\cdot\|$ *is Gâteaux differentiable off zero, then* J_p *is single-valued and norm-to-weak* continuous.*
(f) *If the norm* $\|\cdot\|$ *is Fréchet differentiable off zero, then* J_p *is single-valued and norm-to-norm continuous.*
(g) *The norm* $\|\cdot\|$ *of the Banach space* X *is uniformly convex (resp. uniformly smooth) if and only if the dual norm* $\|\cdot\|_*$ *is uniformly smooth (resp. uniformly convex).*
(h) *If the dual norm* $\|\cdot\|_*$ *is Gâteaux differentiable off zero, then the norm* $\|\cdot\|$ *is strictly convex.*
(i) *If* $(X, \|\cdot\|)$ *is a reflexive Banach space, then* $(X, \|\cdot\|)$ *is smooth if and only if* $(X^*, \|\cdot\|_*)$ *is strictly convex.*

Corollary 2.1 (*See [1]*)**.** *Let* $(X, \|\cdot\|)$ *be a reflexive Banach space whose both norm* $\|\cdot\|$ *and dual norm* $\|\cdot\|_*$ *are strictly convex. If* $J_p : X \to X^*$ *and* $J_q^* : X^* \to X$ *are the duality mappings with* $p^{-1} + q^{-1} = 1$ *and* $p, q > 1$*, then* $J_p^{-1} = J_q^*$*.*

3. Moreau s-envelope

Let X be a normed space and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two extended real-valued functions. *The infimal convolution function* of f and g is defined by

$$(f \Box g)(x) = \inf_{y \in X} \{f(y) + g(x - y)\} \quad \text{for all } x \in X.$$

When the infimum is attained, one says that $(f \Box g)(x)$ is *exact*.

In particular, when $g = \frac{1}{2\lambda}\|\cdot\|^2$, we obtain the important concepts named the Moreau envelope and proximal mapping.

Definition 3.1. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and $\lambda > 0$ be a real positive number. *The Moreau envelope function* $e_\lambda f$ and *the proximal mapping* $P_\lambda f$ with index λ of the function f are defined by

$$e_\lambda f(x) := \inf_{y \in X} \Big(f(y) + \frac{1}{2\lambda}\|x - y\|^2 \Big),$$
$$P_\lambda f(x) := \operatorname*{argmin}_{y \in X} \Big(f(y) + \frac{1}{2\lambda}\|x - y\|^2 \Big),$$

for all $x \in X$. More generally, for any real $s > 1$, we define *the Moreau s-envelope* $e_{\lambda,s} f$ and *the set-valued s-proximal mapping* $P_{\lambda,s} f$ *of index* λ as

$$e_{\lambda,s} f(x) := \inf_{y \in X} \Big(f(y) + \frac{1}{s\lambda}\|x - y\|^s \Big),$$
$$P_{\lambda,s} f(x) := \operatorname*{argmin}_{y \in X} \Big(f(y) + \frac{1}{s\lambda}\|x - y\|^s \Big).$$

Given $\varepsilon > 0$, we will also have to consider the Moreau truncated envelope

$$e_{\lambda,s,\varepsilon} f(x) := \inf_{\|y\| \le \varepsilon} \Big(f(y) + \frac{1}{s\lambda}\|x - y\|^s \Big) = e_{\lambda,s}(f + \delta_{\varepsilon\mathbb{B}})(x)$$

and the truncated proximal mapping

$$P_{\lambda,s} f(x) := \operatorname*{argmin}_{\|y\| \le \varepsilon} \Big(f(y) + \frac{1}{s\lambda}\|x - y\|^s \Big) = P_{\lambda,s}(f + \delta_{\mathbb{B}})(x).$$

In the following lemma (see [10]) we state the relationship between the Fréchet subdifferential of the functions f, g and the Fréchet subdifferential of their infimal convolution.

Lemma 3.1 (*See [10]*). *Let X be a normed space and $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two extended real-valued functions. Let $\overline{x}$ be a point where $(f \Box g)(\overline{x})$ is finite and attained at $\overline{y}$. Then*

$$\partial_F (f \Box g)(\overline{x}) \subset \partial_F f(\overline{y}) \cap \partial_F g(\overline{x} - \overline{y}).$$

In particular, for any $\overline{y} \in P_{\lambda,s} f(\overline{x})$,

$$\partial_F e_{\lambda,s} f(\overline{x}) \subset \partial_F f(\overline{y}) \cap \partial_F \left(\frac{1}{s\lambda} \| \cdot \|^s \right) (\overline{x} - \overline{y}).$$

The following theorem provides some first general properties of Moreau s-envelopes.

Theorem 3.1. *Let $(X, \| \cdot \|)$ be a Banach space. Let a real $s > 1$ and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. Assume that there are reals $\alpha, \beta > 0, \gamma \in \mathbb{R}$ such that*

$$f(x) \geq -\alpha \|x\|^s - \beta \|x\| + \gamma, \quad \textit{for all } x \in X. \tag{3.1}$$

Then for any $\lambda \in]0, \frac{1}{s\alpha}[$, one has:

(a) *For any real number $r > 0$, there exists a real $\tau > 0$ (depending on r and λ) such that for every $x \in B(0, r)$*

$$e_{\lambda,s} f(x) = \inf_{x' \in B(0,\tau)} \left(f(x') + \frac{1}{s\lambda} \|x - x'\|^s \right);$$

(b) *$e_{\lambda,s} f$ is Lipschitzian on any ball $B(0, r)$ with a Lipschitz constant $L \geq \frac{(r+\tau)^{s-1}}{\lambda}$, where τ is as in* (a)*;*
(c) *$e_{\lambda,s} f \uparrow \overline{f}$ as $\lambda \downarrow 0$, where $\overline{f}$ denotes the lower semicontinuous hull of f;*
(d) *If the norm $\| \cdot \|$ is Fréchet differentiable off zero and has the sequential Kadec–Klee property, if the function f is lower semicontinuous, and if $(X, \| \cdot \|)$ is reflexive, then for any $x \in \operatorname{Dom} \partial_F e_{\lambda,s} f$, the set $P_{\lambda,s} f(x)$ is nonempty and for every $x' \in P_{\lambda,s} f(x)$ one has*

$$\partial_F e_{\lambda,s} f(x) = \left\{ \frac{1}{\lambda} J_s(x - x') \right\} \quad \textit{and} \quad \frac{1}{\lambda} J_s(x - x') \in \partial_F f(x');$$

further one has a better Lipschitz constant L than in (b) for $e_{\lambda,s} f$ on $B(0, r)$ with $L \geq \frac{(r+\tau)^{s-1}}{s\lambda}$.

Proof. Fix any $\lambda \in]0, \frac{1}{s\alpha}[$.

(a) One has by definition of $e_{\lambda,s} f$

$$e_{\lambda,s} f(x) \leq \frac{1}{s\lambda} \|x\|^s + f(0).$$

Fix a real $\rho > 0$ and take $x' \in X$ such that

$$f(x') + \frac{1}{s\lambda} \|x' - x\|^s \leq e_{\lambda,s} f(x) + \rho,$$

and hence

$$\frac{1}{s\lambda} \|x' - x\|^s - \alpha \|x'\|^s - \beta \|x'\| + \gamma \leq \frac{1}{s\lambda} \|x\|^s + f(0) + \rho. \tag{3.2}$$

Observe that

$$\begin{aligned}\frac{1}{s\lambda}\|x'-x\|^s-\alpha\|x'\|^s &= \left(\frac{1}{s\lambda}-\alpha\right)\|x'\|^s+\frac{1}{s\lambda}\|x'-x\|^s-\frac{1}{s\lambda}\|x'\|^s\\ &= \frac{1-s\lambda\alpha}{s\lambda}\|x'\|^s+\frac{1}{\lambda}\left(\frac{1}{s}\|x'-x\|^s-\frac{1}{s}\|x'\|^s\right).\end{aligned}$$

Put $\Delta := \frac{1}{s}\|x'-x\|^s-\frac{1}{s}\|x'\|^s$ and consider the function φ defined on $[0,1]$ by

$$\varphi(t) := \frac{1}{s}\|x'-tx\|^s.$$

Since the norm $\|\cdot\|$ is convex, the continuous convex function φ has a right-hand derivative at every real t, and moreover

$$\varphi'_+(t)=\|x'-tx\|^{s-1}N'(x'-tx;-x),$$

where $N'(y;v) := \lim_{\theta\downarrow 0}\theta^{-1}\big(N(y+\theta v)-N(y)\big)$ denotes the directional derivative of the convex function $N := \|\cdot\|$ at the point y. Note also that $N'(x'-tx;-x)\geq -\|-x\|=-\|x\|$ according to the properties of the norm. It results from this and Proposition 2.1(b) that

$$\begin{aligned}\Delta := \varphi(1)-\varphi(0)=\int_0^1\varphi'_+(t)\,dt &= \int_0^1\|x'-tx\|^{s-1}N'(x'-tx;-x)\,dt\\ &\geq -\|x\|\int_0^1\|x'-tx\|^{s-1}\,dt\\ &\geq -\|x\|\int_0^1(\|x'\|+t\|x\|)^{s-1}\,dt.\end{aligned}$$

Further, one knows that for any non-negative numbers μ,ν,σ

$$(\mu+\nu)^\sigma\leq 2^\sigma(\mu^\sigma+\nu^\sigma),$$

then

$$\Delta\geq -2^{s-1}\|x\|\int_0^1(\|x'\|^{s-1}+t^{s-1}\|x\|^{s-1})\,dt,$$

or equivalently

$$\Delta\geq -2^{s-1}\|x\|\,\|x'\|^{s-1}-\frac{2^{s-1}}{s}\|x\|^s.$$

It ensues that

$$\frac{1}{s\lambda}\|x'-x\|^s-\alpha\|x'\|^s\geq\frac{1-s\lambda\alpha}{s\lambda}\|x'\|^s-\frac{2^{s-1}}{\lambda}\|x\|\,\|x'\|^{s-1}-\frac{2^{s-1}}{s\lambda}\|x\|^s. \tag{3.3}$$

Using (3.2) we get

$$\frac{1-s\lambda\alpha}{s\lambda}\|x'\|^s-\frac{2^{s-1}}{\lambda}\|x\|\,\|x'\|^{s-1}-\beta\|x'\|-\frac{2^{s-1}}{s\lambda}\|x\|^s\leq\frac{1}{s\lambda}\|x\|^s+f(0)+\rho-\gamma.$$

Putting $\eta := (1 - s\lambda\alpha)^{-1} > 0$, the previous inequality can be written as

$$\|x'\|^s - 2^{s-1}s\eta\|x\|\,\|x'\|^{s-1} - s\lambda\eta\beta\|x'\| \le (1 + 2^{s-1})\eta\|x\|^s + s\eta\lambda(f(0) + \rho - \gamma).$$

Fix now $x \in B(0, r)$ and take any $x' \in X$. The latter inequality yields

$$\|x'\|^s - 2^{s-1}sr\eta\|x'\|^{s-1} - s\lambda\eta\beta\|x'\| \le (1 + 2^{s-1})r^s\eta + s\eta\lambda(f(0) + \rho - \gamma),$$

so putting

$$a := 2^{s-1}sr\eta > 0, \qquad b := s\lambda\eta\beta > 0, \qquad c := (1 + 2^{s-1})r^s\eta + s\eta\lambda(f(0) + \rho - \gamma),$$

we obtain

$$\|x'\|^s - a\|x'\|^{s-1} - b\|x'\| \le c.$$

It is not difficult to see that the set

$$\{t \ge 0 : t^s - at^{s-1} - bt \le c\},$$

is bounded. Indeed, suppose that there exists a sequence $(t_n)_n$ of positive reals tending to $+\infty$ such that $t_n^s - at_n^{s-1} - bt_n \le c$, hence

$$t_n \le a + \frac{b}{t_n^{s-2}} + \frac{c}{t_n^{s-1}}.$$

(i) If $s > 2$ then, since $t_n \to \infty$ we obtain that $a \ge +\infty$, which is a contradiction.

(ii) If $1 < s \le 2$ then, since $t_n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that for every $n \ge n_0$, we have $t_n > 1$, then $t_n \ge t_n^{s-1}$ because $s - 1 \le 1$, hence
$$t_n^s - (a + b)t_n \le t_n^s - at_n^{s-1} - bt_n \le c \Longrightarrow t_n^{s-1} \le a + b + \frac{c}{t_n},$$
with $s - 1 > 0$. Taking the limit we get $a + b \ge +\infty$, which is a contradiction. Consequently, there exists a real $\tau > 0$ (depending on r and λ) such that $\|x'\| < \tau$, which justifies the assertion in (a).

(b) First, we must show that $e_{\lambda,s}f$ is finite. Thanks to (3.1) we write

$$f(y) + \frac{1}{s\lambda}\|x - y\|^s \ge \frac{1}{s\lambda}\|x - y\|^s - \alpha\|y\|^s - \beta\|y\| + \gamma. \tag{3.4}$$

With $\omega := \frac{1}{s\lambda}\|x - y\|^s - \alpha\|y\|^s$, we know by (3.3) in the proof of (a) that

$$\omega \ge \frac{1 - s\lambda\alpha}{s\lambda}\|y\|^s - \frac{2^{s-1}}{\lambda}\|x\|\,\|y\|^{s-1} - \frac{2^{s-1}}{s\lambda}\|x\|^s.$$

Setting $\nu := -\frac{2^{s-1}}{s\lambda}\|x\|^s + \gamma$, from (3.4) we obtain that for any $y \in X$

$$f(y) + \frac{1}{s\lambda}\|x - y\|^s \ge \frac{1 - s\lambda\alpha}{s\lambda}\|y\|^s - \frac{2^{s-1}}{\lambda}\|x\|\,\|y\|^{s-1} - \beta\|y\| + \nu. \tag{3.5}$$

Take any real $r > 0$. According to (a), there exists a real $\tau > 0$ such that for any $x \in B(0, r)$

$$e_{\lambda,s} f(x) = \inf_{y \in B(0,\tau)} \left(f(y) + \frac{1}{s\lambda} \|x - y\|^s \right).$$

It ensues from (3.5) that

$$\begin{aligned} e_{\lambda,s} f(x) &= \inf_{y \in B(0,\tau)} \left(f(y) + \frac{1}{s\lambda} \|x - y\|^s \right) \\ &\geq \inf_{y \in B(0,\tau)} \left(\frac{1 - s\lambda\alpha}{s\lambda} \|y\|^s - 2^{s-1} \frac{r}{\lambda} \|y\|^{s-1} - \beta \|y\| + \nu \right) \\ &\geq \inf_{y \in B(0,\tau)} \left(-2^{s-1} \frac{r}{\lambda} \|y\|^{s-1} - \beta \|y\| + \nu \right) \\ &\geq -2^{s-1} \frac{r}{\lambda} \tau^{s-1} - \beta\tau + \nu. \end{aligned}$$

It results that $e_{\lambda,s} f$ is finite.
Now, taking $\lambda \in]0, \frac{1}{s\alpha}[$ and $x \in B(0, r)$ then by (a)

$$e_{\lambda,s} f(x) = \inf_{x' \in B(0,\tau)} \left(f(x') + \frac{1}{s\lambda} \|x - x'\|^s \right).$$

In view of the Lipschitz property of $e_{\lambda,s} f$, we note by the mean value theorem that for any $0 \leq t_1 \leq t_2$, there exists a real $c \in [t_1, t_2]$ such that

$$t_2^s - t_1^s = \left| t_2^s - t_1^s \right| \leq s c^{s-1} \left| t_2 - t_1 \right|,$$

then, for any $x, y \in B(0, r)$ and $z \in B(0, \tau)$, there exists c between $\|y - z\|$ and $\|x - z\|$ such that

$$\begin{aligned} \left| \, \|x - z\|^s - \|y - z\|^s \right| &\leq s c^{s-1} \left| \|x - z\| - \|y - z\| \, \right| \\ &\leq s(r + \tau)^{s-1} \|x - y\|. \end{aligned}$$

Fixing $x' \in B(0, \tau)$ and putting $g_{x'}(x) := f(x') + \frac{1}{s\lambda} \|x - x'\|^s$, from the latter inequality we get that for any $x, y \in B(0, r)$

$$g_{x'}(x) - g_{x'}(y) = \frac{1}{s\lambda} (\|x - x'\|^s - \|y - x'\|^s) \leq \frac{1}{\lambda} (r + \tau)^{s-1} \|x - y\|,$$

which implies that

$$\inf_{x' \in \tau\mathbb{B}} g_{x'}(x) \leq \inf_{x' \in \tau\mathbb{B}} g_{x'}(y) + \frac{(r + \tau)^{s-1}}{\lambda} \|x - y\|,$$

then $e_{\lambda,s} f(x) - e_{\lambda,s} f(y) \leq \frac{(r+\tau)^{s-1}}{\lambda} \|x - y\|$. This means that the function $e_{\lambda,s} f$ is Lipschitzian on $r\mathbb{B}$ with the constant $L \geq \frac{(r+\tau)^{s-1}}{\lambda}$.

(c) According to (b) above, for every $\lambda \in]0, \frac{1}{s\alpha}[$, the function $e_{\lambda,s} f$ is finite. Then for any fixed $x \in X$, there exists $y_\lambda \in X$ (depending on x) such that

$$e_{\lambda,s} f(x) \leq f(y_\lambda) + \frac{1}{s\lambda} \|x - y_\lambda\|^s \leq e_{\lambda,s} f(x) + \lambda. \tag{3.6}$$

On the other hand, the net $(e_{\lambda,s}f)_\lambda$ is decreasing and then $\lim_{\lambda\downarrow 0} e_{\lambda,s}f = \sup_{\lambda\in]0,\frac{1}{s\alpha}[} e_{\lambda,s}f$. We put $g(\cdot) := \sup_{\lambda\in]0,\frac{1}{s\alpha}[} e_{\lambda,s}f(\cdot)$ and we prove that $g(\cdot) = \overline{f}(\cdot)$. We have by (3.6)

$$\frac{1}{s\lambda}\|x-y_\lambda\|^s - \alpha\|y_\lambda\|^s - \beta\|y_\lambda\| + \gamma \le g(x) + \lambda, \tag{3.7}$$

hence by (3.3)

$$\frac{1-s\lambda\alpha}{s\lambda}\|y_\lambda\|^s - \frac{2^{s-1}}{\lambda}\|x\|\,\|y_\lambda\|^{s-1} - \beta\|y_\lambda\| - \frac{2^{s-1}}{s\lambda}\|x\|^s \le g(x) + \frac{1}{s\alpha} - \gamma,$$

which entails as in the proof of (a) that there exists a real $r > 0$ such that

$$\|y_\lambda\| \le r \quad \text{for all } \lambda \in \left]0, \frac{1}{s\alpha}\right[.$$

It results from (3.7) that

$$\|x-y_\lambda\|^s \le s\lambda(g(x) + \lambda + \alpha r^s + r\beta - \gamma),$$

which means that y_λ converges to x as $\lambda \downarrow 0$. Then, since $f(y_\lambda) \le g(x) + \lambda$ according to (3.6), we obtain

$$\overline{f}(x) := \liminf_{u\to x} f(u) \le \liminf_{\lambda\downarrow 0} f(y_\lambda) \le g(x),$$

so $\overline{f}(\cdot) \le g(\cdot)$. On the other hand, by definition $f(\cdot) \ge g(\cdot)$ and as $e_{\lambda,s}f$ is locally Lipschitzian so continuous, the function g is lsc. Since $\overline{f}$ is the greatest lower semicontinuous function which minorizes f, it ensues that $\overline{f} \ge g$, which ensures that $g(\cdot) = \overline{f}(\cdot)$ as desired.

(d) We follow the main ideas in [8]. Let $x \in \operatorname{Dom}\partial_F f$ and let $\zeta \in \partial_F e_{\lambda,s}f(x)$. Fix a sequence (t_n) in $]0,1[$ with $t_n \downarrow 0$ and a sequence $(y_n)_n$ in X such that

$$f(y_n) + \frac{1}{s\lambda}\|x-y_n\|^s \le e_{\lambda,s}f(x) + t_n^2, \tag{3.8}$$

so

$$\frac{1}{s\lambda}\|x-y_n\|^s - \alpha\|y_n\|^s - \beta\|y_n\| + \gamma \le \frac{1}{s\lambda}\|x\|^s + f(0) + t_n^2.$$

According to the proof of (a), the sequence $(y_n)_n$ is bounded, then we can extract a subsequence (that we do not relabel) converging weakly to some $\overline{y} \in X$ and such that $\|x-y_n\| \longrightarrow \eta$ for some real η. We show that $\eta = \|x-\overline{y}\|$. By the weak lower semicontinuity of the norm we have $\eta \ge \|x-\overline{y}\|$. Put now $x_n := x - t_n(x-y_n)$ and note that $x_n \to x$ since $(y_n)_n$ is bounded. As $\zeta \in \partial_F e_{\lambda,s}f(x)$, for any real $\varepsilon > 0$, there exists a real number $r > 0$ such that for n sufficiently large

$$\begin{aligned}\langle \zeta, y_n - x\rangle &\le t_n^{-1}\Big(e_{\lambda,s}f(x - t_n(x-y_n)) - e_{\lambda,s}f(x)\Big) + \varepsilon\|x-y_n\| \\ &\le t_n^{-1}\left(f(y_n) + \frac{1}{s\lambda}\|(1-t_n)(x-y_n)\|^s - f(y_n) - \frac{1}{s\lambda}\|x-y_n\|^s + t_n^2\right) + \varepsilon\|x-y_n\|.\end{aligned}$$

Let

$$\tau_n := \frac{1}{s}\|(1-t_n)(x-y_n)\|^s - \frac{1}{s}\|x-y_n\|^s = \frac{1}{s}\|x-y_n-t_n(x-y_n)\|^s - \frac{1}{s}\|x-y_n\|^s,$$

and

$$\varphi_n(\theta) := \frac{1}{s}\|x - y_n - t_n\theta(x - y_n)\|^s.$$

Then, by (c) and (b) in Proposition 2.1

$$\begin{aligned}
\tau_n &= \varphi_n(1) - \varphi_n(0) = \int_0^1 \varphi_n'(\theta)\,d\theta \\
&= -t_n \left\langle \int_0^1 J_s\big(x - y_n - t_n\theta(x - y_n)\big)\,d\theta, x - y_n \right\rangle \\
&= -t_n \left\langle \int_0^1 J_s\big((1 - t_n\theta)(x - y_n)\big)\,d\theta, x - y_n \right\rangle \\
&= -t_n\langle J_s(x - y_n), x - y_n\rangle \int_0^1 (1 - t_n\theta)^{s-1}\,d\theta \\
&= \frac{1}{s}\|x - y_n\|^s((1 - t_n)^s - 1),
\end{aligned}$$

so

$$\begin{aligned}
\langle \zeta, y_n - x\rangle &\le t_n^{-1}\Big(\frac{1}{s\lambda}\|x - y_n\|^s\big((1 - t_n)^s - 1\big) + t_n^2\Big) + \varepsilon\|x - y_n\| \\
&= \frac{1}{s\lambda}\|x - y_n\|^s t_n^{-1}\big((1 - t_n)^s - 1\big) + t_n + \varepsilon\|x - y_n\|.
\end{aligned}$$

Putting $h(t) := (1 - t)^s$ for $0 \le t \le 1$, we have $h'(t) = -s(1 - t)^{s-1}$, and hence

$$\langle \zeta, y_n - x\rangle \le \frac{1}{s\lambda}\|x - y_n\|^s t_n^{-1}(h(t_n) - h(0)) + t_n + \varepsilon\|x - y_n\|.$$

Taking the limit as $n \to \infty$, we get that for every $\varepsilon > 0$

$$\langle \zeta, \overline{y} - x\rangle \le \frac{1}{s\lambda}\eta^s h'(0) + \varepsilon\eta,$$

which gives $\langle \zeta, \overline{y} - x\rangle \le -\frac{1}{\lambda}\eta^s$, then

$$\frac{\eta^s}{\lambda} \le \|\zeta\|\,\|x - \overline{y}\|. \tag{3.9}$$

Now for $x' := x + t_n z, z \in X$, and for n sufficiently large

$$\begin{aligned}
\langle \zeta, z\rangle &\le t_n^{-1}\Big(e_{\lambda,s}f(x + t_n z) - e_{\lambda,s}f(x)\Big) + \varepsilon\|z\| \\
&\le t_n^{-1}\Big(f(y_n) + \frac{1}{s\lambda}\|x - y_n + t_n z\|^s - f(y_n) - \frac{1}{s\lambda}\|x - y_n\|^s + t_n^2\Big) + \varepsilon\|z\|.
\end{aligned}$$

Putting $\sigma_n := \frac{1}{s}\|x - y_n + t_n z\|^s - \frac{1}{s}\|x - y_n\|^s$ and $\psi_n(\theta) := \frac{1}{s}\|x - y_n + t_n z\theta\|^s$, we have by Proposition 2.1(b)

$$\begin{aligned}
\sigma_n &= \psi_n(1) - \psi_n(0) = \int_0^1 \psi_n'(\theta)\,d\theta = \int_0^1 \langle J_s(x - y_n + \theta t_n z), t_n z\rangle\,d\theta \\
&\le t_n\|z\| \int_0^1 \|x - y_n + \theta t_n z\|^{s-1}\,d\theta,
\end{aligned}$$

hence

$$\langle \zeta, z\rangle \le \frac{1}{\lambda}\|z\| \int_0^1 \|x - y_n + \theta t_n z\|^{s-1}\, d\theta + t_n + \varepsilon\|z\|.$$

Noting as $n \to \infty$ that $\|x - y_n + \theta t_n z\| \longrightarrow \eta$ since $t_n \downarrow 0$ and $\|x - y_n\| \to \eta$, it follows from the latter inequality that for every $\varepsilon > 0$

$$\langle \zeta, z\rangle \le \frac{1}{\lambda}\|z\|\eta^{s-1} + \varepsilon\|z\|,$$

then $\langle \zeta, z\rangle \le \frac{1}{\lambda}\|z\|\eta^{s-1}$, so

$$\|\zeta\| \le \frac{\eta^{s-1}}{\lambda}.$$

According to (3.9), we obtain that $\eta \le \|x - \overline{y}\|$, and hence $\eta = \|x - \overline{y}\|$. Consequently

$$\|x - y_n\| \to \|x - \overline{y}\|,$$

and since $(y_n)_n$ converges weakly to $\overline{y}$ and the norm has the sequential Kadec–Klee property, we find that $(y_n)_n$ converges strongly to $\overline{y}$. Passing to the limit in (3.8) and using the lower semicontinuity of f, we get

$$f(\overline{y}) + \frac{1}{s\lambda}\|x - \overline{y}\|^s \le e_{\lambda,s} f(x),$$

hence $e_{\lambda,s} f(x) = f(\overline{y}) + \frac{1}{s\lambda}\|x - \overline{y}\|^s$, which proves that $P_{\lambda,s}(x) \neq \emptyset$ and $\overline{y} \in P_{\lambda,s}(x)$. Lemma 3.1 gives that for any $x' \in P_{\lambda,s}(x)$

$$\partial_F e_{\lambda,s} f(x) \subset \partial_F f(x') \cap \partial_F \left(\frac{1}{s\lambda}\|\cdot\|^s\right)(x - x').$$

Since the norm is Fréchet differentiable off zero, then $\partial_F(\frac{1}{s}\|\cdot\|)^s(\cdot) = J_s(\cdot)$ is a singleton, hence

$$\partial_F e_{\lambda,s} f(x) = \frac{1}{\lambda} J_s(x - x') \quad \text{and} \quad \frac{1}{\lambda} J_s(x - x') \in \partial_F f(x').$$

We can also obtain a better constant of Lipschitz via (b). Indeed, one has

$$\zeta = \partial_F e_{\lambda,s} f(x) = \frac{1}{s\lambda} J_s(x - x'),$$

then

$$\|\zeta\| \le \frac{1}{s\lambda}\|J_s(x - x')\| = \frac{1}{\lambda}\|x - x'\|^{s-1} \le \frac{(r+\tau)^{s-1}}{s\lambda}.$$

According to Theorem 2.1 of [32], we obtain $L \ge \frac{(r+\tau)^{s-1}}{s\lambda}$. □

The next proposition establishes the equivalence between the $\mathcal{C}^1$ property of $e_{\lambda,s} f$ and the single-valuedness and continuity of $P_{\lambda,s} f$.

Proposition 3.1. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space whose norm $\|\cdot\|$ is Fréchet differentiable off zero and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lsc function satisfying the condition (3.1). Let $\lambda > 0$ be a positive real number and U be an open subset of X. The following properties are equivalent:*

(a) *$e_{\lambda,s}f$ is $\mathcal{C}^1$ on U,*
(b) *$P_{\lambda,s}f$ is nonempty, single-valued and continuous on U.*

When these properties hold, one has

$$De_{\lambda,s}f(x) = \lambda^{-1}J_s(x - P_{\lambda,s}f(x)).$$

Proof. Suppose that (a) holds. According to the assertion (c) of Proposition 2.2, the norm $\|\cdot\|$ has the Kadec–Klee property, then from Theorem 3.1(d), $P_{\lambda,s}f(x)$ is nonempty for any $x \in U$. Let $\overline{x} \in U$ where $e_{\lambda,s}f(\overline{x})$ is exact and is attained at some $\overline{y}$, i.e., $\overline{y} \in P_{\lambda,s}f(\overline{x})$. Then from Lemma 3.1

$$\partial_F e_{\lambda,s}f(\overline{x}) \subset \partial_F f(\overline{y}) \cap \partial_F \left(\frac{1}{s\lambda}\|\cdot\|^s\right)(\overline{x} - \overline{y}),$$

and hence $\partial_F e_{\lambda,s}f(\overline{x}) \subset \{\lambda^{-1}J_s(\overline{x} - \overline{y})\}$. The $\mathcal{C}^1$ property of $e_{\lambda,s}f$ on U ensures that

$$De_{\lambda,s}f(\overline{x}) = \lambda^{-1}J_s(\overline{x} - \overline{y}),$$

and since J_s is bijective, we obtain $\overline{y} = \overline{x} - \lambda J_s^{-1}(De_{\lambda,s}f(\overline{x}))$. Thus for any $x \in U$, it results that

$$P_{\lambda,s}f(x) = x - \lambda J_s^{-1}(De_{\lambda,s}f(x)).$$

Keeping in mind that the norm $\|\cdot\|$ is Fréchet differentiable off zero and using Corollary 2.1 and assertions (f) and (g) in Proposition 2.2 we conclude that $P_{\lambda,s}f$ is continuous on U.

Suppose now that (b) holds. This ensures that, for any $x \in U$, there exists $p(x) \in X$ such that $P_{\lambda,s}f(x) = p(x)$, so by Lemma 3.1

$$\partial_F e_{\lambda,s}f(x) \subset \partial_F f(p(x)) \cap \left\{\lambda^{-1}J_s(x - p(x))\right\} \quad \text{for all } x \in U. \tag{3.10}$$

We show that $e_{\lambda,s}f$ is $\mathcal{C}^1$ on U. Let $x \in U$ and let $\xi \in \partial_L e_{\lambda,s}f(x)$. Then by definition $\xi = {}^{w^*}\lim \xi_n$ with $\xi \in \partial_F e_{\lambda,s}f(x_n)$ and $x_n \to_f x$. For n large enough, $x_n \in U$ so by (3.10) we get $\xi_n = \lambda^{-1}J_s(x_n - P_{\lambda,s}f(x_n))$, and the continuity of $P_{\lambda,s}f$ and J_s ensures that $J_s(x_n - P_{\lambda,s}f(x_n)) \to J_s(x - P_{\lambda,s}f(x))$ and $\xi = \lambda^{-1}J_s(x - P_{\lambda,s}f(x))$. It ensues that

$$\partial_L e_{\lambda,s}f(x) \subset \left\{\lambda^{-1}J_s(x - P_{\lambda,s}f(x))\right\} \quad \text{for any } x \in U.$$

Moreover, for any fixed $x \in U$ we know that $\partial_L e_{\lambda,s}f(x) \neq \emptyset$ according to the local Lipschitz continuity of $e_{\lambda,s}f$ (see (2.3)), hence $\partial_L e_{\lambda,s}f(x) = \left\{\lambda^{-1}J_s(x - P_{\lambda,s}f(x))\right\}$, and then by (2.3)

$$\partial_C e_{\lambda,s}f(x) = \overline{co}^* \partial_L e_{\lambda,s}f(x) = \left\{\lambda^{-1}J_s(x - P_{\lambda,s}f(x))\right\}.$$

The single-valuedness of $\partial_C e_{\lambda,s}f(x)$ entails that $e_{\lambda,s}f$ is Gâteaux differentiable on U with $D_G e_{\lambda,s}f(x) = \lambda^{-1}J_s(x - P_{\lambda,s}f(x))$. The continuity of J_s and $P_{\lambda,s}$ entails that $D_G e_{\lambda,s}f$ is continuous on U and hence $e_{\lambda,s}f$ is $\mathcal{C}^1$ on U. $\square$

4. Differentiability of the Moreau envelope

Recall that a set-valued operator $T : X \rightrightarrows X^*$ is (r, σ)-*hypomonotone*, for some reals $\sigma > 1, r \geq 0$, whenever

$$\langle x_2^* - x_1^*, x_2 - x_1\rangle \geq -r\|x_2 - x_1\|^\sigma \quad \text{for all } x_i \in \operatorname{Dom} T \text{ and } x_i^* \in T(x_i),\ i = 1, 2.$$

When $\sigma = 2$, one says that T is r-*hypomonotone*.

Definition 4.1. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function and ε, r be two positive real numbers. The (ε, r)-*truncation* of $\partial_C f$ at $\overline{x} \in X$ is the set-valued operator $T^f_{\varepsilon,r,\overline{x}} : X \rightrightarrows X^*$ whose graph is defined by

$$\operatorname{gph} T^f_{\varepsilon,r,\overline{x}} := \{(x, x^*) \in X \times X^* : x^* \in \partial_C f(x), \|x - \overline{x}\| \le \varepsilon, \|x^*\| \le r\}.$$

When there is no risk of ambiguity, $T^f_{\varepsilon,r,\overline{x}}$ will be denoted by $T_{\varepsilon,r}$.

For a function f which is s-lower regular near the origin, the next proposition will establish the connexion between the truncation of $\partial_C f$ and the Moreau truncated proximal mapping of f in view of the Hölder continuity near zero of that truncated proximal mapping. The proof of the proposition requires two lemmas. The first lemma proves a hypomonotonicity property of the truncation of $\partial_C f$.

Lemma 4.1. *Let $s > 0$ and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function on the Banach space X which is finite at $\overline{x} \in X$. Let $\varepsilon > 0$ be such that f is lower semicontinuous and s-lower regular on an open convex set containing $B[\overline{x}, \varepsilon]$ with a constant $c \ge 0$. Then, given any positive real $c' > c$ and $c_0 := 1/(4c')$, the truncated set-valued operator T_{ε,rc_0} in Definition 4.1 is $(r, s + 1)$-hypomonotone, for every real $r \ge 4c'$.*

Proof. Since f is s-lower regular on an open convex set containing $B[\overline{x}, \varepsilon]$ with the constant $c \ge 0$, we have

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \ge -c(2 + \|x_1^*\| + \|x_2^*\|)\|x_2 - x_1\|^{s+1}, \tag{4.1}$$

for all $(x_i, x_i^*) \in \operatorname{gph} \partial_C f$ with $\|x_i - \overline{x}\| \le \varepsilon, i = 1, 2$. Fix any real $c' > c$ and $c_0 := \frac{1}{4c'}$, and fix also any real $r \ge 4c'$. Let $(x_i, x_i^*) \in \operatorname{gph} T_{\varepsilon,rc_0},\ i = 1, 2$, which means that

$$x_i^* \in \partial_C f(x_i), \quad \|x_i - \overline{x}\| \le \varepsilon \text{ and } \|x_i^*\| \le rc_0.$$

Then, if $\|x_1^*\| + \|x_2^*\| \le 2$, putting this in (4.1), we obtain

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \ge -4c'\|x_2 - x_1\|^{s+1} \ge -r\|x_2 - x_1\|^{s+1}.$$

Else,

$$\begin{aligned}\langle x_2^* - x_1^*, x_2 - x_1 \rangle &\ge -c(2 + \|x_1^*\| + \|x_2^*\|)\|x_2 - x_1\|^{s+1} \\ &\ge -2c(\|x_1^*\| + \|x_2^*\|)\|x_2 - x_1\|^{s+1}\end{aligned}$$

hence the inequality $\|x_i^*\| \le c_0 r$ ensures

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \ge -4cc_0 r\|x_2 - x_1\|^{s+1} \ge -r\|x_2 - x_1\|^{s+1}.$$

The $(r, s + 1)$-hypomonotonicity of the (set-valued) operator T_{ε,rc_0} is then established. □

The second lemma provides, near the origin, a partial connexion between Moreau truncated proximal mappings and appropriate truncations of $\partial_C f$.

Lemma 4.2. *Let $(X, \|\cdot\|)$ be a Banach space with a Gâteaux differentiable norm off zero and let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function with $0 \in \operatorname{dom} f$. Let $\varepsilon_0 > 0$ be such that f is lower semicontinuous on $B(0, \varepsilon_0)$ and bounded from below on $B(0, \varepsilon_0)$ and let $s > 0$. Then, taking a real $\varepsilon' \in]0, \varepsilon_0[$ such that*

$$\varepsilon := (\varepsilon')^{\frac{1}{s}} < \varepsilon_0, \tag{4.2}$$

there exists a real $r_0 > 0$ such that, for any real $r \ge r_0$,

$$P_{\frac{1}{r},s+1,\varepsilon} f(u) \subset \left(I + \frac{1}{r} J_{s+1}^{-1} \circ T_{\varepsilon,r\varepsilon'}\right)^{-1}(u) \quad \text{for all } u \in B\left(0, \frac{\varepsilon}{4}\right),$$

where $T_{\varepsilon,r\varepsilon'} := T^f_{\varepsilon,r\varepsilon',\overline{x}}$, with $\overline{x} = 0$, is the truncated operator in Definition 4.1.

Proof. Let $\varepsilon > 0$ be as in the statement, hence in particular, f is bounded from below on $B[0, \varepsilon]$. Let $u \in B(0, \frac{\varepsilon}{4})$. By definition

$$e_{\frac{1}{r},s+1,\varepsilon} f(u) = \inf_{\|y\| \le \varepsilon} \Big(f(y) + \frac{r}{s+1} \|u - y\|^{s+1} \Big), \tag{4.3}$$

so from Lemma 2.1 with $\beta = \theta = x^* = 0$ there exists a real $r_0 > 0$ such that for any $r \ge r_0$

$$e_{\frac{1}{r},s+1,\varepsilon} f(u) = \inf_{y \in B\left(0, \frac{3\varepsilon}{4}\right)} \Big(f(y) + \frac{r}{s+1} \|u - y\|^{s+1} \Big), \tag{4.4}$$

and any point attaining the infimum in (4.3) (whenever such point exists) must belong to $B(0, \frac{3\varepsilon}{4})$. Consequently, for any $y \in P_{\frac{1}{r},s+1,\varepsilon} f(u)$, we have

$$\|y\| < \frac{3\varepsilon}{4}, \tag{4.5}$$

and the latter inequality combined with (4.4) entails according to the Fermat rule in Section 2

$$0 \in \partial_C \left(f + \frac{r}{s+1} \|u - \cdot\|^{s+1} \right) (y),$$

and since $J_{s+1}(x)$ is the derivative at x of the continuous convex function $\frac{1}{s+1} \| \cdot \|^{s+1}$ we obtain by the subdifferential sum rule (2.2) and by (2.4)

$$0 \in \partial_C f(y) + r J_{s+1}(y - u) \text{ or equivalently } r J_{s+1}(u - y) \in \partial_C f(y).$$

On the other hand, using (4.5) and Proposition 2.1(b) we see that

$$\|r J_{s+1}(u - y)\| = r\|u - y\|^s \le r(\|u\| + \|y\|)^s < r \left(\frac{\varepsilon}{4} + \frac{3\varepsilon}{4} \right)^s = r\varepsilon^s = r\varepsilon',$$

hence

$$r J_{s+1}(u - y) \in T_{\varepsilon, r\varepsilon'}(y) \text{ or equivalently } J_{s+1}(u - y) \in \frac{1}{r} T_{\varepsilon, r\varepsilon'}(y).$$

This means

$$y \in \left(I + \frac{1}{r} J_{s+1}^{-1} \circ T_{\varepsilon, r\varepsilon'} \right)^{-1} (u),$$

which is the inclusion of the lemma. $\square$

Remark 4.1. When $s = 1$, that is, f is c-pln, we take $\varepsilon = \varepsilon'$.

Proposition 4.1. *Let $(X, \| \cdot \|)$ be a uniformly convex Banach space whose norm is Fréchet differentiable off zero and admits a modulus of convexity of power type $s + 1$ with $s \ge 1$. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function on the Banach space X which is finite at 0 and let $\varepsilon_0 > 0$ be such that f is lower semicontinuous and s-lower regular on an open convex set containing $B[0, \varepsilon_0]$ with a constant $c \ge 0$ and such that f is bounded from below on $B[0, \varepsilon_0]$. Let any reals $c_0 \in]0, 1/(4c)[$ and $\varepsilon > 0$ with*

$$\varepsilon^s < \min\{\varepsilon_0, (\varepsilon_0)^s, c_0 L_1\},$$

where L_1 is defined in (2.7). Then, there exists $\lambda_0 > 0$ such that, for any positive real $\lambda < \lambda_0$, one has

$$P_{\lambda,s+1,\varepsilon} f(u) = \left(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon^s}{\lambda}} \right)^{-1} (u) \quad \text{for all } u \in B\left(0, \frac{\varepsilon}{4}\right),$$

where $T_{\varepsilon,\varepsilon^s/\lambda} := T^f_{\varepsilon,\varepsilon^s/\lambda,\overline{x}}$, with $\overline{x} = 0$, is the truncated operator of $\partial_C f$ in Definition 4.1.

Further, both mappings of the latter equality are nonempty, single-valued and $\frac{1}{s+1}$-Hölderian on $B(0, \frac{\varepsilon}{4})$ with

$$\|P_{\lambda,s+1,\varepsilon}(u_2) - P_{\lambda,s+1,\varepsilon}(u_1)\| \le K_\varepsilon \|u_2 - u_1\|^{1/s+1} \quad \textit{for all } u_1, u_2 \in B(0, \varepsilon/4),$$

where $K_\varepsilon := \left(\frac{c_0(\mu L_1 + 2\varepsilon^s)}{c_0 L_1 - \varepsilon^s}\right)^{\frac{1}{s+1}}$ *and* $\mu := (s+1)(\frac{5\varepsilon}{2})^s$.

Proof. Fix any positive real $c_0 < \frac{1}{4c}$ and any positive real ε as in the statement, and put $\varepsilon' := (\varepsilon)^s$. Consider the real $r_0 > 0$ (depending on ε) provided by Lemma 4.2. Applying that lemma with $\lambda_0 := \frac{1}{r_0}$ ensures, for any $\lambda \in]0, \lambda_0[$, that

$$P_{\lambda,s+1,\varepsilon} f(u) \subset \left(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}}\right)^{-1}(u) \quad \text{for all } u \in B\left(0, \frac{\varepsilon}{4}\right). \tag{4.6}$$

Since the function f is s-lower regular on an open convex set containing $B[0,\varepsilon]$, it follows from Lemma 4.1 that the (set-valued) operator T_{ε, rc_0} is $(r, s+1)$-hypomonotone for any real $r \ge 1/c_0$, i.e.,

$$\langle x_2^* - x_1^*, x_2 - x_1 \rangle \ge -r\|x_2 - x_1\|^{s+1} \tag{4.7}$$

whenever $x_i^* \in \partial_C f(x_i)$, $\|x_i^*\| \le c_0 r$, $\|x_i\| \le \varepsilon$, $i = 1, 2$.

We claim that the operator $T_{\varepsilon, r\varepsilon'}$ is $(\frac{r\varepsilon'}{c_0}, s+1)$-hypomonotone, for all real $r \ge 1/\varepsilon'$. Indeed, taking $x_i^* \in \partial_C f(x_i)$ with $\|x_i^*\| \le r\varepsilon'$ and $\|x_i\| \le \varepsilon$, $i = 1, 2$, we have

$$\|x_i^*\| \le c_0 \frac{r\varepsilon'}{c_0} \quad \text{and} \quad r \ge \frac{1}{\varepsilon'},$$

then

$$\|x_i^*\| \le c_0 \frac{r\varepsilon'}{c_0} \quad \text{and} \quad \frac{r\varepsilon'}{c_0} \ge \frac{1}{c_0},$$

hence applying (4.7) with $(r\varepsilon')/c_0$ in place of r we get the desired result. Consequently for $\lambda_0 := \min\{\varepsilon', \frac{1}{r_0}\}$ we find, for any $\lambda \in]0, \lambda_0[$ that $T_{\varepsilon, \frac{\varepsilon'}{\lambda}}$ is $(\frac{\varepsilon'}{c_0 \lambda}, s+1)$-hypomonotone. The rest of the proof is developed in three steps with λ fixed in $]0, \lambda_0[$.

Step 1. Hölder property of $\left(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}}\right)^{-1}$ on $B(0, \frac{\varepsilon}{4}) \cap \operatorname{Dom} P_{\lambda,s+1,\varepsilon} f$.

Take $x_i \in \left(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}}\right)^{-1}(u_i)$ with $u_i \in B(0, \frac{\varepsilon}{4}) \cap \operatorname{Dom} P_{\lambda,s+1,\varepsilon} f$, hence

$$\frac{1}{\lambda} J_{s+1}(u_i - x_i) \in T_{\varepsilon, \frac{\varepsilon'}{\lambda}}(x_i). \tag{4.8}$$

According to the $(\frac{\varepsilon'}{c_0\lambda}, s+1)$-hypomonotonicity of $T_{\varepsilon, \frac{\varepsilon'}{\lambda}}$ we have

$$\left\langle \frac{1}{\lambda} J_{s+1}(u_2 - x_2) - \frac{1}{\lambda} J_{s+1}(u_1 - x_1), x_2 - x_1 \right\rangle \ge -\frac{\varepsilon'}{c_0 \lambda}\|x_2 - x_1\|^{s+1},$$

which implies

$$\begin{aligned} &\langle J_{s+1}(u_2 - x_2) - J_{s+1}(u_1 - x_1), u_1 - x_1 - (u_2 - x_2)\rangle \\ &\quad \ge \langle J_{s+1}(u_2 - x_2) - J_{s+1}(u_1 - x_1), u_1 - u_2\rangle - \frac{\varepsilon'}{c_0}\|x_2 - x_1\|^{s+1}. \end{aligned} \tag{4.9}$$

On the other hand, (4.8) and Proposition 2.1(b) give

$$\|x_i\| \le \varepsilon \quad \text{and} \quad \|u_i - x_i\|^s \le \varepsilon'. \tag{4.10}$$

Further, the norm $\|\cdot\|$ being uniformly convex with a modulus of convexity of power type $s+1$, we know by (2.7) that

$$\begin{aligned}\langle J_{s+1}(u_2 - x_2) - J_{s+1}(u_1 - x_1), u_2 - x_2 - (u_1 - x_1)\rangle &\ge L_1 \|u_2 - x_2 - u_1 + x_1\|^{s+1} \\ &\ge L_1 \big|\|x_1 - x_2\| - \|u_1 - u_2\|\big|^{s+1}.\end{aligned}$$

Putting

$$a := \|x_1 - x_2\|, \qquad b := \|u_1 - u_2\|, \qquad \varphi(t) := |a - bt|^{s+1}$$

and noting by (4.10) that $a \le 2\varepsilon$, $b \le \frac{\varepsilon}{2}$, it results that

$$\begin{aligned}\big|\|x_1 - x_2\| - \|u_1 - u_2\|\big|^{s+1} - \|x_1 - x_2\|^{s+1} = \varphi(1) - \varphi(0) = \int_0^1 \varphi'(t)dt \\ = -(s+1)b \int_0^1 |a - bt|^s \ge -(s+1)b(a+b)^s \ge -(s+1)\left(\frac{5\varepsilon}{2}\right)^s b.\end{aligned}$$

For $\mu := (s+1)(\frac{5\varepsilon}{2})^s$, we obtain

$$\langle J_{s+1}(u_2 - x_2) - J_{s+1}(u_1 - x_1), u_2 - x_2 - (u_1 - x_1)\rangle \ge L_1 \left(\|x_1 - x_2\|^{s+1} - \mu\|u_1 - u_2\|\right),$$

hence using (4.9) and (4.10) it results that

$$\begin{aligned}L_1 \left(\mu\|u_1 - u_2\| - \|x_1 - x_2\|^{s+1}\right) &\ge \langle J_{s+1}(u_2 - x_2) - J_{s+1}(u_1 - x_1), u_1 - x_1 - (u_2 - x_2)\rangle \\ &\ge \langle J_{s+1}(u_2 - x_2) - J_{s+1}(u_1 - x_1), u_1 - u_2\rangle - \frac{\varepsilon'}{c_0}\|x_2 - x_1\|^{s+1} \\ &\ge -\left(\|J_{s+1}(u_2 - x_2)\| + \|J_{s+1}(u_1 - x_1)\|\right)\|u_1 - u_2\| - \frac{\varepsilon'}{c_0}\|x_2 - x_1\|^{s+1} \\ &= -\left(\|u_2 - x_2\|^s + \|u_1 - x_1\|^s\right)\|u_1 - u_2\| - \frac{\varepsilon'}{c_0}\|x_2 - x_1\|^{s+1} \\ &\ge -2\varepsilon'\|u_1 - u_2\| - \frac{\varepsilon'}{c_0}\|x_1 - x_2\|^{s+1},\end{aligned}$$

hence

$$\frac{c_0 L_1 - \varepsilon'}{c_0}\|x_1 - x_2\|^{s+1} \le (\mu L_1 + 2\varepsilon')\|u_1 - u_2\|.$$

Since $\varepsilon' < c_0 L_1$, we find

$$\|x_1 - x_2\| \le K'_{\varepsilon'}\|u_1 - u_2\|^{\frac{1}{s+1}}, \tag{4.11}$$

with $K'_{\varepsilon'} := \left(\frac{c_0(\mu L_1 + 2\varepsilon')}{c_0 L_1 - \varepsilon'}\right)^{\frac{1}{s+1}}$.

Step 2. Let us show the inclusion $B(0, \frac{\varepsilon}{4}) \subset \mathrm{Dom}\, P_{\lambda,s+1,\varepsilon} f$.

Fix any $u \in B(0, \frac{\varepsilon}{4})$ and $\lambda \in]0, \lambda_0[$. According to the density of Fréchet subdifferentiability points of $e_{\lambda,s+1,\varepsilon} f$, for any integer $n \geq 1$, there exists some point $u_n \in B(0, \frac{\varepsilon}{4})$ with $u_n \in \mathrm{Dom}\, \partial_F e_{\lambda,s+1,\varepsilon} f \cap B(u, \frac{1}{n})$. Theorem 3.1(b) ensures that there exists some $x_n \in P_{\lambda,s+1,\varepsilon} f(u_n)$, then $x_n \in \left(I + \lambda J^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}}\right)^{-1}(u_n)$ thanks to (4.6), hence by (4.11), for any $n, m \geq 1$,

$$\|x_n - x_m\| \leq K_{\varepsilon'} \|u_n - u_m\|^{\frac{1}{s+1}},$$

which ensures that $(x_n)_n$ is a Cauchy sequence. Denoting by x the limit of $(x_n)_n$, we have $\|x\| \leq \varepsilon$ because $\|x_n\| \leq \varepsilon$ according to the inclusion $x_n \in P_{\lambda,s+1,\varepsilon} f(u_n)$. Further, the same inclusion $x_n \in P_{\lambda,s+1,\varepsilon} f(u_n)$ tells us that $e_{\lambda,s+1,\varepsilon} f(u_n) = f(x_n) + \frac{1}{\lambda(1+s)} \|u_n - x_n\|^{s+1}$ and thus

$$\liminf_n e_{\lambda,s+1,\varepsilon} f(u_n) = \liminf_n \left(f(x_n) + \frac{1}{\lambda(1+s)} \|u_n - x_n\|^{s+1} \right).$$

Observing that $u_n \to u$ and recalling that $x_n \to x$, the continuity property of $e_{\lambda,s+1,\varepsilon} f$ (see Theorem 3.1(d)) and the lower semicontinuity of f on $B[0, \varepsilon_0]$ give

$$e_{\lambda,s+1,\varepsilon} f(u) \geq f(x) + \frac{1}{\lambda(1+s)} \|u - x\|^{s+1},$$

which means that $x \in P_{\lambda,s+1,\varepsilon} f(u)$ and proves the desired inclusion $B(0, \frac{\varepsilon}{4}) \subset \mathrm{Dom}\, P_{\lambda,s+1,\varepsilon} f$.

Step 3. The inequality (4.11) gives that $(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}})^{-1}$ is single-valued on $B(0, \varepsilon/4) \cap \mathrm{Dom}\, P_{\lambda,s+1,\varepsilon}$, and the inclusions $P_{\lambda,s+1,\varepsilon} f(\cdot) \subset (I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}})^{-1}(\cdot)$ from (4.6) and $B(0, \frac{\varepsilon}{4}) \subset \mathrm{Dom}\, P_{\lambda,s+1,\varepsilon} f$ from Step 2 give that $P_{\lambda,s+1,\varepsilon} f$ is nonempty single-valued and $\frac{1}{s+1}$-Hölderian on $B(0, \frac{\varepsilon}{4})$ with coefficient $K_{\varepsilon'}$. Consequently

$$P_{\lambda,s+1,\varepsilon} f(u) = \left(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon, \frac{\varepsilon'}{\lambda}}\right)^{-1}(u) \quad \text{for all } u \in B\left(0, \frac{\varepsilon}{4}\right),$$

and the Hölder property of $P_{\lambda,s+1,\varepsilon} f$ in the proposition holds true in $B(0, \frac{\varepsilon}{4})$ with the constant

$$K_\varepsilon := K'_{\varepsilon'} = \left(\frac{c_0(\mu L_1 + 2\varepsilon^s)}{c_0 L_1 - \varepsilon^s} \right)^{\frac{1}{s+1}}.$$

This finishes the proof of the proposition. □

In place of the truncated proximal mapping we focus now on the proximal mapping itself of a localization near zero of f.

Proposition 4.2. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space whose norm admits a modulus of convexity of power type $s+1$ with $s \geq 1$ and a modulus of smoothness of power type $\alpha + 1$ with $0 < \alpha \leq 1$. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function which is finite at 0 and let $c \geq 0$ and $\varepsilon_0 > 0$ be such that f is lower semicontinuous and s-lower regular with constant c on an open convex set containing $B[0, \varepsilon_0]$ and such that f is bounded from below on $B[0, \varepsilon_0]$. Let any reals $c_0 \in]0, 1/(4c)[$ and $\varepsilon > 0$ with*

$$\varepsilon^s < \min\{\varepsilon_0, (\varepsilon_0)^s, c_0 L_1\},$$

where L_1 is defined in (2.7).

Define $\overline{f}(\cdot) := f(\cdot) + \delta(., B[0, \varepsilon])$. *Then there exists* $\lambda_0 > 0$ *such that for any* $0 < \lambda < \lambda_0$, *one has the following properties:*

(a) $P_{\lambda,s+1}\overline{f}$ *is nonempty, single-valued and* $\frac{1}{s+1}$*-Hölderian on* $B(0, \frac{\varepsilon}{4})$.
(b) *The function* $e_{\lambda,s+1}\overline{f}$ *is* $\mathcal{C}^{1,\frac{\alpha}{s+1}}$ *on* $B(0, \frac{\varepsilon}{4})$ *and for all* $u \in B(0, \frac{\varepsilon}{4})$

$$De_{\lambda,s+1}\overline{f}(u) = \lambda^{-1} J_{s+1}(u - P_{\lambda,s+1}\overline{f}(u)).$$

Proof. (a) For any $u \in X$ we know that $P_{\lambda,s+1}\overline{f}(u) = P_{\lambda,s+1,\varepsilon} f(u)$. According to Proposition 4.1 we obtain that $P_{\lambda,s+1}\overline{f}$ is nonempty, single-valued and $\frac{1}{s+1}$-Hölderian on $B(0, \frac{\varepsilon}{4})$ with the constant K_ε in the statement of that proposition.

(b) Since $P_{\lambda,s+1}\overline{f}$ is nonempty, single-valued and continuous on $B(0, \frac{\varepsilon}{4})$, Proposition 3.1 tells us that the function $e_{\lambda,s+1}\overline{f}$ is $\mathcal{C}^1$ on $B(0, \frac{\varepsilon}{4})$ and that

$$De_{\lambda,s+1}\overline{f}(u) = \lambda^{-1} J_{s+1}(u - P_{\lambda,s+1}\overline{f}(u)) \quad \text{for all } u \in B\left(0, \frac{\varepsilon}{4}\right).$$

Now fix any $x, y \in B(0, \frac{\varepsilon}{4})$, and observe by (2.8) that

$$\begin{aligned}
&\|De_{\lambda,s+1}\overline{f}(x) - De_{\lambda,s+1}\overline{f}(y)\| \\
&\quad = \frac{1}{\lambda}\|J_{s+1}(x - P_{\lambda,s+1}\overline{f}(x)) - J_{s+1}(y - P_{\lambda,s+1}\overline{f}(y))\| \\
&\quad \le \frac{L_2}{\lambda} \max\{\|x - P_{\lambda,s+1}\overline{f}(x)\|, \|y - P_{\lambda,s+1}\overline{f}(y)\|\}^{s-\alpha} \|x - y + P_{\lambda,s+1}\overline{f}(y) - P_{\lambda,s+1}\overline{f}(x)\|^\alpha.
\end{aligned}$$

Since $P_{\lambda,s+1}\overline{f}(\cdot)$ is $\frac{1}{s+1}$-Hölderian on $B(0, \frac{\varepsilon}{4})$, it is bounded therein, so

$$K := \frac{L_2}{\lambda} \sup_{u \in B(0,\varepsilon/4)} \|u - P_{\lambda,s+1}\overline{f}(u)\|^{s-\alpha} < +\infty.$$

Therefore,

$$\|De_{\lambda,s+1}\overline{f}(x) - De_{\lambda,s+1}\overline{f}(y)\| \le K(\|x - y\| + \|P_{\lambda,s+1}\overline{f}(x) - P_{\lambda,s+1}\overline{f}(y)\|)^\alpha.$$

This and the Hölderian property of $P_{\lambda,s+1}\overline{f}$ give

$$\begin{aligned}
\|De_{\lambda,s+1}\overline{f}(x) - De_{\lambda,s+1}\overline{f}(y)\| &\le K\left(\|x - y\| + K_\varepsilon \|x - y\|^{\frac{1}{s+1}}\right)^\alpha \\
&= K\left(\|x - y\|^{\frac{s}{s+1}} \|x - y\|^{\frac{1}{s+1}} + K_\varepsilon \|x - y\|^{\frac{1}{s+1}}\right)^\alpha \\
&\le K\left(\left(\frac{\varepsilon}{2}\right)^{\frac{s}{s+1}} \|x - y\|^{\frac{1}{s+1}} + K_\varepsilon \|x - y\|^{\frac{1}{s+1}}\right)^\alpha \\
&= K\left(\left(\frac{\varepsilon}{2}\right)^{\frac{s}{s+1}} + K_\varepsilon\right)^\alpha \|x - y\|^{\frac{\alpha}{s+1}}.
\end{aligned}$$

The function $De_{\lambda,s+1}\overline{f}$ is then $\frac{\alpha}{s+1}$-Hölderian over $B(0, \frac{\varepsilon}{4})$ and the proof of the proposition is finished. $\square$

Now we establish the $\mathcal{C}^{1,\frac{\alpha}{s+1}}$ property of $e_{\lambda,s+1}\overline{f}$ around a point $u_0 \in \operatorname{dom} f$.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space whose norm $\|\cdot\|$ admits a modulus of convexity of power type $s+1$ with $s \geq 1$ and a modulus of smoothness of power type $\alpha+1$ with $0 < \alpha \leq 1$. Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a function and let $c \geq 0$ and $\eta > 0$ be such that f is lower semicontinuous and s-lower regular with constant c on an open convex set containing $B[u_0, \eta]$ and such that f is bounded from below on $B[u_0, \eta]$. Let any positive real $c_0 < \frac{1}{4c}$ and any real $r_0 \in]0, \min\{\eta, 2(c_0L_1)^{\frac{1}{s}}, \left(2\left(\frac{3}{4}\right)^{\frac{1}{s}}\right)^{\frac{s}{s-1}}\}[$, where L_1 is the constant defined in (2.7). Let $x_0 \in \operatorname{dom} f$ be such that $\|x_0 - u_0\| \leq \frac{r_0}{16}$ and let $\overline{f}(\cdot) := f(\cdot) + \delta(., B[x_0, \frac{r_0}{2}])$. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in]0, \lambda_0[$, one has*

(a) *$e_{\lambda,s+1}\overline{f}$ is $\mathcal{C}^{1,\frac{\alpha}{s+1}}$ on $B(u_0, \frac{r_0}{16})$;*
(b) *$P_{\lambda,s+1}\overline{f}$ is nonempty single-valued and $\frac{1}{s+1}$-Hölderian on $B(u_0, \frac{r_0}{16})$, with*

$$\|P_{\lambda,s+1}\overline{f}(x) - P_{\lambda,s+1}\overline{f}(y)\| \leq K \|x-y\|^{1/(s+1)} \quad \text{for all } x, y \in B\left(u_0, \frac{r_0}{16}\right),$$

where $K := \left(\frac{c_0(L_1+r_0)}{c_0L_1-(r_0/2)}\right)^{1/2}$.

Proof. Let us define for all $x \in X$

$$F(x) := f(x+x_0) + \delta(x+x_0, B[u_0, r_0]) - f(x_0) \quad \text{and} \quad \overline{F}(x) = F(x) + \delta(x, B[0, \varepsilon]),$$

with $\varepsilon := \frac{r_0}{2}$. Clearly, F is proper, lower semicontinuous and $F(0) = 0$.

Step 1. Let us show the s-lower regularity of F on $B(0, \frac{3r_0}{4})$.
For any $x \in B(0, \frac{3r_0}{4})$, we observe that on the one hand

$$\partial_C F(x) = \partial_C\big(f(\cdot + x_0) + \delta(\cdot + x_0, B[u_0, r_0])\big)(x),$$

and on the other hand $\|x + x_0 - u_0\| \leq (3r_0/4) + (r_0/16) < r_0$. The latter inequality implies

$$f(x+x_0) + \delta(x+x_0, B[u_0, r_0]) = f(x+x_0) \quad \text{for all } x \in B\left(0, \frac{3r_0}{4}\right),$$

hence for all x in the open ball $B(0, \frac{3r_0}{4})$

$$\partial_C\big(f(\cdot + x_0) + \delta(\cdot + x_0, B[u_0, r_0])\big)(x) = \partial_C f(\cdot + x_0)(x),$$

which gives

$$\partial_C F(x) = \partial_C f(x + x_0).$$

Now fix any $x \in B(0, \frac{3r_0}{4}) \cap \operatorname{Dom} \partial_C F$ and any $x^* \in \partial_C F(x)$, so $x^* \in \partial_C f(x + x_0)$ thanks to the latter equality above. For every $y \in B(0, \frac{3r_0}{4})$, according to the s-lower regularity of f on $B(u_0, r_0) \subset B(u_0, \eta)$ and to the inclusions $x + x_0 \in B(u_0, r_0)$ and $y + x_0 \in B(u_0, r_0)$, we have

$$f(y+x_0) \geq f(x+x_0) + \langle x^*, y-x\rangle - c(1+\|x^*\|)\|y-x\|^{s+1}, \tag{4.12}$$

or equivalently

$$f(y+x_0) - f(x_0) - (f(x+x_0) - f(x_0)) \geq \langle x^*, y-x\rangle - c(1+\|x^*\|)\|y-x\|^{s+1},$$

that is,

$$F(y) - F(x) \geq \langle x^*, y - x\rangle - c(1 + \|x^*\|)\|y - x\|^{s+1}.$$

This confirms the s-lower regularity of F on $B(0, \frac{3r_0}{4})$.

Step 2. Let us show that $\inf_{B[0,\frac{3r_0}{4}]} F$ is finite.

Indeed we have

$$\begin{aligned}
\inf_{x\in B[0,\frac{3r_0}{4}]} F(x) &= \inf_{x\in B\left[0,\frac{3r_0}{4}\right]} (f(x + x_0) - f(x_0) + \delta(x + x_0, B[u_0, r_0])) \\
&= \inf_{z\in B\left[x_0,\frac{3r_0}{4}\right]} (f(z) + \delta(z, B[u_0, r_0])) - f(x_0) \\
&= \inf_{z\in B\left[x_0,\frac{3r_0}{4}\right]} f(z) - f(x_0) \in \mathbb{R},
\end{aligned}$$

where the latter infimum is finite since $B[x_0, \frac{3r_0}{4}] \subset B[u_0, \eta]$ and $x_0 \in \operatorname{dom} f$.

Step 3. Take $\varepsilon_0 := \frac{3r_0}{4}$ and note that $\varepsilon^s < \min\{\varepsilon_0, \varepsilon_0^s, c_0L_1\}$ according to the assumption concerning r_0. Then the properties of Step 1 and Step 2, combined with Propositions 4.1 and 4.2 ensure the existence of a real $\lambda_0 > 0$ such that for any $\lambda \in]0, \lambda_0[$

$$P_{\lambda,s+1}\overline{F}(\cdot) = \left(I + \lambda J_{s+1}^{-1} \circ T_{\varepsilon,\frac{\varepsilon^s}{\lambda}}\right)^{-1} \quad \text{on } B\left(0, \frac{\varepsilon}{4}\right),$$

with both mappings being nonempty, single-valued and $\frac{1}{s+1}$-Hölderian on the open ball $B(0, \frac{\varepsilon}{4}) = B(0, \frac{r_0}{8})$ with the constant $K_\varepsilon := \left(\frac{c_0(L_1+2\varepsilon^s)}{c_0L_1-\varepsilon^s}\right)^{\frac{1}{2}}$, and $e_{\lambda,s+1}\overline{F}$ being $\mathcal{C}^{1,\frac{\alpha}{s+1}}$ on $B(0, \frac{r_0}{8})$ with

$$De_{\lambda,s+1}\overline{F}(u) = \lambda^{-1}J_{s+1}(u - P_{\lambda,s+1}\overline{F}(u)) \quad \text{for all } u \in B\left(0, \frac{\varepsilon}{4}\right). \tag{4.13}$$

On the other hand, for all $x \in X$ and $\lambda > 0$, we observe that

$$\begin{aligned}
e_{\lambda,s+1}\overline{F}(x) &= \inf_{y\in X}\left(F(y) + \delta(y, B[0, \varepsilon]) + \frac{1}{(s + 1)\lambda}\|x - y\|^{s+1}\right) \\
&= \inf_{y\in X}\left(f(y + x_0) - f(x_0) + \delta(y + x_0, B[u_0, r_0]) + \delta(y + x_0, B[x_0, \varepsilon]) + \frac{1}{(s + 1)\lambda}\|x - y\|^{s+1}\right) \\
&= -f(x_0) + \inf_{z\in X}\left(f(z) + \delta(z, B[u_0, r_0]) + \delta(z, B[x_0, \varepsilon]) + \frac{1}{(s + 1)\lambda}\|x + x_0 - z\|^{s+1}\right).
\end{aligned}$$

Since $B[x_0, \varepsilon] \subset B[u_0, r_0]$, the latter yields

$$\begin{aligned}
e_{\lambda,s+1}\overline{F}(x) &= -f(x_0) + \inf_{z\in X}\left(f(z) + \delta(z, B[u_0, r_0] \cap B[x_0, \varepsilon]) + \frac{1}{(s + 1)\lambda}\|x + x_0 - z\|^{s+1}\right) \\
&= -f(x_0) + \inf_{z\in X}\left(f(z) + \delta(z, B[x_0, \varepsilon]) + \frac{1}{(s + 1)\lambda}\|x + x_0 - z\|^{s+1}\right) \\
&= -f(x_0) + \inf_{z\in X}\left(\overline{f}(z) + \frac{1}{(s + 1)\lambda}\|x + x_0 - z\|^{s+1}\right) \\
&= -f(x_0) + e_{\lambda,s+1}\overline{f}(x + x_0),
\end{aligned}$$

which gives

$$e_{\lambda,s+1}\overline{f}(x) = e_{\lambda,s+1}\overline{F}(x - x_0) + f(x_0). \tag{4.14}$$

The $\mathcal{C}^{1,\frac{\alpha}{s+1}}$ property of $e_{\lambda,s+1}\overline{F}$ on $B(0, \frac{r_0}{8})$ with (4.14) entails that $e_{\lambda,s+1}\overline{f}$ is $\mathcal{C}^{1,\frac{\alpha}{s+1}}$ on $B(x_0, \frac{r_0}{8})$ and since $B(u_0, \frac{r_0}{16}) \subset B(x_0, \frac{r_0}{8})$ we get

$$e_{\lambda,s+1}\overline{f} \text{ is } \mathcal{C}^{1,\frac{\alpha}{s+1}} \text{ on } B\left(u_0, \frac{r_0}{16}\right).$$

Hence by Proposition 3.1, $P_{\lambda,s+1}\overline{f}$ is nonempty single-valued and continuous on $B(u_0, \frac{r_0}{16})$ with

$$De_{\lambda,s+1}\overline{f}(x) = \lambda^{-1}J_{s+1}(x - P_{\lambda,s+1}\overline{f}(x)) \quad \text{for all } x \in B\left(u_0, \frac{r_0}{16}\right). \tag{4.15}$$

For any $x \in B(u_0, \frac{r_0}{16})$, it results from (4.14) that $De_{\lambda,s+1}\overline{f}(x) = De_{\lambda,s+1}\overline{F}(x - x_0)$, which means by (4.13) and (4.15) that

$$\lambda^{-1}J_{s+1}(x - P_{\lambda,s+1}\overline{f}(x)) = \lambda^{-1}J_{s+1}(x - x_0 - P_{\lambda,s+1}\overline{F}(x - x_0)),$$

or equivalently $x - P_{\lambda,s+1}\overline{f}(x) = x - x_0 - P_{\lambda,s+1}\overline{F}(x - x_0)$. Consequently,

$$P_{\lambda,s+1}\overline{f}(x) = P_{\lambda,s+1}\overline{F}(x - x_0) + x_0 \quad \text{for all } x \in B\left(u_0, \frac{r_0}{16}\right),$$

so, for all $x, y \in B(u_0, \frac{r_0}{16})$,

$$\begin{aligned} \|P_{\lambda,s+1}\overline{f}(x) - P_{\lambda,s+1}\overline{f}(y)\| &= \|P_{\lambda,s+1}\overline{F}(x - x_0) - P_{\lambda,s+1}\overline{F}(y - x_0)\| \\ &\le K_\varepsilon \|x - y\|^{\frac{1}{s+1}}, \end{aligned}$$

which translates the Hölder property of $P_{\lambda,s+1}\overline{f}$ on $B(u_0, \frac{r_0}{16})$ and finishes the proof of theorem. □

Remark 4.2. $e_{\lambda,s+1}\overline{f}$ is $\frac{s+\alpha+1}{s+1}$-lower regular on $B(u_0, \frac{r_0}{16})$ according to the property (a) of the last theorem.

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