



Scalar field equation with non-local diffusion

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Abstract. In this paper we are interested on the existence of ground state solutions for fractional field equations of the form

$$\begin{cases} (I - \Delta)^\alpha u = f(x, u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad \lim_{|x| \rightarrow \infty} u(x) = 0,$$

where $\alpha \in (0, 1)$ and f is an appropriate super-linear sub-critical nonlinearity. We prove regularity, exponential decay and symmetry properties for these solutions. We also prove the existence of infinitely many bound states and, through a non-local Pohozaev identity, we prove nonexistence results in the supercritical case.

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1. Introduction

We are interested in the study of the nonlinear scalar field equation with fractional diffusion

$$\begin{cases} (I - \Delta)^\alpha u = f(x, u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad \lim_{|x| \rightarrow \infty} u(x) = 0, \quad (1.1)$$

where $\alpha \in (0, 1)$ and f is an appropriate super-linear sub-critical nonlinearity.

Equation (1.1) arises in the study of standing waves for the Schrödinger–Klein–Gordon equation

$$i \frac{\partial \psi}{\partial t} = (I - \Delta)^\alpha \psi - \psi - f(x, \psi) \quad (1.2)$$

describing the the behaviour of bosons, spin-0 particles in relativistic fields, see the work by Lieb and Thirring [14] and Lieb and Yau [15]. Actually $\psi(x, t) = \exp(-it)u(x)$ satisfies (1.2) if and only is u satisfies (1.1). The mean field dynamics of boson stars modelled by the pseudo-relativistic Hartree equations have been studied by Elgart and Schlein [6] and Frohlich et al. [9]. In the context of fractional quantum mechanics, non-linear fractional Schrödinger

equation has been proposed by Laskin [12,13] as a result of expanding the Feynman path integral, from the Brownian-like to the Lévy-like quantum mechanical paths.

In a recent paper, Coti Zelati and Nolasco [4] considered Eq. (1.1) for the particular case of $\alpha = \frac{1}{2}$ and for a power nonlinearity. They studied the equation by an extension technique developed by Silvestre and Caffarelli [2] that transforms the problem into a local one by adding one variable. They also obtained regularity and exponential decay of the solution and their nonlinearity admits a nonlocal term. See also Tan et al. [24] where some of these results are also proved.

In this paper we consider Eq. (1.1) working directly with the nonlocal operator in the appropriate Sobolev space. In doing so, we follow some ideas used in [7] in the study of nonlinear Schrödinger equation with the fractional laplacian, but there are some interesting novelties. On one hand for the study of exponential decay we cannot use super and sub-solutions directly as in [7], since we were not able to obtain these comparison functions easily. Instead of that, we consider a method based on a weighted space and the Banach fixed point theorem and comparison theorem. On the other hand, here we obtain a non-local version of the Pohozaev identity from which we can prove a non-existence theorem for positive solutions in the super-critical case.

In what follows we introduce our precise hypotheses on the nonlinearity f . We assume:

(f0) $f: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

(f1) $f(x, s) \geq 0$ if $s \geq 0$ and $f(x, s) = 0$ if $s \leq 0$, for all $x \in \mathbb{R}^N$.

(f2) The function

$$s \mapsto \frac{f(x, s)}{s} \quad \text{is increasing for } s > 0 \text{ and all } x \in \mathbb{R}^N.$$

(f3) There exists $p \in \left(1, \frac{N+2\alpha}{N-2\alpha}\right)$ and $C > 0$ such that

$$f(x, s) \leq C|s|^p \quad \text{for all } s \in \mathbb{R}, x \in \mathbb{R}^N.$$

(f4) There exists $\theta > 2$ such that, for all $s > 0, x \in \mathbb{R}^N$,

$$0 < \theta F(x, s) \leq sf(x, s),$$

where $F(x, s) = \int_0^s f(t) dt$.

(f5) There are continuous functions \bar{f} and a , defined in \mathbb{R} and \mathbb{R}^N respectively, such that \bar{f} satisfies (f0)–(f4) and

$$0 \leq f(x, s) - \bar{f}(s) \leq a(x)(|s| + |s|^p) \quad \text{for all } s \in \mathbb{R}, x \in \mathbb{R}^N,$$

$$\lim_{|x| \rightarrow \infty} a(x) = 0$$

and

$$|\{x \in \mathbb{R}^N : f(x, s) > \bar{f}(s) \forall s > 0\}| > 0,$$

where $|\cdot|$ stands for the Lebesgue measure.

The operator $(I - \Delta)^\alpha$ is defined by its Fourier transform

$$\mathcal{F}((I - \Delta)^\alpha u)(\xi) = (1 + |\xi|^2)^\alpha \hat{u}(\xi).$$

Now we state our main theorems. In the first place we obtain ground states for (1.1)

Theorem 1.1. *Assuming $\alpha \in (0, 1)$ and that the nonlinearity f satisfies (f0)–(f5) Eq. (1.1) possesses a non-negative weak solution u . Moreover, the function u is continuous, positive and it satisfies*

$$0 < u(x) \leq C e^{-\frac{|x|}{2}}, \tag{1.3}$$

where $C > 0$ is an appropriate constant. In case $f(u) = u^p$ the solution u is radially symmetric.

The proof of the existence of a weak solution of (1.1) is undertaken variationally using the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha |\hat{u}(\xi)|^2 d\xi - \int_{\mathbb{R}^N} F(x, u(x)) dx, \tag{1.4}$$

defined on the Sobolev space $H^\alpha(\mathbb{R}^N)$. See Sect. 2 where we also define weak solutions for (1.1) as the critical points of I . In proving the existence part of Theorem 1.1 we use the mountain pass theorem in combination with a comparison argument for avoiding non-convergent Palais–Smale sequences, as devised by Rabinowitz [19]. In order to prove the decay estimate, after proving regularity of the solution, we devise a fixed point scheme that somehow resemble the spirit of the Slaggie–Wichmann method for the two-body eigenfunctions, see the review of Hislop [11].

Regarding the hypotheses in Theorem 1.1, we are not claiming optimality at all. Since its statement includes various results, we could have broken it into different theorems and for each one state more general hypothesis. In particular, (f0) can be relaxed for existence assuming only measurability in the x -variable, but continuity is necessary to prove the decay. Assumption (f1) could also be relaxed using a usual trick of redefining f as being zero for negative arguments and then prove the solution is non-negative. Hypothesis (f3) can be made more general, but then the regularity proof gets more involved, see [7].

Our second theorem on the existence multiple solutions states as follows.

Theorem 1.2. *If $N \geq 2$ and $1 < p < \frac{N+2\alpha}{N-2\alpha}$, the equation*

$$(I - \Delta)^\alpha u = |u|^{p-1}u \tag{1.5}$$

has infinitely many weak solutions in $H^\alpha(\mathbb{R}^N) \cap C^\mu(\mathbb{R}^N)$, for some $\mu > 0$.

The proof of this theorem is based on Lusternik–Schnirelmann genus.

Our third the theorem states the non-existence of solution in the supercritical case.

Theorem 1.3. *If $p \geq \frac{N+2\alpha}{N-2\alpha}$, then the problem*

$$\begin{cases} (I - \Delta)^\alpha u = u^p & \text{in } \mathbb{R}^N \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \quad \lim_{|x| \rightarrow \infty} u(x) = 0 \tag{1.6}$$

has no non-trivial weak solution in $L^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, with $q \geq 1$.

Recently, a great attention has been focused on the study of problems involving fractional elliptic operators, from a pure mathematical point of view as well as from concrete applications, since these operators naturally arise in many different contexts, such as obstacle problems, financial mathematics, phase transitions, anomalous diffusions, crystal dislocations, soft thin films, semipermeable membranes, flame propagations, conservation laws, ultra relativistic limits of quantum mechanics, quasi-geostrophic flows, minimal surfaces, materials science and water waves. The literature is too wide to attempt a reasonable list of references here, so we derive the reader to the work by Di Nezza et al. [5], where a more extensive bibliography and an introduction to the subject are given. Regarding the non-linear fractional Schrödinger equation, see the works in Coti Zelati and Nolasco [4], Tan et al. [24], Cheng [3], Felmer et al. [7], Felmer and Torres [8], Guo and Xu [10], Servadei and Valdinoci [20, 21].

This paper is organized as follow. In Sect. 2 we present some preliminary regarding the framework in which we study the problem. Then we prove the existence of a non-negative weak solution of Eq. (1.1). In Sect. 3 we complete the proof of Theorem 1.1 obtaining regularity, decay and symmetry of the ground state. Section 4 is devoted to the proof of the multiplicity theorem. Finally in Sect. 5 we prove the Pohozaev identity and the non-existence theorem for positive solutions.

2. Existence of a ground state

We consider the Hilbert space

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha |\hat{u}(\xi)|^2 d\xi < \infty \right\}$$

endowed with the inner product

$$\langle u, v \rangle_\alpha = \int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha \hat{u} \bar{\hat{v}} d\xi.$$

The following results will be useful; for its proof see Theorem 2.8.4 in Ziemer [25] and also [7].

Lemma 2.1. *Let $q \in [2, \frac{2N}{N-2\alpha}]$. Then there exists $C > 0$ such that*

$$\|u\|_q \leq C \|u\|_\alpha \quad \forall u \in H^\alpha(\mathbb{R}^N).$$

If further $q < \frac{2N}{N-2\alpha}$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain, then every bounded sequence in $H^\alpha(\mathbb{R}^N)$ has a convergent subsequence in $L^q(\Omega)$.

The next lemma is due to Lions [16].

Lemma 2.2. *Let $N \geq 2$, $q \in (2, \frac{2N}{N-2\alpha})$ and $\{u_n\}$ a bounded sequence in $H^\alpha(\mathbb{R}^N)$ satisfying*

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} u_n(x)^2 dx = 0$$

for some $R > 0$. Then $u_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$.

We say that $u \in H^\alpha(\mathbb{R}^N)$ is a weak solution of the equation $(I - \Delta)^\alpha u = f(x, u)$ if

$$\int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi = \int_{\mathbb{R}^N} f(x, u(x))v(x) dx \quad \forall v \in H^\alpha(\mathbb{R}^N).$$

So we will look for critical points of the functional I defined in (1.4). Assuming (f0) and (f3) I is well defined, and using Sobolev embedding (Theorem 3.1) and the properties of the Nemytskii operator it can be proved that I is of class C^1 in $H^\alpha(\mathbb{R}^N)$.

We consider the Nehari manifold

$$\Lambda = \{u \in H^\alpha(\mathbb{R}^N) \setminus \{0\} : I'(u)u = 0\}$$

and define

$$c^* = \inf_{u \in \Lambda} I(u).$$

We observe that if $u \in \Lambda$, by (f1), $u_+ \neq 0$. And by (f2), given $u \in H^\alpha(\mathbb{R}^N)$ such that $u_+ \neq 0$, the function $t \in \mathbb{R}_+ \mapsto I(tu)$ has a unique maximum $t(u)$ and $t(u)u \in \Lambda$. Therefore

$$c^* = \inf_{u \in H^\alpha(\mathbb{R}^N) \setminus \{0\}} \sup_{t \geq 0} I(tu).$$

We define now

$$\Gamma = \{g \in C([0, 1], H^\alpha(\mathbb{R}^N)) : g(0) = 0, I(g(1)) < 0\}$$

and

$$c = \inf_{g \in \Gamma} \sup_{t \in [0, 1]} I(g(t)).$$

Using (f2) and (f3), it is easy to check that $c > 0$.

Lemma 2.3. $c = c^*$.

Proof. Let $u \in H^\alpha(\mathbb{R}^N) \setminus \{0\}$ such that $u_+ \neq 0$. We may define $g_u(t) = tTu$ with $T > 0$ large enough so that $I(Tu) < 0$. Then $g \in \Gamma$ and

$$c \leq \sup_{t \in [0, 1]} I(tTu) \leq \sup_{t \geq 0} I(tu).$$

Thus $c \leq c^*$. For the other inequality it suffices to show that for all $g \in \Gamma$ there exists $t \in (0, 1)$ such that $g(t) \in \Lambda$. First we notice that if $I'(u)u \geq 0$, by (f4),

$$\begin{aligned} I(u) &\geq \frac{1}{2} \int_{\mathbb{R}^N} f(x, u(x))u(x) dx - \int_{\mathbb{R}^N} F(x, u(x)) dx \\ &\geq \left(\frac{\theta}{2} - 1\right) \int_{\mathbb{R}^N} F(x, u(x)) dx \geq 0. \end{aligned}$$

So, if we assume that $I'(g(t))g(t) > 0$ for all $t \in (0, 1)$, then $I(g(t)) \geq 0$ for all $t \in (0, 1)$, contradicting that $I(g(1)) < 0$. \square

If \bar{f} does not depend on x , we define $\bar{I}, \bar{\Lambda}, \bar{\Gamma}$ and \bar{c} replacing f by \bar{f} .

Theorem 2.1. *If \bar{f} satisfies (f0)–(f4), then \bar{I} has a critical point with critical value \bar{c} .*

Proof. By the Ekeland variational principle (see [18]) there is a sequence $(u_n)_{n \in \mathbb{N}} \subset H^\alpha(\mathbb{R}^N)$ such that

$$\bar{I}(u_n) \rightarrow c \quad \text{and} \quad \bar{I}'(u_n) \rightarrow 0.$$

By (f4), given $\varepsilon > 0$, for all n large enough,

$$\left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\alpha^2 \leq \bar{I}(u_n) - \frac{1}{\theta} \bar{I}'(u_n)u_n \leq c + \varepsilon + \|u_n\|_\alpha.$$

So (u_n) is a bounded sequence in $H^\alpha(\mathbb{R}^N)$. Then, using Lemma 2.1, (u_n) has a subsequence that converges in $L^{p+1}_{loc}(\mathbb{R}^N)$ and weakly in $H^\alpha(\mathbb{R}^N)$ to some function u . For this subsequence and for every $\varphi \in \mathcal{D}(\mathbb{R}^N)$,

$$\bar{I}'(u_n)\varphi \rightarrow \bar{I}'(u)\varphi = 0.$$

It only remains to see that $\bar{I}(u) = c$. Using (f4) again, for all $R > 0$,

$$\begin{aligned} \bar{I}(u_n) - \frac{1}{2} \bar{I}'(u_n)u_n &= \int_{\mathbb{R}^N} \left(\frac{1}{2} \bar{f}(u_n)u_n - \bar{F}(u_n) \right) dx \\ &\geq \int_{B_R} \left(\frac{1}{2} \bar{f}(u_n)u_n - \bar{F}(u_n) \right) dx. \end{aligned}$$

Taking limit as $n \rightarrow \infty$,

$$c \geq \int_{B_R} \left(\frac{1}{2} \bar{f}(u)u - \bar{F}(u) \right) dx.$$

This holds for all $R > 0$, so we can take the integral on \mathbb{R}^N and we get $\bar{I}(u) \leq c$. For the other inequality it suffices to see that $u \neq 0$. Using Lemma 2.2, there is a sequence $\{y_n\} \subset \mathbb{R}^N$, $R > 0$ and $\beta > 0$ such that

$$\int_{B_R(y_n)} u_n(x)^2 dx > \beta.$$

In fact, assuming the contrary, we have $u_n \rightarrow 0$ in $L^{p+1}(\mathbb{R}^N)$. But using (f3), we get, for n large enough and some constant $A > 0$,

$$\bar{c} \leq \bar{I}(u_n) - \frac{1}{2} \bar{I}'(u_n)u_n = \int_{\mathbb{R}^N} \left(\frac{1}{2} \bar{f}(u_n)u_n - \bar{F}(u_n) \right) dx \leq A \int_{\mathbb{R}^N} |u_n|^{p+1} dx,$$

which implies $\bar{c} = 0$, providing a contradiction.

We may then define $\tilde{u}(x) = u(y_n + x)$ and we use the discussion given above to find that $u = w\text{-}\lim \tilde{u}_n$ is a nontrivial critical point of \bar{I} . \square

Theorem 2.2. *If f satisfies (f0)–(f5), then I has a critical point with critical value $c < \bar{c}$.*

Proof. We consider a sequence $u_n \in \Lambda$ such that $\lim_{n \rightarrow \infty} I(u_n) = c$. Let $g_n = g_{u_n}$, as defined in the proof of Lemma 2.3. We use Ekeland variational principle and find sequences $t_n \in [0, 1]$ and $w_n \in H^\alpha$ such that

$$\lim_{n \rightarrow \infty} I(w_n) = c, \quad \lim_{n \rightarrow \infty} I'(w_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|w_n - g_n(t_n)\|_\alpha = 0.$$

Proceeding as in the proof of Theorem 2.1, we find a subsequence of w_n , we keep calling w_n , that converges weakly to w and satisfies

$$\int_{B_R(y_n)} w_n(x)^2 dx > \beta, \quad \forall n \in \mathbb{N}$$

for some $R, \beta > 0$ and $y_n \in \mathbb{R}^N$. If (y_n) has a bounded subsequence, the latter implies $w \neq 0$ and the result follows. Let us assume then, that $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$. And also that, for given $r > 0$,

$$\lim_{n \rightarrow \infty} \int_{B_r(0)} w_n(x)^2 dx = 0,$$

since the contrary implies that $w \neq 0$. We first obtain that $c < \bar{c}$. Let \bar{w} be a nontrivial critical point of \bar{I} given by Theorem 2.1. Let

$$A = \{x \in \mathbb{R}^N : f(x, s) > \bar{f}(s) \ \forall s > 0\}.$$

By (f5) and the fact that \bar{w} is non-zero, we may find $y \in \mathbb{R}^N$ such that the function w_y , defined as $w_y(x) = w(x + y)$ satisfies

$$|\{x \in A : |w_y(x)| > 0\}| > 0.$$

And so

$$\bar{c} = \bar{I}(w_y) \geq \bar{I}(tw_y) > I(tw_y)$$

for all $t > 0$. Choosing $t^* > 0$ such that $I(t^*w_y) = \sup_{t>0} I(tw_y)$, we find $t^*w_y \in \Lambda$ and conclude that

$$\bar{c} > I(t^*w_y) \geq \inf_{v \in \Lambda} I(v) = c.$$

From (f5), we have, for all $t > 0$,

$$\begin{aligned} I(tu_n) &= \bar{I}(tu_n) + \int_{\mathbb{R}^N} (\bar{F}(tu_n) - F(x, tu_n)) dx \\ &\geq \bar{I}(tu_n) - \int_{\mathbb{R}^N} Ca(x) (|tu_n|^2 + |tu_n|^{p+1}) dx. \end{aligned}$$

Let $\varepsilon > 0$. Using (f5) and since (u_n) is a bounded sequence in H^α , there exists $R > 0$ such that

$$\int_{B_R^c} Ca(x) (|tu_n|^2 + |tu_n|^{p+1}) dx \leq \varepsilon,$$

for t bounded. On the other hand, since $\|w_n - g_n(t_n)\|_\alpha \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \int_{B_R} Ca(x) (|tu_n|^2 + |tu_n|^{p+1}) dx = 0.$$

Choosing $t = t^*$ so that $\bar{I}(t^*u_n) = \max_{t \geq 0} \bar{I}(tu_n)$, we see that $c \geq \bar{c} - \varepsilon$. If $\varepsilon > 0$ is chosen small enough, this contradicts the fact that $c < \bar{c}$. \square

3. Some qualitative properties of solutions

In this section we study regularity of solutions to Eq. (1.1), in particular we show that the solution found in Theorem 2.2 is continuous. Moreover, in case the nonlinearity is a power, we get regularity for solutions of Eq. (1.5). Next we study decay properties of positive solutions of (1.1).

We start with some considerations of spaces more general than H^α . Let $\beta \geq 0$ and $q \in [1, \infty]$. We define

$$\mathcal{L}_\beta^q = \{u \in L^q(\mathbb{R}^N) : \mathcal{F}^{-1}[(1 + |\xi|^2)^{\beta/2} \hat{u}] \in L^q(\mathbb{R}^N)\}.$$

The following results will be useful to study the regularity of the solution (see [7, 23]).

Proposition 3.1. *Let $\alpha, \beta \geq 0$, $(I - \Delta)^\alpha$ is an isomorphism of $\mathcal{L}_{\beta+2\alpha}^q$ to \mathcal{L}_β^q . If $\alpha \geq 0$ and $\beta > 0$, $(I - \Delta)^\alpha$ is an isomorphism of $C^{\beta+2\alpha}(\mathbb{R}^N)$ to $C^\beta(\mathbb{R}^N)$, where $C^\gamma(\mathbb{R}^N) = C^{k, \gamma-k}(\mathbb{R}^N)$ and $k = \max\{j \in \mathbb{N} : j \leq \gamma\}$.*

- Theorem 3.1.** (1) *If $1 < p \leq q \leq \frac{Np}{N-\alpha p} < \infty$, then \mathcal{L}_α^p is continuously embedded in L^q .*
 (2) *If $1 \leq q < \frac{N}{N-\alpha}$, then \mathcal{L}_α^1 is continuously embedded in L^q .*
 (3) *If $1 + \frac{N}{p} < \alpha \leq 2$ and $0 < \mu \leq \alpha - \frac{N}{p} - 1$, then \mathcal{L}_α^p is continuously embedded in $C^{1, \mu}$.*
 (4) *If $\frac{N}{p} < \alpha \leq 2$, $\alpha - \frac{N}{p} < 1$ and $0 < \mu \leq \alpha - \frac{N}{p}$, then \mathcal{L}_α^p is continuously embedded in $C^{0, \mu}$.*

Theorem 3.2. *If f satisfies (f0) and (f3) and u is a weak solution of the equation $(I - \Delta)^{\alpha}u = f(x, u)$, then $u \in L^{q_0}(\mathbb{R}^N) \cap C^{0, \mu}(\mathbb{R}^N)$ for some $q_0 \in [2, \infty)$ and $\mu \in (0, 1)$. Moreover, $|u(x)| \rightarrow 0$ as $|x| \rightarrow \infty$.*

Proof. We know that $u \in \mathcal{L}_\alpha^2$ and so $u \in L^{q_0}(\mathbb{R}^N)$, where $q_0 = \frac{2N}{N-2\alpha}$. By (f3), $f(\cdot, u(\cdot)) \in L^{p_1}(\mathbb{R}^N)$ with $p_1 = \frac{q_0}{p}$. Since $(I - \Delta)^{\alpha}u = f(x, u)$, we have $(I - \Delta)^{\alpha}u \in L^{p_1}(\mathbb{R}^N)$ and therefore $u \in \mathcal{L}_{2\alpha}^{p_1}$. We have three cases: (1) $p_1 < \frac{N}{2\alpha}$, (2) $p_1 = \frac{N}{2\alpha}$ and (3) $p_1 > \frac{N}{2\alpha}$.

In case (1) we use the Sobolev embedding to get $u \in L^{q_1}(\mathbb{R}^N)$, where $q_1 = \frac{Np_1}{N-2\alpha p_1}$ and as we did before, $u \in \mathcal{L}_{2\alpha}^{p_2}$, $p_2 = \frac{q_1}{p}$. We can again consider three cases, but now for p_2 . If $p_2 < \frac{N}{2\alpha}$, then $u \in L^{q_2}(\mathbb{R}^N)$, $q_2 = \frac{Np_2}{N-2\alpha p_2}$. Repeating this procedure, we define a sequence (q_j) such that

$$\frac{1}{q_{j+1}} = \sum_{i=0}^j p^i \left(\frac{1}{q_1} - \frac{1}{q_0} \right) + \frac{1}{q_1}.$$

Since $1 < p < \frac{N+2\alpha}{N-2\alpha}$, $q_1 > q_0$ and then the right hand side above becomes negative for j large. Let j be the smallest natural so that the sum is non-positive. Then $p_{j+1} = \frac{N}{2\alpha}$ or $p_{j+1} > \frac{N}{2\alpha}$.

If $p_{j+1} > \frac{N}{2\alpha}$, then $u \in \mathcal{L}_{2\alpha}^{p_{j+1}}$ and by Sobolev embedding we may choose $0 < \mu < \min \left\{ 2\alpha - \frac{N}{p_{j+1}}, 1 \right\}$ so that $u \in C^{0, \mu}(\mathbb{R}^N)$.

If $p_{j+1} = \frac{N}{2\tilde{\alpha}}$, we notice that $u \in \mathcal{L}_{2\tilde{\alpha}}^{p_{j+1}}$ for $\tilde{\alpha} < \alpha$. Then $p_{j+1} < \frac{N}{2\tilde{\alpha}}$ and we make another iteration. If $\tilde{\alpha}$ is close enough to α , we have $p_{j+2} > \frac{N}{2\tilde{\alpha}}$ and we complete the argument.

Finally we observe that the fact that $u \in L^{q_0}(\mathbb{R}^N) \cap C^{0,\mu}(\mathbb{R}^N)$ implies $|u(x)| \rightarrow 0$ as $|x| \rightarrow \infty$. □

Proposition 3.2. *If f satisfies (f0), (f1) and (f3) and u is a weak solution of the equation $(I - \Delta)^\alpha u = f(x, u)$, then u is strictly positive.*

Proof. We know that $u \in \mathcal{L}_{2\alpha}^{p_1}$, $p_1 = \frac{2N}{p(N-2\alpha)}$ and $(I - \Delta)^\alpha u(x) = f(x, u(x))$. So we may write

$$u = (I - \Delta)^{-\alpha} f(\cdot, u(\cdot)) = G_{2\alpha} * f(\cdot, u(\cdot)).$$

where $G_{2\alpha}(x) = \frac{1}{(4\pi)^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\pi|x|^2/t} e^{-t/4\pi} t^{-N/2-1+\alpha} dt$ (see [23, p. 132]). If we take u_- as a test function,

$$\int_{\mathbb{R}^N} u_-(x)^2 dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} G_{2\alpha}(y) f(x - y, u(x - y)) u_-(x) dy dx \leq 0.$$

And therefore, $u_- = 0$ a.e. By Theorem 3.2, u is continuous, so for all $x \in \mathbb{R}^N$

$$u(x) = \int_{\mathbb{R}^N} G_{2\alpha}(y) f(x - y, u(x - y)) dy.$$

Since $G_{2\alpha}$ is strictly positive and $u_+ \neq 0$, we conclude that $u > 0$. □

Remark 3.1. Since $\int G_{2\alpha} = 1$, this kernel defines a probability measure on \mathbb{R}^N and we can use Jensen’s inequality for convex functions. In particular, if $1 \leq q < \infty$ and $v \in L^q$, then for all $x \in \mathbb{R}^N$

$$|G_{2\alpha} * v(x)|^q = \left| \int_{\mathbb{R}^N} G_{2\alpha}(x - y) v(y) dy \right|^q \leq \int_{\mathbb{R}^N} G_{2\alpha}(x - y) |v(y)|^q dy.$$

This allows to show that $\|G_{2\alpha} * v\|_q \leq \|v\|_q$.

Next we prove a stronger regularity result for non-negative solutions in the case of power nonlinearity.

Proposition 3.3. *If $p > 1$, $q > \frac{Np}{2\alpha}$ and $u \in L^q(\mathbb{R}^N)$ is a non-negative solution of (1.5), then $u \in L^r(\mathbb{R}^N)$ for all $r \geq 1$ and there exists $\gamma \in (0, 2\alpha)$ so that $u \in C^{1+\gamma+2\alpha}(\mathbb{R}^N)$.*

Proof. We know that $u \in L^q(\mathbb{R}^N)$ implies $u \in \mathcal{L}_{2\alpha}^{q/p}$ and since $\frac{q}{p} > \frac{N}{2\alpha}$, by Sobolev embedding, we get $u \in C^\mu(\mathbb{R}^N)$ for some $\mu \in (0, 2\alpha - \frac{Np}{q})$. This implies $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$, so $u \in L^r(\mathbb{R}^N)$ for all $r \geq q$. We observe now that $u = G_{2\alpha} * u^p$ and $u^p \in L^r$ for all $r \geq \max\{1, \frac{q}{p}\}$, so $u \in L^r$. Iterating this procedure we conclude that $u \in L^r$ for all $r \geq 1$. Applying the mean value theorem to the function t^p , we get $u^p \in C^\mu(\mathbb{R}^N)$ and then $u \in C^{\mu+2\alpha}(\mathbb{R}^N)$. Again we iterate this procedure until $\mu + 2k\alpha > 1$, so $u \in C^{1+\gamma}(\mathbb{R}^N)$ where $\gamma \in (0, 2\alpha)$. If $p \geq 2$, we may apply the mean value theorem to the derivatives of u , and if $p < 2$, we observe that the function t^{p-1} is of class $C^{p-1}(\mathbb{R}^N)$. In both cases we get $u^p \in C^{1+\gamma}(\mathbb{R}^N)$ and therefore $u \in C^{1+\gamma+2\alpha}(\mathbb{R}^N)$. □

In what follows we study the asymptotic decay of solutions for Eq. (1.1). For that purpose we consider $q \geq 1$ and we define the following complete metric space

$$X = \left\{ v \in C(\mathbb{R}^N) : v \geq 0 \wedge \|v\|_X := \sup_{x \in \mathbb{R}^N} |e^{\frac{|x|}{2q}} v(x)| < \infty \right\}$$

endowed with the distance given by the norm $\|\cdot\|_X$ and consider the operator

$$\begin{aligned} F_\varphi : X &\longrightarrow X \\ v &\longmapsto G_{2\alpha} * \varphi(v) \end{aligned}$$

where φ is an appropriate function. □

Lemma 3.1. *Let $w \in L^q(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be a positive function that vanishes at infinity, $h(x) = \min \left\{ w(x), 1, \frac{1}{2} \|G_{2\alpha} * (e^{-\frac{|\cdot|}{2q}})\|_X^{-1} \right\}$ and $\varphi(v) = hv + \chi_{B_r}$ where χ_{B_r} is the characteristic function of the ball of radius $r > 0$ and r is large enough so that $h = w$ in B_r^c . Then the operator F_φ is well defined and has a unique fixed point \bar{v} that is strictly positive and satisfies the equation*

$$(I - \Delta)^\alpha \bar{v} = h\bar{v} + \chi_{B_r}.$$

Proof. The operator is well defined because $F_\varphi(v)$ is positive and continuous. And

$$\begin{aligned} (e^{\frac{|x|}{2q}} F_\varphi(v)(x))^q &= \left(e^{\frac{|x|}{2q}} \int_{\mathbb{R}^N} G_{2\alpha}(x-y)[h(y)v(y) + \chi_{B_r}(y)] dy \right)^q \\ &\leq e^{\frac{|x|}{2}} \int_{\mathbb{R}^N} G_{2\alpha}(x-y)[h(y)v(y) + \chi_{B_r}(y)]^q dy \\ &\leq ce^{\frac{|x|}{2}} \int_{\mathbb{R}^N} G_{2\alpha}(x-y)h(y)^q v(y)^q dy \\ &\quad + ce^{\frac{|x|}{2}} \int_{B_r} G_{2\alpha}(x-y) dy. \end{aligned}$$

We take now $R > 0$ such that $G_{2\alpha}(x) \leq ce^{-\frac{|x|}{2}}$ if $|x| \geq R$. Then

$$\begin{aligned} &e^{\frac{|x|}{2}} \int_{|x-y| \geq R} G_{2\alpha}(x-y)h(y)^q v(y)^q dy \\ &\leq ce^{\frac{|x|}{2}} \int_{|x-y| \geq R} e^{-\frac{|x-y|}{2}} h(y)^q v(y)^q dy \\ &\leq ce^{\frac{|x|}{2}} \int_{|x-y| \geq R} e^{-\frac{|x|}{2}} e^{\frac{|y|}{2}} h(y)^q e^{-\frac{|y|}{2}} (e^{\frac{|y|}{2q}} v(y))^q dy \\ &\leq c \|v\|_X^q \|h\|_q^q. \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & e^{\frac{|x|}{2}} \int_{|x-y|<R} G_{2\alpha}(x-y)h(y)^q v(y)^q dy \\
 & \leq c \|v\|_X^q e^{\frac{|x|}{2}} \int_{|x-y|<R} G_{2\alpha}(x-y)h(y)^q e^{-\frac{|y|}{2}} dy \\
 & \leq c \|v\|_X^q e^{\frac{|x|}{2}} \int_{|x-y|<R} G_{2\alpha}(x-y)h(y)^q e^{-\frac{|x|}{2}} dy \\
 & \leq c \|v\|_X^q \|h\|_\infty^q.
 \end{aligned}$$

We notice now that

$$e^{\frac{|x|}{2}} \int_{B_r} G_{2\alpha}(x-y) dy \leq \begin{cases} e^{\frac{R+r}{2}} & \text{if } |x| < R+r \\ \int_{B_r} e^{-\frac{|y|}{2}} dy & \text{if } |x| \geq R+r. \end{cases}$$

We conclude that $\|F_\varphi(v)\|_X^q < \infty$, that is $\|F_\varphi(v)\|_X < \infty$ and therefore F_φ is well defined. Finally we prove that F_φ is a contraction,

$$\begin{aligned}
 & e^{\frac{|x|}{2q}} |F_\varphi(v_1)(x) - F_\varphi(v_2)(x)| \\
 & \leq e^{\frac{|x|}{2q}} \int_{\mathbb{R}^N} G_{2\alpha}(x-y)h(y)|v_1(y) - v_2(y)| dy \\
 & \leq \|h\|_\infty \|v_1 - v_2\|_X e^{\frac{|x|}{2q}} \int_{\mathbb{R}^N} G_{2\alpha}(x-y)e^{-\frac{|y|}{2q}} dy \\
 & \leq \|h\|_\infty \|v_1 - v_2\|_X \|G_{2\alpha} * (e^{-\frac{|\cdot|}{2q}})\|_X \leq \frac{1}{2} \|v_1 - v_2\|_X.
 \end{aligned}$$

This implies the existence of a unique fixed point \bar{v} that satisfies

$$\bar{v} = G_{2\alpha} * (h\bar{v} + \chi_{B_r}) > 0.$$

□

Theorem 3.3. *Let $p > 1$, $q = \max\{1, (p - 1)^{-1}\}$ and $u \in L^1(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ a positive solution of $(I - \Delta)^\alpha u = f(x, u)$ that vanishes at infinity. If f satisfies (f_0) and (f_3) , then there exist a constant $c > 0$ such that*

$$u(x) \leq ce^{-\frac{|x|}{2q}} \quad \forall x \in \mathbb{R}^N.$$

Proof. We take φ like in the lemma with $w(x) = \frac{f(x, u(x))}{u(x)}$ and let v be the fixed point of F_φ . We have

$$(I - \Delta)^\alpha v = hv + \chi_{B_r} \quad \text{and} \quad (I - \Delta)^\alpha u = f(x, u).$$

We define the functions $W(x) = u(x) - cv(x)$ and $g(x) = (I - \Delta)^\alpha W(x) - h(x)W(x)$ where $c > 0$ is large enough so that $W \leq 0$ in B_r and $g \leq 0$ in \mathbb{R}^N . Since W is continuous and vanishes at infinity, if we assume that $W \not\leq 0$ in B_r^c , this implies the existence of a global positive maximum $\bar{x} \in B_r^c$. We notice that

$$\begin{aligned} W(\bar{x}) &= G_{2\alpha} * (g + hW)(\bar{x}) \leq G_{2\alpha} * (hW)(\bar{x}) \\ &= \int_{\mathbb{R}^N} G_{2\alpha}(\bar{x} - y)h(y)W(y) dy. \end{aligned}$$

Observing that $\int G_{2\alpha} = 1$,

$$\int_{\mathbb{R}^N} G_{2\alpha}(\bar{x} - y)(W(\bar{x}) - h(y)W(y)) dy \leq 0.$$

Since $h \leq 1$, this contradicts the fact that \bar{x} is a global maximum, in fact $W(\bar{x}) - h(y)W(y)$ is non-negative and non-trivial. Therefore $W \leq 0$ in \mathbb{R}^N and the result follows. \square

Proof of decay estimate in Theorem 1.1. Once we apply Theorem 3.3, we find that u has exponential decay, so that $w = \frac{f(x, u(x))}{u(x)} \in L^1$. Thus Lemma 3.1 and Theorem 3.3 can be proved with $q = 1$, regardless the value of $p > 1$. \square

Ma and Chen [17] proved that if u is a positive solution of $u = G_{2\alpha} * u^p$ and $u \in L^q(\mathbb{R}^N)$ for $q > \max\left\{p, \frac{N(p-1)}{2\alpha}\right\}$, then u is radially symmetric and decreasing about some point.

4. Existence of infinitely many radially symmetric solutions

Now we establish the existence of infinitely many solutions of the equation $(I - \Delta)^\alpha u = |u|^{p-1}u$. We follow the same argument as in [24]. Let \mathcal{M} be a manifold on a Hilbert space H and J a C^1 functional defined on H . We say that $J|_{\mathcal{M}}$ satisfies the positive Palais–Smale condition $[(PS)^+]$ if for $0 < c_1 < c_2$ and every sequence $\{w_n\} \subseteq \mathcal{M}$ such that $c_1 \leq J(w_n) \leq c_2$ and $\|J'|_{\mathcal{M}}(w_n)\| \rightarrow 0$, there exists a convergent subsequence of $\{w_n\}$.

We consider

$$\mathcal{M} = \{v \in H_r^\alpha : \|v\|_\alpha = 1\}$$

where

$$H_r^\alpha = \{v \in H^\alpha : v \text{ is radially symmetric}\}$$

and

$$J(v) = \frac{1}{p+1} \int_{\mathbb{R}^N} |v(x)|^{p+1} dx = \frac{1}{p+1} \|v\|_{p+1}^{p+1}.$$

Lemma 4.1. *If $N \geq 2$, $J|_{\mathcal{M}}$ satisfies the $(PS)^+$ condition.*

Proof. Let $\{u_n\} \subseteq \mathcal{M}$ be a sequence such that $0 < c_1 \leq J(u_n) \leq c_2$ and $\|J'|_{\mathcal{M}}(u_n)\| \rightarrow 0$. Since $\{u_n\}$ is bounded in H^α , we may find a subsequence that converges in $L_r^{p+1}(\mathbb{R}^N)$ (see [22, p. 656]) and weakly in $H^\alpha(\mathbb{R}^n)$ to some function u . We keep calling this subsequence $\{u_n\}$ and then,

$$J(u_n) \rightarrow J(u) \quad \text{and} \quad J'(u_n)v - J'(u_n)u_n \langle u_n, v \rangle \rightarrow 0 \quad \forall v \in H^\alpha(\mathbb{R}^n).$$

Taking $v = u$ we get $\|u\|_{p+1}^{p+1} = \|u\|_{p+1}^{p+1} \|u\|_\alpha^2$. This implies $\|u\|_\alpha = 1$ because $J(u_n) \geq c_1$. Therefore $\|u_n\|_\alpha \rightarrow \|u\|_\alpha$ and we conclude that $\{u_n\}$ converges strongly to u . \square

Let $\Sigma(\mathcal{M})$ denote the set of compact and symmetric subsets of \mathcal{M} . The genus $\gamma(A)$ of a set $A \in \Sigma(\mathcal{M})$ is defined as the least integer $n \geq 1$ such that there exists an odd continuous mapping $\phi: A \rightarrow S^{n-1}$. We set $\gamma(A) = \infty$ if such an integer does not exist. For $k \geq 1$, we define $\Gamma_k = \{A \in \Sigma(\mathcal{M}): \gamma(A) \geq k\}$. We will need the following results (see [1]).

Proposition 4.1. *Let $J: H \rightarrow \mathbb{R}$ be an even functional of class C^1 . Suppose that J is bounded from above on \mathcal{M} and $J|_{\mathcal{M}}$ satisfies the $(PS)^+$ condition. Let*

$$b_k = \sup_{A \in \Gamma_k} \inf_{v \in A} J(v).$$

Then $b_1 \geq b_2 \geq \dots \geq b_k \geq \dots$ and b_k is a critical value of J if $b_k > 0$.

Proposition 4.2. *Let $K_b = \{v \in \mathcal{M}: J(v) = b, J'|_{\mathcal{M}}(v) = 0\}$. Under the hypotheses of Proposition 4.1, suppose that $b_k = b_{k+1} = \dots = b_{k+r-1} = b$. Then $\gamma(K_b) \geq r$. In particular, if $r \geq 2$ there exists infinitely many critical points of $J|_{\mathcal{M}}$ with critical value b .*

Remark 4.1. By Propositions 4.1 and 4.2, under the conditions of Proposition 4.1, there always exists infinitely many critical points of $J|_{\mathcal{M}}$.

For $k \geq 1$ we consider

$$\pi_{k-1} = \left\{ l = (l_1, \dots, l_k) \in \mathbb{R}^k : \sum_{i=1}^k |l_i| = 1 \right\}.$$

This set satisfies $\gamma(\pi_{k-1}) = k$ because it is homeomorphic to S^{k-1} .

Lemma 4.2. *Let $q \in (2, \frac{2N}{N-2})$. For all $k \geq 1$, there exists a constant $R = R(k) > 0$ and an odd continuous mapping $\tau: \pi_{k-1} \rightarrow H_0^1(B_R)$ such that*

- (i) $\tau(l)$ is a radial function for all $l \in \pi_{k-1}$ and $0 \notin \tau(\pi_{k-1})$.
- (ii) For $v \in \tau(\pi_{k-1})$, $\|v\|_q \geq 1$.

Lemma 4.3. *For all $k \geq 1$, $b_k > 0$.*

Proof. We use Lemma 4.2 with $q = p + 1 < \frac{2N}{N-2\alpha} < \frac{2N}{N-2}$ and the embeddings $H_0^1(B_R) \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow H^\alpha(\mathbb{R}^N)$ to find a mapping $\varphi: \pi_{k-1} \rightarrow H^\alpha(\mathbb{R}^N)$ that satisfies conditions (i) and (ii) of Lemma 4.2. Thus we may define an odd continuous mapping $\psi: \pi_{k-1} \rightarrow \mathcal{M}$ as

$$\psi(l) = \frac{\varphi(l)}{\|\varphi(l)\|_\alpha}.$$

Let $A_k = \psi(\pi_{k-1})$. We see that $A_k \in \Gamma_k$. Since $\varphi(\pi_{k-1})$ is compact, there is a constant $M > 0$ such that $\|\varphi(l)\|_\alpha \leq M$ for all $l \in \pi_{k-1}$ and then,

$$\inf_{v \in A_k} J(v) = \inf_{l \in \pi_{k-1}} \frac{1}{p+1} \int_{\mathbb{R}^N} \frac{|\varphi(l)(x)|^{p+1}}{\|\varphi(l)\|_\alpha^{p+1}} dx \geq \frac{1}{M^{p+1}(p+1)} > 0.$$

Therefore $b_k > 0$. □

Proof of Theorem 1.2. We use Sobolev embedding to see that J is bounded on \mathcal{M} . By Lemma 4.1, $J|_{\mathcal{M}}$ satisfies the $(PS)^+$ condition and by Lemma 4.3 and Propositions 4.1 and 4.2, it has infinitely many critical points. The result follows from noticing that such a critical point satisfies

$$\int_{\mathbb{R}^N} (1 + |\xi|^2)^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} \, d\xi = \lambda \int_{\mathbb{R}^N} |u(x)|^{p-1} u(x) v(x) \, dx \quad \forall v \in H_r^\alpha(\mathbb{R}^N)$$

for some $\lambda \in \mathbb{R}$. But this equality also holds for all $v \in H^\alpha(\mathbb{R}^N)$, in fact we have

$$\int_{\mathbb{R}^N} |u(x)|^{p-1} u(x) \varphi(x) \, dx = \langle G_{2\alpha} * (|u|^{p-1} u), \varphi \rangle_\alpha = 0$$

for all $\varphi \in H^\alpha(\mathbb{R}^N)$ which is H^α -orthogonal to $H_r^\alpha(\mathbb{R}^N)$. Finally, taking $u = v$ we get $\lambda > 0$ and we find the weak solution by rescaling. \square

Remark 4.2. We observe that Theorem 3.2 holds for $f(x, s) = |s|^{p-1} s$. Therefore the equation has infinitely many solutions in $L^2(\mathbb{R}^N) \cap C^\mu(\mathbb{R}^N)$.

5. Pohozaev identity and non-existence result

This section is devoted to prove Theorem 1.3 on the non-existence of solutions to (1.6) with supercritical nonlinearity. We start with the proof of a differential identity involving the elliptic operator in our equation. We denote by $\mathcal{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decaying functions.

Proposition 5.1. *For every function $\varphi \in \mathcal{S}(\mathbb{R}^N)$ we have*

$$(I - \Delta)^\alpha (x \cdot \nabla \varphi) = x \cdot \nabla [(I - \Delta)^\alpha \varphi] + 2\alpha(I - \Delta)^\alpha \varphi - 2\alpha(I - \Delta)^{\alpha-1} \varphi.$$

Proof. We start with the basic properties of the Fourier transform, for any $\varphi \in \mathcal{S}(\mathbb{R}^N)$,

$$\mathcal{F}\left(\frac{\partial \varphi}{\partial x_k}\right) = i \xi_k \hat{\varphi} \quad \text{and} \quad \mathcal{F}(x_k \varphi) = i \frac{\partial}{\partial \xi_k} \hat{\varphi},$$

from where we see that

$$\begin{aligned} \mathcal{F}(x \cdot \nabla \varphi)(\xi) &= \sum_{k=1}^N \mathcal{F}\left(\frac{\partial}{\partial x_k}(x_k \varphi) - \varphi\right) \\ &= i \sum_{k=1}^N \xi_k \mathcal{F}(x_k \varphi) - N \hat{\varphi} \\ &= -(\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi}). \end{aligned} \tag{5.1}$$

Therefore,

$$\mathcal{F}((I - \Delta)^\alpha (x \cdot \nabla \varphi))(\xi) = -(1 + |\xi|^2)^\alpha (\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi}). \tag{5.2}$$

Using (5.2) and differentiating, we see that

$$\begin{aligned}
 & \mathcal{F}(x \cdot \nabla[(I - \Delta)^\alpha \varphi])(\xi) \\
 &= -(\xi \cdot \nabla \mathcal{F}((I - \Delta)^\alpha \varphi) + N \mathcal{F}((I - \Delta)^\alpha \varphi)) \\
 &= -(\xi \cdot \nabla[(1 + |\xi|^2)^\alpha \hat{\varphi}] + N(1 + |\xi|^2)^\alpha \hat{\varphi}) \\
 &= -(\xi \cdot [\alpha(1 + |\xi|^2)^{\alpha-1} 2\hat{\varphi}\xi + (1 + |\xi|^2)^\alpha \nabla \hat{\varphi}] + N(1 + |\xi|^2)^\alpha \hat{\varphi}) \\
 &= -2\alpha(1 + |\xi|^2)^\alpha \hat{\varphi} + 2\alpha(1 + |\xi|^2)^{\alpha-1} \hat{\varphi} - (1 + |\xi|^2)^\alpha (\xi \cdot \nabla \hat{\varphi} + N \hat{\varphi}). \tag{5.3}
 \end{aligned}$$

Then, from (5.2) and (5.3), we obtain

$$\begin{aligned}
 & \mathcal{F}((I - \Delta)^\alpha (x \cdot \nabla \varphi) - x \cdot \nabla[(I - \Delta)^\alpha \varphi])(\xi) \\
 &= 2\alpha(1 + |\xi|^2)^\alpha \hat{\varphi} + 2\alpha(1 + |\xi|^2)^{\alpha-1} \hat{\varphi} \\
 &= 2\alpha \mathcal{F}((I - \Delta)^\alpha \varphi) - 2\alpha \mathcal{F}((I - \Delta)^{\alpha-1} \varphi),
 \end{aligned}$$

from where the result follows.

Now we are in a position of proving Theorem 1.3.

Proof of Theorem 1.3. Let us assume the contrary, that is, there is a solution u to Eq. (1.6) such that $u \in L^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, with $q \geq 1$. Then we may use Proposition 3.3 to see that $u \in L^1(\mathbb{R}^N) \cap C^{1+\gamma+2\alpha}(\mathbb{R}^N)$ and then, by Theorem 3.3, u has exponential decay. If $\rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho(\frac{x}{\varepsilon})$ is a mollifier, with support of ρ in the ball $B_1(0)$, then $u_\varepsilon = \rho_\varepsilon * u \in \mathcal{S}(\mathbb{R}^N)$. Then, since u satisfies

$$(I - \Delta)^\alpha u = u^p$$

in the sense of tempered distributions, we have

$$(I - \Delta)^\alpha u_\varepsilon = (I - \Delta)^\alpha \rho_\varepsilon * u = \rho_\varepsilon * (I - \Delta)^\alpha u = \rho_\varepsilon * u^p. \tag{5.4}$$

and, since $x \cdot \nabla u_\varepsilon \in \mathcal{S}(\mathbb{R}^N)$, we also have

$$\int_{\mathbb{R}^N} (I - \Delta)^\alpha u(x \cdot \nabla u_\varepsilon) = \int_{\mathbb{R}^N} u^p(x \cdot \nabla u_\varepsilon). \tag{5.5}$$

We consider the left hand side first. Using Proposition 5.1 and (5.4), we have

$$\begin{aligned}
 \int_{\mathbb{R}^N} (I - \Delta)^\alpha u(x \cdot \nabla u_\varepsilon) &= \int_{\mathbb{R}^N} u (I - \Delta)^\alpha (x \cdot \nabla u_\varepsilon) \\
 &= \int_{\mathbb{R}^N} u x \cdot \nabla(\rho_\varepsilon * u^p) + 2\alpha \int_{\mathbb{R}^N} u (\rho_\varepsilon * u^p) \\
 &\quad - 2\alpha \int_{\mathbb{R}^N} u (I - \Delta)^{\alpha-1} u_\varepsilon. \tag{5.6}
 \end{aligned}$$

For the first integral in the third term, using that u is exponentially decaying and differentiable with bounded derivative, we have

$$\begin{aligned}
 \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u x \cdot \nabla(\rho_\varepsilon * u^p) &= \int_{\mathbb{R}^N} u x \cdot \nabla(u^p) \\
 &= \int_{\mathbb{R}^N} x \cdot (\nabla u^{p+1} - u \nabla u^p) \\
 &= \left(-N + \frac{N}{p+1}\right) \int_{\mathbb{R}^N} u^{p+1}. \tag{5.7}
 \end{aligned}$$

For the second integral in the third term in (5.6) we have

$$\lim_{\varepsilon \rightarrow 0} 2\alpha \int_{\mathbb{R}^N} u(\rho_\varepsilon * u^p) = 2\alpha \int_{\mathbb{R}^N} u^{p+1}. \tag{5.8}$$

For the third integral in the third term in (5.6), we use the fact that $u \in L^2(\mathbb{R}^N)$ and that $u_\varepsilon \rightarrow u$ in $L^2(\mathbb{R}^N)$ and that $\alpha \in (0, 1)$, to see that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u(I - \Delta)^{\alpha-1} u_\varepsilon &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \hat{u}(1 + |\xi|^2)^{\alpha-1} \hat{u}_\varepsilon \\ &= \int_{\mathbb{R}^N} \hat{u}^2(1 + |\xi|^2)^{\alpha-1} > 0. \end{aligned} \tag{5.9}$$

On the other hand, for the right hand side of (5.5), using that u is exponentially decaying and differentiable with bounded derivative, we find that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} u^p(x \cdot \nabla u_\varepsilon) &= \int_{\mathbb{R}^N} u^p(x \cdot \nabla u) \\ &= \frac{1}{p+1} \int_{\mathbb{R}^N} x \cdot \nabla u^{p+1} \\ &= -\frac{N}{p+1} \int_{\mathbb{R}^N} u^{p+1}. \end{aligned} \tag{5.10}$$

From Eq. (5.5), using (5.6)–(5.9) for the left hand side and using (5.10) for the right hand side, we finally obtain

$$\left(2\alpha - N + \frac{2N}{p+1}\right) \int_{\mathbb{R}^N} u^{p+1} = 2\alpha \int_{\mathbb{R}^N} (1 + |\xi|^2)^{\alpha-1} \hat{u}(\xi)^2 > 0.$$

But this implies that $2\alpha - N + \frac{2N}{p+1} > 0$ and then $p < \frac{N+2\alpha}{N-2\alpha}$, contradicting the hypothesis. □

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